P 5 Boundary control problems in polyhedral domains (Th. Apel, O. Steinbach, B. Vexler) → AO, NS, IS

The project deals with finite element error estimates and mesh refinement strategies for elliptic boundary control problems (AO) with special focus on the peculiarities coming from the consideration of three-dimensional non-smooth domains (NS), possibly with interfaces separating different materials (IS).

State of the art. The model problems for the project are

$$\text{minimize } \tfrac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \tfrac{\alpha}{2}\|u\|^2_U$$

$$\begin{array}{rccl} \text{subject to} & -\Delta y + y = f \text{ in } \Omega, & \partial_n y = u \text{ on } \partial\Omega & \text{(Neumann control)}, \\ \text{or subject to} & -\Delta y = f \text{ in } \Omega, & y = u \text{ on } \partial\Omega & \text{(Dirichlet control)}, \end{array}$$

where Ω is a non-smooth domain. Box constraints on the control are admissible.

In recent years various discretization and regularization strategies have been described and analyzed for such problems in the literature, see, e.g., [8, 11, 12, 7] for Neumann control problems and [10, 9, 13, 14] for Dirichlet control problems. While these authors either considered sufficiently regular solutions or proved reduced convergence orders for the case of singular solutions, optimal convergence for methods using locally graded meshes were derived in the predecessor project for Neumann control problems in non-smooth domains [5, 15, 16]. A similar analysis for Dirichlet control problems started recently [3]. Let us consider the results for non-smooth domains and adapted triangulations in more detail.

(1) For Neumann control problems with $U = L^2(\Gamma)$, discretization error estimates in the L^2-norm have been derived for variational discretization and for the postprocessing approach in the 2D [5, 15] and in the 3D [16] case where isotropically graded meshes were investigated. L^∞-error estimates which are also of major interest, have been derived in [6] for a distributed control problem for a Dirichlet problem, but not for boundary control.

(2) Neumann control problems were also investigated with regularization in energy space [7, 16] in 2D. Interesting features are that the control tends to infinity near non-smooth parts of the boundary in the unconstrained case such that the control becomes more regular in the constrained case. In the latter case the largest convex interior angle of the domain determines the regularity of the solution. Quasi-uniform meshes yield full convergence order if the maximal interior angle is less than $2\pi/3$; mesh grading is suited to obtain optimal convergence. In the constrained case the theory is not yet complete.

(3) Dirichlet control problems with $U = L^2(\Gamma)$ are analyzed currently in joint work of Th. Apel and J. Pfefferer (postdoc in the IGDK) with M. Mateos, S. Nicaise and A. Rösch [3, 4].

Thesis project to be supervised by Thomas Apel. (1) For Neumann control problems with $U = L^2(\Gamma)$, the task is to prove error estimates in the L^∞-norm in 2D on the basis of [6, 15]. The 3D case is rated too challenging for a PhD student; L^∞-error estimates on graded meshes are not yet proven for the boundary value problem. (2) For Neumann control problems with $U = H^{-1/2}(\Gamma)$, the task is to complete the analysis of the 2D case and to extend it to the 3D case. (3) A third task is the extension of the 2D analysis of Dirichlet control problems with $U = L^2(\Gamma)$ to the 3D case. (4) Numerical tests have shown that mesh grading not only to the singular parts of the boundary but to the whole boundary leads to a convergence rate in the control of almost 2, compared to 1 for general meshes and $\frac{3}{2}$ for superconvergence meshes without this kind of refinement. The task of the student is to analyze this phenomenon.

Further topics. It has been known for long time, [1], that anisotropic mesh grading near edges is sufficient for optimal convergence of Dirichlet boundary value problems if the error is measured in energy or $L^2(\Omega)$-norm, see [2] for a recent contribution. The analysis of such a strategy for the discretization of boundary control problems is a hard open problem since error estimates in $L^\infty(\Omega)$ and $L^2(\Gamma)$ are not yet proven for the boundary value problem.

Parabolic boundary control problems and sparse control problems in non-convex domains can be considered in cooperation with the group of B. Vexler.

State constraints were not mentioned in the description of the project but lead to further challenges of the numerical analysis.

Other state equations are of interest. In particular it is expected that the cooperation with the project on *Volume-surface reaction-diffusion systems: analysis, numerics, control, and optimality issues* (with K. Fellner and B. Vexler) leads to new questions for the numerical analysis.

Elliptic equations with discontinuous coefficients have singular parts in their solution with similar structure as near corners and edges. However the leading singularity can become more severe such that some steps in the numerical analysis cannot be directly transferred to this case.

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