Analysis of shape optimization problems for unsteady fluid-structure interaction

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Abstract

Shape optimization via the method of mappings is investigated for unsteady fluid-structure interaction (FSI) problems that couple the Navier-Stokes equations and the Lamé system. Building on recent existence and regularity theory we prove Fréchet differentiability results for the state with respect to domain variations. These results form an analytical foundation for optimization und inverse problems governed by FSI systems. Our analysis develops a general framework for deriving local-in-time continuity and differentiability results for parameter dependent nonlinear systems of partial differential equations. The main part of the paper is devoted to conducting this analysis for the FSI problem, transformed to a shape reference domain. The underlying shape transformation — actually we work with the corresponding shape displacement instead — represents the shape and the main result proves the Fréchet differentiability of the solution of the FSI system with respect to the shape transformation.

Keywords: Fluid-structure interaction, shape optimization, shape identification, Fréchet differentiability, method of mappings, method of successive approximations, Navier-Stokes equations, Lamé system, local-in-time analysis.

1 Introduction

Shape optimization for fluid-structure interaction (FSI) has many important applications in engineering and other fields. So far, most of the research devoted to this challenging class of optimization problems mainly targeted at numerical approaches, e.g., for biomedical applications [39], naval architecture [38] or wind engineering [28, 46], using direct differentiation [40] or adjoint based gradient computation [26], while a rigorous supporting theory is scarce. In this paper, we build on recent work by Raymond and Vanninathan [44] on the existence and regularity of solutions to an unsteady FSI problem. We extend these results and prove continuity and Fréchet differentiability of the solution of an unsteady Navier-Stokes-Lamé-system with respect to domain variations. Existence and regularity theory for FSI is challenging due to the hyperbolic nature of the elasticity equation, which leads to a lack of regularity that needs to be compensated by hidden regularity results. To the authors' knowledge, analytical results for unsteady FSI models that consider elastic structures in fluids are so far restricted to cases with stationary interfaces [16, 3], a priori known time-dependency of the domain [9], very smooth data [13, 14] or geometrical constraints on the interface [34, 29, 44], whereas differentiability results are only available for steady FSI models [42, 53].

The fluid-structure interaction model that is considered in this paper couples the transient Navier-Stokes equations with the Lamé system and is formulated in a fully Lagrangian framework: Denote by $\check{\Omega}_f(\mathsf{t})$ and $\check{\Omega}_s(\mathsf{t})$, respectively, the domains occupied by the fluid and the solid, respectively, at time t. Further, let $\check{\Omega}(\mathsf{t})$ denote the interior of $\overline{\check{\Omega}}_f(\mathsf{t}) \cup \overline{\check{\Omega}}_s(\mathsf{t})$ and $\check{\Gamma}_i(\mathsf{t}) := \partial\check{\Omega}_f(\mathsf{t}) \cap \partial\check{\Omega}_s(\mathsf{t})$ the fluid-solid interface. In the considered setting, $\check{\Omega}(\mathsf{t}) = \hat{\Omega}$ is time-independent, while $\check{\Omega}_f(\mathsf{t})$, $\check{\Omega}_s(\mathsf{t})$, and $\check{\Gamma}_i(\mathsf{t})$ change with time. The Lagrangian framework uses

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the displacement field induced by the velocity field to obtain a transformation between the reference domain $\hat{\Omega}_f$ and the physical domain $\check{\Omega}_f(\mathsf{t})$. For the solid, the Lagrangian formulation provides the standard framework and the displacement field induces a transformation between the reference domain $\hat{\Omega}_s$ and $\check{\Omega}_s(\mathsf{t})$. Let $T_f > 0$, $\hat{\Gamma}_i := \partial\hat{\Omega}_f \cap \partial\hat{\Omega}_s$, $\hat{\Gamma}_f \subset \partial\hat{\Omega}_f \backslash \hat{\Gamma}_i$, $\hat{\Gamma}_s \subset \partial\hat{\Omega}_s \backslash \hat{\Gamma}_i$ and the space-time cylinders be denoted by $\hat{Q}^T := \hat{\Omega} \times (0,T)$, $\hat{Q}_f^T$, $\hat{Q}_s^T$, $\hat{\Sigma}_i^T := \hat{\Gamma}_i \times (0,T)$, $\hat{\Sigma}_f^T$ and $\hat{\Sigma}_s^T$ for all $0 < T \leqslant T_f$. A coupled Navier-Stokes-Lamé system in Lagrangian coordinates can be written in the form

$$
\begin{aligned}
\partial_{\mathsf{t}} \hat{\mathbf{v}} - \nu\Delta_{\mathsf{y}} \hat{\mathbf{v}} + \nabla_{\mathsf{y}} \hat{p} &= \hat{\mathcal{F}}(\hat{\mathbf{v}}, \hat{p}) && \text{in } \hat{Q}_f^T, \\
\operatorname{div}_{\mathsf{y}} (\hat{\mathbf{v}}) &= \hat{\mathcal{G}}(\hat{\mathbf{v}}) && \text{in } \hat{Q}_f^T, \\
\hat{\mathbf{v}}(\cdot, 0) &= \hat{\mathbf{v}}_0 && \text{in } \hat{\Omega}_f, \\
\hat{\mathbf{v}} &= 0 && \text{on } \hat{\Sigma}_f^T, \\
\hat{\mathbf{v}} &= \partial_{\mathsf{t}} \hat{\mathbf{w}} && \text{on } \hat{\Sigma}_i^T, \\
\sigma_{f,\mathsf{y}}(\hat{\mathbf{v}}, \hat{p})\hat{\mathbf{n}}_f &= \sigma_{s,\mathsf{y}}(\hat{\mathbf{w}})\hat{\mathbf{n}}_f + \hat{\mathcal{H}}(\hat{\mathbf{v}}, \hat{p}) && \text{on } \hat{\Sigma}_i^T, \\
\partial_{\mathsf{tt}} \hat{\mathbf{w}} - \operatorname{div}_{\mathsf{y}} (\sigma_{s,\mathsf{y}}(\hat{\mathbf{w}})) &= 0 && \text{in } \hat{Q}_s^T, \\
\hat{\mathbf{w}} &= 0 && \text{on } \hat{\Sigma}_s^T, \\
\hat{\mathbf{w}}(\cdot, 0) = 0, \quad \partial_{\mathsf{t}} \hat{\mathbf{w}}(\cdot, 0) &= \hat{\mathbf{w}}_1 && \text{in } \hat{\Omega}_s.
\end{aligned} \tag{1}
$$

The fluid and solid stress tensors $\sigma_{f,\mathsf{y}}$ and $\sigma_{s,\mathsf{y}}$ are given by

$$\sigma_{f,\mathsf{y}}(\hat{\mathbf{v}}, \hat{p}) := 2\nu\epsilon_{\mathsf{y}}(\hat{\mathbf{v}}) - \hat{p}\mathbf{I}, \quad \text{and} \quad \sigma_{s,\mathsf{y}}(\hat{\mathbf{w}}) = \lambda \operatorname{tr}(\epsilon_{\mathsf{y}}(\hat{\mathbf{w}}))\mathbf{I} + 2\mu\epsilon_{\mathsf{y}}(\hat{\mathbf{w}}),$$

where $\epsilon_{\mathsf{y}}(\cdot) := \frac{1}{2}(D_{\mathsf{y}} \cdot + (D_{\mathsf{y}} \cdot)^\top)$ and λ, μ are Lamé coefficients with $\mu > 0$ and $\lambda + \mu > 0$. Here, $D_{\mathsf{y}} \cdot$ denotes the Jacobian and $\hat{\mathbf{n}}_f$ is the unit outer normal vector of $\hat{\Omega}_f$. The variables $\hat{\mathbf{v}}$, $\hat{p}$ denote the fluid velocity and pressure, $\hat{\mathbf{w}}$ the solid displacement, and $\hat{\mathbf{v}}_0$ as well as $\hat{\mathbf{w}}_1$ appropriate initial conditions. We define the underlying transformation by

$$\hat{\boldsymbol{\chi}}(\cdot, \mathsf{t})|_{\hat{\Omega}_f} : \quad \hat{\Omega}_f \to \check{\Omega}_f(\mathsf{t}), \quad \mathsf{y} \to \mathsf{y} + \int_0^{\mathsf{t}} \hat{\mathbf{v}}(\mathsf{y}, \mathsf{s}) \, \mathrm{d}\mathsf{s}, \quad \widehat{\mathbf{F}}_{\boldsymbol{\chi}} = D_{\mathsf{y}} \hat{\boldsymbol{\chi}} = (\nabla_{\mathsf{y}} \hat{\boldsymbol{\chi}})^\top$$

for any $\mathsf{t} \in (0, T)$ and its inverse $\check{\boldsymbol{\Upsilon}}(\cdot, \mathsf{t}) := (\hat{\boldsymbol{\chi}}(\cdot, \mathsf{t}))^{-1}$ as well as $\widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}} := \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1}$, which exist if $T > 0$ is sufficiently small and the initial data are smooth enough, cf. [44]. Then the right hand side terms read

$$
\begin{aligned}
&\hat{\mathcal{F}}(\hat{\mathbf{v}}, \hat{p}) = \nu \sum_{j,k=1}^{d} (\partial_{\mathsf{x}_j\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_k \circ \hat{\boldsymbol{\chi}}) \, \partial_{\mathsf{y}_k} \hat{\mathbf{v}} + \nu \sum_{i,j,k=1}^{d} (\partial_{\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_i \circ \hat{\boldsymbol{\chi}}) \, (\partial_{\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_k \circ \hat{\boldsymbol{\chi}}) \, \partial_{\mathsf{y}_i\mathsf{y}_k} \hat{\mathbf{v}} - \nu\Delta_{\mathsf{y}} \hat{\mathbf{v}} + (\mathbf{I} - \widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top) \nabla_{\mathsf{y}} \hat{p}, \\
&\hat{\mathcal{H}}(\hat{\mathbf{v}}, \hat{p}) = -\nu(D_{\mathsf{y}} \hat{\mathbf{v}} \widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}} + \widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top (D_{\mathsf{y}} \hat{\mathbf{v}})^\top) \operatorname{cof}(\widehat{\mathbf{F}}_{\boldsymbol{\chi}}) \, \hat{\mathbf{n}}_f + \hat{p} \operatorname{cof}(\widehat{\mathbf{F}}_{\boldsymbol{\chi}}) \, \hat{\mathbf{n}}_f + \nu(D_{\mathsf{y}} \hat{\mathbf{v}} + (D_{\mathsf{y}} \hat{\mathbf{v}})^\top) \, \hat{\mathbf{n}}_f - \hat{p} \, \hat{\mathbf{n}}_f, \\
&\hat{\mathcal{G}}(\hat{\mathbf{v}}) = \operatorname{div}_{\mathsf{y}} \hat{\mathbf{v}} - \det(\widehat{\mathbf{F}}_{\boldsymbol{\chi}}) \, D_{\mathsf{y}} \hat{\mathbf{v}} : \widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top = D_{\mathsf{y}} \hat{\mathbf{v}} : (\mathbf{I} - \det(\widehat{\mathbf{F}}_{\boldsymbol{\chi}}) \, \widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top),
\end{aligned}
$$

where cof denotes the cofactor matrix. We define $\hat{\mathbf{g}}(\hat{\mathbf{v}}) := (\mathbf{I} - \det(\widehat{\mathbf{F}}_{\boldsymbol{\chi}})\widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}})\hat{\mathbf{v}}$, such that $\operatorname{div}_{\mathsf{y}} (\hat{\mathbf{g}}(\hat{\mathbf{v}})) = \hat{\mathcal{G}}(\hat{\mathbf{v}})$ due to Piola's identity.

Shape optimization problems can be analyzed with different, yet closely related, techniques. On the one hand, shape calculus can be used to investigate functionals $\hat{J}(\hat{\Omega})$ that depend on the domain $\hat{\Omega}$. The Eulerian derivative $d\hat{J}(\hat{\Omega}, \hat{V})$ can be represented by the Hadamard-Zolésio shape gradient, which is the representation of the shape gradient as a distribution that is supported on the design boundary and only acts on the normal boundary variation $\hat{V} \cdot \hat{\mathbf{n}}_f$ [15, 43, 49]. If a state equation is involved, then the Eulerian derivative depends on the shape derivative of the state and can also be expressed using an adjoint state. An alternative approach is the method of mappings [4, 22, 33, 19, 41, 47], also called perturbation of identity, which parametrizes the shape by a bi-Lipschitz homeomorphism $\tilde{\boldsymbol{\tau}} : \mathbb{R}^d \to \mathbb{R}^d$ via $\hat{\Omega} = \tilde{\boldsymbol{\tau}}(\tilde{\Omega})$, where $\tilde{\Omega} \subset \mathbb{R}^d$ is a nominal domain (or shape reference domain). Optimization can be performed based on the function $\tilde{J} : \tilde{\boldsymbol{\tau}} \mapsto \hat{J}(\tilde{\boldsymbol{\tau}}(\tilde{\Omega}))$. An underlying state equation is then transformed to $\tilde{\Omega}$ and derivatives of $\tilde{J}$ can be obtained via sensitivities or adjoints. The Hadamard-Zolésio shape gradient representation can be derived from this approach essentially by an integration by parts. The method of mappings directly yields an optimal control setting in Banach spaces. Further, it fits well in the theoretical setting of the FSI model that was introduced above since it also employs the idea of domain transformations. In this paper we use the method of mappings to transform the fully Lagrangian FSI system to a shape reference domain. The major part of the paper is devoted to the study of existence, uniqueness and especially continuity and Fréchet differentiability of solutions to the transformed FSI system with respect to transformations of the domain.

The investigations in this paper have several important connections to inverse problems. Shape identification and other inverse problems for FSI systems have many interesting applications in engineering, e.g., wind turbines [28] and naval structures [52], in hemodynamics [7, 8], i.e., blood flows, and in other fields. Since shape variations belong to the most challenging types of parametric dependencies that can arise in PDEs, our differentiability results for the state with respect to shape variations can be transferred to many other parametric dependencies in FSI systems and often the analysis then would become less complex. The theory and methods of nonlinear inverse problems often make use of Fréchet derivatives of the underlying operator, e.g., in the formulation of (generalized) source conditions [17, 31, 32], in iteratively regularized (Gauß-) Newton methods [31, 32], and in Landweber iterations [17, 23]. This makes the differentiability of the parameter-to-state operator a crucial ingredient. Further, shape optimization problems can be ill-posed without a suitable regularization [24]. For the method of mappings considered here, Tikhonov-type regularizations are often employed to enforce the required smoothness of the transformation. Without additional measures, the shape representation by transformations is not unique, which requires special care when applying optimization methods. One possibility is to prepend an (often linear) smooth injective mapping from a space of unique shape parametrization to the space of domain variations. For instance, shapes can be represented by normal displacements of the design boundary, which then can be extended to corresponding domain displacements by solving a suitable linear elliptic equation.

The outline of the paper is as follows. Section 2 recalls the basic definitions and properties of the function spaces that are used in the analysis. Further, it presents the ideas for proving existence, uniqueness, continuity, and differentiability in a general setting. In Section 3 the main results and analytical tools of [44] are recalled and presented in a suitably adjusted way. Section 4 constitutes the main part of the paper, where the plan developed in Section 2 is carried out and the continuity and differentiability of the solution of the FSI problem with respect to domain transformations are proved.
Throughout the paper the superscripts over the functions correspond to the superscripts of the domains on which they are defined. Furthermore, the spatial coordinates on the physical domain $\check{\Omega}$ are denoted by x, on $\hat{\Omega}, \tilde{\Omega}$ by y, z, respectively. If a result is valid for a general domain the notation Ω is used and the coordinates are denoted by ξ.

2 Preliminaries

We now introduce the required function spaces and their properties and sketch the main ideas used in [44].

2.1 Fractional Sobolev and Sobolev-type spaces

Let $s, r \in [0, \infty)$, $\theta \in (0, 1)$, $X, \tilde{X}, Y, \tilde{Y}, Z$ be separable Hilbert spaces. The analysis is carried out in fractional order Sobolev spaces $H^s((0,T), H^r(\Omega))$ and in anisotropic Sobolev spaces $H^{r,s}(Q^T)$. The vector-valued versions are denoted by $H^s((0,T), H^r(\Omega)^d)$ and $H^{r,s}(Q^T)^d$. For more details on these spaces the reader is referred to [36, Ch. 1, Sec. 9], [37, Ch. 4, Sec. 2] and [20, Sec. 2]. These references assume Ω to be a bounded, open domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, with smooth boundary $\partial\Omega = \Gamma$ of class $\mathcal{C}^\infty$. However, these results adapt to the setting in [44].
The fractional order Sobolev spaces $H^s((0,T), X)$ can be endowed with the norm

$$|\cdot|_{H^s((0,T),X)} = (\|\cdot\|^2_{H^m((0,T),X)} + |\partial^m_{\mathsf{t}} \cdot|^2_{\sigma,(0,T),X})^{\frac{1}{2}}, \tag{2}$$

where m, σ are chosen such that $s = m + \sigma$, $m \in \mathbb{N}_0$ and for $0 < \sigma < 1$ the semi-norm $|\cdot|_{\sigma,(0,T),X}$ is defined by

$$|\cdot|^2_{\sigma,(0,T),X} = \int_0^T \int_0^T \frac{\|\cdot(\mathsf{t}) - \cdot(\mathsf{s})\|^2_X}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} \, \mathrm{d}\mathsf{s}\, \mathrm{d}\mathsf{t}. \tag{3}$$

The choice of the norm on the spaces $H^s((0,T), X)$ is crucial for the theoretical analysis which requires the knowledge of the T-dependency of appearing constants. More precisely, for $-\infty < T_1 < T_2 < \infty$ and $T_f \geqslant T$, the spaces $H^s((T_1, T_2), X)$ and the recursively defined subspaces

$$Y^s_{(T_1,T_2)} := \begin{cases} \{u \in H^s((T_1, T_2), X)\} & \text{if } s \in [0, \frac{1}{2}), \\ \{u \in H^s((T_1, T_2), X) \ : \ u(T_1) = 0\} & \text{if } s \in (\frac{1}{2}, 1], \\ \{u \in H^s((T_1, T_2), X) \ : \ u(T_1) = 0, \ \partial_\mathsf{t}\, u \in Y^{s-1}_{(T_1,T_2)}\} & \text{if } s \in (1, 2]\backslash\{\frac{3}{2}\}, \end{cases}$$

are endowed with a norm $\|\cdot\|_{H^s((T_1,T_2),X)}$ such that

P1 for all $s \geqslant 1$ such that $s + \frac{1}{2} \notin \mathbb{N}$,

$$\|\cdot\|_{H^s((T_1,T_2),X)} = (\|\cdot\|^2_{L^2((T_1,T_2),X)} + \|\partial_\mathsf{t}\,(\cdot)\|^2_{H^{s-1}((T_1,T_2),X)})^{\frac{1}{2}}.$$

and $\|\cdot\|_{H^0((T_1,T_2),X)} = \|\cdot\|_{L^2((T_1,T_2),X)}$, where $\|\cdot\|_{L^2((T_1,T_2),X)}$ denotes the standard $L^2((T_1, T_2), X)$-norm.

P2 for all $s \geqslant 0$ such that $s + \frac{1}{2} \notin \mathbb{N}$, there exist constants $c_{\Delta T},\ C_{\Delta T} > 0$ depending on $\Delta T = T_2 - T_1$ such that

$$c_{\Delta T}|\cdot|_{H^s((T_1,T_2),X)} \leqslant \|\cdot\|_{H^s((T_1,T_2),X)} \leqslant C_{\Delta T}|\cdot|_{H^s((T_1,T_2),X)}.$$

P3 for all $s \geqslant 0$ such that $s + \frac{1}{2} \notin \mathbb{N}$, the extension operator Ext defined by

$$\mathrm{Ext}(u)(\mathsf{t}) := \begin{cases} u(\mathsf{t}) & \text{if } \mathsf{t} \in (0, T), \\ 0 & \text{if } \mathsf{t} \in (T - T_f, 0), \end{cases}$$

is continuous as a mapping $Y^s_{(0,T)} \to Y^s_{(T-T_f,T)}$ with a continuity constant that does not depend on T.

P4 for all $s \geqslant 0$ such that $s + \frac{1}{2} \notin \mathbb{N}$, we have

$$\|u\|_{H^s((0,T),X)} \leqslant C\|u\|_{H^s((T-T_f,T),X)}$$

for all $u \in H^s((T - T_f, T), X)$ such that $u|_{(T-T_f,0)} = 0$ with a constant C independent of T.

P5 for all $s \geqslant 0$ such that $s + \frac{1}{2} \notin \mathbb{N}$, the restriction operator R defined by

$$\mathrm{R}(u)(\mathsf{t}) := u(\mathsf{t})$$

is continuous as a mapping $H^s((0, T_f), X) \to H^s((0, T), X)$ with a continuity constant that does not depend on T.

P6 for $s \in [0, 1)\backslash\{\frac{1}{2}\}$ and $\epsilon > 0$ such that $s + \epsilon \in (0, 1]\backslash\{\frac{1}{2}\}$, we have

$$\|u\|_{H^s((0,T),X)} \leqslant CT^\epsilon\|u\|_{H^{s+\epsilon}((0,T),X)}$$

for all $u \in Y^s_{(0,T)}$ with a constant C that does not depend on T.

P7 for $s \in [0, 1]\backslash\{\frac{1}{2}\}$, real, separable Hilbert spaces X_1, X_2 and a linear operator K that is continuous as a mapping from X_1 to X_2, we have

$$\|K(u)\|_{H^s((0,T),X_2)} \leqslant C\|u\|_{H^s((0,T),X_1)}$$

for all $u \in H^s((0, T), X_1)$ with a constant C that does not depend on T.

P8 for all $s \geqslant 0$ such that $s + \frac{1}{2} \notin \mathbb{N}$, $T_1 < T_2$,

$$\|u\|_{H^s((T_1,T_2),X)} = \|\tilde{u}\|_{H^s((0,T_2-T_1),X)},$$

for all $u \in H^s((T_1, T_2), X)$, where $\tilde{u}(\mathsf{t}) := u(\mathsf{t} + T_1)$ for (a.e.) $\mathsf{t} \in (0, T_2 - T_1)$.

The equivalence of the norm to standard norms stated in **P2** and the T-independent norms of extension, see **P3**, and restriction operators according to **P4**, **P5** are needed to show that the norm of solution operators for the considered partial differential equations are independent of T. Explict time dependencies are obtained by estimating right hand side terms using properties **P1**, **P6** and **P7**. Translation invariance of the norm (**P8**) is another desirable property that we will exploit. The case $s + \frac{1}{2} \in \mathbb{N}$ is excluded since the choice $Y^{\frac{1}{2}}_{(T_1,T_2)} = H^{\frac{1}{2}}((T_1, T_2), X)$ violates **P3** due to [36, p.60, Thm. 11.4] and an analogous choice as for $s \in (\frac{1}{2}, 1]$ is not possible since the trace operator is not continuous for $s = \frac{1}{2}$, see [36, p.41, Thm. 9.4].

Lemma 1. *Let X be a separable Hilbert space, $-\infty < T_1 < T_2 < \infty$. There exists a norm $\|\cdot\|_{H^s((T_1,T_2),X)}$ on $H^s((T_1, T_2), X)$ that fulfills **P1**–**P8**.*

Proof. Let

$$\|\cdot\|_{H^0((T_1,T_2),X)} := \|\cdot\|_{L^2((T_1,T_2),X)}$$

and, for $\sigma \in (0, 1)\backslash\{\frac{1}{2}\}$,

$$\|\cdot\|_{H^\sigma((T_1,T_2),X)} := \begin{cases} (|\cdot|^2_{H^\sigma((T_1,T_2),X)} + \frac{1}{\sigma}\int_{T_1}^{T_2}(\mathsf{t} - T_1)^{-2\sigma}\|\cdot(\mathsf{t})\|^2_X d\mathsf{t})^{\frac{1}{2}} & \text{if } \sigma \in (0, \frac{1}{2}), \\ (|\cdot|^2_{H^\sigma((T_1,T_2),X)} + \frac{1}{\sigma}\int_{T_1}^{T_2}(\mathsf{t} - T_1)^{-2\sigma}\|(\cdot - L(\cdot))(\mathsf{t})\|^2_X d\mathsf{t})^{\frac{1}{2}} & \text{if } \sigma \in (\frac{1}{2}, 1), \end{cases}$$

where, for $\sigma \in (\frac{1}{2}, 1)$ and $T_f > 0$, L is chosen as the linear operator defined by

$$L(u)(\mathsf{t}) = \begin{cases} u(T_1)T_f^{-1}(T_f + T_1 - \mathsf{t}) & \text{for } \mathsf{t} \in (T_1, T_1 + T_f), \\ 0 & \text{for } \mathsf{t} \in [T_1 + T_f, \infty). \end{cases}$$

The norm is constructed such that for any $T > 0$ and $u \in H^\sigma_0((0, T), X)$ there holds $\|u\|_{H^\sigma((0,T),X)} = |\mathrm{Ext}(u)|_{H^\sigma((-\infty,T),X)}$. For $s = m + \sigma$, $m > 0$, the norm is chosen such that **P1** holds. [36, Thm. 9.4, 11.2–11.5] and their proofs imply **P2**. **P3**–**P8** can be shown with standard estimates, see Appendix. ☐

Let $s_0, s_1 \in [0, \infty)$, $s_0 > s_1$, and let X, Y and $\tilde{X}, \tilde{Y}$, respectively, be continuously embedded in Hausdorff topological vector spaces V and $\tilde{V}$, respectively. By [2, (3.5)–(3.7), Thm. 3.1, Cor. 4.3], [12, Rem. 3.6], and [6, Thm. 3.4.1] there holds

$$[H^{s_0}((0, T), X), H^{s_1}((0, T), Y)]_\theta = H^{(1-\theta)s_0+\theta s_1}((0, T), [X, Y]_\theta),$$

where $[\cdot, \cdot]_\theta$ denotes the complex interpolation space, cf. [11], [51], [5, p.166], [35, Sec. 0.2.1], and the interpolation inequality, see, e.g., [36, p.19, Prop. 2.3], yields

$$\|\cdot\|_{H^{(1-\theta)s_0+\theta s_1}((0,T),[X,Y]_\theta)} \leqslant C\|\cdot\|^{1-\theta}_{H^{s_0}((0,T),X)}\|\cdot\|^\theta_{H^{s_1}((0,T),Y)} \tag{4}$$

for a constant C that might depend on T. If, in addition, $\theta \in (0, 1)$ and

$$\mathcal{A} \in \mathcal{L}(H^{s_0}((0, T), X), H^{\tilde{s}_0}((0, T), \tilde{X})) \cap \mathcal{L}(H^{s_1}((0, T), Y), H^{\tilde{s}_1}((0, T), \tilde{Y})),$$

then $\mathcal{A} \in \mathcal{L}(H^{(1-\theta)s_0+\theta s_1}((0, T), [X, Y]_\theta), H^{(1-\theta)\tilde{s}_0+\theta\tilde{s}_1}((0, T), [\tilde{X}, \tilde{Y}]_\theta))$ and

$$\begin{aligned} &\|\mathcal{A}\|_{\mathcal{L}(H^{(1-\theta)s_0+\theta s_1}((0,T),[X,Y]_\theta),H^{(1-\theta)\tilde{s}_0+\theta\tilde{s}_1}((0,T),[\tilde{X},\tilde{Y}]_\theta))} \\ &\leqslant C\|\mathcal{A}\|^{1-\theta}_{\mathcal{L}(H^{s_0}((0,T),X),H^{\tilde{s}_0}((0,T),\tilde{X}))}\|\mathcal{A}\|^\theta_{\mathcal{L}(H^{s_1}((0,T),Y),H^{\tilde{s}_1}((0,T),\tilde{Y}))}, \end{aligned} \tag{5}$$

for a constant C that might depend on T, cf., e.g., [11, p.115, 4.].
The spaces $H^{r,s}(Q^T)$ are defined by

$$H^{r,s}(Q^T) = L^2((0, T), H^r(\Omega)) \cap H^s((0, T), L^2(\Omega))$$

and endowed with the norm

$$\|\cdot\|_{H^{r,s}(Q^T)} = (\|\cdot\|^2_{L^2((0,T),H^r(\Omega))} + \|\cdot\|^2_{H^s((0,T),L^2(\Omega))})^{\frac{1}{2}}.$$

For $0 \leqslant r' \leqslant r$, $s' = s(r-r')/r$, the inequality

$$\|\cdot\|_{H^{s'}((0,T),H^{r'}(\Omega))} \leqslant C\|\cdot\|_{H^{r,s}(Q^T)}$$

holds true for a constant $C > 0$ that might depend on T, cf. [20, (2.9)] or [21, (2.7)], which implies

$$\|\cdot\|_{H^{(1-\theta)s}((0,T),H^{\theta r}(\Omega))} \leqslant C\|\cdot\|_{H^{r,s}(Q^T)} \tag{6}$$

for $\theta \in (0,1)$. Trace theorems for the Sobolev-type spaces $H^{r,s}(Q^T)$ imply

$$\|\cdot|_{\Sigma_i^T}\|_{H^{r',s'}(\Sigma_i^T)} \leqslant C\|\cdot\|_{H^{r,s}(Q^T)}, \tag{7}$$

where $C > 0$ is dependent on T, $r > \frac{1}{2}$, $s \geqslant 0$, $r' = r - \frac{1}{2}$ and $s' = (r-\frac{1}{2})\frac{s}{r}$, cf. [37, Chap. 4, Thm. 2.1], [21, Prop. 2.2] or [18, Thm. 3].

2.2 Method of successive approximations

The method of successive approximations is a well known approach for establishing existence and uniqueness results for nonlinear partial differential equations. On an abstract level, the FSI system can be considered as a nonlinear partial differential equation of the form

$$A(y) = 0, \tag{8}$$

where $y \in Y$ and Y is a Banach space. As in [44] we write this in the form $By = \mathcal{F}(y)$, where $\mathcal{F}(y) := By - A(y)$ and B is a linear operator that represents the principal part of the FSI system, i.e., the PDE operator in a linear FSI system. For our setting we will show that the system $By = f$ has a unique solution $y = Sf$, where $S \in \mathcal{L}(W,Y)$, with W being a Banach space. Existence and uniqueness of solutions is now studied via the fixed point equation

$$y = S\mathcal{F}(y). \tag{9}$$

Unique solvability of (9) on a closed subset $\tilde{Y} \subset Y$ can be shown if $y \mapsto S\mathcal{F}(y)$ maps $\tilde{Y}$ into itself and is a contraction on $\tilde{Y}$. This, e.g., is the case if $\|S\|_{\mathcal{L}(W,Y)} \leqslant L_S$ and if $\mathcal{F} : \tilde{Y} \to W$ is Lipschitz continuous with a constant $L_{\mathcal{F}} < \frac{1}{L_S}$. Uniqueness on $\tilde{Y}$ then also follows.

2.3 Framework for continuity and differentiability results

One can extend the considerations of the previous section to an equation

$$A(y,z) = 0 \tag{10}$$

with parameter or control z in a Banach space Z. As before, we consider solutions of the fixed point equation

$$y = S\,\mathcal{F}(y,z), \tag{11}$$

where $\mathcal{F}(y,z) := By - A(y,z)$, B is as in section sec. 2.2 and $S \in \mathcal{L}(W,Y)$ is the solution operator of $By = f$.

Theorem 1. *Let $\tilde{W}, W, Y, Z$ be Banach spaces, $\tilde{W}$ continuously embedded in W, $S \in \mathcal{L}(\tilde{W},Y)$, and $L_S > 0$ a constant such that $\|Sf\|_Y \leqslant L_S\|f\|_W$ for all $f \in \tilde{W}$. Let $\tilde{Z} \subset Z$ be open, $\tilde{Y} \subset Y$ be closed and $\mathcal{F} : \tilde{Y} \times \tilde{Z} \to \tilde{W}$ be an operator. Let there exist constants $L_{\mathcal{F}} \in (0, \frac{1}{L_S})$ and $C > 0$ such that, for all $y, y_1, y_2 \in \tilde{Y}$, $z, z_1, z_2 \in \tilde{Z}$, there hold*

$$\|\mathcal{F}(y_2,z_2) - \mathcal{F}(y_1,z_1)\|_W \leqslant L_{\mathcal{F}}\|y_2 - y_1\|_Y + C\|z_2 - z_1\|_Z, \tag{12}$$

$$S\mathcal{F}(y,z) \in \tilde{Y}. \tag{13}$$

Then, for all $z \in \tilde{Z}$, the system (11) has a unique solution $y(z)$ and $z \mapsto y(z)$ is Lipschitz continuous on $\tilde{Z}$:

$$\|y(z_2) - y(z_1)\|_Y \leqslant \frac{CL_S}{1 - L_S L_{\mathcal{F}}}\|z_2 - z_1\|_Z \quad \forall\ z_1, z_2 \in \tilde{Z}. \tag{14}$$

In addition, let $y(z)$ lie in the relative interior of $\tilde{Y}$ and denote by $\tilde{Y}_L$ the linear subspace parallel to the affine hull $\operatorname{aff}(\tilde{Y})$. Assume that $\mathcal{F}$ is Fréchet differentiable at $(y(z), z)$, where (y, z)-variations are taken in $\tilde{Y}_L \times Z$. Then $y(\cdot)$ is Fréchet differentiable at z. The derivative is given by $y'(z)(h) = \delta_h y(z)$, where $h \in Z$ and $\delta_h y(z) \in \tilde{Y}_L \subset Y$ solves the formally linearized equation

$$\delta_h y(z) = S\delta\mathcal{F}(y(z), z)(\delta_h y(z), h), \tag{15}$$

where $\delta\mathcal{F}(y(z), z)(\delta_h y(z), h) := \mathcal{F}_y(y(z), z)\delta_h y(z) + \mathcal{F}_z(y(z), z)h$.

Proof. For any fixed $z \in \tilde{Z}$, (12) implies the Lipschitz continuity of the mapping $\mathcal{F}(\cdot, z) : \tilde{Y} \to W$. Using (12), (13), and the properties of $\mathcal{F}$, $L_{\mathcal{F}}$ and L_S shows that the map $y \in \tilde{Y} \mapsto S\mathcal{F}(y, z) \in \tilde{Y}$ is a well-defined contraction. The existence of a unique solution $y(z) \in \tilde{Y}$ is thus ensured by the method of successive approximations. Now (14) follows from $\|y(z_2) - y(z_1)\|_Y = \|S(\mathcal{F}(y(z_2), z_2) - \mathcal{F}(y(z_1), z_1))\|_Y \leqslant L_S\|\mathcal{F}(y(z_2), z_2) - \mathcal{F}(y(z_1), z_1)\|_W$ and (12). For showing differentiability, we fix $z \in \tilde{Z}$ and assume that $\mathcal{F}$ is differentiable at $(y(z), z)$ in the way stated in the theorem. Let $h \in Z$ be arbitrarily fixed. Since $y(z)$ is a relative interior point of $\tilde{Y}$, we obtain from (12) that, for all $d_1, d_2 \in \tilde{Y}_L$, there holds:

$$\|\delta\mathcal{F}(y(z), z)(d_2, h) - \delta\mathcal{F}(y(z), z)(d_1, h)\|_W = \|\mathcal{F}_y(y(z), z)(d_2 - d_1)\|_W \leqslant L_{\mathcal{F}}\|d_2 - d_1\|_Y. \tag{16}$$

Thus, since $L_{\mathcal{F}} < \frac{1}{L_S}$, the method of successive approximations applied to the fixed point equation $\delta_h y(z) = S\delta\mathcal{F}(y(z), z)(\delta_h y(z), h)$ posed in $\tilde{Y}_L$, see (15), yields a unique solution $\delta_h y(z) \in \tilde{Y}_L \subset Y$ which by linearity of (15) depends linearly on h. Let $\|h\|_Z$ be sufficiently small. Then $z + h \in \tilde{Z}$ and, as $h \to 0$,

$$\begin{aligned}
&\|\mathcal{F}(y(z+h), z+h) - \mathcal{F}(y(z), z) - \delta\mathcal{F}(y(z), z)(\delta_h y(z), h)\|_W \\
&\quad\leqslant \|\delta\mathcal{F}(y(z), z)(y(z+h) - y(z), h) - \delta\mathcal{F}(y(z), z)(\delta_h y(z), h)\|_W + o(\|y(z+h) - y(z)\|_Y + \|h\|_Z) \\
&\quad\leqslant L_{\mathcal{F}}\|y(z+h) - y(z) - \delta_h y(z)\|_Y + o(\|h\|_Z),
\end{aligned}$$

where (16) is used. Now

$$\begin{aligned}
\|y(z+h) - y(z) - \delta_h y(z)\|_Y &= \|S\mathcal{F}(y(z+h), z+h) - S\mathcal{F}(y(z), z) - S\delta\mathcal{F}(y(z), z)\delta_h y(z)\|_Y \\
&\leqslant L_S\|\mathcal{F}(y(z+h), z+h) - \mathcal{F}(y(z), z) - \delta\mathcal{F}(y(z), z)\delta_h y(z)\|_W \\
&\leqslant L_S L_{\mathcal{F}}\|y(z+h) - y(z) - \delta_h y(z)(z)\|_Y + L_S o(\|h\|_Z) \quad (\|h\|_Z \to 0).
\end{aligned}$$

Therefore,

$$\|y(z+h) - y(z) - \delta_h y(z)\|_Y \leqslant \frac{L_S}{1 - L_S L_{\mathcal{F}}} o(\|h\|_Z) = o(\|h\|_Z) \quad (\|h\|_Z \to 0),$$

which proves the Fréchet differentiability of $z \mapsto y(z)$ at z with $y'(z)h = \delta_h y(z)$. ☐

The rest of the paper is dedicated to the application of this argumentation to shape optimization for the FSI problem via the method of mappings. The parameter z then corresponds to a domain transformation that represents a variation of a reference shape domain.

3 Existence and uniqueness results for Navier-Stokes-Lamé system

In order to have the theoretical tools at hand that will be used for showing differentiability of the state with respect to domain variations the main results of [44] are recalled. Since, in contrast, the analysis will be carried out on a nominal domain $\tilde{\Omega}$ instead of $\hat{\Omega}$, the statements are presented for a general domain $\Omega \in \{\hat{\Omega}, \tilde{\Omega}\}$. We will work under the following Assumption 1 on the unique solvability of the Stokes equations and the elastic wave equation. We will see in Lemma 2 that according to [44] Assumption 1 is satisfied for particular boundary conditions and geometrical settings.

Assumption 1. *Let $\ell \in (\frac{1}{2}, 1)$, $T_f > 0$. In addition, let $\beta > 0$,*

- *Ω, Ω_f, Ω_s be open domains with $\overline{\Omega} = \overline{\Omega_f \cup \Omega_s}$, $\Omega_f \cap \Omega_s = \varnothing$, $\Gamma_i := \partial\Omega_s \cap \partial\Omega_f$, $\Gamma_f \subset \partial\Omega_f \backslash \Gamma_i$, $\Gamma_s \subset \partial\Omega_s \backslash \Gamma_i$,*
- *$D \subset C^\infty(\overline{\Omega})$ be a closed linear subspace, for which the following holds true: If $f \in D$, then $\nabla f \in D^d$; If $f, g \in D$, then $fg \in D$; If $f \in D$, $f > 0$, then $f^{-1} \in D$,*

- $\mathbf{D} = D^d$, $\underline{\mathbf{D}} = D^{d\times d}$, $D_T = C^\infty([0,T],D)$, $\mathbf{D}_T = C^\infty([0,T],\mathbf{D})$, $\underline{\mathbf{D}}_T = C^\infty([0,T],\underline{\mathbf{D}})$,

  $$\begin{aligned}
  &\mathbf{E}_T = \overline{\mathbf{D}_T}^{H^{2+\ell,1+\frac{\ell}{2}}(Q_f^T)^d}, && G_T = \overline{D_T}^{L^2((0,T),H^{1+\ell}(\Omega_f))\cap H^{\frac{\ell}{2}}((0,T),H^1(\Omega_f))},\\
  &\mathbf{F}_T = \overline{\mathbf{D}_T}^{H^{\ell,\frac{\ell}{2}}(Q_f^T)^d}, && P_T = \overline{D_T}^{\{p\in L^2(Q_f^T)\,:\,\nabla p\in H^{\ell,\frac{\ell}{2}}(Q_f^T)^d,\ p|_{\Sigma_i^T}\in H^{\frac12+\ell,\frac14+\frac{\ell}{2}}(\Sigma_i^T)\}},\\
  &\mathbf{G}_T = \overline{\mathbf{D}_T}^{H^{1+\frac{\ell}{2}}((0,T),L^2(\Omega_f)^d)}, && \mathbf{W}_T = \overline{\mathbf{D}_T}^{C^0([0,T],H^{\frac74+\frac{\ell}{2}}(\Omega_s)^d)\cap C^1([0,T],H^{\frac34+\frac{\ell}{2}}(\Omega_s)^d)},\\
  &\mathbf{H}_T = \overline{\mathbf{D}_T}^{H^{\frac12+\ell,\frac14+\frac{\ell}{2}}(\Sigma_i^T)^d}, && \mathbf{N}_T = \overline{\mathbf{D}_T}^{H^\beta((0,T),H^{\frac32+\ell}(\Gamma_s\cup\Gamma_i)^d)\cap H^{\frac12+\ell+\beta}((0,T),H^1(\Gamma_s\cup\Gamma_i)^d)\cap H^{\frac74+\frac{\ell}{2}}((0,T),L^2(\Gamma_s\cup\Gamma_i)^d)},\\
  &\mathbf{V}_0 = \overline{\mathbf{D}}^{H^{1+\ell}(\Omega_f)^d}, && \mathbf{W}_1 = \overline{\mathbf{D}}^{H^{\frac12+\ell+\beta}(\Omega_s)^d},
  \end{aligned}$$

  where, for a Banach space X, $\overline{D}^X$ denotes the closure of D w.r.t. X,

- $\mathbf{v}_0 \in \mathbf{V}_0$, $\mathbf{w}_1 \in \mathbf{W}_1$,

such that

- *Lemma A.4, (6) and (7) are valid.*
- $\mathbf{v}_0|_{\Gamma_f} = 0$, $\operatorname{div}(\mathbf{v}_0) = 0$, $\mathbf{v}_0|_{\Gamma_i} = \mathbf{w}_1|_{\Gamma_i}$ *and* $2\nu(\epsilon(\mathbf{v}_0)\mathbf{n}_f)\cdot\boldsymbol{\tau} = 0$ *on* Γ_i *for any unit vector* $\boldsymbol{\tau}$ *tangent to* Γ_i.
- *for all* $\mathbf{f}\in\mathbf{F}_T$, $\mathbf{h}\in\mathbf{H}_T$, $\mathbf{g}\in\mathbf{G}_T\cap H^1((0,T),H^\ell(\Omega_f)^d)$ *and* $g\in G_T$ *for which the compatibility conditions*

  $$\mathbf{g}|_{\Sigma_f^T} = 0, \quad \mathbf{g}(0) = 0, \quad \textit{and} \quad \mathbf{h}(0) = 0,$$

  are satisfied the system

  $$\begin{aligned}
  \partial_t \mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f} &\quad \textit{in } Q_f^T,\\
  \operatorname{div}(\mathbf{v}) = g = \operatorname{div}(\mathbf{g}) &\quad \textit{in } Q_f^T,\\
  \mathbf{v}(0) = \mathbf{v}_0 &\quad \textit{in } \Omega_f,\\
  \mathbf{v} = 0 &\quad \textit{on } \Sigma_f^T,\\
  \sigma_f(\mathbf{v},p)\mathbf{n}_f = \mathbf{h} &\quad \textit{on } \Sigma_i^T,
  \end{aligned}$$

  admits a unique solution $(\mathbf{v},p)\in\mathbf{V}\times P$ and there exists a constant $C>0$ for which

  $$\begin{aligned}
  &\|\mathbf{v}\|_{H^{2+\ell,1+\frac{\ell}{2}}(Q_f^T)^d} + \|\nabla p\|_{H^{\ell,\frac{\ell}{2}}(Q_f^T)^d} + \|p|_{\Sigma_i^T}\|_{H^{\frac12+\ell,\frac14+\frac{\ell}{2}}(\Sigma_i^T)} \leqslant C(\|\mathbf{v}_0\|_{H^{1+\ell}(\Omega_f)^d} + \|\mathbf{f}\|_{(H^{\ell,\frac{\ell}{2}}(Q_f^T))^d}\\
  &+ \|\mathbf{g}\|_{H^{1+\frac{\ell}{2}}((0,T),L^2(\Omega_f)^d)} + \|g\|_{L^2((0,T),H^{1+\ell}(\Omega_f))\cap H^{\frac{\ell}{2}}((0,T),H^1(\Omega_f))} + \|\mathbf{h}\|_{H^{\frac12+\ell,\frac14+\frac{\ell}{2}}(\Sigma_i^T)^d}).
  \end{aligned}$$

- *for arbitrary $\boldsymbol{\eta}\in\mathbf{N}_T$ for which the compatibility conditions*

  $$\boldsymbol{\eta}(0) = 0, \quad \textit{and} \quad \partial_t\boldsymbol{\eta}(0) = \mathbf{w}_1|_{\Sigma_s^T\cup\Sigma_i^T},$$

  are satisfied the system

  $$\begin{aligned}
  \partial_{tt}\mathbf{w} - \operatorname{div}_y(\sigma_{s,y}(\mathbf{w})) = 0 &\quad \textit{in } Q_s^T,\\
  \mathbf{w} = \boldsymbol{\eta} &\quad \textit{on } \Sigma_s^T\cup\Sigma_i^T,\\
  \mathbf{w}(0) = 0, \quad \partial_t\mathbf{w}(0) = \mathbf{w}_1 &\quad \textit{in } \Omega_s
  \end{aligned}$$

  admits a unique solution $\mathbf{w}\in\mathbf{W}_T$ and there exists a constant $C>0$ for which

  $$\begin{aligned}
  &\|\mathbf{w}\|_{C^0([0,T],H^{\frac74+\frac{\ell}{2}}(\Omega_s)^d)\cap C^1([0,T],H^{\frac34+\frac{\ell}{2}}(\Omega_s)^d)} + \|\sigma_{s,y}(\mathbf{w})\mathbf{n}_f\|_{H^\beta((0,T),H^{\frac12+\ell}(\Gamma_i)^d)\cap H^{\frac34+\frac{\ell}{2}}((0,T),L^2(\Gamma_i)^d)}\\
  &\leqslant C(\|\mathbf{w}_1\|_{H^{\frac12+\ell+\beta}(\Omega_s)^d} + \|\boldsymbol{\eta}\|_{H^\beta((0,T),H^{\frac32+\ell}(\Gamma_s\cup\Gamma_i)^d)\cap H^{\frac12+\ell+\beta}((0,T),H^1(\Gamma_s\cup\Gamma_i)^d)\cap H^{\frac74+\frac{\ell}{2}}((0,T),L^2(\Gamma_s\cup\Gamma_i)^d)}).
  \end{aligned}$$

Remark 1. *The main difficulty in finding a setting that fulfills Assumption 1 is the improved regularity for the normal stress of the Lamé system on the boundary Σ_i^T. [44] provides a setting that fulfills the assumption, see Lemma 2. Another restriction is the validity of the requirements for the fluid system. Existing results require, e.g., that $\overline{\Gamma_i} \cap \overline{\Gamma_f} = \varnothing$, see [21, Def. 7.2, Thm. 7.5].*

Lemma 2. *Let*

$$\begin{aligned}\Omega &= \{\mathsf{y} \in \mathbb{R}^3 \ : \ 0 < \mathsf{y}_1 < L_1, 0 < \mathsf{y}_2 < L_2, 0 < \mathsf{y}_3 < L_3\},\\ \Omega_f &= \{\mathsf{y} \in \mathbb{R}^3 \ : \ 0 < \mathsf{y}_1 < L_1, 0 < \mathsf{y}_2 < L_2, 0 < \mathsf{y}_3 < L_a \text{ or } L_b < \mathsf{y}_3 < L_3\},\\ \Omega_s &= \{\mathsf{y} \in \mathbb{R}^3 \ : \ 0 < \mathsf{y}_1 < L_1, 0 < \mathsf{y}_2 < L_2, L_a < \mathsf{y}_3 < L_b\},\end{aligned} \tag{17}$$

where $0 < L_a < L_b < L_3$. Furthermore, let the interface be defined by $\Gamma_i = \partial\Omega_f \cap \partial\Omega_s$, $\Gamma_f := \{y \in \Omega \ : \ \mathsf{y}_3 = 0 \text{ or } \mathsf{y}_3 = L_3\}$ and periodic boundary conditions be imposed on $\partial\Omega\backslash\Gamma_f$ by setting

$$\begin{aligned} D = C^\infty_\#(\overline{\Omega}) := \{v \in C^\infty(\overline{\Omega}) \ : \ &\exists \bar{v} \in C^\infty(\mathbb{R}^2 \times [0, L_3]) \text{ s.t. } \bar{v}|_\Omega = v,\\ &\bar{v}(\mathsf{y}_1, \mathsf{y}_2, \mathsf{y}_3) = \bar{v}(\mathsf{y}_1 + L_1, \mathsf{y}_2, \mathsf{y}_3), \ \bar{v}(\mathsf{y}_1, \mathsf{y}_2, \mathsf{y}_3) = \bar{v}(\mathsf{y}_1, \mathsf{y}_2 + L_2, \mathsf{y}_3)\}.\end{aligned}$$

Then, $\beta > 0$, $\mathbf{v}_0$ and $\mathbf{w}_1$ can be chosen such that Assumption 1 is fulfilled.

Proof. cf. [44, Sec. 3-4]. □

Remark 2. *As we will see, any geometrical configuration satisfying Assumption 1, for example the one in Lemma 2, can be used as shape reference domain $\tilde{\Omega}$ for shape optimization. The shape reference domain is mapped by a C^1-diffeomorphism to the ALE reference domain $\hat{\Omega}$ for the FSI system. We will show in Theorem 3 that there exists a suitable open neighborhood in $H^{2+\ell}(\tilde{\Omega})^d$ of C^1-diffeomorphisms containing the identity, see (30) and Lemma 4, such that the solution of the corresponding FSI-system pulled-back to the reference shape domain depends continuously differentiable on the transformation.*

For $\ell \in (\frac{1}{2}, 1)$, the function spaces

$$\begin{aligned} &H_T := \overline{D_T}^{H^{\frac{1}{2}+\ell,\frac{1}{4}+\frac{\ell}{2}}(\Sigma_i^T)}, && S_T := \overline{\mathbf{D}_T}^{H^1((0,T),H^{1+\ell}(\Omega_f)) \cap H^{\frac{3}{2}+\frac{\ell}{2}}((0,T),L^2(\Omega_f))},\\ &\mathbf{S}_T := \overline{\mathbf{D}_T}^{H^1((0,T),H^{1+\ell}(\Omega_f)^d) \cap H^{\frac{3}{2}+\frac{\ell}{2}}((0,T),L^2(\Omega_f)^d)}, && \underline{\mathbf{S}}_T := \overline{\underline{\mathbf{D}}_T}^{H^1((0,T),H^{1+\ell}(\Omega_f)^{d\times d}) \cap H^{\frac{3}{2}+\frac{\ell}{2}}((0,T),L^2(\Omega_f)^{d\times d})},\\ &\mathbf{T} := \overline{\mathbf{D}}^{H^{2+\ell}(\Omega)^d}, \end{aligned} \tag{18}$$

the norms

$$\begin{aligned} \|\cdot\|_{\mathbf{E}_T} := \big(&\|\cdot\|^2_{(H^{2+\ell,1+\frac{\ell}{2}}(Q_f^T))^d} + \|\cdot\|^2_{H^1((0,T),H^\ell(\Omega_f)^d))} + \|\cdot\|^2_{H^{\frac{\ell}{2}}((0,T),H^2(\Omega_f)^d)} + \|\cdot|_{\Sigma_i^T}\|^2_{H^{\frac{1}{4}+\frac{\ell}{2}}((0,T),H^1(\Gamma_i)^d)}\\ &+ \|\cdot|_{\Sigma_i^T}\|^2_{H^{\frac{3}{4}+\frac{\ell}{2}}((0,T),L^2(\Gamma_i)^d)} + \|\cdot\|^2_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),H^1(\Omega_f)^d)} + \|\cdot\|^2_{H^{\frac{1}{4}+\frac{\ell}{4}}((0,T),H^{1+\ell}(\Omega_f)^d)}\big)^{\frac{1}{2}},\\ \|\cdot\|_{S_T} := \big(&\|\cdot\|^2_{H^1((0,T),H^{1+\ell}(\Omega_f)) \cap H^{\frac{3}{2}+\frac{\ell}{2}}((0,T),L^2(\Omega_f))} + \|\cdot(0)\|^2_{H^{1+\ell}(\Omega_f)} + \|\partial_{\mathsf{t}} \cdot (0)\|^2_{L^2(\Omega_f)}\big)^{\frac{1}{2}}, \end{aligned} \tag{19}$$

with analogous definitions on the spaces $\mathbf{S}_T$ and $\underline{\mathbf{S}}_T$, and, for $\mathbf{v}_0 \in \mathbf{V}_0$, the metric spaces

$$\begin{aligned} &\mathbf{E}_{T,M_0,\mathbf{v}_0} := \{\mathbf{v} \in \mathbf{E}_T \ : \quad \mathbf{v}(\cdot,0) = \mathbf{v}_0, \ \|\mathbf{v}\|_{\mathbf{E}_T} \leqslant M_0\},\\ &P_{T,M_0,\mathbf{v}_0} := \{p \in P_T \ : \quad \|\nabla p\|_{\mathbf{F}_T} \leqslant M_0, \ \|p|_{\Sigma_i^T}\|_{H^{\frac{1}{2}+\ell,\frac{1}{4}+\frac{\ell}{2}}(\Sigma_i^T)} \leqslant M_0, \ p|_{\Gamma_i\times\{0\}} = 2\nu\epsilon(\mathbf{v}_0)\mathbf{n}_f \cdot \mathbf{n}_f|_{\Gamma_i}\} \end{aligned} \tag{20}$$

are defined. Due to trace theorems and interpolation theorems the modified norms on $\mathbf{E}_T$ and $S_T, \mathbf{S}_T, \underline{\mathbf{S}}_T$ are equivalent to the standard norms on these function spaces. However, the appearing equivalence constant might depend on T without further knowledge about this dependency. Since the dependency of the appearing constants on T is a key point in the theoretical analysis it is therefore necessary to work with the modified norms defined above.

Remark 3. *The following adaptions have been made compared to [44]:*

- *the $\mathbf{E}_T$-norm does not contain the term $\|\cdot\|_{H^{\frac{1}{2}}((0,T),H^{1+\ell}(\Omega_f))}$, which is not needed for estimating the right hand side terms and is not compatible to our choice of the norm, but other norms of interpolation and trace spaces.*
- *In the theoretical setting considered here, it is not guaranteed that $\mathbf{g} \in C([0,T], H^{1+\ell}(\Omega_f)^d)$ which is required in [44, Thm. 5.1] to use a trace theorem and give a meaning to $\mathbf{g}|_{\Sigma_f^T} = 0$. However, the proof of [44, Lem. 4.2] is the only point where $\mathbf{g} \in C([0,T], H^{1+\ell}(\Omega_f)^d)$ is used. A reinspection of the proof shows that $\mathbf{g} \in H^1((0,T), H^\ell(\Omega_f)^d)$ is actually sufficient.*

The following continuity result that is also part of the proof of Theorem 2 corresponds to [44, Thm. 5.1] with the modification that only $\mathbf{g} \in H^1((0,T), H^\ell(\Omega_f)^d)$ is required instead of $\mathbf{g} \in C([0,T], H^{1+\ell}(\Omega_f)^d)$, which is possible by Remark 3. It will be needed for showing the Fréchet-differentiability of the state with respect to domain variations.

Lemma 3. *Let Assumption 1 be fulfilled. Assume that*

$$\mathbf{f} \in \mathbf{F}_T, \quad \mathbf{h} \in \mathbf{H}_T, \quad g \in G_T, \quad \mathbf{g} \in \mathbf{G}_T \cap H^1((0,T), H^\ell(\Omega_f)^d).$$

Furthermore, let

$$\mathbf{g}|_{\Sigma_f^T} = 0, \quad \mathbf{g}(\cdot,0) = 0, \quad \text{and} \quad \mathbf{h}(\cdot,0) = 0.$$

Then, the system

$$\begin{aligned} \partial_t \mathbf{v} - \nu\Delta\mathbf{v} + \nabla p &= \mathbf{f} && \text{in } Q_f^T, \\ \operatorname{div}(\mathbf{v}) = g &= \operatorname{div}(\mathbf{g}) && \text{in } Q_f^T, \\ \mathbf{v}(\cdot,0) &= \mathbf{v}_0 && \text{in } \Omega_f, \\ \mathbf{v} &= 0 && \text{on } \Sigma_f^T, \\ \mathbf{v} &= \partial_t \mathbf{w} && \text{on } \Sigma_i^T, \\ \sigma_f(\mathbf{v},p)\mathbf{n}_f &= \sigma_s(\mathbf{w})\mathbf{n}_f + \mathbf{h} && \text{on } \Sigma_i^T, \\ \partial_{tt}\mathbf{w} - \operatorname{div}(\sigma_s(\mathbf{w})) &= 0 && \text{in } Q_s^T, \\ \mathbf{w} &= 0 && \text{on } \Sigma_s^T, \\ \mathbf{w}(\cdot,0) = 0, \quad \partial_t\mathbf{w}(\cdot,0) &= \mathbf{w}_1 && \text{in } \Omega_s, \end{aligned} \tag{21}$$

admits a unique solution $(\mathbf{v},p,\mathbf{w}) \in \mathbf{E}_T \times P_T \times \mathbf{W}_T$ and the states depend continuously on the initial data and the right hand sides, more precisely,

$$\begin{aligned} &\|\mathbf{v}\|_{\mathbf{E}_T} + \|\nabla p\|_{\mathbf{F}_T} + \|\sigma(\mathbf{w})\mathbf{n}_f\|_{\mathbf{H}_T} + \|p|_{\Sigma_i^T}\|_{H_T} + \|\mathbf{w}\|_{\mathbf{W}_T} \\ &\leqslant C_S(\|\mathbf{v}_0\|_{H^{1+\ell}(\Omega_f)^d} + \|\mathbf{w}_1\|_{H^{\frac{1}{2}+\ell+\beta}(\Omega_s)^d} + \|\mathbf{f}\|_{\mathbf{F}_T} + \|\mathbf{g}\|_{\mathbf{G}_T} + \|g\|_{G_T} + \|\mathbf{h}\|_{\mathbf{H}_T}), \end{aligned}$$

for all $0 < T \leqslant T_f$. The constant C_S depends on T_f but is independent of T.

For obtaining time independent continuity estimates for the Stokes equations and the Lamé system with respect to the right hand sides, the partial differential equations are split into several systems that have either zero initial conditions or the right hand sides are obtained by lifting initial values to the interval $(0,\infty)$ [44, Sections 3, 4]. The systems are extended to the time-interval $(T - T_f, T)$ or $(0, T_f)$ of length T_f. In the first case, the temporal fractional order of the right hand side terms is smaller than $\frac{1}{2}$ or with additional zero initial conditions. Property **P3** of the norm yields continuity of the extension-by-zero-operator $\mathrm{Ext} : Y^s_{(0,T)} \to H^s((T - T_f, T), X)$ with $\|\mathrm{Ext}\|_{\mathcal{L}(Y^s_{(0,T)}, H^s((T-T_f,T),X))} = C$, where $s \in [0, \frac{5}{2}) \backslash \{\frac{1}{2}, \frac{3}{2}\}$, X is a Hilbert space and C is independent of T. Now, the solution theory for the equations can be applied on this extended systems yielding constants C_{T_f} that might depend on T_f but do not depend on T. Due to property **P2** of the norm the equivalence constants of $\|\cdot\|_{H^s((T-T_f,T),X)}$ to an equivalent norm on $H^s((T - T_f, T), X)$ might depend on T_f but not on T (using (6) with $r = 2 + \ell$, $s = 1 + \frac{\ell}{2}$ and (7) explains why we can add norms in the definition of the norm on $\mathbf{E}_T$). Now, by property **P4** of the norm we obtain the estimates on the time interval $(0, T)$ with

constants independent of T. In the second case, the right hand sides can be bounded above by a constant times the norm of the initial values [36, p.22, Remark 3.3], where the appearing constant does not depend on T and we use property **P5** of the norm. In order to obtain an existence and regularity result for the coupled system, a fixed point argument is used [44, Thm. 5.1], which requires property **P6** of the norm. The extension of the local-in-time result to arbitrary time intervals requires **P8**.
The main result of [44] is given by the following theorem (with the same adaptions as in Lemma 3), which shows existence and uniqueness of solutions to the FSI problem if some additional requirements are met.

Theorem 2. *Let Assumption 1 be fulfilled. Denote by $T_f > 0$ a fixed terminal time, let C_S be the constant from Lemma 3, and $K_0 := C_S(\|\mathbf{v}_0\|_{H^{1+\ell}(\Omega_f)^d} + \|\mathbf{w}_1\|_{H^{\frac{1}{2}+\ell+\beta}(\Omega_s)^d})$. Assume that one can find some $0 < T^* < T_f$ such that for all $0 < T \leqslant T^*$ the following estimates hold for arbitrary $M_0 > K_0$, $\mathbf{v}, \mathbf{v}^1, \mathbf{v}^2 \in \mathbf{E}_{T,M_0,\mathbf{v}_0}$ and $p, p^1, p^2 \in P_{T,M_0,\mathbf{v}_0}$:*

$$\mathcal{F}(\mathbf{v},p) \in \mathbf{F}_T, \quad \mathcal{H}(\mathbf{v},p) \in \mathbf{H}_T, \quad \mathcal{G}(\mathbf{v}) \in G_T, \quad \mathbf{g}(\mathbf{v}) \in \mathbf{G}_T \cap H^1((0,T), H^\ell(\Omega_f)^d),$$

and

$$\begin{aligned} &\|\mathcal{F}(\mathbf{v},p)\|_{\mathbf{F}_T} \leqslant C\chi(M_0), \quad \|\mathcal{H}(\mathbf{v},p)\|_{\mathbf{H}_T} \leqslant C\chi(M_0), \\ &\|\mathcal{G}(\mathbf{v})\|_{G_T} \leqslant C\chi(M_0), \qquad \|\mathbf{g}(\mathbf{v})\|_{\mathbf{G}_T} \leqslant C\chi(M_0), \end{aligned} \tag{22}$$

and

$$\begin{aligned} &\|\mathcal{F}(\mathbf{v}^2,p^2) - \mathcal{F}(\mathbf{v}^1,p^1)\|_{\mathbf{F}_T} \leqslant CT^\alpha\chi(M_0)(\|\mathbf{v}^2 - \mathbf{v}^1\|_{\mathbf{E}_T} + \|\nabla p^2 - \nabla p^1\|_{\mathbf{F}_T}), \\ &\|\mathcal{H}(\mathbf{v}^2,p^2) - \mathcal{H}(\mathbf{v}^1,p^1)\|_{\mathbf{H}_T} \leqslant CT^\alpha\chi(M_0)(\|\mathbf{v}^2 - \mathbf{v}^1\|_{\mathbf{E}_T} + \|p^1|_{\Sigma_i^T} - p^2|_{\Sigma_i^T}\|_{H_T}), \\ &\|\mathcal{G}(\mathbf{v}^2) - \mathcal{G}(\mathbf{v}^1)\|_{G_T} \leqslant CT^\alpha\chi(M_0)(\|\mathbf{v}^2 - \mathbf{v}^1\|_{\mathbf{E}_T}), \\ &\|\mathbf{g}(\mathbf{v}^2) - \mathbf{g}(\mathbf{v}^1)\|_{\mathbf{G}_T} \leqslant CT^\alpha\chi(M_0)(\|\mathbf{v}^2 - \mathbf{v}^1\|_{\mathbf{E}_T}), \end{aligned} \tag{23}$$

for some $\alpha > 0$, a positive constant C that does not depend on T but only on T_f and a polynomial χ. Furthermore, let

$$\mathbf{g}(\mathbf{v})|_{\Sigma_f^T} = 0, \quad \mathbf{g}(\mathbf{v})(\cdot,0) = 0, \quad \textit{and} \quad \mathcal{H}(\mathbf{v},p)(\cdot,0) = 0.$$

Then, there exists $T > 0$ and $M_0 < \infty$ such that the system

$$\begin{aligned} \partial_t \mathbf{v} - \nu\Delta \mathbf{v} + \nabla p &= \mathcal{F}(\mathbf{v},p) && \text{in } Q_f^T, \\ \operatorname{div}(\mathbf{v}) = \mathcal{G}(\mathbf{v}) &= \operatorname{div}(\mathbf{g}(\mathbf{v})) && \text{in } Q_f^T, \\ \mathbf{v}(\cdot,0) &= \mathbf{v}_0 && \text{in } \Omega_f, \\ \mathbf{v} &= 0 && \text{on } \Sigma_f^T, \\ \mathbf{v} &= \partial_t \mathbf{w} && \text{on } \Sigma_i^T, \\ \sigma_f(\mathbf{v},p)\mathbf{n}_f &= \sigma_s(\mathbf{w})\mathbf{n}_f + \mathcal{H}(\mathbf{v},p) && \text{on } \Sigma_i^T, \\ \partial_{tt} \mathbf{w} - \operatorname{div}(\sigma_s(\mathbf{w})) &= 0 && \text{in } Q_s^T, \\ \mathbf{w} &= 0 && \text{on } \Sigma_s^T, \\ \mathbf{w}(\cdot,0) = 0, \quad \partial_t \mathbf{w}(\cdot,0) &= \mathbf{w}_1 && \text{in } \Omega_s, \end{aligned} \tag{24}$$

admits a unique solution

$$(\mathbf{v},p,\mathbf{w}) \in \mathbf{E}_{T,M_0,\mathbf{v}_0} \times P_{T,M_0,\mathbf{v}_0} \times \mathbf{W}_T.$$

Proof. This theorem corresponds to a large extent to [44, Thm. 2.1], where the requirements (22) and (23) replace [44, Prop. 6.1]. As a first step the system (24) is reformulated as a fixed point system that can be started with homogeneous $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$. To this end, $(\mathbf{v}^0, p^0, \mathbf{w}^0)$ is introduced as the solution of the system

$$\begin{aligned}
\partial_t \mathbf{v}^0 - \nu\Delta \mathbf{v}^0 + \nabla p^0 &= 0 && \text{in } Q_f^T,\\
\operatorname{div}(\mathbf{v}^0) &= 0 && \text{in } Q_f^T,\\
\mathbf{v}^0(\cdot,0) &= \mathbf{v}_0 && \text{in } \Omega_f,\\
\mathbf{v}^0 &= 0 && \text{on } \Sigma_f^T,\\
\mathbf{v}^0 &= \partial_t \mathbf{w} && \text{on } \Sigma_i^T,\\
\sigma_f(\mathbf{v}^0,p^0)\mathbf{n}_f &= \sigma_s(\mathbf{w}^0)\mathbf{n}_f && \text{on } \Sigma_i^T,\\
\partial_{tt} \mathbf{w}^0 - \operatorname{div}(\sigma_s(\mathbf{w}^0)) &= 0 && \text{in } Q_s^T,\\
\mathbf{w}^0 &= 0 && \text{on } \Sigma_s^T,\\
\mathbf{w}^0(\cdot,0) = 0, \quad \partial_t \mathbf{w}^0(\cdot,0) &= \mathbf{w}_1 && \text{in } \Omega_s,
\end{aligned} \tag{25}$$

that due to Lemma 3 admits for $0 < T \leqslant T_f$ a solution that fulfills

$$\|\mathbf{v}^0\|_{\mathbf{E}_T} + \|\nabla p^0\|_{\mathbf{F}_T} + \|p^0|_{\Sigma_i^T}\|_{H_T} + \|\mathbf{w}^0\|_{\mathbf{W}_T} \leqslant C_S(\|\mathbf{v}_0\|_{H^{1+\ell}(\Omega_f)^d} + \|\mathbf{w}_1\|_{H^{\frac{1}{2}+\ell+\beta}(\Omega_s)^d}) =: K_0,$$

where $C_S > 0$ is a constant that does not depend on T but on T_f. The solution $(\mathbf{v}, p, \mathbf{w})$ of the system (24) then fulfills $\mathbf{v} = \mathbf{u} + \mathbf{v}^0$, $p = q + p^0$ and $\mathbf{w} = \mathbf{z} + \mathbf{w}^0$, where $(\mathbf{u}, q, \mathbf{z})$ is the solution to

$$\begin{aligned}
\partial_t \mathbf{u} - \nu\Delta \mathbf{u} + \nabla q &= \mathcal{F}(\mathbf{u}+\mathbf{v}^0, q+p^0) && \text{in } Q_f^T,\\
\operatorname{div}(\mathbf{u}) = \mathcal{G}(\mathbf{u}+\mathbf{v}^0) &= \operatorname{div}(\mathbf{g}(\mathbf{u}+\mathbf{v}^0)) && \text{in } Q_f^T,\\
\mathbf{u}(\cdot,0) &= 0 && \text{in } \Omega_f,\\
\mathbf{u} &= 0 && \text{on } \Sigma_f^T,\\
\mathbf{u} &= \partial_t \mathbf{z} && \text{on } \Sigma_i^T,\\
\sigma_f(\mathbf{u},q)\mathbf{n}_f &= \sigma_s(\mathbf{z})\mathbf{n}_f + \mathcal{H}(\mathbf{u}+\mathbf{v}^0, q+p^0) && \text{on } \Sigma_i^T,\\
\partial_{tt} \mathbf{z} - \operatorname{div}(\sigma_s(\mathbf{z})) &= 0 && \text{in } Q_s^T,\\
\mathbf{z} &= 0 && \text{on } \Sigma_s^T,\\
\mathbf{z}(\cdot,0) = 0, \quad \partial_t \mathbf{z}(\cdot,0) &= 0 && \text{in } \Omega_s.
\end{aligned} \tag{26}$$

To prove the existence of solutions to the system (24) or the equivalent system (26), the method of successive approximations is used.
Therefore, we show that there exists some $M_0 > K_0$ such that the mapping

$$\mathcal{M}: \quad \mathbf{E}_{T,M_0,\mathbf{v}_0} \times P_{T,M_0,\mathbf{v}_0} \times \mathbf{W}_T \to \mathbf{E}_{T,M_0,\mathbf{v}_0} \times P_{T,M_0,\mathbf{v}_0} \times \mathbf{W}_T, \quad (\mathbf{u},q,\mathbf{z}) \to (\underline{\mathbf{u}}, \underline{q}, \underline{\mathbf{z}}),$$

is well-defined and a contraction with respect to the norm

$$\|(\mathbf{u},q,\mathbf{z})\|_{\mathbf{E}_T\times P_T\times \mathbf{W}_T} := \|\mathbf{u}\|_{\mathbf{E}_T} + \|\nabla q\|_{\mathbf{F}_T} + \|q|_{\Sigma_i^T}\|_{H_T} + \|\mathbf{z}\|_{\mathbf{W}_T},$$

if we choose $T \leqslant T_f$ small enough. Here, $(\underline{\mathbf{u}}, \underline{q}, \underline{\mathbf{z}})$ is defined as the solution of

$$\begin{aligned}
\partial_t \underline{\mathbf{u}} - \nu\Delta \underline{\mathbf{u}} + \nabla \underline{q} &= \mathcal{F}(\mathbf{u}+\mathbf{v}^0, q+p^0) && \text{in } Q_f^T,\\
\operatorname{div}(\underline{\mathbf{u}}) = \mathcal{G}(\mathbf{u}+\mathbf{v}^0) &= \operatorname{div}(\mathbf{g}(\mathbf{u}+\mathbf{v}^0)) && \text{in } Q_f^T,\\
\underline{\mathbf{u}}(\cdot,0) &= 0 && \text{in } \Omega_f,\\
\underline{\mathbf{u}} &= 0 && \text{on } \Sigma_f^T,\\
\underline{\mathbf{u}} &= \partial_t \underline{\mathbf{z}} && \text{on } \Sigma_i^T,\\
\sigma_f(\underline{\mathbf{u}},\underline{q})\mathbf{n}_f &= \sigma_s(\underline{\mathbf{z}})\mathbf{n}_f + \mathcal{H}(\mathbf{u}+\mathbf{v}^0, q+p^0) && \text{on } \Sigma_i^T,\\
\partial_{tt} \underline{\mathbf{z}} - \operatorname{div}(\sigma_s(\underline{\mathbf{z}})) &= 0 && \text{in } Q_s^T,\\
\underline{\mathbf{z}} &= 0 && \text{on } \Sigma_s^T,\\
\underline{\mathbf{z}}(\cdot,0) = 0, \quad \partial_t \underline{\mathbf{z}}(\cdot,0) &= 0 && \text{in } \Omega_s.
\end{aligned} \tag{27}$$

In order to show the contraction property we consider arbitrary $(\mathbf{u}^1, q^1, \mathbf{z}^1), (\mathbf{u}^2, q^2, \mathbf{z}^2) \in \mathbf{E}_{T,M_0,\mathbf{v}_0} \times P_{T,M_0,\mathbf{v}_0} \times \mathbf{W}_T$. Due to Lemma 3 and the inequalities (23) we know that

$$
\begin{aligned}
&\|\mathcal{M}(\mathbf{u}^2, q^2, \mathbf{z}^2) - \mathcal{M}(\mathbf{u}^1, q^1, \mathbf{z}^1)\|_{\mathbf{E}_T \times P_T \times \mathbf{W}_T} \\
&\leqslant C_S(\|\mathcal{F}(\mathbf{u}^2 + \mathbf{v}^0, q^2 + p^0) - \mathcal{F}(\mathbf{u}^1 + \mathbf{v}^0, q^1 + p^0)\|_{\mathbf{F}_T} + \|\mathbf{g}(\mathbf{u}^2 + \mathbf{v}^0) - \mathbf{g}(\mathbf{u}^1 + \mathbf{v}^0)\|_{\mathbf{G}_T} \\
&\qquad + \|\mathcal{G}(\mathbf{u}^2 + \mathbf{v}^0) - \mathcal{G}(\mathbf{u}^1 + \mathbf{v}^0)\|_{G_T} + \|\mathcal{H}(\mathbf{u}^2 + \mathbf{v}^0, q^2 + p^0) - \mathcal{H}(\mathbf{u}^1 + \mathbf{v}^0, q^1 + p^0)\|_{\mathbf{H}_T}) \\
&\leqslant C_S C T^\alpha \chi(M_0) \|(\mathbf{u}^2 + \mathbf{v}^0, q^2 + p^0, 0) - (\mathbf{u}^1 + \mathbf{v}^0, q^1 + p^0, 0)\|_{\mathbf{E}_T \times P_T \times \mathbf{W}_T} \\
&\leqslant C_S C T^\alpha \chi(M_0) \|(\mathbf{u}^2, q^2, 0) - (\mathbf{u}^1, q^1, 0)\|_{\mathbf{E}_T \times P_T \times \mathbf{W}_T},
\end{aligned}
$$

where C is a constant independent of T. If we define $K_1 > 0$ such that

$$\|\mathcal{M}(0,0,0)\|_{\mathbf{E}_T \times P_T \times \mathbf{W}_T} \leqslant K_1,$$

and choose $M_0 > K_1$, then there exists $T > 0$ such that $C_S C T^\alpha \chi(M_0) < 1$ and

$$
\begin{aligned}
\|\mathcal{M}(\mathbf{u}, q, \mathbf{z})\|_{\mathbf{E}_T \times P_T \times \mathbf{W}_T} &\leqslant \|\mathcal{M}(0,0,0)\|_{\mathbf{E}_T \times P_T \times \mathbf{W}_T} + C_S C T^\alpha \chi(M_0) \|(\mathbf{u}, q, 0)\|_{\mathbf{E}_T \times P_T \times \mathbf{W}_T} \\
&\leqslant K_1 + 3 C_S C T^\alpha \chi(M_0) M_0 \leqslant M_0
\end{aligned}
$$

for any $(\mathbf{u}, q, \mathbf{z}) \in \mathbf{E}_{T,M_0,\mathbf{v}_0} \times P_{T,M_0,\mathbf{v}_0} \times \mathbf{W}_T$. Thus, $\mathcal{M}$ is a well-defined contraction and we can apply the fixed point theorem of Banach in order to show existence and uniqueness of the solution to the fixed point equation $\mathcal{M}(\mathbf{u}, q, \mathbf{z}) = (\mathbf{u}, q, \mathbf{z})$ in $\mathbf{E}_{T,M_0,\mathbf{v}_0} \times P_{T,M_0,\mathbf{v}_0} \times \mathbf{W}_T$. □

4 Shape optimization via the method of mappings approach

We consider shape optimization problems governed by the FSI model (1). This results in an optimization problem

$$\min_{(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{w}}, \hat{\Omega}) \in \hat{\mathbf{E}}_T \times \hat{P}_T \times \hat{\mathbf{W}}_T \times \hat{\mathcal{O}}_{ad}} \hat{J}(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{w}}, \hat{\Omega}), \quad \text{s.t. } \hat{E}(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{w}}, \hat{\Omega}) = 0.$$

Here, $\hat{\mathcal{O}}_{ad}$ denotes the set of admissible domains, $\hat{J} : \hat{\mathbf{E}}_T \times \hat{P}_T \times \hat{\mathbf{W}}_T \times \hat{\mathcal{O}}_{ad} \to \mathbb{R}$ is an objective function, and $\hat{E}(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{w}}, \hat{\Omega}) = 0$ if and only if $\hat{\mathbf{v}}$, $\hat{p}$, $\hat{\mathbf{w}}$ fulfill (1), where

$$\hat{E} : \quad \hat{\mathbf{E}}_T \times \hat{P}_T \times \hat{\mathbf{W}}_T \times \hat{\mathcal{O}}_{ad} \to \hat{\mathbf{Z}}_T,$$

and $\hat{\mathbf{Z}}_T$ is a suitable Banach space. There exist different approaches to shape optimization that are closely related to each other. In particular, one can use shape derivatives in the Hadamard-Zolésio sense or one can apply the method of mappings (also called perturbation of identity method) [41]. In this paper, we use the method of mappings for a couple of reasons. It is based on domain transformations of a nominal domain $\tilde{\Omega}$ to represent shapes and thus fits very well to the arbitrary Lagrangian-Eulerian (ALE) approach of which the fully Lagrangian formulation (1) is a special case. Moreover, the method of mappings transforms the shape optimization problem to a nonlinear optimal control problem in a Banach space setting, which is attractive from a theoretical as well as a numerical perspective. The set of admissible domains $\hat{\mathcal{O}}_{ad} := \{\hat{\Omega} \subset \mathbb{R}^d | \ \hat{\Omega} = \tilde{\boldsymbol{\tau}}(\tilde{\Omega}), \tilde{\boldsymbol{\tau}} \in \tilde{\boldsymbol{\mathcal{T}}}_{ad}\}$ comprises all domains that can be obtained by transformation of the shape reference domain $\tilde{\Omega}$ via $\tilde{\boldsymbol{\tau}} \in \tilde{\boldsymbol{\mathcal{T}}}_{ad}$, where $\tilde{\boldsymbol{\mathcal{T}}}_{ad} \subset \tilde{\boldsymbol{\mathcal{T}}}(\tilde{\Omega})$ is a suitable subset of the Banach space $\tilde{\boldsymbol{\mathcal{T}}}(\tilde{\Omega})$ of bicontinuous transformations of $\tilde{\Omega}$. It is convenient to define $\tilde{\mathbf{u}}_\tau := \tilde{\boldsymbol{\tau}} - \mathrm{id}_{\mathbf{z}}$ and

$$\tilde{\mathbf{U}}_{ad} := \{\tilde{\mathbf{u}}_\tau \ : \ \mathbb{R}^d \to \mathbb{R}^d, \text{ s.t. } \mathrm{id}_{\mathbf{z}} + \tilde{\mathbf{u}}_\tau \in \tilde{\boldsymbol{\mathcal{T}}}_{ad}\},$$

and to optimize over $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{U}}_{ad}$ instead of $\boldsymbol{\tau} \in \tilde{\boldsymbol{\mathcal{T}}}_{ad}$. Thus, we obtain the optimization problem

$$\min_{(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{w}}, \tilde{\mathbf{u}}_\tau) \in \tilde{\mathbf{E}}_T \times \tilde{P}_T \times \tilde{\mathbf{W}}_T \times \tilde{\mathbf{U}}_{ad}} \tilde{J}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{w}}, \tilde{\mathbf{u}}_\tau), \quad \text{s.t. } \tilde{E}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{w}}, \tilde{\mathbf{u}}_\tau) = 0,$$

which yields an optimal control setting with the control $\tilde{\mathbf{u}}_\tau$.

Here, $\tilde{\mathbf{v}} := \hat{\mathbf{v}} \circ \tilde{\boldsymbol{\tau}}$, $\tilde{p} := \hat{p} \circ \tilde{\boldsymbol{\tau}}$, $\tilde{\mathbf{w}} := \hat{\mathbf{w}} \circ \tilde{\boldsymbol{\tau}}$, which already requires $\tilde{\mathbf{u}}_\tau \in H^{2+\ell}_{\#}(\tilde{\Omega}_f)^d$ in order to maintain the regularity. Further, $\tilde{E}$ is chosen such that $\tilde{E}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{w}}, \tilde{\mathbf{u}}_\tau) = 0$ if and only if $\hat{E}(\hat{\mathbf{v}} \circ \tilde{\boldsymbol{\tau}}, \hat{p} \circ \tilde{\boldsymbol{\tau}}, \hat{\mathbf{w}} \circ \tilde{\boldsymbol{\tau}}, \tilde{\boldsymbol{\tau}}(\tilde{\Omega})) = 0$. Since the analysis will be carried out on the nominal domain $\tilde{\Omega}$, the geometric assumptions are needed on $\tilde{\Omega}$ instead of $\hat{\Omega}$. We require that two transformations with the same normal displacement of the design boundary part result in the same ALE domains and that the support of the transformation is disjoint from the support of the initial velocity $\mathbf{v}_0$.

4.1 Navier-Stokes-Lamé system on the nominal domain

We apply the method of mappings approach to the FSI problem. In order to maintain the structure required by Theorem 2 we have to ensure that the right hand side of the transformed elasticity equation remains 0. For this purpose, the set of admissible transformations is chosen such that $\tilde{\boldsymbol{\tau}}|_{\tilde{\Omega}_s} = \mathrm{id}_{\mathsf{z}}$ for all $\tilde{\boldsymbol{\tau}} \in \tilde{\boldsymbol{\mathcal{T}}}_{ad}$, i.e. $\tilde{\mathbf{u}}_\tau|_{\tilde{\Omega}_s} = 0$ for all $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{U}}_{ad}$. The transformation of the Navier-Stokes-Lamé system (1) from the reference domain $\hat{\Omega}$ to the shape reference domain $\tilde{\Omega}$ via $\tilde{\boldsymbol{\tau}}$ yields the system

$$
\begin{aligned}
\partial_{\mathsf{t}} \tilde{\mathbf{v}} - \nu \Delta_{\mathsf{z}} \tilde{\mathbf{v}} + \nabla_{\mathsf{z}} \tilde{p} &= \tilde{\mathcal{F}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) && \text{in } \tilde{Q}_f^T, \\
\mathrm{div}_{\mathsf{z}}\,(\tilde{\mathbf{v}}) &= \tilde{\mathcal{G}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) && \text{in } \tilde{Q}_f^T, \\
\tilde{\mathbf{v}}(\cdot, 0) &= \tilde{\mathbf{v}}_0 && \text{in } \tilde{\Omega}_f, \\
\tilde{\mathbf{v}} &= 0 && \text{on } \tilde{\Sigma}_f^T, \\
\tilde{\mathbf{v}} &= \partial_{\mathsf{t}} \tilde{\mathbf{w}} && \text{on } \tilde{\Sigma}_i^T, \\
\sigma_{f,\mathsf{z}}(\tilde{\mathbf{v}}, \tilde{p})\tilde{\mathbf{n}}_f &= \sigma_{s,\mathsf{z}}(\tilde{\mathbf{w}})\tilde{\mathbf{n}}_f + \tilde{\mathcal{H}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) && \text{on } \tilde{\Sigma}_i^T, \\
\partial_{\mathsf{tt}} \tilde{\mathbf{w}} - \mathrm{div}_{\mathsf{z}}\,(\sigma_{s,\mathsf{z}}(\tilde{\mathbf{w}})) &= 0 && \text{in } \tilde{Q}_s^T, \\
\tilde{\mathbf{w}} &= 0 && \text{on } \tilde{\Sigma}_s^T, \\
\tilde{\mathbf{w}}(\cdot, 0) = 0, \quad \partial_{\mathsf{t}} \tilde{\mathbf{w}}(\cdot, 0) &= \tilde{\mathbf{w}}_1 && \text{in } \tilde{\Omega}_s,
\end{aligned}
\tag{28}
$$

where

$$
\sigma_{f,\mathsf{z}}(\tilde{\mathbf{v}}, \tilde{p}) := 2\nu\epsilon_{\mathsf{z}}(\tilde{\mathbf{v}}) - \tilde{p}\mathbf{I}, \quad \sigma_{s,\mathsf{z}}(\tilde{\mathbf{w}}) := \lambda \mathrm{tr}(\epsilon_{\mathsf{z}}(\tilde{\mathbf{w}}))\mathbf{I} + 2\mu\epsilon_{\mathsf{z}}(\tilde{\mathbf{w}}), \quad \epsilon_{\mathsf{z}}(\tilde{\mathbf{w}}) := \frac{1}{2}(D_{\mathsf{z}}\,\tilde{\mathbf{w}} + (D_{\mathsf{z}}\,\tilde{\mathbf{w}})^\top),
$$

$\tilde{\mathbf{v}}_0 = \hat{\mathbf{v}}_0 \circ \tilde{\boldsymbol{\tau}}$, $\tilde{\mathbf{w}}_1 = \hat{\mathbf{w}}_1 \circ \tilde{\boldsymbol{\tau}}$ and the nonlinear terms $\tilde{\mathcal{F}}$, $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{H}}$ are defined by

$$
\begin{aligned}
\tilde{\mathcal{F}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) &= \nu \sum_{j,k,l} (\partial_{\mathsf{x}_j\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_k \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}})(\partial_{\mathsf{y}_k}(\tilde{\boldsymbol{\tau}}^{-1})_l \circ \tilde{\boldsymbol{\tau}})\partial_{\mathsf{z}_l}\tilde{\mathbf{v}} + \nu \sum_{i,k,l} ((\sum_j \partial_{\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_i \partial_{\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_k) \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}})(\partial_{\mathsf{y}_i\mathsf{y}_k}(\tilde{\boldsymbol{\tau}}^{-1})_l \circ \tilde{\boldsymbol{\tau}})\partial_{\mathsf{z}_l}\tilde{\mathbf{v}} \\
&\quad + \nu \sum_{i,k,l,m} ((\sum_j \partial_{\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_i \partial_{\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_k) \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}})((\partial_{\mathsf{y}_k}(\tilde{\boldsymbol{\tau}}^{-1})_l \partial_{\mathsf{y}_i}(\tilde{\boldsymbol{\tau}}^{-1})_m) \circ \tilde{\boldsymbol{\tau}})\partial_{\mathsf{z}_l\mathsf{z}_m}\tilde{\mathbf{v}} - \nu\Delta_{\mathsf{z}}\,\tilde{\mathbf{v}} \\
&\quad + (\mathbf{I} - \tilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top((D_{\mathsf{y}}\,\tilde{\boldsymbol{\tau}}^{-1})^\top \circ \tilde{\boldsymbol{\tau}}))\nabla_{\mathsf{z}}\,\tilde{p}, \\
\tilde{\mathcal{H}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) &= -\nu(D_{\mathsf{z}}\,\tilde{\mathbf{v}}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1}\tilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} + \tilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-T}D_{\mathsf{z}}\,\tilde{\mathbf{v}}^\top)\,\mathrm{cof}(\tilde{\mathbf{F}}_{\boldsymbol{\chi}})\,\mathrm{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})\tilde{\mathbf{n}}_f \\
&\quad + \nu(D_{\mathsf{z}}\,\tilde{\mathbf{v}} + (D_{\mathsf{z}}\,\tilde{\mathbf{v}})^\top)\tilde{\mathbf{n}}_f - \tilde{p}(\mathbf{I} - \mathrm{cof}(\tilde{\mathbf{F}}_{\boldsymbol{\chi}})\,\mathrm{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}))\tilde{\mathbf{n}}_f, \\
\tilde{\mathcal{G}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= D_{\mathsf{z}}\,\tilde{\mathbf{v}} : (\mathbf{I} - \det(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})\det(\tilde{\mathbf{F}}_{\boldsymbol{\chi}})\tilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-T}) = D_{\mathsf{z}}\,\tilde{\mathbf{v}} : (\mathbf{I} - \mathrm{cof}(\tilde{\mathbf{F}}_{\boldsymbol{\chi}})\,\mathrm{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})),
\end{aligned}
$$

where

$$
\tilde{\boldsymbol{\tau}} = \mathrm{id}_{\mathsf{z}} + \tilde{\mathbf{u}}_\tau, \quad \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}} = \hat{\boldsymbol{\chi}} \circ \tilde{\boldsymbol{\tau}}, \quad \tilde{\mathbf{F}}_{\boldsymbol{\chi}} = \hat{\mathbf{F}}_{\boldsymbol{\chi}} \circ \tilde{\boldsymbol{\tau}}, \quad \tilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} = \hat{\mathbf{F}}_{\boldsymbol{\Upsilon}} \circ \tilde{\boldsymbol{\tau}} \tag{29}
$$

and thus $\tilde{\mathbf{F}}_{\boldsymbol{\chi}}(\mathsf{z},\mathsf{t}) := \mathbf{I} + \int_0^{\mathsf{t}} D_{\mathsf{z}}\,\tilde{\mathbf{v}}(\mathsf{z},\mathsf{s})(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}(\mathsf{z}))^{-1}\,\mathrm{ds}$. Moreover, the function $\tilde{\mathbf{g}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = (\mathbf{I} - \mathrm{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^\top \mathrm{cof}(\tilde{\mathbf{F}}_{\boldsymbol{\chi}})^\top)\tilde{\mathbf{v}}$ satisfies $\mathrm{div}_{\mathsf{z}}\,(\tilde{\mathbf{g}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)) = \tilde{\mathcal{G}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)$.
Let $\tilde{\mathbf{T}}$ be defined by (18) and

$$
\tilde{\mathbf{U}} := \{\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{T}} \;:\; \mathrm{supp}(\tilde{\mathbf{u}}_\tau) \cap \mathrm{supp}(\tilde{\mathbf{v}}_0) = \varnothing,\ \tilde{\mathbf{u}}_\tau|_{\tilde{\Omega}_s} = 0\},
$$

which is a closed linear subspace of $H^{2+\ell}(\tilde{\Omega})^d$, be endowed with the norm

$$
\|\cdot\|_{\tilde{\mathbf{U}}} = \|\cdot\|_{H^{2+\ell}(\tilde{\Omega})^d}.
$$

Furthermore, let $\alpha_1 > \|\mathbf{I}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}}$. We consider solutions of the FSI problem for transformations $\mathrm{id}_{\mathsf{z}} + \tilde{\mathbf{u}}_\tau$ induced by displacements $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{V}}$, where

$$
\begin{aligned}
\tilde{\mathbf{V}} := \{\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{U}} \;:\; &\mathrm{id}_{\mathsf{z}} + \tilde{\mathbf{u}}_\tau \text{ can be extended to an orientation-preserving } C^1\text{-diffeomorphism} \\
&\tilde{\boldsymbol{\tau}}_{\mathbb{R}^d} : \mathbb{R}^d \to \mathbb{R}^d \text{ with } \tilde{\boldsymbol{\tau}}_{\mathbb{R}^d} - \mathrm{id}_{\mathsf{z}} \in H^{2+\ell}(\mathbb{R}^d)^d, \\
&\|D_{\mathsf{z}}\,(\mathrm{id}_{\mathsf{z}} + \tilde{\mathbf{u}}_\tau)\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} < \alpha_1,\ \|(D_{\mathsf{z}}\,(\mathrm{id}_{\mathsf{z}} + \tilde{\mathbf{u}}_\tau))^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} < \alpha_1\},
\end{aligned}
\tag{30}
$$

which by Lemma 4 is an open subset of $\tilde{\mathbf{U}}$. In particular, if $\tilde{\mathbf{U}}_{ad} \subset \tilde{\mathbf{V}}$, then our results will hold at any admissible design displacement. Alternatively, the current design of the ALE domain could be viewed as the reference shape domain, making it correspond to $\tilde{\mathbf{u}}_\tau = 0$, and our results then can be applied to study continuity and differentiability w.r.t. variations of this domain.

Remark 4. *1. In [27, Thm. 4.1] it is shown that C^1-diffeomorphisms map bounded Lipschitz domains to bounded Lipschitz domains. Therefore, for all $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{V}}$, $(\mathrm{id}_\mathsf{z} + \tilde{\mathbf{u}}_\tau)(\tilde{\Omega})$ is a bounded Lipschitz domain.*

2. The requirements on the $\mathbb{R}^d$-extended transformations in the definition of the set on the right hand side of (30) allow to apply [30, Lem. B.5, B.6] showing that they map $H^s(\mathbb{R}^d)$-functions to $H^s(\mathbb{R}^d)$-functions for all $0 \leqslant s \leqslant 2 + \ell$. Furthermore, by [30, Cor. 2.1], there exist constants $M > 0$ and $\omega > 0$ such that

$$\|D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_{\mathbb{R}^d}\|_{L^\infty(\mathbb{R}^d)^{d\times d}} < M, \quad \|(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_{\mathbb{R}^d})^{-1}\|_{L^\infty(\mathbb{R}^d)^{d\times d}} < M, \quad \inf_{\mathsf{z}\in\mathbb{R}^d} \det(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_{\mathbb{R}^d}(\mathsf{z})) > \omega. \tag{31}$$

Lemma 4. *For any $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{V}}$ there exists $\rho = \rho(\tilde{\mathbf{u}}_\tau) > 0$ such that $\tilde{\mathbf{v}}_\tau \in \tilde{\mathbf{V}}$ holds for all $\tilde{\mathbf{v}}_\tau \in \tilde{\mathbf{U}}$, $\|\tilde{\mathbf{v}}_\tau - \tilde{\mathbf{u}}_\tau\|_{\tilde{\mathbf{U}}} \leqslant \rho$.*

Proof. Let $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{V}}$ be arbitrary and set $\tilde{\boldsymbol{\tau}} = \mathrm{id}_\mathsf{z} + \tilde{\mathbf{u}}_\tau$. For $\tilde{\mathbf{v}}_\tau \in \tilde{\mathbf{V}}$ we use the notation $\tilde{\boldsymbol{\tau}}_v = \mathrm{id}_\mathsf{z} + \tilde{\mathbf{v}}_\tau$. It has to be verified that there exists $\rho > 0$ such that for all $\tilde{\mathbf{v}}_\tau \in \tilde{\mathbf{V}}$ with $\|\tilde{\mathbf{v}}_\tau - \tilde{\mathbf{u}}_\tau\|_{\tilde{\mathbf{U}}} \leqslant \rho$ the following holds: $\tilde{\boldsymbol{\tau}}_v$ can be extended to an orientation-preserving C^1-diffeomorphism $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}$ of $\mathbb{R}^d$ satisfying $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d} - \mathrm{id}_\mathsf{z} \in H^{2+\ell}(\mathbb{R}^d)^d$, $\|D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_v\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} < \alpha_1$, and $\|(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_v)^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} < \alpha_1$.

The set $\Omega_{\tilde{\mathbf{u}}_\tau} := \tilde{\boldsymbol{\tau}}(\tilde{\Omega})$ is a bounded Lipschitz domain by the definition of $\tilde{\mathbf{V}}$ and Remark 4. Using, e.g., [50, Thm. 5, p. 181] combined with interpolation, there exists a bounded linear extension operator $H^{2+\ell}(\Omega_{\tilde{\mathbf{u}}_\tau})^d \to H^{2+\ell}(\mathbb{R}^d)^d$. Further, the embeddings $H^{2+\ell}(\Omega_{\tilde{\mathbf{u}}_\tau})^d \subset W^{1,\infty}(\Omega_{\tilde{\mathbf{u}}_\tau})^d$ and $H^{2+\ell}(\mathbb{R}^d)^d \subset W^{1,\infty}(\mathbb{R}^d)^d$ are continuous.

Now $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{V}}$ implies $\|D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} =: \alpha_1' < \alpha_1$. Hence, we obtain as required $\|D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_v\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \leqslant \|D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} + \|D_\mathsf{z}\,(\tilde{\mathbf{v}}_\tau - \tilde{\mathbf{u}}_\tau)\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \leqslant \alpha_1' + \rho < \alpha_1$ for ρ sufficiently small.

Denote by $\tilde{\boldsymbol{\tau}}_{\mathbb{R}^d} \in H^{2+\ell}(\mathbb{R}^d)^d$ the orientation-preserving C^1-diffeomorphism that extends $\tilde{\boldsymbol{\tau}}$. Then, by part 2 of Remark 4, there exist constants $M > 0$ and $\omega > 0$ such that (31) holds.

We use the extension operator to obtain $\mathbf{h}_\tau \in H^{2+\ell}(\mathbb{R}^d)^d$ with $\mathbf{h}_\tau|_{\tilde{\Omega}} = \tilde{\mathbf{v}}_\tau - \tilde{\mathbf{u}}_\tau$, $\|\mathbf{h}_\tau\|_{H^{2+\ell}(\mathbb{R}^d)^d} \leqslant C\|\tilde{\mathbf{v}}_\tau - \tilde{\mathbf{u}}_\tau\|_{H^{2+\ell}(\tilde{\Omega})^d}$, and $\|\mathbf{h}_\tau\|_{W^{1,\infty}(\mathbb{R}^d)^d} \leqslant C\|\tilde{\mathbf{v}}_\tau - \tilde{\mathbf{u}}_\tau\|_{H^{2+\ell}(\tilde{\Omega})^d}$. Setting $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d} = \tilde{\boldsymbol{\tau}}_{\mathbb{R}^d} + \mathbf{h}_\tau$, there holds $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}|_{\tilde{\Omega}} = \tilde{\boldsymbol{\tau}}_v$ and $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d} - \mathrm{id}_\mathsf{z} = (\tilde{\boldsymbol{\tau}}_{\mathbb{R}^d} - \mathrm{id}_\mathsf{z}) + \mathbf{h}_\tau \in H^{2+\ell}(\mathbb{R}^d)^d$. By a Sobolev embedding we obtain also that $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}$ is C^1.

Since $W^{1,\infty}(\mathbb{R}^d)$ and $C^{0,1}(\mathbb{R}^d)$, are equal with equivalent norms, see [25, Thm. 4.1, Rem. 4.2], there there exists $c' > 0$ such that any $f \in W^{1,\infty}(\mathbb{R}^d)^d$ has a Lipschitz continuous representative with modulus $\leqslant c'\|f\|_{W^{1,\infty}(\mathbb{R}^d)^d}$.

We now show that $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d} : \mathbb{R}^d \to \mathbb{R}^d$ is bijective. In fact for any fixed $\mathsf{z}' \in \mathbb{R}^d$, the equation $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}(\mathsf{z}) = \mathsf{z}'$ can be written as

$$\mathsf{z} = \tilde{\boldsymbol{\tau}}_{\mathbb{R}^d}^{-1}(\mathsf{z}' - \mathbf{h}_\tau(\mathsf{z})) =: \mathcal{A}(\mathsf{z}'; \mathsf{z}).$$

For sufficiently small ρ, the map $\mathcal{A}(\mathsf{z}'; \cdot)$ is a contraction since, for any $\mathsf{z}_1, \mathsf{z}_2 \in \mathbb{R}^d$, by using (31)

$$\begin{aligned}
\|\tilde{\boldsymbol{\tau}}_{\mathbb{R}^d}^{-1}(\mathsf{z}' - \mathbf{h}_\tau(\mathsf{z}_1)) - \tilde{\boldsymbol{\tau}}_{\mathbb{R}^d}^{-1}(\mathsf{z}' - \mathbf{h}_\tau(\mathsf{z}_2))\| &\leqslant c'\|(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_{\mathbb{R}^d})^{-1}\|_{L^\infty(\mathbb{R}^d)^{d\times d}}\|\mathbf{h}_\tau(\mathsf{z}_1) - \mathbf{h}_\tau(\mathsf{z}_2)\| \\
&\leqslant Mc'\|\mathbf{h}_\tau\|_{W^{1,\infty}(\mathbb{R}^d)^d}\|\mathsf{z}_1 - \mathsf{z}_2\| \leqslant CMc'\rho\|\mathsf{z}_1 - \mathsf{z}_2\|.
\end{aligned}$$

Hence, by the Banach fixed point theorem, if ρ is sufficiently small, then for any $\mathsf{z}' \in \mathbb{R}^d$ there exists a unique $\mathsf{z} \in \mathbb{R}^d$ with $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}(\mathsf{z}) = \mathsf{z}'$.

We show next that $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}^{-1}$ is C^1. From (31) and $\|\mathbf{h}_\tau\|_{W^{1,\infty}(\mathbb{R}^d)^d} \leqslant C\rho$ we obtain a constant $C' > 0$ with

$$\begin{aligned}
\inf_{\mathsf{z}\in\mathbb{R}^d} \det(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}(\mathsf{z})) &\geqslant \omega - \|\det(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}) - \det(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_{\mathbb{R}^d})\|_{L^\infty(\mathbb{R}^d)} \geqslant \omega - C'\|D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d} - D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_{\mathbb{R}^d}\|_{L^\infty(\mathbb{R}^d)^{d\times d}} \\
&\geqslant \omega - C'\|\mathbf{h}_\tau\|_{W^{1,\infty}(\mathbb{R}^d)^d} \geqslant \omega - CC'\rho.
\end{aligned}$$

Hence, for $\rho > 0$ small enough we obtain $\det(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}(\mathsf{z})) > \omega/2$ for all $\mathsf{z} \in \mathbb{R}^d$ and thus $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}^{-1}$ is C^1 by the inverse function theorem.

We have shown that for $\rho > 0$ small enough $\det(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}) \geqslant \omega/2$. Now $(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_v)^{-1} = 1/\det(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_v)\mathrm{cof}(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_v)^\top$. Since by Lemma A.4 products of functions in $H^{1+\ell}(\tilde{\Omega}_f)$ are again in $H^{1+\ell}(\tilde{\Omega}_f)$, we have $\det(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_v), \mathrm{cof}(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_v) \in H^{1+\ell}(\tilde{\Omega}_f)$ and since $\det(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_v) \geqslant \omega/2 > 0$ by [45, pp. 336 and 297] also $1/\det(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_v) \in H^{1+\ell}(\tilde{\Omega}_f)$. Hence, $(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}}_v)^{-1} \in H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}$ for $\|\tilde{\mathbf{v}}_\tau - \tilde{\mathbf{u}}_\tau\|_{\tilde{\mathbf{U}}} \leqslant \rho$.

Finally, with a constant $C' > 0$ we obtain

$$\begin{aligned}\|(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v)^{-1} - (D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} &= \|(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v)^{-1}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}} - D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v)(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \\ &\leqslant C'\|(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v)^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}}\|(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}}\|D_{\mathsf{z}}\,(\tilde{\mathbf{v}}_\tau - \tilde{\mathbf{u}}_\tau)\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \\ &\leqslant \alpha_1 C'(\|(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v)^{-1} - (D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} + \alpha_1)\rho,\end{aligned}$$

from which $\|(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v)^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} < \alpha_1$ follows if ρ is chosen sufficiently small.

□

Let with $\rho = \rho(0)$ according to Lemma 4

$$\tilde{\mathbf{V}}_\rho := \{\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{V}} \;:\; \|\tilde{\mathbf{u}}_\tau\|_{\tilde{\mathbf{U}}} < \rho\}. \tag{32}$$

Then $\tilde{\mathbf{V}}_\rho$ is by Lemma 4 an open subset of $\tilde{\mathbf{U}}$ and we will study the differentiability of the solution of (28) on $\tilde{\mathbf{V}}_\rho$ at $\tilde{\mathbf{u}}_\tau = 0$.

The choice of the space of admissible transformations restricts the shape optimization to the optimal design of the fluid domain, but keeps the interface in the Lagrangian frame fixed. The boundedness properties of $\tilde{\mathbf{V}}$ allow us to establish estimates of the right hand sides in (28). The following Lemma is a helpful tool that takes the special structure of the right hand side terms into account.

Lemma 5. *Let $T > 0$, $k \in \mathbb{N}$, $k \geqslant 2$, X, X_j, Y, W_n, Z be real, separable Hilbert spaces, $1 \leqslant j \leqslant k$, $2 \leqslant n \leqslant k-1$, $s_1 \in [0,1]\backslash\{\frac{1}{2}\}$, $s_i \in (\frac{1}{2}, 1]$ for $2 \leqslant i \leqslant k$ and $0 \leqslant s \leqslant \min_j s_j$. Let $m_1 : X_1 \times W_2 \to X$, $m_l : X_l \times W_{l+1} \to W_l$ for $2 \leqslant l \leqslant k-2$ and $m_{k-1} : X_{k-1} \times X_k \to W_{k-1}$ be continuous bilinear forms, $m : \times_{j=1}^{k} X_j \to X$ be defined by $m(x_1, \ldots, x_k) = m_1(x_1, m_2(x_2, \ldots))$ and $\mathcal{T}_j : Y \times Z \to S_j$, where $S_j := H^{s_j}((0,T), X_j)$ is endowed with the norm*

- $\|\cdot\|_{S_j} := \|\cdot\|_{H^{s_j}((0,T),X_j)}$, *if $s_j \in [0, \frac{1}{2})$,*
- $\|\cdot\|_{S_j} := (\|\cdot\|^2_{H^{s_j}((0,T),X_j)} + \|\cdot(0)\|^2_{X_j})^{\frac{1}{2}}$, *if $s_j \in (\frac{1}{2}, 1]$,*

and $S := H^s((0,T), X)$ be endowed with the analogously defined norm $\|\cdot\|_S$. Furthermore, let $\mathcal{T} : Y \times Z \to S$ be defined by

$$\mathcal{T}(y,z) = m(\mathcal{T}_1(y,z), \ldots, \mathcal{T}_k(y,z)).$$

1. *Let $M_j > 0$, $\tilde{Y} \subset Y$ and $\tilde{Z} \subset Z$ be such that $\|\mathcal{T}_j(y,z)\|_{S_j} \leqslant M_j$ for all $(y,z) \in \tilde{Y} \times \tilde{Z}$, $1 \leqslant j \leqslant k$. Then, there exists a constant $C > 0$ that is independent of T such that $\|\mathcal{T}(y,z)\|_S \leqslant C\Pi_j M_j$ for all $(y,z) \in \tilde{Y} \times \tilde{Z}$.*
2. *Let in addition to 1. $\mathcal{T}_j : Y \times Z \to S_j$ be Lipschitz continuous on $\tilde{Y} \times \tilde{Z}$ for all $1 \leqslant j \leqslant k$, i.e., there exist $M_{j,1}, M_{j,2} > 0$ such that $\|\mathcal{T}_j(y_2,z_2) - \mathcal{T}_j(y_1,z_1)\|_{S_j} \leqslant M_{j,1}\|y_2 - y_1\|_Y + M_{j,2}\|z_2 - z_1\|_Z$ for arbitrary $y_1, y_2 \in \tilde{Y}$ and $z_1, z_2 \in \tilde{Z}$. Then, $\|\mathcal{T}(y_2,z_2) - \mathcal{T}(y_1,z_1)\|_{H^s((0,T),X)} \leqslant C(\max_j(M_{j,1}\Pi_{n\neq j}M_n)\|y_2 - y_1\|_Y + \max_j(M_{j,2}\Pi_{n\neq j}M_n)\|z_2 - z_1\|_Z)$ with a constant $C > 0$ that is independent of T.*
3. *Let (y_1, z_1) be an element of the relative interior of $\tilde{Y} \times \tilde{Z}$ and $\mathcal{T}_j : \tilde{Y} \times \tilde{Z} \to S_j$ be Fréchet differentiable in (y_1, z_1) for all $1 \leqslant j \leqslant k$. Then, $\mathcal{T} : \tilde{Y} \times \tilde{Z} \to S$ is Fréchet differentiable in (y_1, z_1).*

Proof. By recursively applying Lemmas A.4 and A.1 it can be verified that $m : \Pi_j S_j \to H^s((0,T), X)$ is a continuous multilinear form that fulfills

$$\|m(x_1, \ldots, x_k)\|_{H^s((0,T),X)} \leqslant C\Pi_j\|x_j\|_{S_j},$$

where C is a constant independent of T. Assertion 1 follow immediately if one directly uses the continuity properties of m in order to estimate the norms at the initial value $\mathsf{t} = 0$. Further, for $y_1, y_2 \in \tilde{Y}$, $z_1, z_2 \in \tilde{Z}$ we have

$$\begin{aligned}&m(\mathcal{T}_1(y_2,z_2), \ldots, \mathcal{T}_k(y_2,z_2)) - m(\mathcal{T}_1(y_1,z_1), \ldots, \mathcal{T}_k(y_1,z_1)) \\ &= m((\mathcal{T}_1(y_2,z_2) - \mathcal{T}_1(y_1,z_1)), \mathcal{T}_2(y_2,z_2), \mathcal{T}_3(y_2,z_2) \ldots, \mathcal{T}_k(y_2,z_2)) \\ &\quad + m(\mathcal{T}_1(y_1,z_1), (\mathcal{T}_2(y_2,z_2) - \mathcal{T}_2(y_1,z_1)), \mathcal{T}_3(y_2,z_2), \ldots, \mathcal{T}_k(y_2,z_2)) \\ &\quad + \cdots + m(\mathcal{T}_1(y_1,z_1), \ldots, \mathcal{T}_{k-1}(y_1,z_1), (\mathcal{T}_k(y_2,z_2) - \mathcal{T}_k(y_1,z_1))),\end{aligned}$$

which implies

$$\begin{aligned}
&\|m(\mathcal{T}_1(y_2,z_2),\dots,\mathcal{T}_k(y_2,z_2)) - m(\mathcal{T}_1(y_1,z_1),\dots,\mathcal{T}_k(y_1,z_1))\|_S \\
&\leqslant C\sum_{j=1}^{k}((\Pi_{n<j}\|\mathcal{T}_n(y_1,z_1)\|_{S_n})(\Pi_{n>j}\|\mathcal{T}_n(y_2,z_2)\|_{S_n})\|\mathcal{T}_j(y_2,z_2)-\mathcal{T}_j(y_1,z_1)\|_{S_j}) \\
&\leqslant C(\max_j(M_{j,1}\Pi_{n\neq j}M_n)\|y_2-y_1\|_Y + \max_j(M_{j,2}\Pi_{n\neq j}M_n)\|z_2-z_1\|_Z)
\end{aligned}$$

for a generic constant C independent of T and therefore assertion 2. Since a continuous multilinear form is infinitely differentiable 3 follows with the chain rule. □

Lemma 6. *Let $T > 0$, $k \in \mathbb{N}$, $X_1, X_2, X_{j,1}, X_{j,2}, Y, Z$ be real, separable Hilbert spaces, $1 \leqslant j \leqslant k$, $s_1 \in [0,1]\backslash\{\frac{1}{2}\}$, $s_i \in (\frac{1}{2}, 1]$ for $2 \leqslant i \leqslant k$. Let m be a k-linear form that is recursively constructed via bilinear forms as in Lemma 5 such that $m : \times_{j=1}^{k} X_{j,1} \to X_1$ and $m : \times_{j=1}^{k} X_{j,1+\delta_{jl}} \to X_2$ are continuous for all $1 \leqslant l \leqslant k$, where δ_{jl} denotes the Kronecker delta. Let $0 \leqslant s \leqslant \min_j s_j$ and*

$$S_j := H^1((0,T), X_{j,1}) \cap H^{1+s_j}((0,T), X_{j,2})$$

be endowed with the norm

- $\|\cdot\|_{S_j} := (\|\cdot\|^2_{H^1((0,T),X_{j,1})\cap H^{1+s_j}((0,T),X_{j,2})} + \|\cdot(0)\|^2_{X_{j,1}})^{\frac{1}{2}}$, *if $s_j \in [0, \frac{1}{2})$.*
- $\|\cdot\|_{S_j} := (\|\cdot\|^2_{H^1((0,T),X_{j,1})\cap H^{1+s_j}((0,T),X_{j,2})} + \|\cdot(0)\|^2_{X_{j,1}} + \|\partial_t(\cdot)(0)\|^2_{X_{j,2}})^{\frac{1}{2}}$, *if $s_j \in (\frac{1}{2}, 1]$.*

and $S := H^1((0,T), X_1) \cap H^{1+s}((0,T), X_2)$ be endowed with the analogously defined norm $\|\cdot\|_S$. Further, let $\mathcal{T}_j : Y \times Z \to S_j$ and $\mathcal{T} : Y \times Z \to S$ be defined by

$$\mathcal{T}(y,z) = m(\mathcal{T}_1(y,z),\dots,\mathcal{T}_k(y,z)).$$

Then,

1. *Let $M_j > 0$, $\tilde{Y} \subset Y$ and $\tilde{Z} \subset Z$ be such that $\|\mathcal{T}_j(y,z)\|_{S_j} \leqslant M_j$ for all $(y,z) \in \tilde{Y} \times \tilde{Z}$, $1 \leqslant j \leqslant k$. Then, there exists a constant $C > 0$ that is independent of T such that $\|\mathcal{T}(y,z)\|_S \leqslant C\Pi_j M_j$ for all $(y,z) \in \tilde{Y} \times \tilde{Z}$.*
2. *Let in addition to 1. $\mathcal{T}_j : Y \times Z \to S_j$ be Lipschitz continuous on $\tilde{Y} \times \tilde{Z}$ for all $1 \leqslant j \leqslant k$, i.e., there exist $M_{j,1}, M_{j,2} > 0$ such that $\|\mathcal{T}_j(y_2,z_2) - \mathcal{T}_j(y_1,z_1)\|_{S_j} \leqslant M_{j,1}\|y_2 - y_1\|_Y + M_{j,2}\|z_2 - z_1\|_Z$ for arbitrary $y_1, y_2 \in \tilde{Y}$ and $z_1, z_2 \in \tilde{Z}$. Then, $\|\mathcal{T}(y_2,z_2) - \mathcal{T}(y_1,z_1)\|_S \leqslant C(\max_j(M_{j,1}\Pi_{n\neq j}M_n)\|y_2 - y_1\|_Y + \max_j(M_{j,2}\Pi_{n\neq j}M_n)\|z_2 - z_1\|_Z)$ with a constant $C > 0$ that is independent of T.*
3. *Let (y_1, z_1) be an element of the relative interior of $\tilde{Y} \times \tilde{Z}$ and $\mathcal{T}_j : \tilde{Y} \times \tilde{Z} \to S_j$ be Fréchet differentiable in (y_1, z_1) for all $1 \leqslant j \leqslant k$. Then, $\mathcal{T} : \tilde{Y} \times \tilde{Z} \to S$ is Fréchet differentiable in (y_1, z_1).*

Proof. We recursively apply Lemma A.1 in order to get continuity of $m : \times_{j=1}^{k} S_j \to L^2((0,T), X_1)$, $\partial_t m : \times_{j=1}^{k} S_j \to L^2((0,T), X_1)$, as well as, $\partial_t m : \times_{j=1}^{k} S_j \to H^s((0,T), X_2)$ and use that

$$\partial_t m(x_1,\dots,x_k) = m(\partial_t x_1, x_2,\dots,x_k) + m(x_1, \partial_t x_2,\dots,x_k) + \cdots + m(x_1, x_2,\dots,\partial_t x_k).$$

It holds

$$\|m(x_1,\dots,x_k)\|_{L^2((0,T),X_1)} \leqslant C\|x_1\|_{L^2((0,T),X_{1,1})}\Pi_{i=2}^{k}\|x_i\|_{\hat{S}_i},$$

$$\|m(x_1,\dots,\partial_t x_j,\dots,x_k)\|_{L^2((0,T),X_1)} \leqslant C\|\partial_t x_j\|_{L^2((0,T),X_{j,1})}\Pi_{i\neq j}\|x_i\|_{\hat{S}_i},$$

where $\hat{S}_j := H^1((0,T), X_{j,1})$ is endowed with the norm $\|\cdot\|_{\hat{S}_j} := (\|\cdot\|^2_{H^1((0,T),X_{j,1})} + \|\cdot(0)\|^2_{X_{j,1}})^{\frac{1}{2}}$ for $1 \leqslant j \leqslant k$. Furthermore, there holds

$$\|m(x_1,\dots,\partial_t x_j,\dots,x_k)\|_{H^s((0,T),X_2)} \leqslant C\|\partial_t x_j\|_{\tilde{S}_j}\Pi_{i\neq j}\|x_i\|_{\hat{S}_i},$$

where $\tilde{S}_j := H^{s_j}((0,T), X_{j,2})$ is endowed with the norm

- $\|\cdot\|_{\tilde S_j} := \|\cdot\|_{H^{s_j}((0,T),X_{j,2})}$, if $s_j \in [0, \frac{1}{2})$.
- $\|\cdot\|_{\tilde S_j} := (\|\cdot\|^2_{H^{s_j}((0,T),X_{j,2})} + \|\cdot(0)\|^2_{X_{j,2}})^{\frac{1}{2}}$, if $s_j \in (\frac{1}{2}, 1]$.

In order to show the boundedness in the norm $\|\cdot\|_S$ the initial values have to be bounded appropriately. However, this is ensured by the continuity properties of the multilinear form m. Moreover, property **P1** of the norm is used. The assertions now follow directly as in Lemma 5. □

Furthermore, the following results will be needed for establishing the required right hand side estimates.

Lemma 7. *1. Let $\tilde f, \tilde g \in \tilde S_T$. Then, $\tilde f \tilde g \in \tilde S_T$ and $\|\tilde f \tilde g\|_{\tilde S_T} \leqslant C \|\tilde f\|_{\tilde S_T} \|\tilde g\|_{\tilde S_T}$ with a constant C that is independent of T.*

2. Let $\tilde f \in \tilde S_T$. If $\tilde f \geqslant \omega > 0$ holds a.e. on $\tilde\Omega_f^T$ with a constant $\omega > 0$ then $\tilde f^{-1} \in \tilde S_T$ and

$$\|\tilde f^{-1}\|_{\tilde S_T} \leqslant C(1 + \|\tilde f\|_{\tilde S_T})^{10} \|\tilde f\|_{\tilde S_T}$$

for a constant C that is independent of T.

Proof. 1. The bilinear form $m(x_1, x_2) := x_1 \cdot x_2$ is by Lemma A.4 continuous as a mapping $L^2(\tilde\Omega_f) \times H^{1+\ell}(\tilde\Omega_f) \to L^2(\tilde\Omega_f)$ and as a mapping $H^{1+\ell}(\tilde\Omega_f) \times H^{1+\ell}(\tilde\Omega_f) \to H^{1+\ell}(\tilde\Omega_f)$. Therefore, Lemma 6 implies $\|\tilde f \tilde g\|_{\tilde S_T} \leqslant C \|\tilde f\|_{\tilde S_T} \|\tilde g\|_{\tilde S_T}$ for a constant C that is independent of T. Here, we recall that the norm on $\tilde S_T$ is defined by (19).

2. By [44, Lemma A.7] we know that

$$\|\tilde f^{-1}\|_{L^2((0,T),H^{1+\ell}(\tilde\Omega_f)} \leqslant C(1 + \|\tilde f\|_{H^1((0,T),H^{1+\ell}(\tilde\Omega_f))} + \|\tilde f(\cdot,0)\|_{H^{1+\ell}(\tilde\Omega_f)}) \|\tilde f\|_{L^2((0,T),H^{1+\ell}(\tilde\Omega_f))},$$
$$\|\partial_t \tilde f^{-1}\|_{L^2((0,T),H^{1+\ell}(\tilde\Omega_f)} \leqslant C(1 + \|\tilde f\|_{H^1((0,T),H^{1+\ell}(\tilde\Omega_f))} + \|\tilde f(\cdot,0)\|_{H^{1+\ell}(\tilde\Omega_f)})^4 \|\tilde f\|_{H^1((0,T),H^{1+\ell}(\tilde\Omega_f))}.$$

for a constant C independent of T. The proof of this Lemma also shows that

$$\|\tilde f^{-1}(\cdot,0)\|_{H^{1+\ell}(\tilde\Omega_f)} \leqslant C(1 + \|\tilde f(\cdot,0)\|_{H^{1+\ell}(\tilde\Omega_f)}) \|\tilde f(\cdot,0)\|_{H^{1+\ell}(\tilde\Omega_f)}.$$

Let C now denote a generic constant independent of T. In order to bound $\|\partial_t \tilde f^{-1}(\cdot,0)\|_{L^2(\tilde\Omega_f)}$, we consider $G \in \mathcal{C}^\infty(\mathbb{R})$ such that $G(0) = 0$ and $G(x) = x^{-1}$ for all $x \geqslant \omega$. Then,

$$\begin{aligned}
\|\partial_t \tilde f^{-1}(\cdot,0)\|^2_{L^2(\tilde\Omega_f)} &= \|\partial_t G(\tilde f)(\cdot,0)\|^2_{L^2(\tilde\Omega_f)} = \|G'(\tilde f)(\cdot,0)\partial_t \tilde f(\cdot,0)\|^2_{L^2(\tilde\Omega_f)} \\
&= \int_{\tilde\Omega_f} (G'(\tilde f)(\mathsf{z},0)\partial_t \tilde f(\mathsf{z},0))^2 \, \mathrm{d}\mathsf{z} \\
&\leqslant \sup_{\mathsf{z}\in\tilde\Omega_f} |G'(\tilde f)(\mathsf{z},0)| \|\partial_t \tilde f(\cdot,0)\|^2_{L^2(\tilde\Omega_f)} \leqslant C \|\partial_t \tilde f(\cdot,0)\|^2_{L^2(\tilde\Omega_f)}.
\end{aligned}$$

These estimates imply

$$\begin{aligned}
\|\tilde f^{-1}\|_{H^1((0,T),H^{1+\ell}(\tilde\Omega_f)} &\leqslant C(1 + \|\tilde f\|_{H^1((0,T),H^{1+\ell}(\tilde\Omega_f))} + \|\tilde f(\cdot,0)\|_{H^{1+\ell}(\tilde\Omega_f)})^4 \|\tilde f\|_{H^1((0,T),H^{1+\ell}(\tilde\Omega_f))} \\
&\leqslant C(1 + \|\tilde f\|_{H^1((0,T),H^{1+\ell}(\tilde\Omega_f))} + \|\tilde f(\cdot,0)\|_{H^{1+\ell}(\tilde\Omega_f)})^5.
\end{aligned}$$

Now,

$$\|\tilde f^{-1}\|_{L^2((0,T),L^2(\tilde\Omega_f))} \leqslant C \|\tilde f^{-1}\|_{L^2((0,T),H^{1+\ell}(\tilde\Omega_f))}$$

for a constant independent of T and it remains to estimate $\|\partial_t \tilde f^{-1}\|_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),L^2(\tilde\Omega_f))}$. We obtain with Lemma A.1, 2.

$$\begin{aligned}
&\|\partial_t \tilde f^{-1}\|_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),L^2(\tilde\Omega_f))} = \|\tilde f^{-2}\partial_t \tilde f\|_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),L^2(\tilde\Omega_f))} \\
&\leqslant C(\|\tilde f^{-2}\|_{H^1((0,T),H^{1+\ell}(\tilde\Omega_f))} + \|\tilde f^{-2}(\cdot,0)\|_{H^{1+\ell}(\tilde\Omega_f)})(\|\partial_t \tilde f\|_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),L^2(\tilde\Omega_f))} + \|\partial_t \tilde f(\cdot,0)\|_{L^2(\tilde\Omega_f)}) \\
&\leqslant C(\|\tilde f^{-1}\|_{H^1((0,T),H^{1+\ell}(\tilde\Omega_f))} + \|\tilde f^{-1}(\cdot,0)\|_{H^{1+\ell}(\tilde\Omega_f)})^2(\|\partial_t \tilde f\|_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),L^2(\tilde\Omega_f))} + \|\partial_t \tilde f(\cdot,0)\|_{L^2(\tilde\Omega_f)}) \\
&\leqslant C(1 + \|\tilde f\|_{H^1((0,T),H^{1+\ell}(\tilde\Omega_f))} + \|\tilde f(\cdot,0)\|_{H^{1+\ell}(\tilde\Omega_f)})^{10}(\|\tilde f\|_{H^{\frac{3}{2}+\frac{\ell}{2}}((0,T),L^2(\tilde\Omega_f))} + \|\partial_t \tilde f(\cdot,0)\|_{L^2(\tilde\Omega_f)}).
\end{aligned}$$

Combining the estimates implies the assertion. □

Lemmas 5, 6 and 7 allow to estimate products of rational functions in terms of the norms of the factors. We start by estimating the appearing factors.
Since $\mathsf{z} = \tilde{\boldsymbol{\tau}}^{-1}(\tilde{\boldsymbol{\tau}}(\mathsf{z}))$, it follows that $\mathbf{I} = (D_{\mathsf{y}}\,\tilde{\boldsymbol{\tau}}^{-1} \circ \tilde{\boldsymbol{\tau}})D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}$ and

$$D_{\mathsf{y}}\,\tilde{\boldsymbol{\tau}}^{-1} \circ \tilde{\boldsymbol{\tau}} = (D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1}. \tag{33}$$

Furthermore, for arbitrary invertible matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{d\times d}$ one has

$$\begin{aligned}&\mathbf{A}^{-1} - \mathbf{B}^{-1} = \mathbf{B}^{-1}(\mathbf{B}-\mathbf{A})\mathbf{A}^{-1},\\ &\mathbf{A}^{-1} - \mathbf{B}^{-1} - \mathbf{B}^{-1}(\mathbf{B}-\mathbf{A})\mathbf{B}^{-1} = \mathbf{B}^{-1}(\mathbf{B}-\mathbf{A})(\mathbf{A}^{-1}-\mathbf{B}^{-1}) = \mathbf{B}^{-1}(\mathbf{B}-\mathbf{A})\mathbf{B}^{-1}(\mathbf{B}-\mathbf{A})\mathbf{A}^{-1}.\end{aligned} \tag{34}$$

Let $\tilde{\mathbf{u}}_\tau^i \in \tilde{\mathbf{V}}$, then $\tilde{\boldsymbol{\tau}}^i := \mathrm{id}_{\mathsf{z}} + \tilde{\mathbf{u}}_\tau^i$, $i = 1, 2$, satisfy by Lemma A.4, (34) and the definition of $\tilde{\mathbf{V}}$

$$\begin{aligned}&\|(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^1)^{-1} - (D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^2)^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \leqslant C\|\tilde{\boldsymbol{\tau}}^1 - \tilde{\boldsymbol{\tau}}^2\|_{H^{2+\ell}(\tilde{\Omega}_f)^d} = C\|\tilde{\mathbf{u}}_\tau^1 - \tilde{\mathbf{u}}_\tau^2\|_{H^{2+\ell}(\tilde{\Omega}_f)^d},\\ &\|(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^1)^{-1} - (D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^2)^{-1} + (D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^2)^{-1}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^1 - D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^2)(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^2)^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \leqslant C\|\tilde{\mathbf{u}}_\tau^1 - \tilde{\mathbf{u}}_\tau^2\|^2_{H^{2+\ell}(\tilde{\Omega}_f)^d}.\end{aligned} \tag{35}$$

We define analogously to $\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}$ in (29)

$$\widetilde{\mathbf{F}}^i_{\boldsymbol{\chi}}(\mathsf{z},\mathsf{t}) = \widetilde{\mathbf{F}}^i_{\boldsymbol{\chi}}(\mathsf{z},\mathsf{t};\tilde{\mathbf{v}}^i,\tilde{\mathbf{u}}^i_\tau) := \mathbf{I} + \int_0^{\mathsf{t}} D_{\mathsf{z}}\,\tilde{\mathbf{v}}^i(\mathsf{z},\mathsf{s})(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^i(\mathsf{z}))^{-1}\,\mathrm{d}\mathsf{s},\ i \in \{1,2\}. \tag{36}$$

Lemma 8. *Let Assumption 1 be satisfied. Let $M_0 > 0$, $\alpha \in (0,1)$ and $\alpha_1 > 0$. Then, there exists $T_\alpha > 0$ such that $\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}(\cdot,\mathsf{t})$ is invertible, and $\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}(\cdot,\mathsf{t})) \geqslant \alpha$ for all $\mathsf{t} \in (0,T_\alpha)$ and for all $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{V}}$ and $\tilde{\mathbf{v}} \in \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0}$. In addition, for each of the following terms, there exists a constant $C > 0$ independent of T such that for all $0 < T < T_\alpha$, $\tilde{\mathbf{v}}, \tilde{\mathbf{v}}^1, \tilde{\mathbf{v}}^2 \in \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0}$, $\tilde{\mathbf{u}}_\tau, \tilde{\mathbf{u}}^1_\tau, \tilde{\mathbf{u}}^2_\tau \in \tilde{\mathbf{V}}$ we have*

1. *(a)* $\widetilde{\mathbf{F}}_{\boldsymbol{\chi}} \in \underline{\tilde{\mathbf{S}}}_T$, $\|\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}\|_{\underline{\tilde{\mathbf{S}}}_T} \leqslant C(1+M_0)$,

   (b) $\|\widetilde{\mathbf{F}}^2_{\boldsymbol{\chi}} - \widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}}\|_{\underline{\tilde{\mathbf{S}}}_T} \leqslant C\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + C(1+M_0)\|\tilde{\mathbf{u}}^2_\tau - \tilde{\mathbf{u}}^1_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}$,

   (c) The mapping $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho \to \underline{\tilde{\mathbf{S}}}_T$, $(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) \mapsto \widetilde{\mathbf{F}}_{\boldsymbol{\chi}}$ *is Fréchet differentiable on the relative interior of* $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$.

2. *(a)* $\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}) \in \underline{\tilde{\mathbf{S}}}_T$, $\|\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\|_{\underline{\tilde{\mathbf{S}}}_T} \leqslant C(1+M_0^2)$,

   (b) $\|\operatorname{cof}(\widetilde{\mathbf{F}}^2_{\boldsymbol{\chi}}) - \operatorname{cof}(\widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}})\|_{\underline{\tilde{\mathbf{S}}}_T} \leqslant C(1+M_0)(\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + (1+M_0)\|\tilde{\mathbf{u}}^2_\tau - \tilde{\mathbf{u}}^1_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d})$,

   (c) The mapping $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho \to \underline{\tilde{\mathbf{S}}}_T$, $(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) \mapsto \operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})$ *is Fréchet differentiable on the relative interior of* $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$.

3. *(a)* $\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}) \in \tilde{S}_T$, $\|\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\|_{\tilde{S}_T} \leqslant C(1+M_0^3)$,

   (b) $\|\det(\widetilde{\mathbf{F}}^2_{\boldsymbol{\chi}}) - \det(\widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}})\|_{\tilde{S}_T} \leqslant C(1+M_0^2)(\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + (1+M_0)\|\tilde{\mathbf{u}}^2_\tau - \tilde{\mathbf{u}}^1_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d})$,

   (c) The mapping $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho \to \underline{\tilde{\mathbf{S}}}_T$, $(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) \mapsto \det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})$ *is Fréchet differentiable on the relative interior of* $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$.

4. *(a)* $(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))^{-1} \in \tilde{S}_T$, $\|(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))^{-1}\|_{\tilde{S}_T} \leqslant C(1+M_0^{33})$,

   (b) $\|(\det(\widetilde{\mathbf{F}}^2_{\boldsymbol{\chi}}))^{-1} - (\det(\widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}}))^{-1}\|_{\tilde{S}_T} \leqslant C(1+M_0^{68})(\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + (1+M_0)\|\tilde{\mathbf{u}}^2_\tau - \tilde{\mathbf{u}}^1_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d})$,

   (c) The mapping $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho \to \underline{\tilde{\mathbf{S}}}_T$, $(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) \mapsto (\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))^{-1}$ *is Fréchet differentiable on the relative interior of* $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$.

5. *(a)* $\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} \in \underline{\tilde{\mathbf{S}}}_T$, $\|\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}\|_{\underline{\tilde{\mathbf{S}}}_T} \leqslant C(1+M_0^{35})$,

   (b) $\|\widetilde{\mathbf{F}}^2_{\boldsymbol{\Upsilon}} - \widetilde{\mathbf{F}}^1_{\boldsymbol{\Upsilon}}\|_{\underline{\tilde{\mathbf{S}}}_T} \leqslant C(1+M_0^{70})(\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + (1+M_0)\|\tilde{\mathbf{u}}^2_\tau - \tilde{\mathbf{u}}^1_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d})$,

   (c) The mapping $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho \to \underline{\tilde{\mathbf{S}}}_T$, $(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) \mapsto \widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}$ *is Fréchet differentiable on the relative interior of* $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$.

6. (a) $\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}(\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})^T \in \underline{\tilde{\mathbf{S}}}_T$, $\|\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}(\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})^T\|_{\underline{\tilde{\mathbf{S}}}_T} \leqslant C(1+M_0^{70})$,
   (b) $\|\widetilde{\mathbf{F}}^2_{\boldsymbol{\Upsilon}}(\widetilde{\mathbf{F}}^2_{\boldsymbol{\Upsilon}})^T - \widetilde{\mathbf{F}}^1_{\boldsymbol{\Upsilon}}(\widetilde{\mathbf{F}}^1_{\boldsymbol{\Upsilon}})^T\|_{\underline{\tilde{\mathbf{S}}}_T} \leqslant C(1+M_0^{105})(\|\tilde{\mathbf{v}}^2-\tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + (1+M_0)\|\tilde{\mathbf{u}}^2_\tau - \tilde{\mathbf{u}}^1_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d})$,
   (c) *The mapping* $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho \to \underline{\tilde{\mathbf{S}}}_T$, $(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) \mapsto \widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}(\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})^T$ *is Fréchet differentiable on the relative interior of* $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$.
7. (a) $(\partial_{\mathsf{x}_j\mathsf{x}_k}\check{\boldsymbol{\Upsilon}}) \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}} \in H^1((0,T),H^\ell(\tilde{\Omega}_f)^d)$, $\|(\partial_{\mathsf{x}_j\mathsf{x}_k}\check{\boldsymbol{\Upsilon}}) \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}\|_{H^1((0,T),H^\ell(\tilde{\Omega}_f)^d)} \leqslant C(1+M_0^{70})$,
   (b) $\|(\partial_{\mathsf{x}_j\mathsf{x}_k}\check{\boldsymbol{\Upsilon}}^2 \circ \tilde{\boldsymbol{\chi}}^2_{\boldsymbol{\tau}} - \partial_{\mathsf{x}_j\mathsf{x}_k}\check{\boldsymbol{\Upsilon}}^1 \circ \tilde{\boldsymbol{\chi}}^1_{\boldsymbol{\tau}}\|_{H^1((0,T),H^\ell(\tilde{\Omega}_f)^d)}$
   $\leqslant C(1+M_0^{105})(\|\tilde{\mathbf{v}}^2-\tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + (1+M_0)\|\tilde{\mathbf{u}}^2_\tau - \tilde{\mathbf{u}}^1_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d})$,
   (c) *The mapping* $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho \to H^1((0,T),H^\ell(\tilde{\Omega}_f)^d)$, $(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) \mapsto (\partial_{\mathsf{x}_j\mathsf{x}_k}\check{\boldsymbol{\Upsilon}}) \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}$ *is Fréchet differentiable on the relative interior of* $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$.

Proof. In order to show the existence of the required $T_\alpha > 0$ we consider $\widetilde{\mathbf{F}}_{\boldsymbol{\chi}} - \mathbf{I} = \int_0^{\mathsf{t}} D_{\mathsf{z}}\,\tilde{\mathbf{v}}(\mathsf{z},\mathsf{s})(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}(\mathsf{z}))^{-1}\,\mathsf{ds}$ and estimate with Lemma A.4

$$\|\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}(\cdot,\mathsf{t}) - \mathbf{I}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \leqslant C\alpha_1 \int_0^{\mathsf{t}} \|D_{\mathsf{z}}\,\tilde{\mathbf{v}}(\cdot,\mathsf{s})\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}}\,\mathsf{ds}.$$

Thus, since $H^{1+\ell}(\tilde{\Omega}_f)^{d\times d} \hookrightarrow \mathcal{C}(\overline{\tilde{\Omega}}_f)^{d\times d}$, we have

$$\|\widetilde{\mathbf{F}}_{\boldsymbol{\chi}} - \mathbf{I}\|_{\mathcal{C}(\overline{\tilde{Q}_f^T})^{d\times d}} \leqslant CT^{\frac{1}{2}}\alpha_1 M_0$$

for a constant C independent of T. Since $\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}(\cdot,0)) = \det(\mathbf{I}) = 1$, we can find T_α such that $\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}(\cdot,\mathsf{t})$ is invertible and $\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}(\cdot,\mathsf{t})) \geqslant \alpha$ for all $\mathsf{t} \in [0,T_\alpha]$, all $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{V}}$, and all $\tilde{\mathbf{v}} \in \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0}$.
Now, let $0 < T < T_\alpha$. Consider the multilinear form $m(x_1,\dots,x_k) = x_1 \cdot \ldots \cdot x_k$ for $k \in \mathbb{N}$, which is by Lemma A.4 continuous as a mapping $L^2(\tilde{\Omega}_f) \times H^{1+\ell}(\tilde{\Omega}_f) \times \cdots \times H^{1+\ell}(\tilde{\Omega}_f) \to L^2(\tilde{\Omega}_f)$ and as a mapping $H^{1+\ell}(\tilde{\Omega}_f) \times H^{1+\ell}(\tilde{\Omega}_f) \times \cdots \times H^{1+\ell}(\tilde{\Omega}_f) \to H^{1+\ell}(\tilde{\Omega}_f)$. The terms we have to estimate are obtained by inserting operators $\mathcal{T}_j : \tilde{\mathbf{E}}_T \times \tilde{\mathbf{V}} \to \tilde{S}_T$, $(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) \mapsto \mathcal{T}_j(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)$ in the multilinear form. If they are bounded, continuous and Fréchet differentiable for $1 \leqslant j \leqslant k$ and arbitrary $(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) \in \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$, we can use Lemma 6 to show the claims of the lemma. If we have to estimate vector or matrix valued quantities, we use the argumentation for every component. In the following, C denotes a generic constant independent of T.

1. Consider $\widetilde{\mathbf{F}}_{\boldsymbol{\chi}} - \mathbf{I} = m(\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau),\mathcal{T}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau))$ with $\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) = \int_0^{\mathsf{t}} D_{\mathsf{z}}\,\tilde{\mathbf{v}}(\mathsf{s})\,\mathsf{ds}$ and $\mathcal{T}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) = (D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1}$. We have $\|\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{\underline{\tilde{\mathbf{S}}}_T} \leqslant C(1+\|\tilde{\mathbf{v}}\|_{\tilde{\mathbf{E}}_T})$, since $\|\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)(0)\|_{H^{1+\ell}(\Omega)^{d\times d}} = 0$ and $\|\partial_{\mathsf{t}}\,\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)(0)\|_{L^2(\Omega)^{d\times d}} = \|D_{\mathsf{z}}\,\tilde{\mathbf{v}}_0\|_{L^2(\Omega)^{d\times d}}$, as well as, with **P7**,

$$\|\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)(\cdot,\mathsf{t})\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \leqslant T^{\frac{1}{2}}\|D_{\mathsf{z}}\,\tilde{\mathbf{v}}\|_{L^2((0,T),H^{1+\ell}(\tilde{\Omega}_f)^{d\times d})} \leqslant T^{\frac{1}{2}}\|\tilde{\mathbf{v}}\|_{\tilde{\mathbf{E}}_T},$$
$$\|\partial_{\mathsf{t}}\,\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{L^2((0,T),H^{1+\ell}(\tilde{\Omega}_f)^{d\times d})} = \|D_{\mathsf{z}}\,\tilde{\mathbf{v}}\|_{L^2((0,T),H^{1+\ell}(\tilde{\Omega}_f)^{d\times d})} \leqslant \|\tilde{\mathbf{v}}\|_{\tilde{\mathbf{E}}_T},$$
$$\|\partial_{\mathsf{t}}\,\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),L^2(\tilde{\Omega}_f)^{d\times d})} = \|D_{\mathsf{z}}\,\tilde{\mathbf{v}}\|_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),L^2(\tilde{\Omega}_f)^{d\times d})} \leqslant \|\tilde{\mathbf{v}}\|_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),H^1(\tilde{\Omega}_f)^{d\times d})} \leqslant \|\tilde{\mathbf{v}}\|_{\tilde{\mathbf{E}}_T},$$

   for almost every $\mathsf{t} \in (0,T)$ due to the definition of $\|\cdot\|_{\tilde{\mathbf{E}}_T}$. Boundedness follows with property **P1** of the norm. Fréchet differentiability and continuity now follow by linearity of $\mathcal{T}_1$ and due to

$$(\mathcal{T}_1(\tilde{\mathbf{v}}^2,\tilde{\mathbf{u}}^2_\tau) - \mathcal{T}_1(\tilde{\mathbf{v}}^1,\tilde{\mathbf{u}}^1_\tau))(0) = \partial_{\mathsf{t}}\,(\mathcal{T}_1(\tilde{\mathbf{v}}^2,\tilde{\mathbf{u}}^2_\tau) - \mathcal{T}_1(\tilde{\mathbf{v}}^1,\tilde{\mathbf{u}}^1_\tau))(0) = 0$$

   for all $(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) \in \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}$. Note that $\mathcal{T}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)$ is independent of $\tilde{\mathbf{v}}$ and depends linearly on $(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1}$ with $\|\mathcal{T}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{\underline{\tilde{\mathbf{S}}}_T} \leqslant C\|(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}}$. Hence, boundedness, continuity and differentiability follow from the definition of $\tilde{\mathbf{V}}$, (32) and (35).

2. Each component of the cofactor matrix $\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})$ can be written as a finite sum of terms $a \cdot x_1 \cdot x_2$, where x_1, x_2 denote components of the matrix $\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}$ and $a \in \{-1,1\}$. Therefore, $\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})$ is a sum of bilinear forms with factors $\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) := a(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})_{i,j}$ and $\mathcal{T}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) := (\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})_{k,l}$ for $i,j,k,l \in \{1,2,3\}$. Due to the estimates in 1.(a) we know that $\|\mathcal{T}_i(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{\tilde{S}_T} \leqslant C(1+M_0)$ for $i \in \{1,2\}$, and, therefore, $\|\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\|_{\underline{\tilde{\mathbf{S}}}_T} \leqslant C(1+M_0^2)$. 1.(b) yields $\|\mathcal{T}_i(\tilde{\mathbf{v}}^2,\tilde{\mathbf{u}}^2_\tau) - \mathcal{T}_i(\tilde{\mathbf{v}}^1,\tilde{\mathbf{u}}^1_\tau)\|_{\tilde{S}_T} \leqslant C\|\tilde{\mathbf{v}}^2-\tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + C(1+M_0)\|\tilde{\mathbf{u}}^2_\tau-\tilde{\mathbf{u}}^1_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}$, $i \in \{1,2\}$. Therefore, the continuity estimate and Fréchet differentiability follow from Lemma 6.

3. Since $\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})$ is a polynomial of degree 3 in the components of the matrix $\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}$, the assertions can be proved similar to 2.

4. (a) Since $\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})$ is a cubic polynomial in the components of $\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}$ and $\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})(\cdot,\mathsf{t}) \geqslant \alpha > 0$ for all $\mathsf{t} \in [0, T_\alpha]$, the assertion follows from Lemma 7, 2., which implies

$$\|(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))^{-1}\|_{\tilde{S}_T} \leqslant C(1 + \|\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\|_{\tilde{S}_T})^{10}\|\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\|_{\tilde{S}_T}.$$

Now, 3.(a) implies that

$$\|(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))^{-1}\|_{\tilde{S}_T} \leqslant C(1 + M_0^3)^{10}(1 + M_0^3) \leqslant C(1 + M_0^{33}).$$

(b) The difference

$$(\det(\widetilde{\mathbf{F}}^2_{\boldsymbol{\chi}}))^{-1} - (\det(\widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}}))^{-1} = -(\det(\widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}}))^{-1}(\det(\widetilde{\mathbf{F}}^2_{\boldsymbol{\chi}}))^{-1}(\det(\widetilde{\mathbf{F}}^2_{\boldsymbol{\chi}}) - \det(\widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}}))$$

is a 3-linear form with factors $\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) := (\det(\widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}}))^{-1}$, $\mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) := (\det(\widetilde{\mathbf{F}}^2_{\boldsymbol{\chi}}))^{-1}$ and $\mathcal{T}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) := -(\det(\widetilde{\mathbf{F}}^2_{\boldsymbol{\chi}}) - \det(\widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}}))$. Lemma 7, 1. now yields

$$\|(\det(\widetilde{\mathbf{F}}^2_{\boldsymbol{\chi}}))^{-1} - (\det(\widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}}))^{-1}\|_{\tilde{S}_T} \leqslant C\|(\det(\widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}}))^{-1}\|_{\tilde{S}_T}\|(\det(\widetilde{\mathbf{F}}^2_{\boldsymbol{\chi}}))^{-1}\|_{\tilde{S}_T}\|\det(\widetilde{\mathbf{F}}^2_{\boldsymbol{\chi}}) - \det(\widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}})\|_{\tilde{S}_T}.$$

The estimates 3.(b) and 4.(a) now imply

$$\begin{aligned}&\|(\det(\widetilde{\mathbf{F}}^2_{\boldsymbol{\chi}}))^{-1} - (\det(\widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}}))^{-1}\|_{\tilde{S}_T} \\ &\leqslant C(1 + M_0^{33})(1 + M_0^{33})(1 + M_0^2)(\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + (1 + M_0)\|\tilde{\mathbf{u}}^2_\tau - \tilde{\mathbf{u}}^1_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}).\end{aligned}$$

(c) Let $(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) \in \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$ be arbitrary. Then by 2. and 4.(a) we have $\|(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))^{-1}\|_{\tilde{S}_T} \leqslant C(1 + M_0^{33})$ and $\|\mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\|_{\underline{\tilde{\mathbf{S}}}_T} \leqslant C(1 + M_0^2)$. Hence, Lemma 7 yields

$$\|\widetilde{\mathbf{F}}^{-1}_{\boldsymbol{\chi}}\|_{\underline{\tilde{\mathbf{S}}}_T} = \|(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))^{-1}\mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\|_{\underline{\tilde{\mathbf{S}}}_T} \leqslant \|(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))^{-1}\|_{\tilde{S}_T}\|\mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\|_{\underline{\tilde{\mathbf{S}}}_T} \leqslant C(1 + M_0^{35}). \quad (37)$$

Now $\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^{-1} = \det(\widetilde{\mathbf{F}}^{-1}_{\boldsymbol{\chi}})$, thus it suffices by 1., 3. and the chain rule to show that $(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}^1_\tau) \in \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho \mapsto (\widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}})^{-1} \in \underline{\tilde{\mathbf{S}}}_T$ is Fréchet differentiable at $(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}^1_\tau)$. This follows from (34), (37) and Lemma 7, since with $\mathbf{A} = \widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}}$

$$\|\mathbf{A}^{-1} - \widetilde{\mathbf{F}}^{-1}_{\boldsymbol{\chi}} - \widetilde{\mathbf{F}}^{-1}_{\boldsymbol{\chi}}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}} - \mathbf{A})\widetilde{\mathbf{F}}^{-1}_{\boldsymbol{\chi}}\|_{\underline{\tilde{\mathbf{S}}}_T} = \|\widetilde{\mathbf{F}}^{-1}_{\boldsymbol{\chi}}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}} - \mathbf{A})\widetilde{\mathbf{F}}^{-1}_{\boldsymbol{\chi}}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}} - \mathbf{A})\mathbf{A}^{-1}\|_{\underline{\tilde{\mathbf{S}}}_T} \leqslant C(1 + M_0^{35})^3\|\widetilde{\mathbf{F}}_{\boldsymbol{\chi}} - \mathbf{A}\|^2_{\underline{\tilde{\mathbf{S}}}_T},$$

which yields with 1. the Fréchet differentiability.

5. Since $\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} = (\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^{-1} = (\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))^{-1}\mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^\top$, we can prove the result via multilinear forms and use Lemma 7, 1. .

6. Again, the assertions can be shown via multilinear forms.

7. From $\tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}^{-1} \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}} = \mathrm{id}_z$, it follows that

$$\mathbf{I} = D_{\mathsf{z}}\,(\tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}^{-1} \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}) = D_{\mathsf{x}}\tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}^{-1} \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}} D_{\mathsf{z}}\,\tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}.$$

Therefore, since $D_{\mathsf{z}}\,\tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}} = \widetilde{\mathbf{F}}_{\boldsymbol{\chi}} D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}$, we have

$$D_{\mathsf{x}}\tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}^{-1} \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}} = (\widetilde{\mathbf{F}}_{\boldsymbol{\chi}} D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1} = D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^{-1}\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}. \quad (38)$$

Furthermore, we have $(\widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}})_{l,k} = (\widehat{\mathbf{F}}^{-1}_{\boldsymbol{\chi}})_{l,k} = (\partial_{\mathsf{x}_k}\check{\boldsymbol{\Upsilon}}_l) \circ \hat{\boldsymbol{\chi}}$, which implies

$$(\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})_{l,k} = (\partial_{\mathsf{x}_k}\check{\boldsymbol{\Upsilon}}_l) \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}. \quad (39)$$

Thus, $(\partial_{\mathsf{x}_j\mathsf{x}_k}\check{\boldsymbol{\Upsilon}}_l)\circ\tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}} = \partial_{\mathsf{x}_j}(\tilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}\circ\tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}^{-1})_{l,k}\circ\tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}$ and with (38) we obtain

$$\partial_{\mathsf{x}_j}(\tilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}\circ\tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}^{-1})_{l,k}\circ\tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}} = \sum_m(\partial_{\mathsf{z}_m}\tilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})_{l,k}\partial_{\mathsf{x}_j}(\tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}^{-1})_m\circ\tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}} = \sum_{m,i}(\partial_{\mathsf{z}_m}\tilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})_{l,k}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^{-1})_{m,i}(\tilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})_{i,j}$$

and each summand is the composition of a multilinear form $m(x_1,x_2,x_3) = x_1\cdot x_2\cdot x_3$, which is by Lemma A.4 continuous as a mapping $H^\ell(\tilde{\Omega}_f)\times H^{1+\ell}(\tilde{\Omega}_f)\times\cdots\times H^{1+\ell}(\tilde{\Omega}_f)\to H^\ell(\tilde{\Omega}_f)$ with an operator $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0}\times\tilde{\mathbf{V}}\to H^1((0,T),H^\ell(\tilde{\Omega}_f))\times H^1((0,T),H^{1+\ell}(\tilde{\Omega}_f))\times H^1((0,T),H^{1+\ell}(\tilde{\Omega}_f))$ that by (35), **P7** and 5. is bounded and continuous on $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0}\times\tilde{\mathbf{V}}_\rho$ as well as Fréchet differentiable on $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0}\times\tilde{\mathbf{V}}_\rho$. Now, we can apply Lemma 5 to conclude the proof.

□

Remark 5. *Lemma 8 and its proof could be extended in a canonical way from Fréchet differentiability to continuous differentiability.*

With the above Lemmas the required right hand side estimates can be established.

Lemma 9. *Let Assumption 1 be satisfied. Let $T_f > 0$ and $\rho = \rho(0)$ be given by Lemma 4. There exist $0 < T^* \leqslant T_f$, $\alpha_1 > 0$, as well as, for each of the following terms, a constant $C > 0$ independent of T but dependent on T_f and a polynomial χ such that for all $0 < T < T^*$, $\tilde{\mathbf{v}},\tilde{\mathbf{v}}^1,\tilde{\mathbf{v}}^2\in\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0}$, $\tilde{p},\tilde{p}^1,\tilde{p}^2\in\tilde{P}_{T,M_0,\tilde{\mathbf{v}}_0}$ and $\tilde{\mathbf{u}}_\tau,\tilde{\mathbf{u}}_\tau^1,\tilde{\mathbf{u}}_\tau^2\in\tilde{\mathbf{V}}_\rho$ we have*

$$\tilde{\mathcal{F}}(\tilde{\mathbf{v}},\tilde{p},\tilde{\mathbf{u}}_\tau)\in\tilde{\mathbf{F}}_T,\quad \tilde{\mathcal{H}}(\tilde{\mathbf{v}},\tilde{p},\tilde{\mathbf{u}}_\tau)\in\tilde{\mathbf{H}}_T,\quad \tilde{\mathcal{G}}(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\in\tilde{G}_T,\quad \tilde{\mathbf{g}}(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\in\tilde{\mathbf{G}}_T\cap H^1((0,T),H^\ell(\tilde{\Omega}_f)^d),$$

$$\tilde{\mathbf{g}}(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)|_{\Sigma_f^T} = 0,\quad \tilde{\mathbf{g}}(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)(\cdot,0) = 0,\quad \textit{and}\quad \tilde{\mathcal{H}}(\tilde{\mathbf{v}},\tilde{p},\tilde{\mathbf{u}}_\tau)(\cdot,0) = 0, \tag{40}$$

as well as,

$$\begin{aligned}
&\|\tilde{\mathcal{F}}(\tilde{\mathbf{v}},\tilde{p},\tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{F}}_T}\leqslant C\chi(M_0)(T^{1-\ell}+\rho), && \|\tilde{\mathcal{H}}(\tilde{\mathbf{v}},\tilde{p},\tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{H}}_T}\leqslant C\chi(M_0)(T^{1-\ell}+\rho),\\
&\|\tilde{\mathcal{G}}(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{\tilde{G}_T}\leqslant C\chi(M_0)(T^{1-\ell}+\rho), && \|\tilde{\mathbf{g}}(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{G}}_T}\leqslant C\chi(M_0)(1+\rho),
\end{aligned}$$

and

$$\begin{aligned}
&\|\tilde{\mathcal{F}}(\tilde{\mathbf{v}}^2,\tilde{p}^2,\tilde{\mathbf{u}}_\tau^2)-\tilde{\mathcal{F}}(\tilde{\mathbf{v}}^1,\tilde{p}^1,\tilde{\mathbf{u}}_\tau^1)\|_{\tilde{\mathbf{F}}_T}\\
&\qquad\leqslant C\chi(M_0)((T^{1-\ell}+\rho)(\|\tilde{\mathbf{v}}^2-\tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T}+\|\nabla_{\mathsf{z}}\tilde{p}^2-\nabla_{\mathsf{z}}\tilde{p}^1\|_{\tilde{\mathbf{F}}_T})+\|\tilde{\mathbf{u}}_\tau^2-\tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}),\\
&\|\tilde{\mathcal{H}}(\tilde{\mathbf{v}}^2,\tilde{p}^2,\tilde{\mathbf{u}}_\tau^2)-\tilde{\mathcal{H}}(\tilde{\mathbf{v}}^1,\tilde{p}^1,\tilde{\mathbf{u}}_\tau^1)\|_{\tilde{\mathbf{H}}_T}\\
&\qquad\leqslant C\chi(M_0)((T^{1-\ell}+\rho)(\|\tilde{\mathbf{v}}^2-\tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T}+\|\tilde{p}^2|_{\tilde{\Sigma}_i^T}-\tilde{p}^1|_{\tilde{\Sigma}_i^T}\|_{H_T})+\|\tilde{\mathbf{u}}_\tau^2-\tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}),\\
&\|\tilde{\mathcal{G}}(\tilde{\mathbf{v}}^2,\tilde{\mathbf{u}}_\tau^2)-\tilde{\mathcal{G}}(\tilde{\mathbf{v}}^1,\tilde{\mathbf{u}}_\tau^1)\|_{\tilde{G}_T}\leqslant C\chi(M_0)((T^{1-\ell}+\rho)\|\tilde{\mathbf{v}}^2-\tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T}+\|\tilde{\mathbf{u}}_\tau^2-\tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}),\\
&\|\tilde{\mathbf{g}}(\tilde{\mathbf{v}}^2,\tilde{\mathbf{u}}_\tau^2)-\tilde{\mathbf{g}}(\tilde{\mathbf{v}}^1,\tilde{\mathbf{u}}_\tau^1)\|_{\tilde{\mathbf{G}}_T}\leqslant C\chi(M_0)((T^{\frac14-\frac{\ell}{4}}+\rho)\|\tilde{\mathbf{v}}^2-\tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T}+\|\tilde{\mathbf{u}}_\tau^2-\tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}),
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\mathcal{F}}:\ \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0}\times\tilde{P}_{T,M_0,\tilde{\mathbf{v}}_0}\times\tilde{\mathbf{V}}_\rho\to\tilde{\mathbf{F}}_T, && \tilde{\mathcal{H}}:\ \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0}\times\tilde{P}_{T,M_0,\tilde{\mathbf{v}}_0}\times\tilde{\mathbf{V}}_\rho\to\tilde{\mathbf{H}}_T,\\
&\tilde{\mathcal{G}}:\ \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0}\times\tilde{\mathbf{V}}_\rho\to\tilde{G}_T, && \tilde{\mathbf{g}}:\ \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0}\times\tilde{\mathbf{V}}_\rho\to\tilde{\mathbf{G}}_T
\end{aligned}$$

are Fréchet differentiable on the relative interior of $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0}\times\tilde{P}_{T,M_0,\tilde{\mathbf{v}}_0}\times\tilde{\mathbf{V}}_\rho$ and $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0}\times\tilde{\mathbf{V}}_\rho$, respectively.

Proof. The compatibility conditions (40) are fulfilled, due to the choice of $\tilde{\mathbf{V}}$, which ensures that $\operatorname{supp}\tilde{\mathbf{u}}_\tau\cap\operatorname{supp}\tilde{\mathbf{v}}_0 = \varnothing$ and therefore $\tilde{\mathcal{H}}(\tilde{\mathbf{v}},\tilde{p},\tilde{\mathbf{u}}_\tau)(\cdot,0) = 0$ and $\tilde{\mathbf{g}}(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)(\cdot,0) = 0$. The boundary condition on Σ_f^T ensures that $\tilde{\mathbf{g}}(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)|_{\Sigma_f^T} = 0$. The right hand side terms $\tilde{\mathcal{F}}$, $\tilde{\mathcal{H}}$, $\tilde{\mathcal{G}}$ and $\tilde{\mathbf{g}}$ are sums of multilinear forms as introduced in Lemma 5 and 6. In Lemma 8 boundedness, continuity and Fréchet differentiability of the corresponding factors are shown. Thus, it suffices to establish an appropriate boundedness estimate such that the product of the appearing M_j in Lemma 5 have the structure $\tilde{C}(T^\alpha+\|\tilde{\mathbf{u}}_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d})$ for a suitable $\alpha\geqslant 0$ and $\tilde{C}$ which is independent of T. The explicit time dependency is obtained by using the extension and restriction properties **P3**,

P4 and **P5** of the norm and by using Lemma **P6**. The time dependency for the corresponding constants $M_{j,1}$ and $M_{j,2}$ follows with similar arguments. The desired continuity estimates, as well as, Fréchet differentiability can be deduced from Lemma 5 if (33) and thus

$$(\partial_{\mathsf{y}_i} D_{\mathsf{y}}\, \tilde{\boldsymbol{\tau}}^{-1}) \circ \tilde{\boldsymbol{\tau}} = (\partial_{\mathsf{y}_i}(D_{\mathsf{y}}\, \tilde{\boldsymbol{\tau}}^{-1} - \mathbf{I})) \circ \tilde{\boldsymbol{\tau}} = \sum_m \partial_{\mathsf{z}_m}((D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})^{-1} - \mathbf{I})(\partial_{\mathsf{y}_i} \tilde{\boldsymbol{\tau}}_m^{-1}) \circ \tilde{\boldsymbol{\tau}},$$

are kept in mind, which by Lemma A.4 and the definition of $\tilde{\mathbf{V}}$ implies

$$\|(\partial_{\mathsf{y}_i} D_{\mathsf{y}}\, \tilde{\boldsymbol{\tau}}^{-1}) \circ \tilde{\boldsymbol{\tau}}\|_{H^\ell(\tilde{\Omega}_f)} \leqslant C\|(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)}\|(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})^{-1} - \mathbf{I}\|_{H^{1+\ell}(\tilde{\Omega}_f)} \leqslant C\alpha_1(1+\alpha_1). \tag{41}$$

Moreover, since for arbitrary matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{d\times d}$ the cofactor-matrix is a polynomial of degree $d-1$ in every entry, we have that

$$\operatorname{cof}(\mathbf{A}) - \operatorname{cof}(\mathbf{B}) \leqslant \sum_{i,j} \chi_{i,j}(\mathbf{A}, \mathbf{B})(\mathbf{A} - \mathbf{B})_{i,j},$$

where $\chi_{i,j}$ is a polynomial of degree $d-2$ in the entries of $\mathbf{A}$ and $\mathbf{B}$ for $1 \leqslant i, j \leqslant 3$. Thus,

$$\|\operatorname{cof}(\mathbf{A}) - \operatorname{cof}(\mathbf{B})\|_{H^{1+\ell}(\Omega)^{d\times d}} \leqslant C(\|\mathbf{A}\|^{d-2}_{H^{1+\ell}(\Omega)^{d\times d}} + \|\mathbf{B}\|^{d-2}_{H^{1+\ell}(\Omega)^{d\times d}})\|\mathbf{A} - \mathbf{B}\|_{H^{1+\ell}(\Omega)^{d\times d}},$$

and for $\tilde{\mathbf{u}}_\tau, \tilde{\mathbf{u}}_\tau^1, \tilde{\mathbf{u}}_\tau^2 \in \tilde{\mathbf{V}}$ we have

$$\|\operatorname{cof}(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \leqslant C \tag{42}$$

$$\|\operatorname{cof}(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}}^1) - \operatorname{cof}(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}}^2)\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \leqslant C\alpha_1^{d-2}\|\tilde{\boldsymbol{\tau}}^1 - \tilde{\boldsymbol{\tau}}^2\|_{H^{2+\ell}(\tilde{\Omega}_f)^d} \leqslant C\alpha_1^{d-2}\|\tilde{\mathbf{u}}_\tau^1 - \tilde{\mathbf{u}}_\tau^2\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}. \tag{43}$$

We show boundedness of $\tilde{\mathcal{F}}$, $\tilde{\mathcal{H}}$, $\tilde{\mathcal{G}}$ and $\tilde{\mathbf{g}}$. In order to obtain the estimates we have to split the terms such that the initial values of selected factors vanish at $\mathsf{t} = 0$. To this end, we decompose

$$\begin{aligned}
\tilde{\mathcal{F}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) &= \tilde{\mathcal{F}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{F}}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{F}}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{F}}_4(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{F}}_5(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{F}}_6(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau),\\
\tilde{\mathcal{F}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= \nu \sum_{j,k,l} (\partial_{\mathsf{x}_j\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_k \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}})(\partial_{\mathsf{y}_k}(\tilde{\boldsymbol{\tau}}^{-1})_l \circ \tilde{\boldsymbol{\tau}})\partial_{\mathsf{z}_l} \tilde{\mathbf{v}},\\
\tilde{\mathcal{F}}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= \nu \sum_{i,k,l} ((\sum_j \partial_{\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_i \partial_{\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_k - \delta_{i,k}) \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}})(\partial_{\mathsf{y}_i\mathsf{y}_k}(\tilde{\boldsymbol{\tau}}^{-1})_l \circ \tilde{\boldsymbol{\tau}})\partial_{\mathsf{z}_l} \tilde{\mathbf{v}},\\
\tilde{\mathcal{F}}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= \nu \sum_{i,k,l,m} ((\sum_j \partial_{\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_i \partial_{\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_k - \delta_{i,k}) \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}})((\partial_{\mathsf{y}_k}(\tilde{\boldsymbol{\tau}}^{-1})_l \partial_{\mathsf{y}_i}(\tilde{\boldsymbol{\tau}}^{-1})_m) \circ \tilde{\boldsymbol{\tau}})\partial_{\mathsf{z}_l\mathsf{z}_m} \tilde{\mathbf{v}},\\
\tilde{\mathcal{F}}_4(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= \nu \sum_{k,l} (\partial_{\mathsf{y}_k\mathsf{y}_k}(\tilde{\boldsymbol{\tau}}^{-1})_l \circ \tilde{\boldsymbol{\tau}})\partial_{\mathsf{z}_l} \tilde{\mathbf{v}} + \nu \sum_{k,l,m} ((\partial_{\mathsf{y}_k}(\tilde{\boldsymbol{\tau}}^{-1})_l \partial_{\mathsf{y}_k}(\tilde{\boldsymbol{\tau}}^{-1})_m) \circ \tilde{\boldsymbol{\tau}} - \delta_{l,m})\partial_{\mathsf{z}_l\mathsf{z}_m} \tilde{\mathbf{v}},\\
\tilde{\mathcal{F}}_5(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) &= (\mathbf{I} - \widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top)\nabla_{\mathsf{z}}\, \tilde{p},\\
\tilde{\mathcal{F}}_6(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) &= \widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top(\mathbf{I} - (D_{\mathsf{y}}\, \tilde{\boldsymbol{\tau}}^{-1})^\top \circ \tilde{\boldsymbol{\tau}})\nabla_{\mathsf{z}}\, \tilde{p},
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{H}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) = {}& \tilde{\mathcal{H}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{H}}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{H}}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{H}}_4(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{H}}_5(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\\
&+ \tilde{\mathcal{H}}_6(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{H}}_7(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{H}}_8(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{H}}_9(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{H}}_{10}(\tilde{p}, \tilde{\mathbf{u}}_\tau),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\mathcal{H}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= -\nu D_{\mathsf{z}}\, \tilde{\mathbf{v}}(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})^{-1}\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}(\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}) - \mathbf{I})\operatorname{cof}(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})\tilde{\mathbf{n}}_f,\\
\tilde{\mathcal{H}}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= -\nu \widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})^{-\top} D_{\mathsf{z}}\, \tilde{\mathbf{v}}^\top(\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}) - \mathbf{I})\operatorname{cof}(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})\tilde{\mathbf{n}}_f,\\
\tilde{\mathcal{H}}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= -\nu D_{\mathsf{z}}\, \tilde{\mathbf{v}}(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})^{-1}(\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} - \mathbf{I})\operatorname{cof}(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})\tilde{\mathbf{n}}_f,\\
\tilde{\mathcal{H}}_4(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= -\nu(\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} - \mathbf{I})^\top(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})^{-\top} D_{\mathsf{z}}\, \tilde{\mathbf{v}}^\top \operatorname{cof}(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})\tilde{\mathbf{n}}_f,\\
\tilde{\mathcal{H}}_5(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= -\nu D_{\mathsf{z}}\, \tilde{\mathbf{v}}(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})^{-1}(\operatorname{cof}(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}}) - \mathbf{I})\tilde{\mathbf{n}}_f,\\
\tilde{\mathcal{H}}_6(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= -\nu(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})^{-\top} D_{\mathsf{z}}\, \tilde{\mathbf{v}}^\top(\operatorname{cof}(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}}) - \mathbf{I})\tilde{\mathbf{n}}_f,\\
\tilde{\mathcal{H}}_7(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= -\nu D_{\mathsf{z}}\, \tilde{\mathbf{v}}((D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})^{-1} - \mathbf{I})\tilde{\mathbf{n}}_f,\\
\tilde{\mathcal{H}}_8(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= -\nu((D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})^{-\top} - \mathbf{I})D_{\mathsf{z}}\, \tilde{\mathbf{v}}^\top \tilde{\mathbf{n}}_f,\\
\tilde{\mathcal{H}}_9(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) &= -\tilde{p}(\mathbf{I} - \operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))\operatorname{cof}(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})\tilde{\mathbf{n}}_f,\\
\tilde{\mathcal{H}}_{10}(\tilde{p}, \tilde{\mathbf{u}}_\tau) &= -\tilde{p}(\mathbf{I} - \operatorname{cof}(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}}))\tilde{\mathbf{n}}_f,
\end{aligned}$$

$$\begin{aligned}\tilde{\mathcal{G}}(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) &= D_{\mathsf{z}}\,\tilde{\mathbf{v}} : ((\mathbf{I}-\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))\operatorname{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})) + D_{\mathsf{z}}\,\tilde{\mathbf{v}} : (\mathbf{I}-\operatorname{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}))\\ &=: \tilde{\mathcal{G}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{G}}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau).\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathbf{g}}(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) &= \operatorname{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^\top(\mathbf{I}-\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^\top)\tilde{\mathbf{v}} + (\mathbf{I}-\operatorname{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^\top)\tilde{\mathbf{v}}\\ &=: \tilde{\mathbf{g}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) + \tilde{\mathbf{g}}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau).\end{aligned}$$

Since the ideas for the estimates for the different summands of $\tilde{\mathcal{F}}$, $\tilde{\mathcal{H}}$, $\tilde{\mathcal{G}}$ and $\tilde{\mathbf{g}}$ are similar we just present the proofs for $\tilde{\mathcal{F}}_2$, $\tilde{\mathcal{F}}_6$, $\tilde{\mathcal{H}}_1$, $\tilde{\mathcal{G}}_1$ and $\tilde{\mathbf{g}}_1$. Let C denote a generic constant independent of T. In the following argumentation we frequently use Lemma A.4 in order to ensure that $X_1,\dots,X_k$ are chosen such that multilinear forms $m(x_1,\dots,x_k) := x_1\cdot\ldots\cdot x_k$ fulfill the requirements of Lemma 5. The notation S_i, M_i, $M_{i,1}$, $M_{i,2}$, s_i for $i\in\{1,\dots,k\}$ is defined by Lemma 5.

• Estimation of $\tilde{\mathbf{g}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)$:

To apply Lemma 5 we use property **P1**, which implies $\|\cdot\|^2_{\tilde{\mathbf{G}}_T} = \|\cdot\|^2_{L^2((0,T),L^2(\tilde{\Omega}_f)^d)} + \|\partial_{\mathsf{t}}\cdot\|^2_{H^{\frac{\ell}{2}}((0,T),L^2(\tilde{\Omega}_f)^d)}$, and estimate $\tilde{\mathbf{g}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)$ and $\partial_{\mathsf{t}}\tilde{\mathbf{g}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)$ separately.

1. $\tilde{\mathbf{g}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)$ is a multilinear form with factors $\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) = \tilde{\mathbf{v}}$, $\mathcal{T}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) = \mathbf{I}-\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^\top$, $\mathcal{T}_3(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) = \operatorname{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^\top$. With Lemma 5, $s=s_1=0$, $s_2=\ell$, $s_3=1$, $X=L^2(\tilde{\Omega}_f)^d$, $X_1 = H^{2+\ell}(\tilde{\Omega}_f)^d$, $X_2 = H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}$, $X_3 = H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}$, we obtain

$$\|\tilde{\mathbf{g}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{L^2((0,T),L^2(\Omega_f)^d)} \leqslant CM_0(1+M_0)T^{1-\ell},$$

since by **P6**, (42) and Lemmas A.3, A.4, 8

$$\begin{aligned}&\|\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_1} \leqslant \|\tilde{\mathbf{v}}\|_{\tilde{\mathbf{E}}_T} \leqslant M_0,\\ &\|\mathcal{T}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_2} \leqslant CT^{1-\ell}\|\mathbf{I}-\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\|_{H^1((0,T),X_2)} \leqslant CT^{1-\ell}(1+M_0^2),\\ &\|\mathcal{T}_3(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_3} = (\|\operatorname{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})\|^2_{X_3} + \|\operatorname{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})\|^2_{L^2((0,T),X_3)})^{\frac12} \leqslant C\|\operatorname{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})\|_{X_3} \leqslant C,\end{aligned}\tag{44}$$

i.e., $M_1 = M_0$, $M_2 = CT^{1-\ell}(1+M_0^2)$ and $M_3 = C$ in the notation of Lemma 5. Using in addition (43) gives

$$\begin{aligned}&\|\mathcal{T}_1(\tilde{\mathbf{v}}^2,\tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_1(\tilde{\mathbf{v}}^1,\tilde{\mathbf{u}}_\tau^1)\|_{S_1} \leqslant \|\tilde{\mathbf{v}}^2-\tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T},\\ &\|\mathcal{T}_2(\tilde{\mathbf{v}}^2,\tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_2(\tilde{\mathbf{v}}^1,\tilde{\mathbf{u}}_\tau^1)\|_{S_2} \leqslant CT^{1-\ell}\|\operatorname{cof}(\widetilde{\mathbf{F}}^2_{\boldsymbol{\chi}}) - \operatorname{cof}(\widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}})\|_{\tilde{\mathbf{E}}_T}\\ &\qquad \leqslant CT^{1-\ell}(1+M_0)(\|\tilde{\mathbf{v}}^2-\tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + (1+M_0)\|\tilde{\mathbf{u}}_\tau^2-\tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}),\\ &\|\mathcal{T}_3(\tilde{\mathbf{v}}^2,\tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_3(\tilde{\mathbf{v}}^1,\tilde{\mathbf{u}}_\tau^1)\|_{S_3} \leqslant C\|\operatorname{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^2) - \operatorname{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^1)\|_{X_3} \leqslant C\|\tilde{\mathbf{u}}_\tau^2-\tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}.\end{aligned}$$

Hence, $M_{1,1}=1$, $M_{1,2}=0$, $M_{2,1} = CT^{1-\ell}(1+M_0)$, $M_{2,2} = CT^{1-\ell}(1+M_0)^2$, $M_{3,2}=C$, $M_{3,1}=0$ and Lemma 5 yields for a polynomial χ

$$\|\tilde{\mathbf{g}}^1(\tilde{\mathbf{v}}^2,\tilde{\mathbf{u}}_\tau^2) - \tilde{\mathbf{g}}^1(\tilde{\mathbf{v}}^1,\tilde{\mathbf{u}}_\tau^1)\|_{L^2((0,T),L^2(\Omega_f)^d)} \leqslant C\chi(M_0)T^{1-\ell}(\|\tilde{\mathbf{v}}^2-\tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + \|\tilde{\mathbf{u}}_\tau^2-\tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d})$$

and Fréchet differentiability of $\tilde{\mathbf{g}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau):\ \tilde{\mathbf{E}}_T\times\tilde{\mathbf{V}} \to L^2((0,T),L^2(\tilde{\Omega}_f)^d)$ on $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0}\times\tilde{\mathbf{V}}_\rho$.

2. $\partial_{\mathsf{t}}\tilde{\mathbf{g}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) = -\operatorname{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^\top\partial_{\mathsf{t}}\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^\top\tilde{\mathbf{v}} + \operatorname{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^\top(\mathbf{I}-\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^\top)\partial_{\mathsf{t}}\,\tilde{\mathbf{v}}$ is a sum of multilinear forms. We exemplarily estimate the first term. Here, $\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) = \tilde{\mathbf{v}}$, $\mathcal{T}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) = -\partial_{\mathsf{t}}\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^\top$ and $\mathcal{T}_3(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) = \operatorname{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^\top$. Choose $s=s_1=\frac{\ell}{2}$, $s_2 = \frac12+\frac{\ell}{4}$, $s_3=1$, $X=X_2=L^2(\tilde{\Omega}_f)^d$, $X_1 = H^{1+\ell}(\tilde{\Omega}_f)^d$, $X_3 = H^{1+\ell}(\tilde{\Omega}_f)^d$. With Lemmas A.2, 8 we obtain

$$\begin{aligned}&\|\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_1} \leqslant C(\|\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)(0)\|_{H^{1+\ell}(\tilde{\Omega}_f)^d} + T^{\frac14-\frac{\ell}{4}}\|\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{E}}_T}) \leqslant C(1+M_0),\\ &\|\mathcal{T}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_2} \leqslant C(\|\mathcal{T}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)(0))\|_{X_2} + T^{\frac{\ell}{4}}\|\mathcal{T}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{H^{\frac12+\frac{\ell}{2}}((0,T),L^2(\tilde{\Omega}_f)^d)}) \leqslant C(1+M_0^2),\\ &\|\mathcal{T}_3(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_3} \leqslant C,\end{aligned}\tag{45}$$

where we use for the second term that $0 = \partial_{\mathsf{t}}\,(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}) = \partial_{\mathsf{t}}\,\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} + \widetilde{\mathbf{F}}_{\boldsymbol{\chi}}\partial_{\mathsf{t}}\,\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}$ and thus with (36)

$$\begin{aligned}\partial_{\mathsf{t}}(\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^\top)(0) &= \partial_{\mathsf{t}}\,(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})(0) = (\partial_{\mathsf{t}}\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} + \det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\partial_{\mathsf{t}}\,\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})(0)\\ &= (\operatorname{tr}(\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^\top\partial_{\mathsf{t}}\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} - \det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}D_{\mathsf{z}}\,\tilde{\mathbf{v}}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1}\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})(0)\\ &= \operatorname{tr}(D_{\mathsf{z}}\,\tilde{\mathbf{v}}_0(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1})\mathbf{I} - D_{\mathsf{z}}\,\tilde{\mathbf{v}}_0(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1}.\end{aligned}$$

Since $(\mathcal{T}_i(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_i(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1))(0) = 0$ for $i \in \{1,2,3\}$, analogous to (45), we obtain with (43)

$$\begin{aligned}
&\|\mathcal{T}_1(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_1(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1)\|_{S_1} \leqslant CT^{\frac{1}{4}-\frac{\ell}{4}} \|\mathcal{T}_1(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_1(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1)\|_{\tilde{\mathbf{E}}_T},\\
&\|\mathcal{T}_2(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_2(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1)\|_{S_2} \leqslant CT^{\frac{\ell}{4}} \|\mathcal{T}_2(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_2(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1)\|_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),L^2(\tilde{\Omega}_f)^d)},\\
&\|\mathcal{T}_3(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_3(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1)\|_{S_3} \leqslant C\|\tilde{\mathbf{u}}_\tau^2 - \tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}.
\end{aligned}$$

Continuity and Fréchet differentiability follow now by Lemmas 5, 8. Finally, $\tilde{\mathbf{g}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) \in H^1((0,T), H^\ell(\tilde{\Omega}_f)^d)$, since $\mathbf{v} \in H^1((0,T), H^\ell(\tilde{\Omega}_f)^d)$, $\tilde{\boldsymbol{\tau}} \in H^{2+\ell}(\tilde{\Omega}_f)^d$ and $(\mathbf{I} - \mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})) \in H^1((0,T), H^{1+\ell}(\tilde{\Omega}_f)^{d\times d})$.

• Bound for $\|\tilde{\mathcal{F}}_6(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{F}}_T}$:

$\tilde{\mathcal{F}}_6(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau)$ is a multilinear form with factors $\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) = \nabla_{\mathsf{z}}\, \tilde{p}$, $\mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) = \mathbf{I} - (D_{\mathsf{y}}\, \tilde{\boldsymbol{\tau}}^{-1})^\top \circ \tilde{\boldsymbol{\tau}} = \mathbf{I} - (D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})^{-\top}$ due to (33) and $\mathcal{T}_3(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) = \widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top$.

1. $\|\tilde{\mathcal{F}}_6(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau)\|_{L^2((0,T),H^\ell(\tilde{\Omega}_f)^d)}$:

Choose $s = s_1 = 0$, $s_2 = 1$, $s_3 = \ell$, $X = X_1 = H^\ell(\tilde{\Omega}_f)^d$, $X_2 = X_3 = H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}$. $\|\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau)\|_{S_1} \leqslant M_0$ follows by (20). With (35) and Lemma A.3 we obtain

$$\|\mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau)\|_{S_2} \leqslant C\|\tilde{\mathbf{u}}_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d} \leqslant C\rho \tag{46}$$

since $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{V}}_\rho$. $\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}(0) = \mathbf{I}$ and Lemmas A.2, 8 imply

$$\|\mathcal{T}_3(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau)\|_{S_3} \leqslant C(1 + T^{1-\ell}\|\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}\|_{H^1((0,T),X_3)}) \leqslant C(1 + M_0^{35}). \tag{47}$$

With (35) and Lemma 8 we have

$$\begin{aligned}
&\|\mathcal{T}_1(\tilde{\mathbf{v}}^2, \tilde{p}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_1(\tilde{\mathbf{v}}^1, \tilde{p}^1, \tilde{\mathbf{u}}_\tau^1)\|_{S_1} \leqslant \|\nabla_{\mathsf{z}}\, \tilde{p}^2 - \nabla_{\mathsf{z}}\, \tilde{p}^1\|_{\tilde{\mathbf{F}}_T},\\
&\|\mathcal{T}_2(\tilde{\mathbf{v}}^2, \tilde{p}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_2(\tilde{\mathbf{v}}^1, \tilde{p}^1, \tilde{\mathbf{u}}_\tau^1)\|_{S_2} \leqslant C\|\tilde{\mathbf{u}}_\tau^2 - \tilde{\mathbf{u}}_\tau^1\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}},\\
&\|\mathcal{T}_3(\tilde{\mathbf{v}}^2, \tilde{p}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_3(\tilde{\mathbf{v}}^1, \tilde{p}^1, \tilde{\mathbf{u}}_\tau^1)\|_{S_3} \leqslant C(1 + M_0^{70})T^{1-\ell}(\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + (1 + M_0)\|\tilde{\mathbf{u}}_\tau^2 - \tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}).
\end{aligned}$$

2. $\|\tilde{\mathcal{F}}_6(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau)\|_{H^{\frac{\ell}{2}}((0,T),L^2(\tilde{\Omega}_f)^d)}$:

Let $s = s_1 = \frac{\ell}{2}$, $s_2 = 1$, $s_3 = \ell$, $X = X_1 = L^2(\tilde{\Omega}_f)^d$, $X_2 = X_3 = H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}$. With (20), (46), (47) and Lemmas A.4, 8 we obtain the same bounds as before.
Thus, with Lemma 5, $\|\tilde{\mathcal{F}}_6(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{F}}_T} \leqslant C(1 + M_0^{36})\rho$ and

$$\|\tilde{\mathcal{F}}_6(\tilde{\mathbf{v}}^2, \tilde{p}^2, \tilde{\mathbf{u}}_\tau^2) - \tilde{\mathcal{F}}_6(\tilde{\mathbf{v}}^1, \tilde{p}^1, \tilde{\mathbf{u}}_\tau^1)\|_{\tilde{\mathbf{F}}_T} \leqslant C\chi(M_0)(T^{1-\ell}\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + \rho\|\nabla_{\mathsf{z}}\, \tilde{p}^2 - \nabla_{\mathsf{z}}\, \tilde{p}^1\|_{\tilde{\mathbf{F}}_T} + \|\tilde{\mathbf{u}}_\tau^2 - \tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}),$$

where χ is a polynomial.
As seen in the previous estimates, due to Lemma 5, the derivation of the continuity estimates and Fréchet differentiability is straightforward if one knows how to show boundedness of the multilinear forms. We thus only address boundedness in the following.

• Bound for $\|\tilde{\mathcal{F}}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{F}}_T}$:

$\tilde{\mathcal{F}}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)$ is a sum of multilinear forms with factors $\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = \partial_{\mathsf{z}_l}\tilde{\mathbf{v}}$, $\mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = \nu\partial_{\mathsf{y}_i}\partial_{\mathsf{y}_k}(\tilde{\boldsymbol{\tau}}^{-1})_l \circ \tilde{\boldsymbol{\tau}}$, $\mathcal{T}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = (\sum_j (\partial_{\mathsf{x}_j}\check{\boldsymbol{\Upsilon}}_i \partial_{\mathsf{x}_j}\check{\boldsymbol{\Upsilon}}_k) - \delta_{i,k}) \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}$ for $i,k,l \in \{1,\ldots,d\}$ with $\mathcal{T}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)(0) = 0$.

1. $\|\tilde{\mathcal{F}}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{L^2((0,T),H^\ell(\tilde{\Omega}_f)^d)}$:

Boundedness, continuity and Fréchet differentiability are obtained with Lemma 5 for $s = s_1 = 0$, $s_2 = 1$, $s_3 = \ell$, $X = H^\ell(\tilde{\Omega}_f)^d$, $X_1 = H^{1+\ell}(\tilde{\Omega}_f)^d$, $X_2 = H^\ell(\tilde{\Omega}_f)$, $X_3 = H^{1+\ell}(\tilde{\Omega}_f)$ and Lemma 8. By **P6** and (39) we obtain

$$\|\mathcal{T}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{S_3} \leqslant CT^{1-\ell}\|\mathcal{T}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{H^1((0,T),X_3)} \leqslant CT^{1-\ell}(1 + \|\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}(\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})^\top\|_{\underline{\tilde{\mathbf{S}}}_T}) \leqslant CT^{1-\ell}(1 + M_0^{70}), \tag{48}$$

(41) and Lemma A.3 imply $\|\mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{S_2} \leqslant C$ and, with **P7**,

$$\|\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{H^{s_1}((0,T),X_1)} \leqslant C\|\tilde{\mathbf{v}}\|_{L^2((0,T),H^{2+\ell}(\tilde{\Omega}_f)^d)} \leqslant CM_0.$$

2. $\|\tilde{\mathcal{F}}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{H^{\frac{\ell}{2}}((0,T),L^2(\tilde{\Omega}_f)^d)}$:
Choose $s = s_1 = \frac{\ell}{2}$, $s_2 = 1$, $s_3 = \ell$, $X = L^2(\tilde{\Omega}_f)^d$, $X_1 = H^1(\tilde{\Omega}_f)^d$, $X_2 = H^\ell(\tilde{\Omega}_f)$, $X_3 = H^{1+\ell}(\tilde{\Omega}_f)$ and use (41), (48), **P6**, **P7** and Lemmas A.3, A.4, 5 and 8. We obtain $\|\tilde{\mathcal{F}}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{F}}_T} \leqslant CT^{1-\ell}(1+M_0^{71})$.

• Bound for $\|\tilde{\mathcal{H}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{H}}_T}$:

$\tilde{\mathcal{H}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)$ is a multilinear form with factors $\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) = D_{\mathbf{z}}\,\tilde{\mathbf{v}}$, $\mathcal{T}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) = (D_{\mathbf{z}}\,\tilde{\boldsymbol{\tau}})^{-1}$, $\mathcal{T}_3(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) = \widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}$, $\mathcal{T}_4(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) = (\mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}) - \mathbf{I})$, $\mathcal{T}_5(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) = \nu\mathrm{cof}(D_{\mathbf{z}}\,\tilde{\boldsymbol{\tau}})\tilde{\mathbf{n}}_f$ and $\mathcal{T}_4(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)(0) = 0$. Due to Lemma A.4 on Γ_i, which can locally be mapped to bounded open domains on $\mathbb{R}^{d-1}$, Lemma 5 can be applied. **P7** and boundedness of the trace operator yield

$$\|\mathcal{T}_j(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{H^\alpha((0,T),H^{\frac{1}{2}+\ell}(\tilde{\Gamma}_i)^{d\times d})} \leqslant C\|\mathcal{T}_j(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{H^\alpha((0,T),H^{1+\ell}(\tilde{\Omega}_f)^{d\times d})}$$

for $\alpha \in [0,1]\backslash\{\frac{1}{2}\}$, $j \in \{1,\ldots,5\}$.

1. $\|\tilde{\mathcal{H}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{L^2((0,T),H^{\frac{1}{2}+\ell}(\tilde{\Gamma}_i)^d)}$:
Choose $s = s_1 = 0$, $s_2 = s_5 = 1$, $s_3 = s_4 = \ell$ and $X_1 = X_2 = X_3 = X_4 = H^{\frac{1}{2}+\ell}(\tilde{\Gamma}_i)^{d\times d}$ and $X = X_5 = H^{\frac{1}{2}+\ell}(\tilde{\Gamma}_i)^d$. We have with **P7** and the definition of $\tilde{\mathbf{V}}$

$$\|\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_1} \leqslant C\|\tilde{\mathbf{v}}\|_{L^2((0,T),H^{2+\ell}(\tilde{\Omega}_f)^d)} \leqslant CM_0, \quad \|\mathcal{T}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_2} \leqslant C\|(D_{\mathbf{z}}\,\tilde{\boldsymbol{\tau}})^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \leqslant C\alpha_1.$$

Lemma A.2 and 8 imply

$$\|\mathcal{T}_3(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_3} \leqslant C(T^{1-\ell}\|\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}\|_{H^1((0,T),H^{1+\ell}(\tilde{\Omega}_f)^{d\times d})} + \|\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}(0)\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}}) \leqslant C(1+M_0^{35}).$$

Moreover, (42) yields $\|\mathcal{T}_5(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_5} \leqslant C\|\mathrm{cof}(D_{\mathbf{z}}\,\tilde{\boldsymbol{\tau}})\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \leqslant C$. Finally, **P6** implies

$$\|\mathcal{T}_4(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_4} \leqslant C\|\mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}){-}\mathbf{I}\|_{H^\ell((0,T),H^{1+\ell}(\tilde{\Omega}_f)^{d\times d})} \leqslant CT^{1-\ell}\|\mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}){-}\mathbf{I}\|_{H^1((0,T),H^{1+\ell}(\tilde{\Omega}_f)^{d\times d})} \leqslant CT^{1-\ell}(1{+}M_0^2).$$

2. $\|\tilde{\mathcal{H}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{H^{\frac{1}{4}+\frac{\ell}{2}}((0,T),L^2(\tilde{\Gamma}_i)^d)}$:
Let $s = s_1 = s_3 = s_4 = \frac{1}{4}+\frac{\ell}{2}$, $s_2 = s_5 = 1$, $X = L^2(\tilde{\Gamma}_i)^d$, $X_1 = L^2(\tilde{\Gamma}_i)^{d\times d}$, $X_2 = X_3 = X_4 = H^{\frac{1}{2}+\ell}(\tilde{\Gamma}_i)^{d\times d}$, $X_5 = H^{\frac{1}{2}+\ell}(\tilde{\Gamma}_i)^d$. The estimates for $\|\mathcal{T}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_2}$ and $\|\mathcal{T}_5(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_5}$ are as above. Since $\|D_{\mathbf{z}}\cdot|_{\tilde{\Sigma}_i^T}\|_{H^{\frac{1}{4}+\frac{\ell}{2}}((0,T),L^2(\tilde{\Gamma}_i)^{d\times d})}$ appears in the definition of $\|\cdot\|_{\tilde{\mathbf{E}}_T}$ we have $\|\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_1} \leqslant CM_0$. Lemma A.2 and 8 yield

$$\|\mathcal{T}_3(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_3} \leqslant C(T^{\frac{3}{4}-\frac{\ell}{2}}\|\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}\|_{H^1((0,T),H^{1+\ell}(\tilde{\Omega}_f)^{d\times d})} + \|\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}(0)\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}}) \leqslant C(1+M_0^{35}).$$

Lemma A.2 and **P6** imply

$$\|\mathcal{T}_4(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_4} \leqslant CT^{\frac{3}{4}-\frac{\ell}{2}}\|\mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}) - \mathbf{I}\|_{H^1((0,T),H^{1+\ell}(\tilde{\Omega}_f)^{d\times d})} \leqslant CT^{\frac{3}{4}-\frac{\ell}{2}}(1+M_0^2).$$

Hence, application of Lemma 5 in both cases yields $\|\tilde{\mathcal{H}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{H}}_T} \leqslant CT^{1-\ell}(1+M_0^{38})$.

• Estimation of $\tilde{\mathcal{G}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)$:

$\tilde{\mathcal{G}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)$ is a sum of multilinear forms with factors $\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) = (D_{\mathbf{z}}\,\tilde{\mathbf{v}})_{i,j}$, $\mathcal{T}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) = (\mathbf{I} - \mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))_{i,k}$, $\mathcal{T}_3(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) = (\mathrm{cof}(D_{\mathbf{z}}\,\tilde{\boldsymbol{\tau}}))_{k,j}$ with $i,k,j \in \{1,\ldots,d\}$.
1. $\|\tilde{\mathcal{G}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{L^2((0,T),H^{1+\ell}(\tilde{\Omega}_f))}$:
Choose $s = s_1 = 0$, $s_2 = \ell$, $s_3 = 1$, $X = X_1 = X_2 = X_3 = H^{1+\ell}(\tilde{\Omega}_f)$. $\|\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{L^2((0,T),H^{1+\ell}(\tilde{\Omega}_f))} \leqslant C\|\tilde{\mathbf{v}}\|_{\tilde{\mathbf{E}}_T} \leqslant CM_0$ due to **P7** and (20), and with (44) we obtain the bound $\|\tilde{\mathcal{G}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{L^2((0,T),H^{1+\ell}(\tilde{\Omega}_f))} \leqslant CT^{1-\ell}M_0(1{+}M_0^2)$.
2. $\|\tilde{\mathcal{G}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{H^{\frac{\ell}{2}}((0,T),H^1(\tilde{\Omega}_f))}$:
We choose $s = s_1 = \frac{\ell}{2}$, $s_2 = \ell$, $s_3 = 1$, $X = X_1 = H^1(\tilde{\Omega}_f)$, $X_2 = X_3 = H^{1+\ell}(\tilde{\Omega}_f)$. **P7** and (20) yield $\|\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_1} \leqslant C\|\tilde{\mathbf{v}}\|_{H^{\frac{\ell}{2}}((0,T),H^2(\tilde{\Omega}_f)^d)} \leqslant CM_0$. Thus, with (44) and Lemmas A.4, 5, 8 we obtain $\|\tilde{\mathcal{G}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{H^{\frac{\ell}{2}}((0,T),H^1(\tilde{\Omega}_f))} \leqslant CT^{1-\ell}M_0(1+M_0^2)$. □

Theorem 3. *Let Assumption 1 be fulfilled. Then, there exist $\epsilon_l > 0$, $T_l > 0$ and $M_l > 0$ such that for all $0 < T \leqslant T_l$ and for arbitrary $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{V}}_{\epsilon_l}$ the system* (28) *admits a unique solution* $\tilde{\mathbf{y}}(\tilde{\mathbf{u}}_\tau) := (\tilde{\mathbf{v}}(\tilde{\mathbf{u}}_\tau), \tilde{p}(\tilde{\mathbf{u}}_\tau), \tilde{\mathbf{w}}(\tilde{\mathbf{u}}_\tau))$

on the relative interior of $\tilde{\mathbf{E}}_{T,M_l,\tilde{\mathbf{v}}_0} \times \tilde{P}_{T,M_l,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{W}}_T$*. The mapping* $\tilde{\mathbf{u}}_\tau \mapsto \tilde{\mathbf{y}}(\tilde{\mathbf{u}}_\tau)$ *is continuous and Fréchet differentiable on the interior of* $\tilde{\mathbf{V}}_{\epsilon_l}$ *and, for* $\mathbf{h} \in \tilde{\mathbf{U}}$ *the derivative* $\tilde{\mathbf{y}}(\tilde{\mathbf{u}}_\tau)'\mathbf{h} := \delta_h\tilde{\mathbf{y}} = (\delta_h\tilde{\mathbf{v}}, \delta_h\tilde{p}, \delta_h\tilde{\mathbf{w}})$ *is given as the solution of the system*

$$\begin{aligned}
\partial_{\mathsf{t}}\,\delta_h\tilde{\mathbf{v}} - \nu\Delta_{\mathsf{z}}\,\delta_h\tilde{\mathbf{v}} + \nabla_{\mathsf{z}}\,\delta_h\tilde{p} &= (\tilde{\mathcal{F}}(\tilde{\mathbf{v}}(\tilde{\mathbf{u}}_\tau), \tilde{p}(\tilde{\mathbf{u}}_\tau), \tilde{\mathbf{u}}_\tau))_{\mathbf{v}}\delta_h\tilde{\mathbf{v}} \\
+(\tilde{\mathcal{F}}(\tilde{\mathbf{v}}(\tilde{\mathbf{u}}_\tau), \tilde{p}(\tilde{\mathbf{u}}_\tau), \tilde{\mathbf{u}}_\tau))_p\delta_h\tilde{p} &+ (\tilde{\mathcal{F}}(\tilde{\mathbf{v}}(\tilde{\mathbf{u}}_\tau), \tilde{p}(\tilde{\mathbf{u}}_\tau), \tilde{\mathbf{u}}_\tau))_{\boldsymbol{u}_\tau}\tilde{\mathbf{h}} \quad \text{in } \tilde{Q}_f^T, \\
\mathrm{div}_{\mathsf{z}}\,(\delta_h\tilde{\mathbf{v}}) &= (\tilde{\mathcal{G}}(\tilde{\mathbf{v}}(\tilde{\mathbf{u}}_\tau), \tilde{\mathbf{u}}_\tau))_{\mathbf{v}}\delta_h\tilde{\mathbf{v}} + (\tilde{\mathcal{G}}(\tilde{\mathbf{v}}(\tilde{\mathbf{u}}_\tau), \tilde{\mathbf{u}}_\tau))_{\boldsymbol{u}_\tau}\tilde{\mathbf{h}} \quad \text{in } \tilde{Q}_f^T, \\
\delta_h\tilde{\mathbf{v}}(\cdot, 0) &= 0 \quad \text{in } \tilde{\Omega}_f, \\
\delta_h\tilde{\mathbf{v}} &= 0 \quad \text{on } \tilde{\Sigma}_f^T \\
\delta_h\tilde{\mathbf{v}} &= \partial_{\mathsf{t}}\,\delta_h\tilde{\mathbf{w}} \quad \text{on } \tilde{\Sigma}_i^T, \\
\sigma_{f,\mathsf{z}}(\delta_h\tilde{\mathbf{v}}, \delta_h\tilde{p})\tilde{\mathbf{n}}_f &= \sigma_{s,\mathsf{z}}(\delta_h\tilde{\mathbf{w}})\tilde{\mathbf{n}}_f + (\tilde{\mathcal{H}}(\tilde{\mathbf{v}}(\tilde{\mathbf{u}}_\tau), \tilde{p}(\tilde{\mathbf{u}}_\tau), \tilde{\mathbf{u}}_\tau))_{\mathbf{v}}\delta_h\tilde{\mathbf{v}} \\
+(\tilde{\mathcal{H}}(\tilde{\mathbf{v}}(\tilde{\mathbf{u}}_\tau), \tilde{p}(\tilde{\mathbf{u}}_\tau), \tilde{\mathbf{u}}_\tau))_p\delta_h\tilde{p} &+ (\tilde{\mathcal{H}}(\tilde{\mathbf{v}}(\tilde{\mathbf{u}}_\tau), \tilde{p}(\tilde{\mathbf{u}}_\tau), \tilde{\mathbf{u}}_\tau))_{\boldsymbol{u}_\tau}\tilde{\mathbf{h}} \quad \text{on } \tilde{\Sigma}_i^T, \\
\partial_{\mathsf{tt}}\,\delta_h\tilde{\mathbf{w}} - \mathrm{div}_{\mathsf{z}}\,(\sigma_{s,\mathsf{z}}(\delta_h\tilde{\mathbf{w}})) &= 0 \quad \text{in } \tilde{Q}_s^T, \\
\delta_h\tilde{\mathbf{w}} &= 0 \quad \text{on } \tilde{\Sigma}_s^T, \\
\delta_h\tilde{\mathbf{w}}(\cdot, 0) = 0, \quad \partial_{\mathsf{t}}\,\delta_h\tilde{\mathbf{w}}(\cdot, 0) &= 0 \quad \text{in } \tilde{\Omega}_s.
\end{aligned} \tag{49}$$

Proof. In the notation of Theorem 1, choose $y = (\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{w}})$, $z = \tilde{\mathbf{u}}_\tau$, $Y = \tilde{\mathbf{E}}_T \times \tilde{P}_T \times \tilde{\mathbf{W}}_T$, $Z = \tilde{\mathbf{U}}$, $\mathcal{F}(y, z) = (\tilde{\mathcal{F}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau), \tilde{\mathcal{H}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau), \tilde{\mathcal{G}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau), \tilde{\mathbf{g}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau), \tilde{\mathbf{v}}_0, \tilde{\mathbf{w}}_1)$. Furthermore, let $W := \tilde{\mathbf{F}}_T \times \tilde{\mathbf{H}}_T \times \tilde{G}_T \times \tilde{\mathbf{G}}_T \times \tilde{\mathbf{V}}_0 \times \tilde{\mathbf{W}}_1$, and $\tilde{W} := \tilde{\mathbf{F}}_T \times \{\tilde{\mathbf{h}} \in \tilde{\mathbf{H}}_T \;:\; \tilde{\mathbf{h}}(0) = 0\} \times \tilde{G}_T \times \{\tilde{\mathbf{g}} \in \tilde{\mathbf{G}}_T \cap H^1((0,T), H^\ell(\tilde{\Omega}_f)^d) \;:\; \tilde{\mathbf{g}}|_{\tilde{\Sigma}_f^T} = 0,\ \mathbf{g}(0) = 0\} \times \{(\tilde{\mathbf{v}}_0, \tilde{\mathbf{w}}_1) \in \tilde{\mathbf{V}}_0 \times \tilde{\mathbf{W}}_1 \;:\; \tilde{\mathbf{v}}_0|_{\tilde{\Gamma}_f} = 0,\ \mathrm{div}_{\mathsf{z}}\,(\tilde{\mathbf{v}}_0) = 0,\ \tilde{\mathbf{v}}_0|_{\tilde{\Gamma}_i} = \tilde{\mathbf{w}}_1|_{\tilde{\Gamma}_i},\ 2\nu(\epsilon_{\mathsf{z}}(\tilde{\mathbf{v}}_0)) \cdot \boldsymbol{\tau} = 0$ on $\tilde{\Gamma}_i$ for any unit vector $\boldsymbol{\tau}$ tangent to $\tilde{\Gamma}_i\}$, let $\rho = \rho(0)$ be given by Lemma 4. Lemma 3 defines the operator S and yields $T_f > 0$ and $L_S = C_S > 0$ such that $S \in \mathcal{L}(\tilde{W}, \tilde{\mathbf{E}}_T \times \tilde{P}_T \times \tilde{\mathbf{W}}_T)$ for $0 < T \leqslant T_f$ and $\|Sf\|_Y \leqslant L_S\|f\|_W$ for all $f \in \tilde{W}$. Theorem 2 and Lemma 9 yield constants $M_0 > 0$, $T_0 \in (0, T_f)$ and $\epsilon = \min(\rho, T_0^{\frac{1}{4}-\frac{l}{4}})$ such that, for all $0 < T \leqslant T_0$ and $z \in \tilde{\mathbf{V}}_\epsilon$ there exists a unique solution $y_0(z) \in \tilde{\mathbf{E}}_{T,M_0,\mathbf{v}_0} \times P_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{W}}_T$, which is a subset of the relative interior of $\tilde{\mathbf{E}}_{T,M_l,\tilde{\mathbf{v}}_0} \times \tilde{P}_{T,M_l,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{W}}_T$ for $M_l > M_0$. Furthermore, Theorem 2 and Lemma 9 yield $T_l \in (0, T_0]$ and $\epsilon_l \leqslant \min(\rho, T_l^{\frac{1}{4}-\frac{l}{4}})$ such that for all $z \in \tilde{\mathbf{V}}_{\epsilon_l}$ there exists a unique solution $y_l(z) \in \tilde{\mathbf{E}}_{T,M_l,\tilde{\mathbf{v}}_0} \times P_{T,M_l,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{W}}_T$ and the required boundedness, continuity and Fréchet differentiability results are fulfilled for the choices $\tilde{Y} = \tilde{\mathbf{E}}_{T,M_l,\tilde{\mathbf{v}}_0} \times \tilde{P}_{T,M_l,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{W}}_T$ and $\tilde{Z} = \tilde{\mathbf{V}}_{\epsilon_l}$ and $0 < T \leqslant T_l$. Furthermore, the proof of Theorem 2 implies that $S\mathcal{F}(y, z) \in \tilde{Y}$ for $(y, z) \in \tilde{Y} \times \tilde{Z}$. Since $y_0(z) \in \tilde{\mathbf{E}}_{T,M_l,\tilde{\mathbf{v}}_0} \times P_{T,M_l,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{W}}_T$ and the solution is unique, we have $y_0(z) = y_l(z)$ for all $z \in \tilde{Z}$. Thus, $y_l(z)$ is in the relative interior of $\tilde{Y}$ and Theorem 1 can be applied. ☐

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A Appendix

The subsequent Lemma is a concise version of [44, Lemma A.1].

Lemma A.1. *Let* X, Y, Z *be real, separable Hilbert spaces and* m *be a bounded bilinear mapping from* $X \times Y$ *into* Z*. Further, let* $f \in H^{s_1}((0,T), X)$ *and* $g \in H^{s_2}((0,T), Y)$ *with* $s_1, s_2 \geqslant 0$*. Then the following holds.*

1. *If* $\frac{1}{2} < s_1 \leqslant 1$, $0 \leqslant s_2 < \frac{1}{2}$, *then* $m(f, g)$ *belongs to* $H^{s_2}((0,T), Z)$ *and*

$$\|m(f,g)\|_{H^{s_2}((0,T),Z)} \leqslant C_{s_1,s_2}(\|f\|_{H^{s_1}((0,T),X)} + \|f(0)\|_X)\|g\|_{H^{s_2}((0,T),Y)},$$

for all $0 \leqslant T \leqslant T_f$*, where* C_{s_1,s_2} *is independent of* T*.*

2. *If* $\frac{1}{2} < s_1 \leqslant s_2 \leqslant 1$*, then* $m(f,g)$ *belongs to* $H^{s_1}((0,T),Z)$ *and*

$$\|m(f,g)\|_{H^{s_1}((0,T),Z)} \leqslant C_{s_1,s_2}(\|f\|_{H^{s_1}((0,T),X)} + \|f(0)\|_X)(\|g\|_{H^{s_2}((0,T),Y)} + \|g(0)\|_Y),$$

for all $0 \leqslant T \leqslant T_f$*, where* C_{s_1,s_2} *is independent of* T*.*

Proof. We prove 2., 1. follows with similar arguments. Let $f_0 \in H^1((-\infty,\infty),X)$ and $g_0 \in H^1((-\infty,\infty),Y)$ be such that $f_0(0) = f(0)$, $g_0(0) = g(0)$ and for $-\infty < a < b < \infty$,

$$\|f_0\|_{H^1((a,b),X)} \leqslant C_0\|f(0)\|_X,$$
$$\|g_0\|_{H^1((a,b),Y)} \leqslant C_0\|g(0)\|_Y,$$

with a constant C_0 independent of $(b-a)$ (extension to $H^1((0,\infty),X)$ and mirroring at $\mathtt{t} = 0$). Let C and C_{T_f} denote generic constants (C_{T_f} is used if the constant might depend on T_f). Using property **P2** of the norm and the first inequality in the proof of [44, Lemma A.1] yields

$$\|m(f,g)\|_{H^{s_1}((a,a+T_f),Z)} \leqslant C_{T_f}\|f\|_{H^{s_1}((a,a+T_f),X)}\|g\|_{H^{s_2}((a,a+T_f),Y)}, \tag{50}$$

(use equivalence of norms with T_f-dependent constants). Now,

$$\|m(f,g)\|_{H^{s_1}((0,T),Z)} \leqslant (\|m(f,g) - m(f_0,g_0)\|_{H^{s_1}((0,T),Z)} + \|m(f_0,g_0)\|_{H^{s_1}((0,T),Z)}).$$

Due to Property **P5** of the norm and (50),

$$\begin{aligned}\|m(f_0,g_0)\|_{H^{s_1}((0,T),Z)} &\leqslant C\|m(f_0,g_0)\|_{H^{s_1}((0,T_f),Z)} \leqslant C_{T_f}\|f_0\|_{H^{s_1}((0,T_f),X)}\|g_0\|_{H^{s_2}((0,T_f),Y)}\\ &\leqslant C_{T_f}\|f(0)\|_X\|g(0)\|_Y.\end{aligned}$$

Furthermore,

$$\begin{aligned}\|m(f,g) - m(f_0,g_0)\|_{H^{s_1}((0,T),Z)} &\leqslant \|m(f-f_0,g-g_0)\|_{H^{s_1}((0,T),Z)}\\ &\quad + \|m(f-f_0,g_0)\|_{H^{s_1}((0,T),Z)} + \|m(f_0,g-g_0)\|_{H^{s_1}((0,T),Z)}.\end{aligned}$$

We know that $(f-f_0)|_{\mathtt{t}=0} = 0$. Due to properties **P3** and **P4** of the norm and with (50),

$$\begin{aligned}\|m(f-f_0,g-g_0)\|_{H^{s_1}((0,T),Z)} &\leqslant C_{T_f}\|\mathrm{Ext}(f-f_0)\|_{H^{s_1}((T-T_f,T),X)}\|\mathrm{Ext}(g-g_0)\|_{H^{s_2}((T-T_f,T),Y)}\\ &\leqslant C_{T_f}\|f-f_0\|_{H^{s_1}((0,T),X)}\|g-g_0\|_{H^{s_2}((0,T),Y)}\\ &\leqslant C_{T_f}(\|f\|_{H^{s_1}((0,T),X)} + \|f_0\|_{H^{s_1}((0,T_f),X)})(\|g\|_{H^{s_2}((0,T),Y)} + \|g_0\|_{H^{s_2}((0,T_f),Y)})\\ &\leqslant C_{T_f}(\|f\|_{H^{s_1}((0,T),X)} + \|f_0\|_{H^1((0,T_f),X)})(\|g\|_{H^{s_2}((0,T),Y)} + \|g_0\|_{H^1((0,T_f),Y)})\\ &\leqslant C_{T_f}(\|f\|_{H^{s_1}((0,T),X)} + \|f(0)\|_X)(\|g\|_{H^{s_2}((0,T),Y)} + \|g(0)\|_Y).\end{aligned}$$

We now estimate $m(f-f_0,g_0)$ using the norm properties **P3**, **P4**:

$$\begin{aligned}\|m(f-f_0,g_0)\|_{H^{s_1}((0,T),Z)} &\leqslant C\|m(\mathrm{Ext}(f-f_0),g_0)\|_{H^{s_1}((T-T_f,T),Z)}\\ &\leqslant C_{T_f}\|\mathrm{Ext}(f-f_0)\|_{H^{s_1}((T-T_f,T),X)}\|g_0\|_{H^{s_2}((T-T_f,T),Y)}\\ &\leqslant C_{T_f}\|f-f_0\|_{H^{s_1}((0,T),X)}\|g_0\|_{H^{s_2}((-T_f,T_f),Y)}\\ &\leqslant C_{T_f}(\|f\|_{H^{s_1}((0,T),X)} + \|f(0)\|_X)\|g(0)\|_Y.\end{aligned}$$

Since $m(f_0,g-g_0)$ can be estimated in the same way, this concludes the proof of 2. ☐

Lemma A.2. *Let* X *be a real, separable Hilbert space and* $\alpha \in [0,1)\backslash\{\frac{1}{2}\}$*. Furthermore, let* $\beta > 0$ *be such that* $\alpha+\beta \in (\frac{1}{2},1]$*,* $c \in X$ *and* $g \in H^{\alpha+\beta}((0,T),X)$ *be such that* $g(0) = c$*. Then, there exists a constant* C *independent of* T *such that*

$$\|g\|_{H^\alpha((0,T),X)} \leqslant C(T^\beta\|g\|_{H^{\alpha+\beta}((0,T),X)} + \|c\|_X).$$

Proof. Let C denote a generic constant independent of T, where $0 < T \leqslant T_f$. There exists $h \in H^1((0,T_f),X)$ such that $h(0) = c$ and $\|h\|_{H^1((0,T_f),X)} \leqslant C\|c\|_X$ e.g., $h(\mathsf{t}) := cT_f^{-1}(T_f - \mathsf{t})$ for $\mathsf{t} \in (0,T_f)$. Set $\tilde{g} = g - h$.
Properties **P5**, **P2**, the definition of h and **P6** yield

$$\begin{aligned}\|g\|_{H^\alpha((0,T),X)} &\leqslant \|\tilde{g}\|_{H^\alpha((0,T),X)} + \|h\|_{H^\alpha((0,T),X)} \leqslant \|\tilde{g}\|_{H^\alpha((0,T),X)} + C\|h\|_{H^1((0,T_f),X)} \\ &\leqslant \|\tilde{g}\|_{H^\alpha((0,T),X)} + C\|c\|_X \leqslant CT^\beta\|\tilde{g}\|_{H^{\alpha+\beta}((0,T),X)} + C\|c\|_X \\ &\leqslant C(T^\beta\|g\|_{H^{\alpha+\beta}((0,T),X)} + \|c\|_X).\end{aligned}$$

□

Lemma A.3. *Let X be a real, separable Hilbert space and $s \geqslant 0$. Let $c \in X$ and $g(\mathsf{t}) = c$ for a.e. $\mathsf{t} \in (0,T)$. Then, $g \in H^s((0,T),X)$ and there exists a constant C independent of T such that $\|g\|_{H^s((0,T),X)} \leqslant C\|c\|_X$.*

Proof. Let $T_f \geqslant T$ and C denote a generic constant independent of T. For $s \geqslant 1$ we have, due to **P1** and $\partial_{\mathsf{t}} g = 0$,

$$\|g\|_{H^s((0,T),X)} = \|g\|_{L^2((0,T),X)} \leqslant T^{\frac{1}{2}}\|c\|_X \leqslant C\|c\|_X. \tag{51}$$

For $s \in [0,1)$, Lemma A.2 and (51) yield

$$\|g\|_{H^s((0,T),X)} \leqslant C(T^{1-s}\|g\|_{H^1((0,T),X)} + \|c\|_X) \leqslant C\|c\|_X.$$

□

Furthermore, the following result corresponds to [21, Prop. B.1 (i)].

Lemma A.4. *Let $\zeta, \upsilon, \omega \in \mathbb{R}$, $f \in H^{\zeta+\upsilon}(\Omega_f)$, and $g \in H^{\zeta+\omega}(\Omega_f)$. Then, there exists $C > 0$ such that*

$$\|fg\|_{H^\zeta(\Omega_f)} \leqslant C\|f\|_{H^{\zeta+\upsilon}(\Omega_f)}\|g\|_{H^{\zeta+\omega}(\Omega_f)},$$

1. *if $\upsilon + \omega + \zeta \geqslant \frac{d}{2}$, $\upsilon > 0$, $\omega > 0$, and $2\zeta > -\upsilon - \omega$,*
2. *or $\upsilon + \omega + \zeta > \frac{d}{2}$, $\upsilon \geqslant 0$, $\omega \geqslant 0$, and $2\zeta \geqslant -\upsilon - \omega$.*

A.1 Proof of Lemma 1

We verify properties **P1 -P8** for the norm defined in the proof of Lemma 1. Let $-\infty < T_1 < T_2 \leqslant \infty$. Let $\|\cdot\|_{H^0((T_1,T_2),X)} := \|\cdot\|_{L^2((T_1,T_2),X)}$. Furthermore, let for $\sigma \in (0,1)\backslash\{\frac{1}{2}\}$,

$$\|\cdot\|_{H^\sigma((T_1,T_2),X)} := \begin{cases} (|\cdot|^2_{H^\sigma((T_1,T_2),X)} + \frac{1}{\sigma}\int_{T_1}^{T_2}(\mathsf{t}-T_1)^{-2\sigma}\|\cdot(\mathsf{t})\|_X^2 d\mathsf{t})^{\frac{1}{2}} & \text{if } \sigma \in (0,\frac{1}{2}), \\ (|\cdot|^2_{H^\sigma((T_1,T_2),X)} + \frac{1}{\sigma}\int_{T_1}^{T_2}(\mathsf{t}-T_1)^{-2\sigma}\|(\cdot - L(\cdot))(\mathsf{t})\|_X^2 d\mathsf{t})^{\frac{1}{2}} & \text{if } \sigma \in (\frac{1}{2},1), \end{cases} \tag{52}$$

where, for $\sigma \in (\frac{1}{2},1)$, L is chosen as a linear operator that is continuous as a mapping

$$L: \quad H^\sigma((T_1,T_2),X) \to H^\sigma((T_1,\infty),X),$$

and such that there exists a constant C independent of $\Delta T := T_2 - T_1$ such that $|L(u)|_{H^\sigma((T_1,\infty),X)} \leqslant C\|u(T_1)\|_X$, and $L(u)(T_1) = u(T_1)$, e.g., for fixed $T_f > 0$,

$$L(u)(\mathsf{t}) = \begin{cases} u(T_1)T_f^{-1}(T_f + T_1 - \mathsf{t}) & \text{for } \mathsf{t} \in (T_1, T_1 + T_f), \\ 0 & \text{for } \mathsf{t} \in [T_1 + T_f, \infty). \end{cases}$$

Remark A.1. *The Sobolev-Slobodeckij norm $|\cdot|_{H^\sigma((T_1,T_2),X)}$ is equivalent to the norm introduced in [36, p.10, Def. 2.1], which is equivalent to the complex interpolation norm due to [36, p. 92, Thm. 14.1 and p. 23, Remark 3.6]. In the Hilbert space setting, complex and real interpolation norms are equivalent due to [12, Thm. 3.3 and Rem. 3.6]. [2, (3.4), (3.5) - (3.7), Corollary 4.3] concludes the argumentation. More details, e.g., the definition of '≐', can be found in [1, Section 5]. For the equivalence of Besov and Sobolev-Slobodeckij spaces for $\sigma \in (0,1)$, the reader is refered to [48, Prop. 2], for the interpolation of Besov spaces to [48, Proof of Thm. 30] and the references therin, [10, p. 194, Thm. 3.4.2].*

Lemma A.5. *Let $\sigma \in (0,1)\backslash\{\frac{1}{2}\}$. Then, there exist $C_{\Delta T} > 0$ that depends on $\Delta T = T_2 - T_1$ such that*

$$|u|_{H^\sigma((T_1,T_2),X)} \leqslant \|u\|_{H^\sigma((T_1,T_2),X)} \leqslant C_{\Delta T}|u|_{H^\sigma((T_1,T_2),X)}$$

for all $u \in H^\sigma((T_1,T_2),X)$.

Proof. The first inequality follows directly from the definition of $\|\cdot\|_{H^\sigma((T_1,T_2),X)}$. The second inequality is a direct consequence of [36, p. 57, Thm. 11.2 and p. 59, Thm. 11.3]. Even though the theorems [36, Thm. 11.2 - Thm. 11.3] are formulated in the scalar valued spaces, the proofs of the theorems, as well as, [36, p. 47, Remark 9.5] imply the validity for X-valued spaces. More precisely, the following holds:

• Let $\sigma \in (0,\frac{1}{2})$ and $u \in H^\sigma((T_1,T_2),X)$:
Due to [36, p. 60, Thm. 11.4] we know that

$$|\tilde{u}|_{H^\sigma(\mathbb{R},X)} \leqslant C_{\Delta T}|u|_{H^\sigma((T_1,T_2),X)}$$

for $\tilde{u}(\mathsf{t}) := \begin{cases} u(\mathsf{t}) & \text{if } \mathsf{t} \in (T_1,T_2), \\ 0 & \text{else.} \end{cases}$ Thus, for $T_3 > T_2$, using [36, p.57, Thm. 11.2],

$$\begin{aligned}\int_{T_1}^{T_2}(\mathsf{t}-T_1)^{-2\sigma}\|u(\mathsf{t})\|_X^2 d\mathsf{t} &\leqslant \int_{T_1}^{T_3}(\mathsf{t}-T_1)^{-2\sigma}\|\tilde{u}(\mathsf{t})\|_X^2 d\mathsf{t} \\ &\leqslant C|\tilde{u}|^2_{H^\sigma((T_1,T_3),X)} \leqslant C|\tilde{u}|^2_{H^\sigma(\mathbb{R},X)} \leqslant C_{\Delta T}|u|^2_{H^\sigma((T_1,T_2),X)}.\end{aligned}$$

• Let $\sigma \in (\frac{1}{2},1)$ and $u \in H^\sigma((T_1,T_2),X)$:
Let $v := u - L(u)$, $v_2(\mathsf{t}) := (T_2-T_1)^{-1}(\mathsf{t}-T_1)v(T_2)$ for $\mathsf{t} \in (T_1,T_2)$ and define $v_1 := v - v_2$. Due to [36, p. 62, Thm. 11.5], $v_1 \in H_0^\sigma((T_1,T_2),X)$ and, by [36, p. 60, Thm. 11.4],

$$|\tilde{v}_1|_{H^\sigma(\mathbb{R},X)} \leqslant C_{\Delta T}|v_1|_{H^\sigma((T_1,T_2),X)},$$

where $\tilde{v}_1(\mathsf{t}) := \begin{cases} v_1(\mathsf{t}) & \text{if } \mathsf{t} \in (T_1,T_2), \\ 0 & \text{else.} \end{cases}$ Furthermore, for $T_3 > T_2$, using [36, p. 59, Thm. 11.3],

$$\begin{aligned}&\int_{T_1}^{T_2}(\mathsf{t}-T_1)^{-2\sigma}\|v_1(\mathsf{t})\|_X^2 d\mathsf{t} \leqslant \int_{T_1}^{T_3}(\mathsf{t}-T_1)^{-2\sigma}\|\tilde{v}_1(\mathsf{t})\|_X^2 d\mathsf{t} \\ &\leqslant C|\tilde{v}_1|^2_{H^\sigma((T_1,T_3),X)} \leqslant C|\tilde{v}_1|^2_{H^\sigma(\mathbb{R},X)} \leqslant C_{\Delta T}|v_1|^2_{H^\sigma((T_1,T_2),X)}\end{aligned} \tag{53}$$

and

$$\begin{aligned}\int_{T_1}^{T_2}(\mathsf{t}-T_1)^{-2\sigma}\|v_2(\mathsf{t})\|_X^2 d\mathsf{t} &= \int_{T_1}^{T_2}(\mathsf{t}-T_1)^{2-2\sigma}(T_2-T_1)^{-2}\|v(T_2)\|_X^2 d\mathsf{t} \\ &= (3-2\sigma)^{-1}(T_2-T_1)^{1-2\sigma}\|v(T_2)\|_X^2 \\ &\leqslant C_{\Delta T}|v|^2_{H^\sigma((T_1,T_2),X)}.\end{aligned} \tag{54}$$

Since $|v_2|_{H^\sigma((T_1,T_2);X)} \leqslant C_{\Delta T}\|v(T_2)\|_X \leqslant C_{\Delta T}|v|_{H^\sigma((T_1,T_2),X)}$ and

$$\begin{aligned}|v|_{H^\sigma((T_1,T_2),X)} &\leqslant |u|_{H^\sigma((T_1,T_2),X)} + |L(u)|_{H^\sigma((T_1,T_2),X)} \\ &\leqslant |u|_{H^\sigma((T_1,T_2),X)} + C\|u(T_1)\|_X \leqslant C_{\Delta T}|u|_{H^\sigma((T_1,T_2),X)},\end{aligned} \tag{55}$$

where the last inequality follows from the equivalence of the Sobolev-Slobodeckij norm to interpolation norms (Remark A.1) and [36, p.41, Proof of Thm. 9.4]. (53), (54) and (55) imply

$$\int_{T_1}^{T_2}(\mathsf{t}-T_1)^{-2\sigma}\|v(\mathsf{t})\|_X^2 d\mathsf{t} \leqslant C_{\Delta T}|u|^2_{H^\sigma((T_1,T_2),X)}.$$

□

Lemma A.6. *Let $T_f \geqslant T$ and $\sigma \in (0,1)\backslash\{\frac{1}{2}\}$. Then, the extension operator*

$$\mathrm{Ext}u := \begin{cases} 0 & \text{for } \mathsf{t} \in (T - T_f, 0), \\ u(\mathsf{t}) & \text{for } \mathsf{t} \in (0, T), \end{cases}$$

is continuous as a mapping $Y^{\sigma}_{(0,T)} \to Y^{\sigma}_{(T-T_f,T)}$ with norm bounded by 1.

Proof. Direct computations show that for $u \in Y^{\sigma}_{(0,T)}$,

$$\|u\|_{H^{\sigma}((0,T),X)} = |\mathrm{Ext}(u)|_{H^{\sigma}((-\infty,T),X)} = \|\mathrm{Ext}(u)\|_{H^{\sigma}((T-T_f,T),X)}. \tag{56}$$

This can be verified as follows: It holds that

$$|\mathrm{Ext}(u)|^2_{H^{\sigma}((-\infty,T),X)} = \int_{-\infty}^{T}\int_{-\infty}^{T} \frac{\|\mathrm{Ext}(u)(\mathsf{t}) - \mathrm{Ext}(u)(\mathsf{s})\|^2_X}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s}d\mathsf{t} + \|\mathrm{Ext}(u)\|^2_{L^2((-\infty,T),X)}.$$

We know that

$$\|\mathrm{Ext}(u)\|_{L^2((-\infty,T),X)} = \|\mathrm{Ext}(u)\|_{L^2((T-T_f,T),X)} = \|u\|_{L^2((0,T),X)},$$

further,

$$\begin{aligned} \int_{-\infty}^{T}\int_{-\infty}^{T} \frac{\|\mathrm{Ext}(u)(\mathsf{t}) - \mathrm{Ext}(u)(\mathsf{s})\|^2_X}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s}d\mathsf{t} &= \int_0^T\int_0^T \frac{\|u(\mathsf{t}) - u(\mathsf{s})\|^2_X}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s}d\mathsf{t} + 2\int_0^T\int_{-\infty}^0 \frac{\|u(\mathsf{t})\|^2_X}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s}d\mathsf{t} \\ &= \int_0^T\int_0^T \frac{\|u(\mathsf{t}) - u(\mathsf{s})\|^2_X}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s}d\mathsf{t} + 2\int_0^T \|u(\mathsf{t})\|^2_X \int_{\mathsf{t}}^{\infty} \mathsf{q}^{-2\sigma-1} d\mathsf{q}d\mathsf{t} \\ &= \int_0^T\int_0^T \frac{\|u(\mathsf{t}) - u(\mathsf{s})\|^2_X}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s}d\mathsf{t} + \frac{1}{\sigma}\int_0^T \mathsf{t}^{-2\sigma}\|u(\mathsf{t})\|^2_X d\mathsf{t}, \end{aligned}$$

and, analogously,

$$\begin{aligned} &\int_{-\infty}^{T}\int_{-\infty}^{T} \frac{\|\mathrm{Ext}(u)(\mathsf{t}) - \mathrm{Ext}(u)(\mathsf{s})\|^2_X}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s}d\mathsf{t} \\ &= \int_{T-T_f}^{T}\int_{T-T_f}^{T} \frac{\|\mathrm{Ext}(u)(\mathsf{t}) - \mathrm{Ext}(u)(\mathsf{s})\|^2_X}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s}d\mathsf{t} + 2\int_{T-T_f}^{T}\int_{-\infty}^{T-T_f} \frac{\|\mathrm{Ext}(u)(\mathsf{t})\|^2_X}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s}d\mathsf{t} \\ &= \int_0^T\int_0^T \frac{\|\mathrm{Ext}(u)(\mathsf{t}) - \mathrm{Ext}(u)(\mathsf{s})\|^2_X}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s}d\mathsf{t} + \frac{1}{\sigma}\int_0^T (\mathsf{t} - (T-T_f))^{-2\sigma}\|\mathrm{Ext}(u)(\mathsf{t})\|^2_X d\mathsf{t}. \end{aligned}$$

□

Lemma A.7. *Let $0 < T \leqslant T_f$, $\sigma \in (0,1)\backslash\{\frac{1}{2}\}$ and X be a Hilbert space. Then, the restriction operator*

$$\mathrm{R}(\cdot)(\mathsf{t}) = \cdot(\mathsf{t})$$

is continuous as a mapping $H^{\sigma}((0,T_f),X) \to H^{\sigma}((0,T),X)$ with norm bounded by 1.

Proof. Follows from the definition of the norm (52). □

Lemma A.8. *Let $T_f \geqslant T > 0$, X be a Hilbert space and $\sigma \in (0,1)\backslash\{\frac{1}{2}\}$. Furthermore, let $u \in H^{\sigma}((T-T_f,T),X)$ be such that $u|_{(T-T_f,0)} = 0$. Then,*

$$\|u\|_{H^{\sigma}((0,T),X)} = \|u\|_{H^{\sigma}((T-T_f,T),X)}.$$

Proof. Follows due to (56). □

Lemma A.9. *Let X be a Hilbert space, $\sigma \in [0,1)\backslash\{\frac{1}{2}\}$ and $\alpha > 0$ be chosen such that $\sigma + \alpha \in (0,1)\backslash\{\frac{1}{2}\}$. Then, there exists a constant C independent of T such that*

$$\|u\|_{H^{\sigma}((0,T),X)} \leqslant CT^{\alpha}\|u\|_{H^{\sigma+\alpha}((0,T),X)},$$

for all $u \in Y^{\sigma+\alpha}_{(0,T)}$.

Proof. We have

$$\|u\|^2_{L^2((0,T),X)} \leqslant T^{2(\sigma+\alpha)} \int_0^T \mathsf{t}^{-2(\sigma+\alpha)} \|u(\mathsf{t})\|^2_X d\mathsf{t} \leqslant T^{2(\sigma+\alpha)} \|u\|^2_{H^{\sigma+\alpha}((0,T),X)}. \tag{57}$$

In addition,

$$\int_0^T \int_0^T \frac{\|u(\mathsf{s}) - u(\mathsf{t})\|^2_X}{|\mathsf{s}-\mathsf{t}|^{2\sigma+1}} d\mathsf{s}d\mathsf{t} \leqslant T^{2\alpha} \int_0^T \int_0^T \frac{\|u(\mathsf{s}) - u(\mathsf{t})\|^2_X}{|\mathsf{s}-\mathsf{t}|^{2(\sigma+\alpha)+1}} d\mathsf{s}d\mathsf{t} \leqslant T^{2\alpha} \|u\|^2_{H^{\sigma+\alpha}((0,T),X)} \tag{58}$$

and

$$\int_0^T \mathsf{t}^{-2\sigma} \|u(\mathsf{t})\|^2_X dt = \int_0^T \mathsf{t}^{2\alpha} \mathsf{t}^{-2(\sigma+\alpha)} \|u(\mathsf{t})\|^2_X dt \leqslant T^{2\alpha} \int_0^T \mathsf{t}^{-2(\sigma+\alpha)} \|u(\mathsf{t})\|^2_X d\mathsf{t}. \tag{59}$$

Combining (57), (58) and (59) yields the assertion. □

Lemma A.10. *Let X be a Hilbert space, $\sigma \in [0,1)\backslash\{\frac{1}{2}\}$. Then, there exists a constant C independent of T such that*

$$\|u\|_{H^\sigma((0,T),X)} \leqslant CT^{1-\sigma} \|u\|_{H^1((0,T),X)},$$

for all $u \in Y^1_{(0,T)}$.

Proof. Let $u \in Y^1_{(0,T)}$. Since $u(0) = 0$ and due to Hölders' inequality,

$$\|u\|^2_{L^2((0,T),X)} = \int_0^T \| \int_0^{\mathsf{t}} \partial_{\mathsf{t}} u(\tau) d\tau \|^2_X d\mathsf{t} \leqslant \int_0^T \mathsf{t} \int_0^{\mathsf{t}} \|\partial_{\mathsf{t}} u(\tau)\|^2_X d\tau d\mathsf{t} \leqslant \frac{1}{2} T^2 \|u\|^2_{H^1((0,T),X)}. \tag{60}$$

Moreover, for $\sigma \in (0, \frac{1}{2})$,

$$\begin{aligned} \int_0^T \int_0^T \frac{\|u(\mathsf{s}) - u(\mathsf{t})\|^2_X}{|\mathsf{s}-\mathsf{t}|^{1+2\sigma}} d\mathsf{s}d\mathsf{t} &= 2 \int_0^T \int_0^{\mathsf{t}} \frac{\|u(\mathsf{s}) - u(\mathsf{t})\|^2_X}{|\mathsf{s}-\mathsf{t}|^{1+2\sigma}} d\mathsf{s}d\mathsf{t} \leqslant 2 \int_0^T \int_0^{\mathsf{t}} (\mathsf{t}-\mathsf{s})^{-2\sigma} \int_s^{\mathsf{t}} \|\partial_{\mathsf{t}} u(\tau)\|^2_X d\tau d\mathsf{s}d\mathsf{t} \\ &\leqslant 2\|u\|^2_{H^1((0,T),X)} \int_0^T \int_0^{\mathsf{t}} (\mathsf{t}-\mathsf{s})^{-2\sigma} d\mathsf{s}d\mathsf{t} = \frac{2}{(1-2\sigma)(2-2\sigma)} T^{2-2\sigma} \|u\|^2_{H^1((0,T),X)}. \end{aligned} \tag{61}$$

and for $\sigma \in (\frac{1}{2}, 1)$,

$$\begin{aligned} &\int_0^T \int_0^T \frac{\|u(\mathsf{s}) - u(\mathsf{t})\|^2_X}{|\mathsf{s}-\mathsf{t}|^{1+2\sigma}} d\mathsf{s}d\mathsf{t} = 2 \int_0^T \int_0^{\mathsf{t}} \frac{\|u(\mathsf{s}) - u(\mathsf{t})\|^2_X}{|\mathsf{s}-\mathsf{t}|^{1+2\sigma}} d\mathsf{s}d\mathsf{t} \leqslant 2 \int_0^T \int_0^{\mathsf{t}} (\mathsf{t}-\mathsf{s})^{-2\sigma} \int_s^{\mathsf{t}} \|\partial_{\mathsf{t}} u(\tau)\|^2_X d\tau d\mathsf{s}d\mathsf{t} \\ &= 2 \int_0^T \int_0^{\mathsf{t}} \int_0^{\tau} (\mathsf{t}-\mathsf{s})^{-2\sigma} \|\partial_{\mathsf{t}} u(\tau)\|^2_X d\mathsf{s} d\tau d\mathsf{t} = 2 \int_0^T \int_0^{\mathsf{t}} (2\sigma-1)^{-1} ((\mathsf{t}-\tau)^{-2\sigma+1} - \mathsf{t}^{-2\sigma+1}) \|\partial_{\mathsf{t}} u(\tau)\|^2_X d\tau d\mathsf{t} \\ &= 2(2\sigma-1)^{-1} \int_0^T \int_\tau^T ((\mathsf{t}-\tau)^{-2\sigma+1} - \mathsf{t}^{-2\sigma+1}) \|\partial_{\mathsf{t}} u(\tau)\|^2_X d\mathsf{t} d\tau \\ &= 2(2\sigma-1)^{-1}(2-2\sigma)^{-1} \int_0^T ((T-\tau)^{2-2\sigma} - T^{2-2\sigma} + \tau^{2-2\sigma}) \|\partial_{\mathsf{t}} u(\tau)\|^2_X d\tau \\ &\leqslant 2(2\sigma-1)^{-1}(2-2\sigma)^{-1} T^{2-2\sigma} \|u\|^2_{H^1((0,T),X)}. \end{aligned} \tag{62}$$

Furthermore, for $\sigma \in (0,1)$,

$$\int_0^T \mathsf{t}^{-2\sigma} \|u(\mathsf{t})\|^2_X d\mathsf{t} = \int_0^T \mathsf{t}^{-2\sigma} \| \int_0^{\mathsf{t}} \partial_{\mathsf{t}} u(\tau) d\tau \|^2_X d\mathsf{t} \leqslant \int_0^T \mathsf{t}^{1-2\sigma} \int_0^{\mathsf{t}} \|\partial_{\mathsf{t}} u(\tau)\|^2_X d\tau d\mathsf{t} \leqslant \frac{1}{2-2\sigma} T^{2-2\sigma} \|u\|^2_{H^1((0,T),X)}. \tag{63}$$

(60), (61), (62) and (63) imply the assertion. □

Lemma A.11. *Let X_1, X_2 be real, separable Hilbert spaces and $\sigma \in (0,1)\backslash\{\frac{1}{2}\}$. Let K be a linear operator that is continuous as a mapping $X_1 \to X_2$ and $f \in H^\sigma((0,T), X_1)$. Then,*

$$\|K(f)\|_{H^\sigma((0,T),X_2)} \leqslant C\|f\|_{H^\sigma((0,T),X_1)}$$

with a constant C independent of T.

Proof. Follows directly from the definition (52) of the norm. □

Lemma A.12. *Let X be a real, separable Hilbert space, $\sigma \in [0,1]\backslash\{\frac{1}{2}\}$ and $T_1 < T_2$. Then,*

$$\|u\|_{H^\sigma((T_1,T_2),X)} = \|\tilde{u}\|_{H^\sigma((0,T_2-T_1),X)},$$

for all $u \in H^\sigma((T_1,T_2),X)$, where $\tilde{u}(\mathsf{t}) := u(\mathsf{t}+T_1)$ for (a.e.) $\mathsf{t} \in (0, T_2 - T_1)$.

Proof. Follows from the definition (52) of the norm and substitution $\tilde{\mathsf{t}} := \mathsf{t} - T_1$. □

For $s = m + \sigma$, $m \in \mathbb{N}$, $\sigma \in [0,1)\backslash\{\frac{1}{2}\}$, define

$$\|\cdot\|_{H^s((T_1,T_2),X)} = \begin{cases} \left(\|\cdot\|^2_{H^{m-1}((T_1,T_2),X)} + \|\partial_{\mathsf{t}}^m(\cdot)\|^2_{H^\sigma((T_1,T_2),X)}\right)^{\frac{1}{2}} & \text{if } \sigma \in (0,1)\backslash\{\frac{1}{2}\}, \\ \|\cdot\|_{H^m((T_1,T_2),X)} & \text{if } \sigma = 0. \end{cases}$$

and with Lemmas A.5 - A.12 it is straightforward to verify properties **P1-P8** of the norm.

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