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IMPROVED ERROR ESTIMATES FOR OPTIMAL CONTROL OF THE STOKES PROBLEM WITH POINTWISE TRACKING IN THREE DIMENSIONS

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Abstract. This work is motivated by recent interest in the topic of pointwise tracking type optimal control problems for the Stokes problem. Pointwise tracking consists of point evaluations in the objective functional which lead to Dirac measures appearing as source terms of the adjoint problem. Considering bounds for the control allows for improved regularity results for the exact solution and improved approximation error estimates of its numerical counterpart. We show a sub-optimal convergence result in three dimensions that nonetheless improves the results known from the literature. Finally, we offer supporting numerical experiments and insights towards optimal approximation error estimates.

1. **Introduction.** For $\Omega \subset \mathbb{R}^3$, an open non-empty convex polyhedral domain, we consider the following pointwise tracking type optimal control problem. Let $\{\vec{x}_i\}_{i\in I} \neq \emptyset$ be a finite subset of Ω and $\{\vec{\xi}_i\}_{i\in I}$ a corresponding set in $\mathbb{R}^3$. We denote the space of controls as $Q = L^2(\Omega)^3$. Then, the pointwise tracking type optimal control problem is given by

$$\text{Minimize} \;\; J(\vec{u},\vec{q}) = \frac{1}{2}\sum_{i\in I}(\vec{u}(\vec{x}_i) - \vec{\xi}_i)^2 + \frac{\alpha}{2}\|\vec{q}\|^2_{L^2(\Omega)} \quad \text{for } \vec{q} \in Q, \text{ subject to}$$

$$-\Delta\vec{u} + \nabla p = \vec{q} \qquad \text{in } \Omega, \tag{1a}$$

$$\nabla\cdot\vec{u} = 0 \qquad \text{in } \Omega, \tag{1b}$$

$$\vec{u} = \vec{0} \qquad \text{on } \partial\Omega, \text{and} \tag{1c}$$

$$\vec{a} \le \vec{q}(x) \le \vec{b} \quad \text{componentwise for a.a } \vec{x} \in \Omega, \tag{1d}$$

for $\vec{a} < \vec{b}$ componentwise, $\vec{a},\vec{b} \in \mathbb{R}^3$ and $\alpha > 0$. We choose p to have zero mean. The space of *admissible* controls fulfilling (1d) is denoted by Q_{ad}. Similar to the corresponding elliptic optimal control problem, which is discussed in [2, 4, 6, 10], it

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quickly follows that the point evaluations of $\vec{u}$ in J lead to singular sources on the right-hand side of the adjoint equation

$$-\Delta\vec{z} + \nabla r = \sum_{i\in I}(\vec{u} - \vec{\xi}_i)\delta_{\vec{x}_i} \qquad \text{in } \Omega, \tag{2a}$$

$$\nabla\cdot\vec{z} = 0 \qquad \text{in } \Omega, \tag{2b}$$

$$\vec{z} = \vec{0} \qquad \text{on } \partial\Omega, \tag{2c}$$

with r having zero mean. We denote the sum on the right-hand side of (2a) as D_Σ to simplify notation.

The Dirac source $\delta_{\vec{x}_i}$ is supported at $\vec{x}_i$. In spite of the singularities on the right-hand side it is possible to derive almost optimal convergence rates for the control in the control-constrained case when the control is discretized using piecewise constant functions.

This analysis is motivated by recent interest in the Stokes pointwise tracking type problem in [14]. Let $h > 0$ denote the discretization parameter describing the maximal mesh size. Using cell-wise constant discretization for the control space, a $O(|\ln h|^3 h)$ convergence rate is proven in [14, Theorem 3, Remark 2] for the control approximation error in two dimensions and $O(|\ln h|h^{1/2})$ in three dimensions based on new weighted stability results in [13]. Similar analysis has already been conducted for the standard Poisson problem in [2, 4, 6, 10]. Furthermore, the authors of [14] give references for potential applications and also discuss an optimal control problem featuring Dirac sources on the right-hand side of the state equation.

Related to this problem are state constrained optimal control problems for the Stokes system as introduced in [12]. State constraints also lead to measure valued right-hand sides of the adjoint equation. The low regularity of the right-hand side motivates an existence and uniqueness approach for the solution of the Stokes problem based on very weak solutions. The case of state constraints has been already discussed at length for the Poisson problem, see for example [8, 24].

Using new results for local pointwise estimates of the Stokes problem in [3] we improve the estimate in three dimensions to $O(|\ln h|^{2/3} h^{5/6})$. The technique we employ is similar to the approach used in [4] but significantly different in some details, in particular in how we handle the behavior of the solution of the adjoint equation close to the singularities. While for the respective Poisson problem the absolute value of the solution of the adjoint equation grows towards infinity the closer it is to a singularity, this does not happen in the case of the Stokes problem for certain parameter settings.

Here we consider Taylor-Hood finite elements of an order greater than or equal to three for the solutions of the discrete state and discrete adjoint state equations. For the control, we consider a variational discretization as in [19] as well as discretization with cell-wise constant functions as in [4, 14].

In the following we begin our analysis by recalling and introducing basic properties of the optimal control problem (1). Next we consider the discretized problem and multiple approximation error results for the quantities involved, finally leading up the approximation error estimates for the control.

2. **Preliminaries and regularity results.** We now introduce basic notation. Throughout this paper, we employ the usual notation for the Lebesgue, Sobolev and Hölder spaces. These spaces can be extended in a straightforward manner to vector functions, with the same notation but with the following modification for the

norm in the non-Hilbert case: if $\vec{u} = (u_1, u_2, u_3)$, we then set

$$\|\vec{u}\|_{L^r(\Omega)} = \left[\int_\Omega |\vec{u}(\vec{x})|^r d\vec{x}\right]^{1/r}$$

where $|\cdot|$ denotes the Euclidean vector norm for vectors or the Frobenius norm for tensors.

We denote by $(\cdot,\cdot)$ the $L^2(\Omega)$ inner product and specify subdomains by subscripts in the case they are not equal to the whole domain. $L^s_0(\Omega)$ contains the $L^s(\Omega)$ function with mean zero for $1 \leq s \leq \infty$. For Sobolev spaces, e.g., $H^1_0(\Omega)^3$ or $W^{1,s}_0(\Omega)^3$ denote the respective Sobolev spaces of functions with zero boundary conditions. Here $W^{-1,s'}(\Omega)^3$ is the dual space of $W^{1,s}_0(\Omega)^3$.

2.1. **Regularity of solutions to state and adjoint state equations.** Next we recall some results for solutions to the state (1a) to (1c) and adjoint equation (2a) to (2c). We consider the solutions to the respective (very) weak formulations.

To keep notation simple, we introduce an auxiliary problem for which we describe all properties not directly related to the optimal control problem. Analysis for this problem will done in the following sections for different right-hand sides $\vec{f}$. Existence and uniqueness of solutions to the weak formulation of the Stokes problem on bounded domains are shown, for example, in [15, Theorem IV.1.1] for $\vec{f} \in H^{-1}(\Omega)^3$, for $\vec{f}$ with even less regularity we give an existence result below. The regularity results for polyhedral domains stated next can be found, e.g., in [23, Chap. 11] and partially in [11]. For $1 < s < \infty$ and $\vec{f} \in W^{-1,s}(\Omega)^3$ let $(\vec{w},\varphi) \in W^{1,s}_0(\Omega)^3 \times L^s_0(\Omega)$ solve

$$a((\vec{w},\varphi),(\vec{v},l)) = (\vec{f},\vec{v}) \quad \forall (\vec{v},l) \in W^{1,s'}_0(\Omega)^3 \times L^{s'}(\Omega), \tag{3}$$

for

$$a((\vec{w},\varphi),(\vec{v},l)) = (\nabla\vec{w},\nabla\vec{v}) - (\varphi,\nabla\cdot\vec{v}) + (\nabla\cdot\vec{w},l)$$

where we choose φ to have zero mean. Then, there holds

$$\|\vec{w}\|_{W^{1,s}(\Omega)} + \|\varphi\|_{L^s(\Omega)} \leq C\|\vec{f}\|_{W^{-1,s}(\Omega)}. \tag{4}$$

Furthermore, for $\vec{f} \in L^2(\Omega)^3$, $(\vec{w},\varphi)$ are elements of $(H^1_0(\Omega) \cap H^2(\Omega))^3 \times H^1(\Omega)$ and

$$\|\vec{w}\|_{H^2(\Omega)} + \|\varphi\|_{H^1(\Omega)} \leq C\|\vec{f}\|_{L^2(\Omega)}. \tag{5}$$

And finally, for $\vec{f} \in L^\infty(\Omega)^3$ and $\sigma \in]0,1[$, depending on the largest interior angle of the domain, we have

$$\|\vec{w}\|_{C^{1,\sigma}(\bar{\Omega})} + \|\varphi\|_{C^{0,\sigma}(\bar{\Omega})} \leq C\|\vec{f}\|_{L^\infty(\Omega)}. \tag{6}$$

A discussion of this result can also be found below [17, Theorem 3].

Result (5) shows that (1) is well defined and we can introduce for $\vec{f} \in L^2(\Omega)^3$ a linear control-to-state mapping for the velocity $S\colon L^2(\Omega)^3 \to C(\bar{\Omega})^3$ and the pressure $S^p\colon L^2(\Omega) \to L^2_0(\Omega) \cap H^1(\Omega)$ such that $S\vec{f} = \vec{w}$ and $S^p\vec{f} = \varphi$ as the components of the solution to (3).

Due to the linearity of S, S^p, the convexity of the cost functional and the fact that $\alpha > 0$, standard arguments as in [26] lead to the existence of a unique solution to the optimal control problem (1).

Next, we discuss the regularity of the solution to the adjoint equation. Since we were not able to locate existence and uniqueness results for $\vec{f} \in W^{-1,s}(\Omega)^3$ with

$1 < s < 3/2$, in particular not for $\vec{f} \in \mathcal{M}(\Omega)^3$, in the available literature on convex polyhedral domains, we apply a result on Lipschitz domains due to Brown and Shen [7, Theorem 2.9]. We state a version of the theorem adapted to our notation.

Proposition 1. *For Ω a bounded Lipschitz domain in $\mathbb{R}^3$ and $f \in W^{-1,s}(\Omega)^3$ with $\tilde{\varepsilon} > 0$ small and $(3+\tilde{\varepsilon})/(2+\tilde{\varepsilon}) < s < 3+\tilde{\varepsilon}$ there exist unique $\vec{w} \in W_0^{1,s}(\Omega)^3$ and a, up to a constant, unique $\varphi \in L^s(\Omega)$ such that*

$$\begin{aligned} -\Delta \vec{w} + \nabla \varphi &= \vec{f} && \text{in } \Omega, \\ \nabla \cdot \vec{w} &= 0 && \text{in } \Omega, \\ \vec{w} &= \vec{0} && \text{on } \partial\Omega. \end{aligned}$$

In particular, this then holds for $\vec{f} \in W^{-1,s}(\Omega)^3$ with $3/2 - \varepsilon < s < 3 + \tilde{\varepsilon}$ for $\varepsilon > 0$ sufficiently small.

Remark 1. On polyhedral domains one may extend the range of s using the regularity results in [23]. For our purposes Proposition 1 is sufficient here.

Having established existence and uniqueness for this kind of right-hand side $\vec{f}$ we are now able to discuss the solution to the adjoint equation. The right-hand side of the adjoint problem (2a) to (2c) consists of a linear combination of regular Borel measures in the space $\mathcal{M}(\Omega)^3$ which can be motivated as the dual space of continuous functions on Ω. By the well-known Sobolev embedding theorem it follows that for $i \in I$, $\delta_{\vec{x}_i} \in W^{-1,s}(\Omega)$ for $s < \frac{3}{2}$. Thus, based on this consideration, Proposition 1 and regularity result (4), we conclude that there exists a solution $(\vec{z}, r) \in W_0^{1,s}(\Omega)^3 \times L_0^s(\Omega)$.

The following result is relevant for deriving approximation error estimates of the adjoint equation. Since the point evaluations in the cost functional are located in the interior of the domain, we are interested in the regularity of a dual problem to (2a) to (2c) in the interior of the domain, away from the boundary. In the following we denote by $\Omega_1 \Subset \Omega_2$ that a domain Ω_1 is contained in Ω_2 and $dist(\Omega_1, \partial\Omega_2) > d > 0$.

Proposition 2 (Interior regularity for Stokes)**.** *Let $(\vec{w}, r) \in H_0^1(\Omega)^3 \times L_0^2(\Omega)$ solve*

$$\begin{aligned} -\Delta \vec{w} + \nabla \varphi &= f \\ \nabla \cdot \vec{w} &= 0 \\ \vec{w} &= 0 \end{aligned}$$

$\vec{f} \in L^\infty(\Omega)$. Then we get for $\Omega_1 \Subset \Omega_2 \Subset \Omega$ the following semi-norm estimate

$$|\vec{w}|_{W^{2,p}(\Omega_1)} + |\varphi|_{W^{1,p}(\Omega_1)} \leq Cp\|f\|_{L^\infty(\Omega)} \quad \text{for all } 1 < p < \infty,$$

with C independent of p.

Proof. The proposition, for the most part, is already available in [15, Theorem IV.4.1] with

$$|\vec{w}|_{W^{2,p}(\Omega_1)} + |\varphi|_{W^{1,p}(\Omega_1)} \leq C(\|\vec{f}\|_{L^p(\Omega_2)} + \|\vec{w}\|_{W^{1,p}(\Omega_2 \setminus \Omega_1)} + \|\varphi\|_{L^p(\Omega_2 \setminus \Omega_1)}).$$

Now, by (6) we can bound $\|\vec{w}\|_{W^{1,p}(\Omega_1 \setminus \Omega_2)} + \|\varphi\|_{L^p(\Omega_1 \setminus \Omega_2)}$ for $p \to \infty$ by $\|\vec{f}\|_{L^\infty(\Omega)}$ and obviously $\|\vec{f}\|_{L^p(\Omega_1)}$ is bounded as well, such that we get

$$\|\vec{w}\|_{W^{2,p}(\Omega_1)} + \|\varphi\|_{W^{1,p}(\Omega_1)} \leq C\|\vec{f}\|_{L^\infty(\Omega)}.$$

It remains to trace the dependency on p of the constant C. Starting from [15, Theorem IV.4.1] we can trace the constant over [15, Theorem IV.2.1] to [15, Theorem II.11.4] and [15, Remark II.11.2] to the form stated in the theorem. $\square$

2.2. **Optimality condition and derivatives.** First we consider derivatives of the cost functional, which also motivates the introduction of the adjoint problem. For $\vec{q} \in Q_{\mathrm{ad}}$ we define the reduced cost functional as $j(\vec{q}) = J(\vec{q}, S\vec{q})$.

Lemma 2.1. *For $\vec{q}, \delta\vec{q} \in Q$, the directional Fréchet derivative of the reduced cost functional j is given by*

$$j'(\vec{q})(\delta\vec{q}) = (\alpha\vec{q} + \vec{z}, \delta\vec{q}),$$

where $\vec{z} \in W^{1,s}(\Omega)^3$ solves

$$\begin{aligned} -\Delta\vec{z} + \nabla r &= \sum_{i\in I}(S\vec{q} - \vec{\xi}_i)\delta_{\vec{x}_i} && \text{in } \Omega,\\ \nabla\cdot\vec{z} &= 0 && \text{in } \Omega,\\ \vec{z} &= \vec{0} && \text{on } \partial\Omega, \end{aligned}$$

which corresponds to (2a) to (2c). The second directional derivative is given for $\vec{q}, \delta\vec{q}, \tau\vec{q} \in Q$ by

$$j''(\vec{q})(\delta\vec{q}, \tau\vec{q}) = \sum_{i\in I} S\delta\vec{q}(\vec{x}_i) S\tau\vec{q}(\vec{x}_i) + \alpha(\delta\vec{q}, \tau\vec{q}).$$

Proof. The explicit derivatives follow directly from the linearity of S and the definition of the Fréchet derivative. $\square$

Using the adjoint equation, it is possible to formulate necessary and sufficient first order optimality conditions following standard arguments which can be found, e.g., in [22, 26].

Lemma 2.2. *A control $\bar{q} \in Q_{ad}$ with associated state $\bar{u} = S\bar{q} \in (H_0^1(\Omega) \cap H^2(\Omega))^3$ is an optimal solution to the problem (1) if and only if there exists an adjoint state $\bar{z} \in (W_0^{1,s}(\Omega))^3$ such that $\bar{u}$ solves (1a) to (1c) with right-hand side $\bar{q}$ and $\bar{z}$ solves (2a) to (2c) with $\vec{u} = \bar{u}$ in the right-hand side where $\bar{q}$ and $\bar{z}$ satisfy the following inequality*

$$(\bar{z} + \alpha\bar{q}, \vec{q} - \bar{q}) \geq 0 \quad \forall \vec{q} \in Q_{ad}. \tag{8}$$

The variational inequality is equivalent to the following projection formula

$$\bar{q} = P_{[\vec{a},\vec{b}]}\left(-\frac{1}{\alpha}\bar{z}\right), \tag{9}$$

where $P_{[\vec{a},\vec{b}]}$ is applied componentwise and defined as $P_{[a,b]}(\vec{v}) = \min(\vec{b}, \max(\vec{a}, \vec{v}))$, with min, max being also applied componentwise and pointwise.

2.3. **Regularity of the optimal solution $\bar{q}$.** We derive a regularity result based on (9) for solutions to (3) with right-hand side $\vec{f} = \vec{\mu} \in \mathcal{M}(\Omega)^3$. This is a well defined application of the regularity result for (3) since $\mathcal{M}(\Omega)^3$ is compactly embedded into $W^{-1,s}(\Omega)^3$ for $s < 3/2$. This follows from an duality argument and the Sobolev embedding for the maximum norm, see, e.g., [1, Theorem 10.10]. More details can be found in Section 3.4.

Lemma 2.3. *Let $\vec{w}$ be the solution of (3) with right-hand side $\vec{\mu} \in \mathcal{M}(\Omega)^3$. Then, $Proj_{[-\vec{M},\vec{M}]}(\vec{w}) \in H_0^1(\Omega)^3$ for every $\vec{M} \in \mathbb{R}_+^3$.*

Proof. We deduce the result similarly to [9, Lemma 3.3] but take into account the additional pressure term. Let $\{\vec{\mu}_k\}_k \subset L^2(\Omega)^3$ be a sequence, such that $\vec{\mu}_k \overset{*}{\rightharpoonup} \vec{\mu}$ and $\|\vec{\mu}_k\|_{L^1(\Omega)} \leq \|\vec{\mu}\|_{\mathcal{M}(\Omega)}$. Then, let $(\vec{w}_k, \varphi_k) \in (H^1_0(\Omega) \cap H^2(\Omega))^3 \times H^1(\Omega)$ be the solution of

$$-\Delta \vec{w}_k + \nabla \varphi_k = \vec{\mu}_k \qquad \text{in } \Omega, \tag{10a}$$

$$\nabla \cdot \vec{w}_k = 0 \qquad \text{in } \Omega, \tag{10b}$$

$$\vec{w}_k = \vec{0} \qquad \text{on } \partial\Omega. \tag{10c}$$

Now since $\mathcal{M}(\Omega)^3$ is compactly embedded into $W^{-1,s}(\Omega)^3$ for $s < 3/2$, it follows $\vec{w}_k \to \vec{w}$ strongly in $W^{1,s}(\Omega)^3$. Now we consider the projection $\vec{w}^M_k = \mathrm{Proj}_{[-\vec{M},\vec{M}]}(\vec{w}_k)$ which by [27, Corollary 2.1.8] is continuous from $W^{1,s}(\Omega)^3 \to W^{1,s}(\Omega)^3$. Thus, we also have $\vec{w}^M_k \to \vec{w}^M$ strongly in $W^{1,s}_0(\Omega)^3$, where $\vec{w}^M$ is defined as $\mathrm{Proj}_{[-\vec{M},\vec{M}]}(\vec{w})$. Using (10a) to (10c) we now can conclude

$$\begin{aligned} \|\nabla \vec{w}^M_k\|^2_{L^2(\Omega)} &= (\nabla \vec{w}^M_k, \nabla \vec{w}^M_k) \leq (\nabla \vec{w}_k, \nabla \vec{w}^M_k) \\ &= (\vec{\mu}_k, \vec{w}^M_k) + (\varphi_k, \nabla \cdot \vec{w}^M_k) \\ &\leq \|\vec{w}^M_k\|_{L^\infty(\Omega)} \|\vec{\mu}_k\|_{L^1(\Omega)} \leq |\vec{M}| \|\delta_{\vec{x}_i}\|_{\mathcal{M}(\Omega)} \end{aligned} \tag{11}$$

and from this that $\{\vec{w}^M_k\}_k$ is bounded in $H^1_0(\Omega)^3$ and there exist $\vec{w}^M \in H^1_0(\Omega)^3$ and a subsequence of $\{\vec{w}^M_k\}_k$ such that $\vec{w}^M_k \rightharpoonup \vec{w}^M$ weakly in $H^1_0(\Omega)^3$. Now due to the strong convergence of $\vec{w}^M_k$ in $W^{1,s}(\Omega)^3$ we get $\vec{w}^M \in H^1_0(\Omega)^3$.

Note that in (11) we made use of [20, Theorem A.1] or [27, Corollary 2.1.8] which guarantee the existence of all weak partial derivatives. In particular those that vanish on neighborhoods on which the projection is active and the function constant. Furthermore we used that the divergence of $\vec{w}_k$ is zero. ☐

Since $\delta_{\vec{x}_i} \in \mathcal{M}(\Omega)$ we can apply this result for $\bar{q} = P_{[\vec{a},\vec{b}]}\left(-\frac{1}{\alpha}\bar{z}\right)$ and we conclude the following corollary.

Corollary 1. *Let $\bar{q}$ be the solution of (1). Then, $\bar{q} \in H^1(\Omega)^3$.*

3. Finite element approximation and estimates.

In the following we introduce finite element spaces for the state and adjoint state equations as well as a discretization of the control space Q and the space of admissible controls Q_{ad}.

3.1. State and control.

Let $\mathcal{T}_h$ be a regular, quasi-uniform family of triangulations of $\bar{\Omega}$, made of closed tetrahedra T, where h is the global mesh-size and $L^2_0(\Omega)$ the space of $L^2(\Omega)$ functions with zero-mean value. Let $\vec{V}_h \subset H^1_0(\Omega)^3$ and $M_h \subset L^2_0(\Omega)$ be a pair of finite element spaces satisfying a uniform discrete inf-sup condition, as, e.g., in [18]. The respective discrete solution associated with the velocity-pressure pair $(\vec{w}, \varphi) \in H^1_0(\Omega)^3 \times L^2_0(\Omega)$ is defined as the pair $(\vec{w}_h, \varphi_h) \in \vec{V}_h \times M_h$ that solves the following equation based on the bilinear form $a(\cdot,\cdot)$

$$a((\vec{w}_h, \varphi_h), (\vec{v}_h, l_h)) = (\vec{f}, \vec{v}_h) \quad \forall (\vec{v}_h, l_h) \in \vec{V}_h \times M_h, \tag{12}$$

for suitable $\vec{f}$. In particular, we require the assumptions in [3, Section 2.4] to hold. A suitable finite element space is given for example by Taylor-Hood finite elements of order greater than or equal to three. More details can be found in [3, Remark 2.10]. The assumptions regarding the finite element space enable us to use [3, Corollary 2.17, Remark 2.18], here stated as the following proposition.

Proposition 3. *For $\Omega_1 \Subset \Omega_2 \Subset \Omega$ with dist$(\bar{\Omega}_1, \partial\Omega_2) \geq d \geq \bar{\kappa} h$ and for $(\vec{w}, \varphi) \in (L^\infty(\Omega_2)^3 \times L^\infty(\Omega_2)) \cap (H_0^1(\Omega)^3 \times L_0^2(\Omega))$ the solution to (3) with $\vec{f} \in L^\infty(\Omega)^3$ and $(\vec{w}_h, \varphi_h)$ the solution to (12), we have*

$$\|\vec{w} - \vec{w}_h\|_{L^\infty(\Omega_1)} \leq \inf_{(\vec{v}_h, l_h) \in \vec{V}_h \times M_h} C|\ln h| \Big(\|\vec{w} - \vec{v}_h\|_{L^\infty(\Omega_2)} + h\|\varphi - l_h\|_{L^\infty(\Omega_2)} \Big) + C_d |\ln h| \Big(h\|\vec{w} - \vec{v}_h\|_{H^1(\Omega)} + \|\vec{w} - \vec{v}_h\|_{L^2(\Omega)} + h\|\varphi - l_h\|_{L^2(\Omega)} \Big).$$

Here, the constant C_d depends on the distance to Ω_1 from $\partial\Omega_2$.

Similar to the exact solution in Section 2 we can define respective control-to-state maps in the discrete case, for the velocity $S_h \colon Q \to \vec{V}_h$ and the pressure $S_h^p \colon Q \to M_h$ such that $S_h \vec{f} = \vec{w}_h$ and $S_h^p \vec{f} = \varphi_h$ as the components of the solution to (12).

The space of discrete admissible controls is given by

$$Q_{\mathrm{ad},h} = Q_h \cap Q_{\mathrm{ad}} \tag{13}$$

where Q_h is the space of piecewise constant functions

$$Q_h = \{\vec{q} \in L^2(\bar{\Omega})^3 \colon \vec{q}|_T \in \mathcal{P}_0(T)^3 \ \forall T \in \mathcal{T}_h\}.$$

For Q_h we now introduce the L^2 projection $\pi_h \colon L^2(\Omega)^3 \to Q_h$ of a function $\vec{q} \in L^2(\Omega)^3$ as $\pi_h \vec{q} \in Q_h$ satisfying

$$(\pi_h \vec{q}, \vec{r}_h) = (\vec{q}, \vec{r}_h) \quad \forall \vec{r}_h \in Q_h. \tag{14}$$

Using orthogonality, π_h can also be characterized as

$$(\pi_h \vec{q})_i = \frac{1}{|K|} \int_K \vec{q}_i \, d\vec{x}_i \quad \text{for } 1 \leq i \leq 3$$

on each cell $K \in \mathcal{T}_h$. Now using Poincaré's inequality [21, Theorem 12.30] on each cell K we get for $1 \leq s < \infty$ and $\vec{q} \in W^{1,s}(\Omega)^3$

$$\|\pi_h \vec{q} - \vec{q}\|_{L^s(K)} \leq Ch \|\nabla \vec{q}\|_{L^s(K)}. \tag{15}$$

Summing up, we conclude

$$\|\pi_h \vec{q} - \vec{q}\|_{L^s(\Omega)} \leq Ch \|\nabla \vec{q}\|_{L^s(\Omega)}. \tag{16}$$

Note that while the convergence result holds for $1 \leq s < \infty$ we still require $\vec{q} \in L^2(\Omega)^3$ to apply property (14).

3.2. **Discrete optimal control problem and optimality conditions.** We can then formulate the discrete version of (1) as

$$\text{Minimize} \ \ J(\vec{u}_h, \vec{q}_h) \quad \text{for } \vec{q}_h \in Q_{\mathrm{ad},h}$$

subject to

$$a((\vec{u}_h, p_h), (\vec{v}_h, l_h)) = (\vec{q}_h, \vec{v}_h) \quad \forall (\vec{v}_h, l_h) \in \vec{V}_h \times M_h. \tag{17a}$$

We have the following adjoint problem

$$a((\vec{z}_h, r_h), (\vec{v}_h, l_h)) = \sum_{i \in I} (\vec{u}_h - \vec{\xi}_i) \vec{v}_h(\vec{x}_i) \quad \forall (\vec{v}_h, l_h) \in \vec{V}_h \times M_h, \tag{18}$$

which can again be motivated by the following derivatives of the objective functional. For $\vec{q} \in Q_{\mathrm{ad}}$ we define the discrete reduced cost functional $j_h(\vec{q}) = J(\vec{q}, S_h \vec{q})$. We get the following first and second derivatives with respect to $\vec{q}$ for j_h.

Lemma 3.1. *For $\vec{q}, \delta\vec{q} \in Q$, the first directional Fréchet derivative of the reduced cost functional j_h is given by*

$$j_h(\vec{q})(\delta\vec{q}) = (\alpha\vec{q}_h + \vec{z}_h, \delta\vec{q}),$$

where $\vec{z}_h \in \vec{V}_h$ solves

$$a((\vec{z}_h, r_h), (\vec{v}_h, l_h)) = \sum_{i\in I} (S_h\vec{q} - \vec{\xi}_i)\vec{v}_h(\vec{x}_i) \quad \forall(\vec{v}_h, l_h) \in \vec{V}_h \times M_h$$

which corresponds to (18). The second directional derivative is given for $\vec{q}, \delta\vec{q}, \tau\vec{q} \in Q$ by

$$j_h''(\vec{q})(\delta\vec{q}, \tau\vec{q}) = \sum_{i\in I} S_h\delta\vec{q}(\vec{x}_i) S_h\tau\vec{q}(\vec{x}_i) + \alpha(\delta\vec{q}, \tau\vec{q}). \tag{19}$$

Proof. The form of the derivative follows as for the continuous case. □

Similarly to the continuous case we then have the following optimality condition.

Lemma 3.2. *A control $\bar{q}_h \in Q_{ad,h}$ with associated state $\bar{u}_h = S_h\bar{q}_h \in \vec{V}_h$ is an optimal solution to the problem (17) if and only if there exists an adjoint state $\bar{z}_h \in \vec{V}_h$ such that $\bar{u}_h$ solves (17a) with right-hand side $\bar{q}_h$ and $\bar{z}_h$ solves (18) with right-hand side $\bar{u}_h$ and $\bar{q}_h$ satisfies the following inequality*

$$(\bar{z}_h + \alpha\bar{q}_h, \vec{q}_h - \bar{q}_h) \geq 0 \quad \forall\vec{q}_h \in Q_{ad,h}. \tag{20}$$

3.3. **Error estimates for the solutions to state and adjoint state equations.** In this section, we consider convergence rates for the discrete Stokes problem with bounded right-hand side.

Lemma 3.3. *Let $\Omega_1 \Subset \Omega_2 \Subset \Omega$, $\vec{w} \in H_0^1(\Omega)^3$ the velocity solution to (3) and $\vec{w}_h$ the respective finite element velocity solution. Then, for $\vec{f} \in L^\infty(\Omega)^3$ there holds*

$$\|\vec{w} - \vec{w}_h\|_{L^\infty(\Omega_1)} \leq C|\ln h|^2 h^2 \|\vec{f}\|_{L^\infty(\Omega)} + C_d|\ln h| h^2 \|\vec{f}\|_{L^2(\Omega)}$$

for $dist(\Omega_1, \partial\Omega_2) \geq d > 0$.

Proof. Due to Proposition 3 we have

$$\begin{aligned} \|\vec{w} - \vec{w}_h\|_{L^\infty(\Omega_1)} \leq \inf_{(\vec{v}_h, l_h)\in\vec{V}_h\times M_h} & C|\ln h|(\|\vec{w} - v_h\|_{L^\infty(\Omega_2)} + h\|\varphi - l_h\|_{L^\infty(\Omega_2)}) \\ & + C_d|\ln h|(h\|\vec{w} - \vec{v}_h\|_{H^1(\Omega)} + \|\vec{w} - \vec{v}_h\|_{L^2(\Omega)} + h\|\varphi - l_h\|_{L^2(\Omega)}). \end{aligned} \tag{21}$$

[5, Corollary 4.4.24] shows that we get the expected convergence rates for finite element functions $(\vec{v}_h, l_h)$ since $(\vec{w}, \varphi)$ are sufficiently regular. In particular, to use nodal interpolation we conclude from (6) that $(\vec{w}, \varphi) \in C^{1,\sigma}(\Omega)^3 \times C^{0,\sigma}$. This then shows the result for the second line in (21) due to $(\vec{w}, \varphi) \in H^2(\Omega)^3 \times H^1(\Omega)$. For the first line we can argue by [5, Corollary 4.4.24] and Proposition 2 that

$$\begin{aligned} \|\vec{w} - v_h\|_{L^\infty(\Omega_0)} + h\|r - l_h\|_{L^\infty(\Omega_0)} & \\ \leq Ch^{2-3/p}(\|\nabla^2\vec{w}\|_{L^p(\Omega_0)} + \|\nabla\varphi\|_{L^p(\Omega_0)}) & \\ \leq Cph^{2-3/p}\|\vec{f}\|_{L^\infty(\Omega)}. & \end{aligned}$$

Choosing $p = |\ln h|$, we get $ph^{2-3/p} \leq C|\ln h|h^2$ and thus follows the result. □

Using this, we can now prove a "dual" result for the adjoint equation.

Lemma 3.4. *For the velocity solution $\vec{z} \in W_0^{1,s}(\Omega)^3$ of (2a) to (2c) and $\hat{z}_h$ the solution of the respective finite element problem with right-hand side D_Σ there holds the following error estimate*

$$\|\vec{z} - \hat{z}_h\|_{L^1(\Omega)} \leq C|\ln h|^2 h^2 (\|\vec{q}\|_{L^2(\Omega)} + \sum_{i\in I} |\vec{\xi}_i|).$$

Proof. Using a dual formulation with $\vec{f} = \mathrm{sgn}(\vec{z} - \hat{z}_h) \in L^\infty(\Omega)^3$ as the right-hand side of (3) we get

$$\begin{aligned}
\|\vec{z} - \hat{z}_h\|_{L^1(\Omega)} &= (\vec{f}, \vec{z} - \hat{z}_h)\\
&= (\nabla\vec{w}, \nabla(\vec{z} - \hat{z}_h)) - (\varphi, \nabla\cdot(\vec{z} - \hat{z}_h))\\
&= (\nabla(\vec{w} - \vec{w}_h), \nabla(\vec{z} - \hat{z}_h)) + (\varphi, \nabla\cdot\hat{z}_h) + (\nabla\vec{w}_h, \nabla(\vec{z} - \hat{z}_h)) \quad (22)\\
&= (\nabla(\vec{w} - \vec{w}_h), \nabla(\vec{z} - \hat{z}_h)) + (\varphi, \nabla\cdot\hat{z}_h) + (\nabla\cdot\vec{w}_h, r - \hat{r}_h)) \quad (23)\\
&= (\nabla(\vec{w} - \vec{w}_h), \nabla\vec{z}) + (\varphi, \nabla\cdot\hat{z}_h)\\
&\quad + (\nabla\cdot\vec{w}_h, r) - (\nabla(\vec{w} - \vec{w}_h), \nabla\hat{z}_h)) \quad (24)\\
&= (\nabla(\vec{w} - \vec{w}_h), \nabla\vec{z}) + (\varphi, \nabla\cdot\hat{z}_h)\\
&\quad + (\nabla\cdot\vec{w}_h, r) - (\varphi - \varphi_h, \nabla\cdot\hat{z}_h)) \quad (25)\\
&= (\nabla(\vec{w} - \vec{w}_h), \nabla\vec{z}) + (\varphi, \nabla\cdot\hat{z}_h) + (\nabla\cdot\vec{w}_h, r) - (\varphi, \nabla\cdot\hat{z}_h)\\
&= (\nabla(\vec{w} - \vec{w}_h), \nabla\vec{z}) - (\nabla\cdot(\vec{w} - \vec{w}_h), r) \quad (26)\\
&= (\vec{w} - \vec{w}_h, D_\Sigma)\\
&\leq \|\vec{w} - \vec{w}_h\|_{L^\infty(\Omega_1)} \sum_{i\in I} |\vec{u}(\vec{x}_i) - \vec{\xi}_i| \|\delta_{\vec{x}_i}\|_{\mathcal{M}(\Omega)}\\
&\leq C|\ln h|^2 h^2 \|\vec{f}\|_{L^\infty(\Omega)} (\|\vec{q}\|_{L^2(\Omega)} + \sum_{i\in I} |\vec{\xi}_i|).
\end{aligned}$$

We used (3), the fact that $\vec{z}$ is divergence free and inserted $\vec{w}_h$ in (22). Next we test (2a) to (2c) and the respective finite element formulation with $\vec{w}_h$ to get (23), use in (24) and (25) that $\vec{w}_h$ is discretely divergence free and test (3) with $\hat{z}_h$. To proceed, we use that $\hat{z}_h$ is discretely divergence free and that $\vec{w}$ is divergence free to arrive at (26) where we apply the weak formulation of (2a) to (2c). Finally, we apply Lemma 3.3 with Ω_1 containing all $\vec{x}_i$ for $i \in I$. □

One also quickly surmises that $\|\vec{q}\|_{L^2(\Omega)} + \sum_{i\in I} |\vec{\xi}_i|$ only depends on the prescribed values $\vec{\xi}_i$ and the control constraints.

3.4. L^2 projection approximation error estimates for adjoint and control.

We start with a convergence result for the L^2 projection of $\bar{z} \in W^{1,s}(\Omega)^3$ with $s < 3/2$. The convergence rate of the projection for a sufficiently regular function is discussed in (16). The question is now one of regularity. To analyze the dependence on s when we consider the error in the L^s norm we choose $s = 3/2 - \varepsilon$ and let the Hölder conjugate s' be given by $(1/s + 1/s' = 1)$. Then, by the Sobolev bound on the supremum norm [1, Theorem 10.10], we have $\vec{v} \in L^\infty(\Omega)^3$ and

$$\|\vec{v}\|_{L^\infty(\Omega)} \leq \Big(\int_{B_R(\vec{x}_0)} \frac{d\vec{x}}{|\vec{x} - \vec{x}_0|^{2s}}\Big)^{1/s} \|\nabla\vec{v}\|_{L^{s'}(\Omega)}.$$

Since Ω is bounded, it is contained in the ball $B_R(\vec{x}_0)$. Transforming to spherical coordinates, we can rewrite this as

$$\begin{aligned}\|\vec{v}\|_{L^\infty(\Omega)} &\le C\Big(\int_0^R \rho^{2-2s} d\rho\Big)^{1/s} \|\nabla \vec{v}\|_{L^{s'}(\Omega)} \\ &\le C\Big(\int_0^R \rho^{-1+2\varepsilon} d\rho\Big)^{1/s} \|\nabla \vec{v}\|_{L^{s'}(\Omega)} \\ &= C\Big(\Big[\frac{1}{2\varepsilon}\rho^{2\varepsilon}\Big]_0^R\Big)^{1/s} \|\nabla \vec{v}\|_{L^{s'}(\Omega)} \\ &\le C\varepsilon^{-1/s}\|\nabla \vec{v}\|_{L^{s'}(\Omega)}.\end{aligned}$$

Using a duality argument we get

$$\begin{aligned}\|D_\Sigma\|_{W^{-1,s}(\Omega)} &= \sup_{\vec{v}\in W_0^{1,s'}(\Omega), \|\vec{v}\|_{W_0^{1,s'}(\Omega)}\le 1} \langle D_\Sigma, \vec{v}\rangle \\ &\le \|D_\Sigma\|_{\mathcal{M}(\Omega)}\|\vec{v}\|_{C_0(\Omega)} \\ &\le \|D_\Sigma\|_{\mathcal{M}(\Omega)}\|\vec{v}\|_{L^\infty(\Omega)} \\ &\le C\varepsilon^{-1/s}\|D_\Sigma\|_{\mathcal{M}(\Omega)}. \end{aligned} \tag{27}$$

These considerations allow us to prove the following lemma.

Lemma 3.5. *Let $\vec{z} \in W_0^{1,s}(\Omega)^3$ be the solution to (2a) and s is defined as above. Then, it holds for the L^2 projection π_h to the space of cellwise constant functions*

$$\|\vec{z} - \pi_h \vec{z}\|_{L^s(\Omega)} \le Ch\varepsilon^{-1/s}\|D_\Sigma\|_{\mathcal{M}(\Omega)}$$

Proof. This follows by applying (16) and (27). □

Since the optimal solution $\bar{q}$ to Problem (1) is given by $\mathrm{Proj}_{[\vec{a},\vec{b}]}(\bar{z})$, we obtain by Lemma 2.3 that $\bar{q} \in (L^\infty(\Omega \cap H_0^1(\Omega))^3$ due to Lemma 2.3, thus motivating the following suboptimal convergence result for the L^2 projection onto cellwise constant functions.

Lemma 3.6. *Let $\bar{q} \in (L^\infty(\Omega \cap H_0^1(\Omega))^3$ be the solution to the optimal control problem Problem (1) and s' as above. Then, it holds for the L^2 projection π_h to the space of cellwise constant function*

$$\|\bar{q} - \pi_h \bar{q}\|_{L^{s'}(\Omega)} \le Ch^{2/s'}\|\nabla \bar{q}\|^{2/s'}_{L^2(\Omega)}$$

Proof. The result follows from an application of (15). To see that we consider $\bar{q} - \pi_h\bar{q}$ on the cell K

$$\begin{aligned}\|\bar{q} - \pi_h\bar{q}\|^{s'}_{L^{s'}(K)} &= \int_K |\bar{q} - \pi_h\bar{q}|^{s'} d\vec{x} \le \|(\bar{q} - \pi_h\bar{q})^{s'-2}\|_{L^\infty(K)}\|\bar{q} - \pi_h\bar{q}\|^2_{L^2(K)} \\ &\le Ch^2\|\nabla\bar{q}\|^2_{L^2(K)}.\end{aligned}$$

Since $\bar{q} \in Q_{\mathrm{ad}}$, summing over all cells gives the conclusion of the lemma. □

Remark 2. As mentioned, we consider this estimate suboptimal, which is due to the fact that the regularity of $\bar{q}$, as derived in Corollary 1, is likely not the best possible regularity result. Compared to the elliptic problem studied in [4], where it was shown that the control actually lies in $W^{1,\infty}(\Omega)$, the Stokes fundamental solutions exhibits large jumps at the singularity depending on the approach direction in certain situations. In particular, one can construct examples such that in every neighborhood of the singularity we can find an open subset where the solution is bounded and thus the projection does not become active for the whole neighborhood, leading to less regularity for the gradient of the projected solution. Based on the behavior of the fundamental solution one can straightforwardly construct optimal control problems also exhibiting this behavior. This is visualized in Fig-

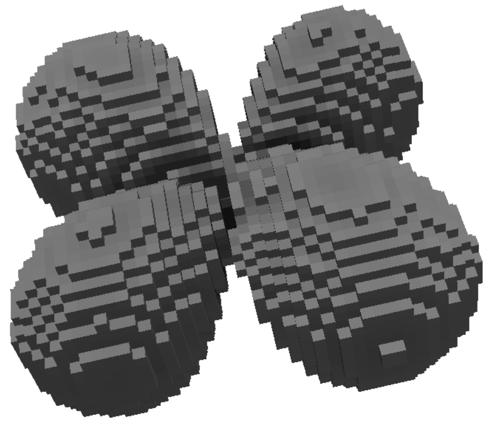

FIGURE 1. Threshold visualization of the first component of a solution $\vec{q}_h$ to Problem (17).

ure 1. Depicted is a neighborhood of a point $\vec{x}_i$ for the first component of $\vec{q}_h$. Only cells where the function value is greater or respectively smaller than a threshold are visible. Note that also for this discrete solution in the neighborhood of $\vec{x}_i$ there are subsets on which the function appears to be bounded, i.e. the thresholds do not become active and the respective cells are not visible.

3.5. **Error estimates for the objective functional.** Next we give an approximation result for difference of the directional Fréchet derivatives of j and j_h.

Lemma 3.7. *For $\vec{q}$, $\delta\vec{q} \in Q_{ad}$, it holds*

$$|j'(\vec{q})(\delta\vec{q}) - j_h'(\vec{q})(\delta\vec{q})| \le C|\ln h|^2 h^2 \|\delta\vec{q}\|_{L^\infty(\Omega)}.$$

Proof. Due to Lemma 2.1 and Lemma 3.1 we get

$$|j'(\vec{q})(\delta\vec{q}) - j_h'(\vec{q})(\delta\vec{q})| = |(\vec{z} - \vec{z}_h, \delta\vec{q})|$$

with $\vec{z}$ and $\vec{z}_h$ defined as in Lemma 2.1 and Lemma 3.1. Defining $\hat{z}_h \in \vec{V}_h$ as the solution of

$$a((\hat{z}_h, \hat{r}_h), (\vec{v}_h, l_h)) = \sum_{i \in I} (S\vec{q} - \vec{\xi}_i)\vec{v}_h(\vec{x}_i) \quad \forall (\vec{v}_h, l_h) \in \vec{V}_h \times M_h \tag{28}$$

we obtain by the triangle inequality

$$|(\vec{z} - \vec{z}_h, \delta\vec{q})| \le |(\vec{z} - \hat{z}_h, \delta\vec{q})| + |(\hat{z}_h - \vec{z}_h, \delta\vec{q})|.$$

We first consider the first term on the right-hand side. Splitting the scalar product and then using Hölder's inequality and Lemma 3.4 gives

$$(\vec{z} - \hat{z}_h, \delta\vec{q}) \le \|\vec{z} - \hat{z}_h\|_{L^1(\Omega)} \|\delta\vec{q}\|_{L^\infty(\Omega)} \le C|\ln h|^2 h^2 \|\delta\vec{q}\|_{L^\infty(\Omega)}. \tag{29}$$

For the second term we use the auxiliary problem (12) with right-hand side $\delta\vec{q}$. Now, since $\hat{z}_h - \vec{z}_h \in \vec{V}_h$ and $(\nabla \cdot (\hat{z}_h - \vec{z}_h), \varphi_h) = 0$, we can write due to (28)

$$(\hat{z}_h - \vec{z}_h, \delta\vec{q}) = a((\hat{z}_h - \vec{z}_h, \hat{r}_h - r_h), (\vec{w}_h, \varphi_h)) = \sum_{i\in I}(S\vec{q} - S_h\vec{q})\vec{w}_h(\vec{x}_i).$$

Since $\vec{x}_i$ for $i \in I$ does not lie on the boundary, we can choose subsets $\Omega_1 \Subset \Omega_2 \Subset \Omega$ which fulfill the requirements of Lemma 3.3. Thus, we can conclude

$$\begin{aligned}(\hat{z}_h - \vec{z}_h, \vec{q} - \vec{q}_h) &\le C\|S\vec{q} - S_h\vec{q}\|_{L^\infty(\Omega_1)}\|\vec{w}_h\|_{L^\infty(\Omega_1)}\\ &\le C\|S\vec{q} - S_h\vec{q}\|_{L^\infty(\Omega_1)}(\|\vec{w}\|_{L^\infty(\Omega_1)} + \|\vec{w} - \vec{w}_h\|_{L^\infty(\Omega_1)})\\ &\le C|\ln h|^2h^2\|\vec{q}\|_{L^\infty(\Omega)}(\|\delta\vec{q}\|_{L^2(\Omega)} + |\ln h|^2h^2\|\delta\vec{q}\|_{L^\infty(\Omega)})\end{aligned}$$

Combined with (29) and the fact the $\vec{q} \in Q_{\mathrm{ad}}$ this proves the lemma. □

Lemma 3.8. *Let $\vec{p}$, $\vec{q}$, $\delta\vec{q} \in L^2(\Omega)$. Then, there holds*

$$|j_h'(\vec{q})(\delta\vec{q}) - j_h'(\vec{p})(\delta\vec{q})| \le C\|\vec{q} - \vec{p}\|_{L^2(\Omega)}\|\delta\vec{q}\|_{L^2(\Omega)}.$$

Proof. To show the result, we first show a maximum norm bound for S_h. Due to [3, Theorem 2.14, Remark 2.18] we obtain, e.g. for $\vec{q} \in L^2(\Omega)^3$

$$\begin{aligned}\|S_h\vec{q}\|_{L^\infty(\Omega)} &\le \|S\vec{q}\|_{L^\infty(\Omega)} + \|S\vec{q} - S_h\vec{q}\|_{L^\infty(\Omega)}\\ &\le \|\vec{q}\|_{L^2(\Omega)} + \inf_{(\vec{v}_h, l_h)\in\vec{V}_h\times M_h} C|\ln h|(\|S\vec{q} - \vec{v}_h\|_{L^\infty(\Omega)}\\ &\quad + h\|S^p\vec{q} - l_h\|_{L^\infty(\Omega)}).\end{aligned}$$

Since $(S\vec{q}, S^p\vec{q}) \in C^{1,\sigma}(\Omega)^3 \times C^{0,\sigma}(\Omega)$ due to (6) we can conclude for $h \le 1$ that

$$\|S_h\vec{q}\|_{L^\infty(\Omega)} \le \|\vec{q}\|_{L^2(\Omega)}(1 + C|\ln h|h) \le C\|\vec{q}\|_{L^2(\Omega)}.$$

With this result in mind and Lemma 3.1 we then see due to the mean value theorem for arbitrary $\vec{\rho} \in Q$ that

$$j_h'(\vec{q})(\delta\vec{q}) - j_h'(\vec{p})(\delta\vec{q}) = j_h''(\vec{\rho})(\vec{q} - \vec{p}, \delta\vec{q}) = \sum_{i\in I} S_h(\vec{q} - \vec{p})(\vec{x}_i)S_h\delta\vec{q}(\vec{x}_i) + \alpha(\vec{q} - \vec{p}, \delta\vec{q})$$

which can be bounded as

$$\begin{aligned}|j_h'(\vec{q})(\delta\vec{q}) - j_h'(\vec{p})(\delta\vec{q})| &\le C\|\vec{q} - \vec{p}\|_{L^\infty(\Omega)}\|S_h\delta\vec{q}\|_{L^\infty(\Omega)} + \alpha|(\vec{q} - \vec{p}, \delta\vec{q})|\\ &\le C\|\vec{q} - \vec{p}\|_{L^2(\Omega)}\|\delta\vec{q}\|_{L^2(\Omega)}.\end{aligned}$$

□

4. **Error estimates for $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}$.** In this section, we discuss approximation error estimates for two types of control discretization. Before considering discretization with piecewise constants as introduced in Section 3, we show a result for the so called variational discretization which was first discussed in [19].

4.1. **Variational Discretization.** Variational discretization means we do *not* discretize the control, i.e., $Q_{ad,h} = Q_{ad}$. It should be noted that the control nonetheless has a discrete structure due to discretization of the adjoint state and the variational inequality (20). The variational discretization allows for a more direct approach when proving the following convergence result, since we can test the discrete optimality condition with the solution to the continuous optimal control problem.

Theorem 4.1. *Let $\bar{q} \in Q_{ad}$ be the solution of Problem (1) and $\bar{q}_h \in Q_{ad}$ the solution of the corresponding discrete Problem (17) with $Q_{ad,h} = Q_{ad}$. Then, it holds*

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \leq C|\ln h|h.$$

Proof. For $\delta\vec{q} = \tau\vec{q} = \bar{q} - \bar{q}_h$ it follows from (19)

$$\alpha\|\bar{q} - \bar{q}_h\|^2_{L^2(\Omega)} \leq j''_h(\vec{\rho})(\bar{q} - \bar{q}_h, \bar{q} - \bar{q}_h)$$

with $\vec{\rho} \in Q$ arbitrary. By the mean value theorem, (20) and (8) it follows

$$\begin{aligned}
\alpha\|\bar{q} - \bar{q}_h\|^2_{L^2(\Omega)} &\leq j''_h(\vec{\rho})(\bar{q} - \bar{q}_h, \bar{q} - \bar{q}_h) \\
&= j'_h(\bar{q})(\bar{q} - \bar{q}_h) - j'_h(\bar{q}_h)(\bar{q} - \bar{q}_h) \\
&= (\vec{z}_h + \alpha\bar{q}, \bar{q} - \bar{q}_h) - (\bar{z}_h + \alpha\bar{q}_h, \bar{q} - \bar{q}_h) \\
&\leq (\vec{z}_h + \alpha\bar{q}, \bar{q} - \bar{q}_h) - (\bar{z} + \alpha\bar{q}, \bar{q} - \bar{q}_h) \\
&= j'_h(\bar{q})(\bar{q} - \bar{q}_h) - j'(\bar{q})(\bar{q} - \bar{q}_h).
\end{aligned}$$

Here $\vec{z}$ and $\vec{z}_h$ are as in Lemma 2.1 and Lemma 3.1 with $\vec{q} = \bar{q}$. Applying Lemma 3.7 shows the result. □

4.2. Discretization with piecewise constant functions.

Theorem 4.2. *Let $\bar{q} \in Q_{ad}$ be the solution of Problem (1) and $\bar{q}_h \in Q_{ad,h}$ the solution of the corresponding discrete Problem (17) with $Q_{ad,h}$ as defined in (13). Then, it holds*

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \leq C|\ln h|^{2/3}h^{5/6}.$$

Proof. Since in this case $Q_{\mathrm{ad}} \neq Q_{\mathrm{ad},h}$ we need to consider the L^2 projection when testing the optimality conditions. To do so we split

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \leq \|\bar{q} - \pi_h\bar{q}\|_{L^2(\Omega)} + \|\pi_h\bar{q} - \bar{q}_h\|_{L^2(\Omega)}.$$

We get that the first term is bounded by $Ch\|\nabla\bar{q}\|_{L^2(\Omega)}$ due to (16) and Lemma 2.3. For the second term we argue as in the variational case with the mean value theorem with $\vec{\rho} \in Q$

$$\begin{aligned}
\alpha\|\pi_h\bar{q} - \bar{q}_h\|^2_{L^2(\Omega)} &\leq j''_h(\vec{\rho})(\pi_h\bar{q} - \bar{q}_h, \pi_h\bar{q} - \bar{q}_h) \\
&= j'_h(\pi_h\bar{q})(\pi_h\bar{q} - \bar{q}_h) - j'_h(\bar{q}_h)(\pi_h\bar{q} - \bar{q}_h) \\
&\leq j'_h(\pi_h\bar{q})(\pi_h\bar{q} - \bar{q}_h) - j'(\bar{q})(\bar{q} - \bar{q}_h),
\end{aligned}$$

where we used the optimality conditions (20) and (8) in the last line. We can further expand this to

$$\begin{aligned}
\alpha\|\pi_h\bar{q} - \bar{q}_h\|^2_{L^2(\Omega)} &= [j'_h(\pi_h\bar{q})(\pi_h\bar{q} - \bar{q}_h) - j'_h(\bar{q})(\pi_h\bar{q} - \bar{q}_h)] \\
&\quad + [j'_h(\bar{q})(\pi_h\bar{q} - \bar{q}_h) - j'(\bar{q})(\pi_h\bar{q} - \bar{q}_h)] - j'(\bar{q})(\bar{q} - \pi_h\bar{q}) \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

For I_1 we can apply Lemma 3.8 and get by Young's inequality

$$I_1 \leq C\|\pi_h\bar{q} - \bar{q}\|_{L^2(\Omega)}\|\pi_h\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \leq Ch^2 + \frac{\alpha}{2}\|\pi_h\bar{q} - \bar{q}_h\|^2_{L^2(\Omega)}.$$

Then, I_2 is dealt with by Lemma 3.7

$$I_2 \leq C|\ln h|^2h^2\|\pi_h\bar{q} - \bar{q}_h\|_{L^\infty(\Omega)} \leq C|\ln h|^2h^2.$$

And finally we apply Lemma 3.5 and Lemma 3.6 to I_3. Recall that we chose $s = 3/(2-\varepsilon)$ which implies $s' > 3$. Now by the L^2 orthogonality of the projection π_h we get

$$\begin{aligned}|I_3| &= |(\alpha\bar{q}+\bar{z}, \bar{q}-\pi_h\bar{q})| = |((\alpha\bar{q}+\bar{z}) - \pi_h(\alpha\bar{q}+\bar{z}), \bar{q}-\pi_h\bar{q})| \\ &\le \|(\alpha\bar{q}+\bar{z}) - \pi_h(\alpha\bar{q}+\bar{z})\|_{L^s(\Omega)}\|\bar{q}-\pi_h\bar{q}\|_{L^{s'}(\Omega)} \\ &\le Ch^{1+2/s'}\varepsilon^{-1/s}.\end{aligned}$$

Next, we simplify the expression for h and ε and choose ε appropriately. For s' we get

$$s' = \frac{3-2\varepsilon}{1-2\varepsilon}$$

and thus for $h^{2/s'}$, $h<1$ and ε small

$$h^{2(1-2\varepsilon)/(3-2\varepsilon)} \le h^{2(1-2\varepsilon)/3} = h^{2/3}h^{-4\varepsilon/3}.$$

We choose $\varepsilon = 1/|\ln h| = -1/\ln(h)$. Then, it follows $h^{-4\varepsilon/3} = e^{4/3}$, implying

$$|I_3| \le C|\ln h|^{\frac{2}{3-2/|\ln h|}}h^{5/3} \le Ch^{5/3}|\ln h|^{2/3}.$$

We conclude that in this convergence estimate, I_3 is the dominating term and therefore the statement of the lemma follows. □

Remark 3. The proof shows that the estimate of I_3 is the limiting factor for the convergence rate estimate. To achieve an optimal convergence rate, one would require a regularity estimate $\bar{q} = P_{\vec{a},\vec{b}}(\bar{z}) \in W^{1,3+\varepsilon}(\Omega)^3$.

5. **Numerical experiments.** We conduct numerical experiments to support the result in Theorem 4.2. The optimal control problems are solved by the optimization library RoDoBo [25] and the finite element toolkit Gascoigne [16]. The empirical convergence rates are computed by comparing solutions with a solution computed on a mesh twice as fine as the finest mesh which we compare.

While in our numerical experiments we consider a slightly different setting than that introduced in Section 3, using local projection stabilization finite element methods on meshes of hexahedral geometry instead of Taylor-Hood finite elements on a triangulation, the results indicate better rates than in Theorem 4.2 for h small enough.

Our results coincide with the output shown in [14, Fig. 2 (Ex.2)] when considering the same example problem ([14, Example 2]) which we introduce next.

Example 1. Let $\Omega = (0,1)^3$, $\vec{a} = (a,a,a)^T$, $\vec{b} = (b,b,b)^T$, $\alpha = 1.99$, $\vec{\xi}_0 = (-1,-1,-1)^T$ and $\vec{x}_0 = (0.5,0.5,0.5)^T$ with $I = \{0\}$ and a and b to be chosen later. Then, we consider the optimal control problem as in Problem (1) but with the state equation

$$\begin{aligned}-\Delta\vec{u} + \nabla p &= \vec{f} + \vec{q} \qquad &&\text{in } \Omega,\\ \nabla\cdot\vec{u} &= 0 &&\text{in } \Omega,\\ \vec{u} &= \vec{0} &&\text{on } \partial\Omega,\end{aligned}$$

with

$$\vec{f} = \frac{1}{\pi}\Delta\text{curl}((\sin(2\pi x_1)\sin(2\pi x_2)\sin(2\pi x_3))^2\vec{e}_1) + \nabla(x_1x_2x_3).$$

Remark 4. This is not precisely the same example as stated in [14, Example 2] because there the authors consider a problem with a slightly different forcing term and inhomogeneous Dirichlet boundary condition but the essential numerical behavior should be unchanged.

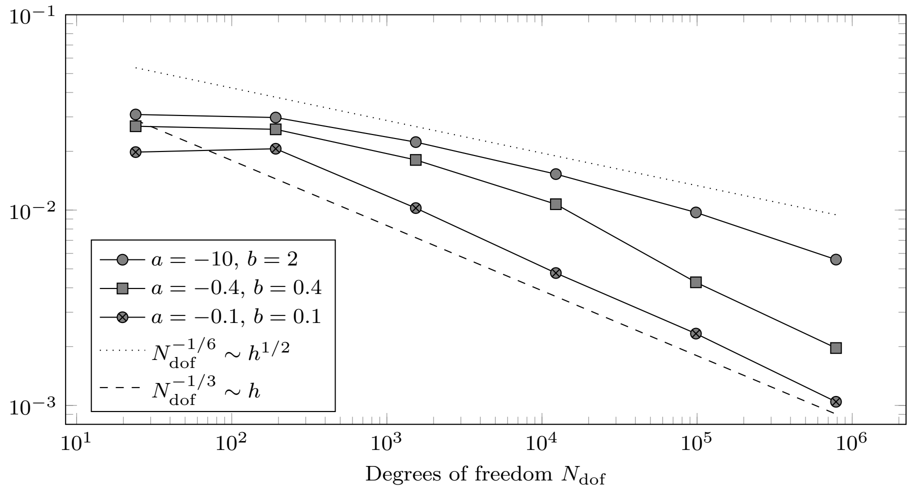

FIGURE 2. Error $\|\bar{q}_n - \bar{q}_h\|_{L^2(\Omega)}$ for cellwise constant control discretization and different choices for the bounds $\vec{a}$ and $\vec{b}$. $\bar{q}_n$ denotes the approximate solution on a finer mesh.

The resulting empirical convergence rates for different bounds $\vec{a}$ and $\vec{b}$ are shown in Figure 2 where N_{dof} corresponds to the number of cells in the mesh and $\bar{q}_n$ to the approximate solution computed on a finer mesh. When the constraints do not become active because h is not small enough, we observe a convergence rate $h^{1/2}$ as in the case $a = -10$, $b = 2$. For the intermediate case $a = -0.4$, $b = 0.4$ we see that as soon as the constraints become active, the convergence rate increases.

Finally, for $a = -0.1$, $b = 0.1$, we immediately observe an empirical convergence rate of $O(h)$ which is faster than the result we have proven in Theorem 4.2. That is likely due to $\bar{q}$ being in $W^{1,3+\varepsilon}$ for $\varepsilon > 0$ which is better than what we have shown with Lemma 2.3. More careful analysis of the impact of $P_{[\vec{a},\vec{b}]}$ on the Stokes fundamental solution might provide additional insights.

Remark 5. Example 1 is well behaved in the sense that the singularities in the adjoint equation do not exhibit the behavior described in Remark 2. Additional tests ran for a modified problem also resulted in a numerical convergence rate of $O(h)$.

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