Error estimates for the postprocessing approach applied to Neumann boundary control problems in polyhedral domains

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Abstract

This paper deals with error estimates for the finite element approximation of Neumann boundary control problems in polyhedral domains. Special emphasis is put on singularities contained in the solution as the computational domain has edges and corners. Thus, we use tailored regularity results in weighted Sobolev spaces which allow to derive sharp convergence results for locally refined meshes. The first main result is an optimal error estimate for linear finite element approximations on the boundary in the $L^2(\Gamma)$-norm for both quasi-uniform and isotropically refined meshes. Later, the approximations of Neumann control problems using the postprocessing approach are investigated, that is, first a fully discrete solution with piecewise linear state and co-state, and piecewise constant controls, is computed and afterwards, an improved control by a pointwise evaluation of the discrete optimality condition is obtained. It is shown that quadratic convergence up to logarithmic factors is achieved for this control approximation if either the singularities are weak enough or the sequence of meshes is refined appropriately.

Keywords: edge and corner singularities, finite element error estimates, local mesh refinement, optimal Neumann boundary control, postprocessing approach

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1 Introduction

Throughout the paper, $\Omega \subset \mathbb{R}^3$ denotes a bounded domain having a polyhedral boundary Γ. For a given desired state $y_d \in L^2(\Omega)$ and some regularization parameter $\alpha > 0$ the control constrained Neumann boundary control problem under consideration reads

$$J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Gamma)} \to \min! \tag{1}$$

subject to

$$\begin{cases} -\Delta y + y = 0 & \text{in } \Omega, \\ \partial_n y = u & \text{on } \Gamma, \end{cases} \tag{2}$$

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$$u \in U_{ad} := \{v \in L^2(\Gamma)\colon u_a \le v \le u_b \quad \text{a.e. on } \Gamma\}. \tag{3}$$

We assume that the control bounds $u_a, u_b \in \mathbb{R}$ are constant. It is already well-known that the pair (y, u) is optimal if and only if some adjoint state p exists satisfying the adjoint problem

$$\begin{aligned} -\Delta p + p &= y - y_d \quad &&\text{in } \Omega, \\ \partial_n p &= 0 &&\text{on } \Gamma, \end{aligned} \tag{4}$$

and the projection formula

$$u = \Pi_{ad}\left(-\frac{1}{\alpha}p\right), \qquad [\Pi_{ad} v](x) := \max\{u_a, \min\{u_b, v(x)\}\}. \tag{5}$$

The present paper deals with error estimates for some computable approximation u_h of the optimal control u. Special emphasis is put on computational domains that are polyhedral. In this case we have in general reduced regularity due to edge and corner singularities contained in the solution and hence, if the singularities are too strong, a reduced convergence rate for finite element approximations. The primal goal of this paper is to restore the convergence rates we would expect on smooth domains. As the circumstances require the meshes have to be refined locally towards the singular points, and of interest are refinement conditions that guarantee optimal convergence.

An intermediate result required to prove estimates for the optimal control problem is an error estimate for the trace of the finite element approximation to the solution of the boundary value problem. While classical techniques like the Aubin-Nitsche method or trace theorems lead at best to a convergence rate of 3/2, the technique developed for planar problems by Apel/Pfefferer/Rösch [2] allows for almost quadratic convergence for quasi-uniform/appropriately refined meshes depending on the singularities. Therein, the proof extends an idea of Schatz/Wahlbin [18], more precisely, a dyadic decomposition around the singular corner is introduced and within each subset the sequence of meshes is quasi-uniform which allows the use of local results. We transferred this idea in our former paper [3] to the three-dimensional case where the estimate

$$\|y - y_h\|_{L^2(\Gamma)} \le ch^2 |\ln h|^{3/2}$$

is shown for the linear finite element approximation y_h of y. However, as we used a dyadic decomposition to all singular points, namely the skeleton of edges, the refinement criterion used for the singular corners is not sharp. In the present paper we show the necessary modifications in order to obtain sharp bounds for the refinement parameters. This basically relies on an additional dyadic decomposition towards the corners. In addition, regularity results in weighted Sobolev spaces are exploited which contain weight functions in their norms that allow to acquire singularities more accurately.

The second goal of this paper is to derive error estimates of the form

$$\|u - u_h\|_{L^2(\Gamma)} \le ch^\beta,$$

for certain approximations u_h of the optimal control u solving (1)–(3). Let us briefly summarize some important milestones on discretization strategies for optimal control problems. The most obvious idea is a full discretization of the optimality system meaning that state, adjoint state and control are sought in some finite-dimensional function space. For a finite element approximation using continuous and piecewise linear functions for the state variables, and piecewise constant functions for the control the convergence rate $\beta = 1$ can be expected [9, 24] for arbitrary polyhedral domains as the control belongs always to $H^1(\Gamma)$. However, away from the transition between active and inactive set the control possesses higher regularity. Hence, one might come up with the idea to use also piecewise linear functions for the control variable, but for control constrained problems this would lead to a convergence rate of at most $\beta = 3/2$ [17] under some structural assumption that we use later on in a similar way as well. Thus, advanced approaches are of interest which might even lead to quadratic convergence and this is indeed possible by taking the projection formula (5) into account so that kinks at the transition between active and inactive set are resolved also in the discretization. One of these approaches is the *variational discretization* introduced by Hinze [11] where the control is not discretized explicitly, but implicitly by means of the projection formula $u_h = \Pi_{ad}(-\alpha^{-1} p_h)$ with p_h the piecewise linear approximation of the adjoint state. We have already investigated error estimates for this approach in our former paper [3] and proved that the convergence rate $\beta = 2$, up to logarithmic factors, can be always achieved when the computational meshes are refined if necessary. With the results of the present paper we can relax the refinement condition used for singular corners.

Another approach on which we will focus in this paper is the *postprocessing* approach based on an idea of Meyer and Rösch [14] who applied the projection formula (5) to the discrete adjoint states of the fully discrete solution with piecewise constant control approximation, to construct piecewise linear controls that can converge quadratically. In a contribution of Mateos and Rösch [12] these results have been extended to Neumann control problems in polygonal domains using quasi-uniform meshes, but the error estimates derived therein are not sharp when the computational domain has corners with interior angle between 90° and 180°. This gap was closed by Apel, Pfefferer and Rösch [2] who made use of sharp finite element error estimates in $L^2(\Gamma)$ whose proof can be also found in this reference. Moreover, they investigate local mesh refinement towards singular corners and derived a refinement criterion which guarantees optimal convergence of the discrete control variable. The present paper extends the results for the postprocessing concept from [2] to the three-dimensional case. In addition to the finite element error estimate in $L^2(\Gamma)$, we have to show a superconvergence result for the midpoint interpolant. As in all contributions on the postprocessing approach this relies on a structural assumption on the active set. For planar problems, this assumption is for instance fulfilled if the number of points where the control switches between the active and inactive set is finite. For three-dimensional problems the control is defined on a two-dimensional manifold and the transition between active and inactive set consists in general of closed curves. Here, we assume that these curves have finite length. A straight-forward application of the techniques used in the two-dimensional case could lead to a suboptimal refinement criterion.

2 Weighted Sobolev spaces and regularity results

In this section we recall some regularity results for the weak solutions of the state and adjoint equations (2) and (4), respectively, which have the form

$$\text{Find } y \in H^1(\Omega)\colon \qquad a(y,v) = \langle f, v\rangle_\Omega + \langle g, v\rangle_\Gamma \qquad \forall v \in H^1(\Omega) \tag{6}$$

with

$$a\colon H^1(\Omega)\times H^1(\Omega) \to \mathbb{R} \qquad a(u,v) := \int_\Omega \left(\nabla u\cdot\nabla v + uv\right),$$

and the dual parings

$$\langle\cdot,\cdot\rangle_\Omega : [H^1(\Omega)]^* \times H^1(\Omega) \to \mathbb{R}, \qquad \langle\cdot,\cdot\rangle_\Gamma : H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma) \to \mathbb{R}.$$

Throughout this paper, $\Omega \subset \mathbb{R}^3$ is a polyhedral domain having corner points $\boldsymbol{c}_j$, $j \in \mathcal{C} := \{1,\ldots,d'\}$ and edges $\boldsymbol{e}_k$, $k \in \mathcal{E} := \{1,\ldots,d\}$.

The solution of (6) possesses singularities in the vicinity of edges and corners. It is known [10] that edge singularities of the form

$$\begin{aligned} &r^{\lambda^{\boldsymbol{e}}}\cos(\lambda^{\boldsymbol{e}}\varphi) && \text{if} \quad \lambda^{\boldsymbol{e}} := \frac{\pi}{\omega_{\boldsymbol{e}}} \neq \mathbb{Z}, \\ &r^{\lambda^{\boldsymbol{e}}}(\ln r\cos(\lambda^{\boldsymbol{e}}\varphi) + \varphi\sin(\lambda^{\boldsymbol{e}}\varphi)) && \text{if} \quad \lambda^{\boldsymbol{e}} := \frac{\pi}{\omega_{\boldsymbol{e}}} \in \mathbb{Z}, \end{aligned}$$

occur, where $\omega_{\boldsymbol{e}}$ is the interior angle at the edge $\boldsymbol{e}$ and (r,φ,z) are cylindrical coordinates chosen in such a way that $\varphi = 0$ and $\varphi = \omega$ correspond to the two faces meeting in $\boldsymbol{e}$. The number $\lambda^{\boldsymbol{e}}$ is called *singular exponent*. In the vicinity of a corner $\boldsymbol{c}$ the solution contains singularities of the form

$$\varrho^{\lambda^{\boldsymbol{c}}} F^{\boldsymbol{c}}(\varphi,\vartheta),$$

where $(\varrho,\varphi,\vartheta)$ are spherical coordinates around the corner $\boldsymbol{c}$. Here, the singular exponent is $\lambda^{\boldsymbol{c}} = -1/2 + \sqrt{1/2 + \mu^{\boldsymbol{c}}}$ and $(\mu^{\boldsymbol{c}}, F^{\boldsymbol{c}})$ denote the second-smallest eigenvalue and its corresponding eigenfunction of the Laplace-Beltrami operator on the surface $S_1(\boldsymbol{c}) \cap \Omega$, see [10]. If $S_1(\boldsymbol{c})$ contains other corners, the domain has to be rescaled appropriately. The eigenvalue $\mu^{\boldsymbol{c}}$ can in general be computed approximately only [16, 23].

The mesh refinement conditions we are going to derive merely depend on the strongest singularity and hence, we define the number

$$\lambda := \min_{k\in\mathcal{E}, j\in\mathcal{C}} \{\lambda^{\boldsymbol{e}_k}, 1/2 + \lambda^{\boldsymbol{c}_j}\} \tag{7}$$

that characterizes the global regularity of the solution of (6). For the equations considered in this paper there holds $\lambda^{\boldsymbol{e}} > 1/2$ and $\lambda^{\boldsymbol{c}} > 0$ and hence, $\lambda > 1/2$.

In the following we will define weighted Sobolev spaces. The weights used in these spaces are the distance functions towards the singular points defined by

$$r_k(x) := \inf_{y\in\boldsymbol{e}_k} |x-y|, \qquad \rho_j(x) := |x - \boldsymbol{c}_j|, \qquad r(x) := \min_{k\in\mathcal{E}} r_k(x).$$

Let $\{U_j\}_{j\in\mathcal{C}}$ be an open covering of Ω such that U_j contains only the corner $\boldsymbol{c}_j$ but no other ones. For a non-negative integer $\ell \in \mathbb{N}_0$, a real number $p \in [1,\infty]$ and vectors $\vec\beta \in \mathbb{R}^{d'}$, $\vec\delta \in \mathbb{R}^d$ the space $W^{\ell,p}_{\vec\beta,\vec\delta}(\Omega)$ is defined as the closure of $C^\infty(\bar\Omega\backslash\{\boldsymbol{c}_1,\ldots,\boldsymbol{c}_{d'}\})$ with respect to the norm

$$\|v\|_{W^{\ell,p}_{\vec\beta,\vec\delta}(\Omega)} := \left(\sum_{|\boldsymbol{\alpha}|\le\ell} \sum_{j\in\mathcal{C}} \int_{\Omega\cap U_j} \rho_j(x)^{p(\beta_j-\ell+|\boldsymbol{\alpha}|)} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}(x)\right)^{p\delta_k} |D^{\boldsymbol{\alpha}} v(x)|^p \right)^{\frac{1}{p}}, \tag{8}$$

if $p \in [1,\infty)$, and

$$\|v\|_{W^{\ell,\infty}_{\vec\beta,\vec\delta}(\Omega)} := \sum_{|\boldsymbol{\alpha}|\le\ell} \max_{j\in\mathcal{C}} \operatorname*{ess\,sup}_{x\in\Omega\cap U_j} \rho_j(x)^{\beta_j-\ell+|\boldsymbol{\alpha}|} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}(x)\right)^{\delta_k} |D^{\boldsymbol{\alpha}} v(x)|.$$

Here, we used the multi-index notation, i. e. $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, which allows us to define generalized partial derivatives by $D^{\boldsymbol{\alpha}} = \partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_z^{\alpha_3}$. Moreover, we write $|\boldsymbol{\alpha}| := \alpha_1 + \alpha_2 + \alpha_3$. The set $X_j \subset \mathcal{E}$ contains the indices of those edges $\boldsymbol{e}_k$ having an endpoint in the corner $\boldsymbol{c}_j$.

When taking the first sum in (8) over all $|\boldsymbol{\alpha}| = \ell$ only we obtain a semi-norm $|\cdot|_{W^{\ell,p}_{\vec\beta,\vec\delta}(\Omega)}$.

In the following we will frequently use these spaces in some subset $G \subset \Omega$. In this case the weights used in the norm definition (8) are still related to the edges and corners of Ω.

Regularity results for the solution of (6) in weighted Sobolev spaces, are proven e. g. in [1, 7, 13, 25]. We recall a result that we have already adapted to our situation in [3]:

Theorem 1. *a) Let $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$. Assume that the edge and corner weights $\vec\delta \in \mathbb{R}^d_+$ and $\vec\beta \in \mathbb{R}^{d'}_+$ satisfy*

$$1 - \lambda^{\boldsymbol{e}_k} < \delta_k < 1 \quad \forall k \in \mathcal{E}, \qquad 1/2 - \lambda^{\boldsymbol{c}_j} < \beta_j < 3/2 \quad \forall j \in \mathcal{C}.$$

Then, the solution of (6) satisfies $D^{\boldsymbol{\alpha}} y \in W^{1,2}_{\vec\beta,\vec\delta}(\Omega)$ for all $|\boldsymbol{\alpha}| = 1$.

b) Let $f \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$ and $g \equiv 0$. Assume that the weights $\vec\delta \in \mathbb{R}^d_+$ and $\vec\beta \in \mathbb{R}^{d'}_+$ satisfy

$$2 - \lambda^{\boldsymbol{e}_k} < \delta_k < 2 \quad \forall k \in \mathcal{E}, \qquad 2 - \lambda^{\boldsymbol{c}_j} < \beta_j \quad \forall j \in \mathcal{C}.$$

Then, the solution of (6) satisfies $D^{\boldsymbol{\alpha}} y \in W^{1,\infty}_{\vec\beta,\vec\delta}(\Omega)$ for all $|\boldsymbol{\alpha}| = 1$.

3 Error estimates for the state equation

This section is devoted to error estimates for the finite element approximation of the solution of (6). We assume that the input data satisfy at least $f \in [H^1(\Omega)]^*$ and $g \in H^{-1/2}(\Gamma)$, but we will demand more regularity later to obtain the desired error estimates. Let $\{\mathcal{T}_h\}_{h>0}$ denote a family of conforming tetrahedral triangulations of the domain Ω. The induced triangulation of the boundary Γ is denoted by $\partial\mathcal{T}_h$. We seek an approximation of (6) in the space of continuous and piecewise linear functions

$$Y_h := \{v_h \in C(\overline{\Omega})\colon v_h|_T \in \mathcal{P}_1 \quad \forall T \in \mathcal{T}_h\}. \tag{9}$$

The approximate solution $y_h \in Y_h$ is then defined via

$$a(y_h, v_h) = \langle f, v_h\rangle_\Omega + \langle g, v_h\rangle_\Gamma \qquad \forall v_h \in Y_h. \tag{10}$$

Due to the occurring singularities in the vicinity of edges and corners we demand additionally that the mesh is refined locally towards the singular points. Therefore, let

$$r_{k,T} := \inf_{x\in T}\inf_{y\in \boldsymbol{e}_k} |x-y|, \qquad \rho_{j,T} := \inf_{x\in T}|x-\boldsymbol{c}_j|, \qquad r_T := \min_{k=1,\ldots,d} r_{k,T}$$

denote the distance between the set $T \subset \Omega$, which will be either an element or a patch containing an element of $\mathcal{T}_h$, and the singular points of Ω. Each element $T \in \mathcal{T}_h$ is assumed to satisfy

$$h_T \sim \begin{cases} h^{1/\mu}, & \text{if } r_T = 0,\\ h r_T^{1-\mu}, & \text{if } r_T > 0,\end{cases} \tag{11}$$

where $\mu \in (1/3, 1]$ is the refinement parameter. The lower bound is required to ensure that the number of nodes is of order $N \sim h^{-3}$ [4]. For the choice $\mu = 1$ the sequence of meshes is quasi-uniform, and the smaller this parameter is the stronger the mesh is refined locally. Thus, we are interested in upper bounds for this parameter such that each choice below this bound leads to optimal convergence of the finite element solutions.

First, we recall a result from our foregoing paper [3].

Theorem 2. *Let $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$. Assume that the family of triangulations $\{\mathcal{T}_h\}_{h>0}$ is refined according to (11) with refinement parameter $1/3 < \mu < \lambda$. Then, the error estimate*

$$\|y - y_h\|_{H^\ell(\Omega)} \le ch^{2-\ell}|y|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega)} \le ch^{2-\ell}\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}\right)$$

holds for $\ell \in \{0,1\}$ with weights $\alpha_j = \max\{0, 1/2 - \lambda^{\boldsymbol{c}_j} + \varepsilon\}$, $j \in \mathcal{C}$, and $\delta_k = \max\{0, 1 - \lambda^{\boldsymbol{e}_k} + \varepsilon\}$, $k \in \mathcal{E}$, and sufficiently small $\varepsilon > 0$.

To derive discretization error estimates for the optimal control problem we need a finite element error estimate in the norm $L^2(\Gamma)$ as the adjoint control-to-state operator maps into this space. We already proved such an estimate in [3] but the refinement criterion derived therein is not sharp with respect to the singular corners. However, the proof requires rigorous modifications.

First, we recall some notation used already in [3]. We define the sets

$$\Omega_R := \{x \in \Omega\colon 0 \le r(x) \le R\}, \qquad \Gamma_R := \partial\Omega_R \cap \Gamma, \tag{12}$$

where the corner and edge singularities have influence on the regularity of the solution. The remaining set, where the distance to the corners and edges is larger than R, is denoted by $\tilde{\Gamma}_R := \Gamma\backslash\Gamma_R$. Without loss of generality we will set $R = 1$ in the following, because the domain Ω can be be rescaled as the circumstances require.

Furthermore, we introduce a dyadic decomposition of Ω_R, more precisely, we bound the distance to the singular bounds by the quantities $d_i := 2^{-i}$, $i = 0, \ldots, I$ and $d_{I+1} = 0$. Let $c_I \ge 1$ be a constant independent of h such that $d_I = c_I\, h^{1/\mu}$ holds. This implies the property $I \sim |\ln h|$. We will fix the constant c_I at the end of the proof of Theorem 7 as the result proved there holds only for sufficiently large c_I. In some steps of our proof, when the constant is unimportant, we will hide it in the generic constant c. The dyadic decomposition of Ω_R we will use in the sequel is defined by

$$\Omega_R = \bigcup_{i=0}^{I} \overline{\Omega}_i \quad \text{with} \quad \Omega_i := \{x \in \Omega_R \colon d_{i+1} < r(x) < d_i\} \quad \text{for } i = 0, \ldots, I.$$

This induces a decomposition of the boundary part Γ_R as well,

$$\Gamma_R = \bigcup_{i=0}^{I} \overline{\Gamma}_i \quad \text{with} \quad \Gamma_i := \partial\Omega_i \cap \Gamma, \quad \text{for } i = 0, \ldots, I. \tag{13}$$

We will further need the patches of Ω_i with its adjacent sets defined by

$$\Omega_i^{(m)} := \operatorname{int}\left(\bar{\Omega}_{\max\{0,i-m\}} \cup \ldots \cup \bar{\Omega}_i \cup \ldots \cup \bar{\Omega}_{\min\{I,i+m\}}\right), \quad m \in \mathbb{N},$$

and we use the abbreviations $\Omega_i' := \Omega_i^{(1)}$, $\Omega_i'' := \Omega_i^{(2)}$.

In order to separate the parts of Ω_i where only edge singularities and where both corner and edge singularities are present we introduce a further decomposition of Ω_i. To each edge $\boldsymbol{e}_k$ we associate a Cartesian coordinate system (x_k, y_k, z_k) so that $\boldsymbol{c}_j = (0,0,0)$ and $\boldsymbol{c}_{j'} = (0,0,L_{\boldsymbol{e}_k})$ are the endpoints of $\boldsymbol{e}_k$. The minimal angle between two edges meeting in a corner $\boldsymbol{c}_j$ is denoted by $\alpha_j := \min_{k,\ell \in X_j} \alpha_{k,\ell}$, where $\alpha_{k,\ell} := \sphericalangle(\boldsymbol{e}_k, \boldsymbol{e}_l)$. We cut off a set with measure d_i^3 at each corner that we denote by

$$\Omega_i^{\boldsymbol{c}} := \bigcup_{k \in X_j} \{x \in \Omega_i \colon z_k(x) < (2+A)d_i\}, \qquad \Gamma_i^{\boldsymbol{c}} := \partial\Omega_i^{\boldsymbol{c}} \cap \Gamma,$$

with $A := 2\min_{j\in\mathcal{C}} \cot\frac{\alpha_j}{2} \sim 1$, see also Figure 1a). By construction we have $|\Gamma_i^{\boldsymbol{c}}| \sim d_i^2$.

The remaining parts of Γ_i are defined as follows. For each edge $\boldsymbol{e} := \boldsymbol{e}_k$ and $i = 0, \ldots, I$ we introduce

$$\Omega_i^{\boldsymbol{e}} := \{x \in \Omega_i \colon z_k(x) \in ((2+A)\,d_i, L_{\boldsymbol{e}} - (2+A)\,d_i)\}.$$

The boundary parts are denoted by $\Gamma_i^{\boldsymbol{e}} := \partial\Omega_i^{\boldsymbol{e}} \cap \Gamma$. We observe that the boundary part Γ_i is covered completely by the sets defined above, i. e.

$$\Gamma_i = \operatorname{int}\left(\bigcup_{j\in\mathcal{C}} \overline{\Gamma_i^{\boldsymbol{c}_j}} \cup \bigcup_{k\in\mathcal{M}} \overline{\Gamma_i^{\boldsymbol{e}_k}}\right) \tag{14}$$

It remains to define appropriate patches

$$\Omega_i^{\boldsymbol{c},(m)} := \bigcup_{k\in X_j} \left\{x \in \Omega_i^{(m)} \colon z_k(x) < (2+m+A)d_i\right\},$$
$$\Omega_i^{\boldsymbol{e},(m)} := \left\{x \in \Omega_i^{(m)} \colon z_k(x) \in ((2-m+A)d_i, L_{\boldsymbol{e}} - (2-m+A)d_i)\right\},$$

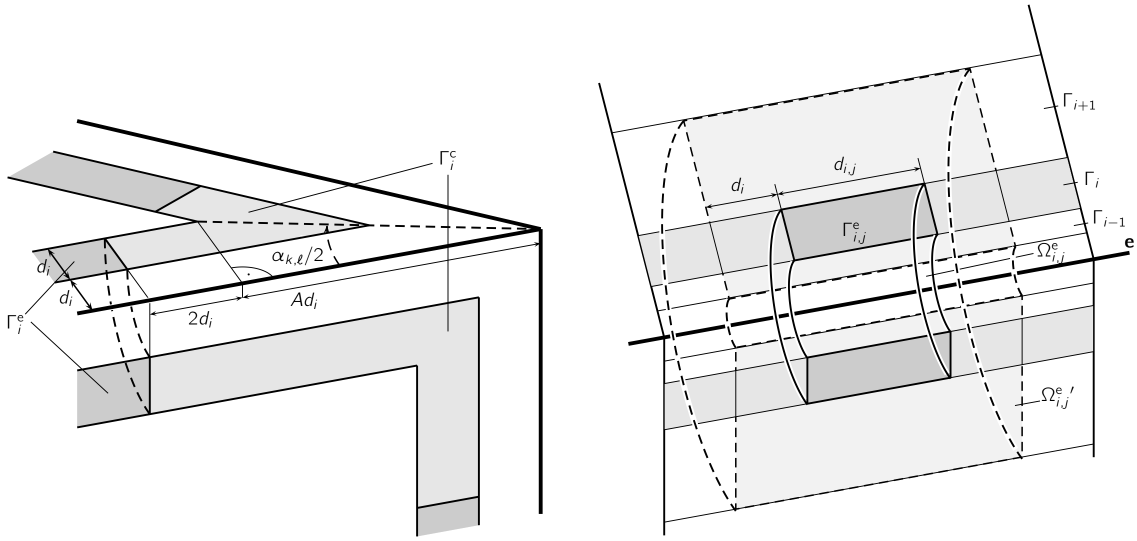
for $m \in \{1,2\}$. We use again the abbreviations

$$\Omega_i^{\boldsymbol{c}\prime} := \Omega_i^{\boldsymbol{c},(1)}, \quad \Omega_i^{\boldsymbol{c}\prime\prime} := \Omega_i^{\boldsymbol{c},(2)}, \quad \Omega_i^{\boldsymbol{e}\prime} := \Omega_i^{\boldsymbol{e},(1)}, \quad \Omega_i^{\boldsymbol{e}\prime\prime} := \Omega_i^{\boldsymbol{e},(2)}.$$

The essential property that we exploit in the following is

$$\operatorname{dist}(\partial\Omega_i^{\boldsymbol{e}\prime} \setminus \Gamma, \partial\Omega_i^{\boldsymbol{e}} \setminus \Gamma) \sim d_i, \qquad \operatorname{dist}(\partial\Omega_i^{\boldsymbol{c}\prime} \setminus \Gamma, \partial\Omega_i^{\boldsymbol{c}} \setminus \Gamma) \sim d_i.$$

Moreover, we require a dyadic decomposition of $\Omega_i^{\boldsymbol{e}}$ and its patches $\Omega_i^{\boldsymbol{e},(m)}$ in order to carve out the influence of the corner singularity. This additional decomposition has not been used in our former paper



(a) Definition of $\Omega_i^{\boldsymbol{e}}$ and $\Omega_i^{\boldsymbol{c}}$

(b) Definition of $\Omega_{i,j}^{\boldsymbol{e}}$ and its patch

Figure 1: Illustration of the domains $\Omega_i^{\boldsymbol{c}}$ and $\Omega_{i,j}^{\boldsymbol{e}}$.

[3] which is the reason why the refinement condition derived therein which is necessary to compensate the corner singularity is too strong. For $j = 0, \ldots, i$ and $m \in \{0, 1, 2\}$ we define

$$\begin{aligned}
\Omega_{i,j}^{\boldsymbol{e},+,(m)} &:= \Big\{x \in \Omega_i^{\boldsymbol{e},(m)} \colon z_k(x) \in \; ((1 + A + 2^j - m)d_i, \\
&\qquad\qquad (1 + A + 2^{j+1} + m)d_i)\Big\}, \\
\Omega_{i,j}^{\boldsymbol{e},-,(m)} &:= \Big\{x \in \Omega_i^{\boldsymbol{e},(m)} \colon z_k(x) \in \; (L_{\boldsymbol{e}} - (1 + A + 2^{j+1} + m)d_i, \\
&\qquad\qquad L_{\boldsymbol{e}} - (1 + A + 2^j - m)d_i)\Big\}, \\
\tilde{\Omega}_i^{\boldsymbol{e},(m)} &:= \Big\{x \in \Omega_i^{\boldsymbol{e},(m)} \colon z_k(x) \in \; ((1 + A + 2^{i+1} - m)d_i, \\
&\qquad\qquad L_{\boldsymbol{e}} - (1 + A + 2^{i+1} - m)d_i)\Big\},
\end{aligned}$$

and we observe that

$$\Omega_i^{\boldsymbol{e},(m)} = \operatorname{int}\left(\bigcup_{j=0}^{i} \overline{\Omega_{i,j}^{\boldsymbol{e},\pm,(m)}} \cup \overline{\tilde{\Omega}_i^{\boldsymbol{e},(m)}}\right).$$

As usual, the boundary parts are denoted by

$$\Gamma_{i,j}^{\boldsymbol{e},\pm} := \partial\Omega_{i,j}^{\boldsymbol{e},\pm} \cap \Gamma, \qquad \tilde{\Gamma}_i^{\boldsymbol{e}} := \partial\tilde{\Omega}_i^{\boldsymbol{e}} \cap \Gamma.$$

One easily confirms that the properties

$$\begin{aligned}
|\Omega_{i,j}^{\boldsymbol{e},\pm,(m)}| &\sim d_i^2 d_{i,j}, & |\tilde{\Omega}_i^{\boldsymbol{e},(m)}| &\sim d_i^2, \\
|\Gamma_{i,j}^{\boldsymbol{e},\pm,(m)}| &\sim d_i d_{i,j}, & |\tilde{\Gamma}_i^{\boldsymbol{e},(m)}| &\sim d_i,
\end{aligned} \tag{15}$$

hold for $i = 0, \ldots, I$ and $j = 0, \ldots, i$, with

$$d_{i,j} := 2^j d_i = 2^{j-i} \leq 1.$$

In the next lemma we will derive interpolation error estimates on the sets $\Omega_i^{\boldsymbol{c}}$ and $\Omega_i^{\boldsymbol{e}}$. The proof of this result relies on local estimates for a quasi-interpolation operator $Z_h \colon W^{1,1}(\Omega) \to Y_h$ exploiting regularity in weighted Sobolev spaces. For an accurate definition of this interpolant we refer to [20]. In this paper the definition is not explicitly needed. In [3, Lemma 4.4] the following result is proven. Let $T \in \mathcal{T}_h$ and $j \in \mathcal{C}$ such that $T \subset U_j$. Then there holds

$$\begin{aligned} &|u - Z_h u|_{H^\ell(T)} \\ &\leq c h_T^{2-\ell} |T|^{\frac{1}{2}-\frac{1}{p}} |u|_{W^{2,p}_{\vec\beta,\vec\delta}(S_T)} \cdot \begin{cases} h_T^{-\beta_j}, & \text{if } \rho_{j,S_T} = 0, \\ h_T^{-\delta_k} \rho_{j,T}^{\delta_k-\beta_j}, & \text{if } r_{k,S_T} = 0,\ \rho_{j,S_T} > 0, \\ \rho_{j,T}^{-\beta_j} \prod_{k \in X_j} \left(\frac{r_{k,T}}{\rho_{j,T}}\right)^{-\delta_k}, & \text{if } r_{k,S_T} > 0\ \forall k \in X_j, \end{cases} \end{aligned} \tag{16}$$

for $\ell \in \{0,1\}$, $p \in (6/5, \infty]$, $\vec\beta \in [0, 5/2 - 3/p)^{d'}$, $\vec\delta \in [0, 5/3 - 2/p)^d$. Here, S_T denotes the union of T and its adjacent elements. We will frequently use the simplified version [3, Lemma 4.4]

$$|u - Z_h u|_{H^\ell(T)} \leq c h_T^{2-\ell} |T|^{\frac{1}{2}-\frac{1}{p}} |u|_{W^{2,p}_{\vec\beta,\vec\delta}(S_T)} \cdot \begin{cases} h_T^{-\kappa_j}, & \text{if } r_{S_T} = 0, \\ r_T^{-\kappa_j}, & \text{if } r_{S_T} > 0, \end{cases} \tag{17}$$

instead, where $\kappa_j := \max\{\beta_j, \max_{k \in X_j} \delta_k\}$.

Lemma 3. *Let some function $u \in H^1(\Omega_i^{(m+1)})$, $m \in \{0,1\}$, be given and assume that the property $D^{\boldsymbol\alpha} u \in W^{1,p}_{\vec\alpha,\vec\delta}(\Omega_i^{(m+1)})$ holds for all $|\boldsymbol\alpha| = 1$ with $p \in [2,\infty]$ and weights $\vec\alpha \in [0, 5/2 - 3/p)^{d'}$, $\vec\delta \in [0, 5/3 - 2/p)^d$. Let $\boldsymbol e := \boldsymbol e_k$, $k \in \mathcal{E}$, and $\boldsymbol c := \boldsymbol c_j$, $j \in \mathcal{C}$, be an arbitrary edge and corner, respectively. Moreover, define the numbers $\kappa_j := \max\{\alpha_j, \max_{k \in X_j} \delta_k\}$, $\tilde\alpha_k := \max\{\alpha_j, \alpha_{j'}\}$ where $j \neq j'$ are the corner indices such that $k \in X_j \cap X_{j'}$, $s_k := 1/2 - 1/p + \delta_k - \tilde\alpha_k$, and $\Theta_\ell := (7/2 - \ell - 3/p)(1 - \mu)$. It is assumed that $s_k \neq 0$ for all $k \in \mathcal{E}$.*

a) For $i = 0, \ldots, I - 2 - m$ there hold the estimates

$$\begin{aligned} |u - Z_h u|_{H^\ell(\Omega_i^{\boldsymbol c,(m)})} &\leq c h^{2-\ell} d_i^{(2-\ell)(1-\mu)+3/2-3/p-\kappa_j} |u|_{W^{2,p}_{\vec\alpha,\vec\delta}(\Omega_i^{\boldsymbol c,(m+1)})}, \\ |u - Z_h u|_{H^\ell(\Omega_i^{\boldsymbol e,(m)})} &\leq c h^{2-\ell} d_i^{(2-\ell)(1-\mu)+1-2/p-\delta_k+[s_k]_-} |u|_{W^{2,p}_{\vec\alpha,\vec\delta}(\Omega_i^{\boldsymbol e,(m+1)})}. \end{aligned}$$

b) For $i = I - 1 - m, \ldots, I$ there hold the estimates

$$\begin{aligned} |u - Z_h u|_{H^\ell(\Omega_i^{\boldsymbol c,(m)})} &\leq c c_I^{[\Theta_\ell - \kappa_j]_+ + 3/2 - 3/p} h^{(7/2-3/p-\ell-\kappa_j)/\mu} |u|_{W^{2,p}_{\vec\alpha,\vec\delta}(\Omega_i^{\boldsymbol c,(m+1)})}, \\ |u - Z_h u|_{H^\ell(\Omega_i^{\boldsymbol e,(m)})} &\leq c c_I^{[\Theta_\ell - \delta_k]_+ + 1 - 2/p} h^{(3-2/p-\ell-\delta_k+[s_k]_-)/\mu} |u|_{W^{2,p}_{\vec\alpha,\vec\delta}(\Omega_i^{\boldsymbol e,(m+1)})}, \end{aligned}$$

where $[a]_+ := \max\{0, a\}$ and $[a]_- := \min\{0, a\}$ for $a \in \mathbb{R}$.

Proof. We only prove the result for $m = 0$ as the case $m = 1$ follows from exactly the same arguments. First, we show the estimate on $\Omega_i^{\boldsymbol c}$ by insertion of local interpolation error estimates into the discrete Hölder inequality

$$|u - Z_h u|^2_{H^\ell(\Omega_i^{\boldsymbol c})} \leq \left(\sum_{T \cap \Omega_i^{\boldsymbol c} \neq \emptyset} 1\right)^{1-2/p} \left(\sum_{T \cap \Omega_i^{\boldsymbol c} \neq \emptyset} |u - Z_h u|^p_{H^\ell(T)}\right)^{2/p}. \tag{18}$$

For the case $i = 0, \ldots, I - 2$, the number of elements intersecting $\Omega_i^{\boldsymbol c}$ can be estimated by

$$\sum_{T \cap \Omega_i^{\boldsymbol c} \neq \emptyset} 1 \leq c \max_{T \cap \Omega_i^{\boldsymbol c} \neq \emptyset} \frac{|\Omega_i^{\boldsymbol c}|}{|T|} \leq c \max_{T \cap \Omega_i^{\boldsymbol c} \neq \emptyset} \frac{d_i^3}{|T|}. \tag{19}$$

For all $T \cap \Omega_i^{\boldsymbol c} \neq \emptyset$ we obtain with the local estimate (17) and the property $r_T \sim d_i$ the estimate

$$|u - Z_h u|_{H^\ell(T)} \leq c h_T^{2-\ell} |T|^{1/2-1/p} d_i^{-\kappa_j} |u|_{W^{2,p}_{\vec\alpha,\vec\delta}(S_T)}. \tag{20}$$

Insertion of (19) into (18) yields for $i = 0, \ldots, I-2$

$$|u - Z_h u|_{H^\ell(\Omega_i^c)} \le c h^{2-\ell} d_i^{(2-\ell)(1-\mu)+3(1/2-1/p)-\kappa_j} |u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(\Omega_i^{c\prime})}. \tag{21}$$

In order to derive the estimate on $\Omega_i^{\boldsymbol{e}}$ we can basically use the same technique. However, we have to decompose the domain $\Omega_i^{\boldsymbol{e}}$ into the subsets defined in (15) first. For all elements $T \subset U_l$ intersecting $\Omega_{i,j}^{\boldsymbol{e},\pm}$ or $\tilde{\Omega}_i^{\boldsymbol{e}}$ we get from (16) and the property $\rho_{l,T} \sim d_{i,j}$ the local estimates

$$\begin{aligned} |u - Z_h u|_{H^\ell(T)} &\le c h^{2-\ell} d_i^{(2-\ell)(1-\mu)-\delta_k} d_{i,j}^{\delta_k - \tilde{\alpha}_k} |T|^{1/2-1/p} |u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(S_T)}, \quad &&\text{if } T \cap \Omega_{i,j}^{\boldsymbol{e},\pm} \ne \emptyset, \\ |u - Z_h u|_{H^\ell(T)} &\le c h^{2-\ell} d_i^{(2-\ell)(1-\mu)-\delta_k} |T|^{1/2-1/p} |u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(S_T)}, &&\text{if } T \cap \tilde{\Omega}_i^{\boldsymbol{e}} \ne \emptyset. \end{aligned} \tag{22}$$

The number of elements which intersect $\Omega_{i,j}^{\boldsymbol{e},\pm}$ and $\tilde{\Omega}_i^{\boldsymbol{e}}$ is of order

$$\sum_{T \cap \Omega_{i,j}^{\boldsymbol{e},\pm} \ne \emptyset} 1 \le c \max_{T \cap \Omega_{i,j}^{\boldsymbol{e},\pm} \ne \emptyset} \frac{d_i^2 d_{i,j}}{|T|} \qquad \text{and} \qquad \sum_{T \cap \tilde{\Omega}_i^{\boldsymbol{e}} \ne \emptyset} 1 \le c \max_{T \cap \tilde{\Omega}_i^{\boldsymbol{e}} \ne \emptyset} \frac{d_i^2}{|T|},$$

respectively, compare also (19). From the Hölder inequality similar to (18) we then obtain

$$\begin{aligned} |u - Z_h u|_{H^\ell(\Omega_i^{\boldsymbol{e}})} &\le c h^{2-\ell} d_i^{(2-\ell)(1-\mu)+1-2/p-\delta_k} \\ &\qquad \times \left(\sum_{j=0}^{i} d_{i,j}^{(1/2-1/p+\delta_k-\tilde{\alpha}_k)p'} \right)^{1/p'} |u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(\Omega_i^{\boldsymbol{e}\prime})}, \end{aligned} \tag{23}$$

where $p^{-1} + p'^{-1} = 1$. The limit value of the geometric series yields

$$\sum_{j=0}^{i} d_{i,j}^{s_k p'} = d_i^{s_k p'} \sum_{j=0}^{i} 2^{j s_k p'} \le c d_i^{s_k p'} (2^{(i+1)s_k p'} - 1) \le c(2^{s_k p'} + d_i^{s_k p'}) \le c d_i^{[s_k]_- p'}, \tag{24}$$

and we conclude from (23) the desired estimate on $\Omega_i^{\boldsymbol{e}}$ for $i = 0, \ldots, I-2$.

Let us now consider the case $i = I-1, I$. We start with an estimate on $\Omega_i^{\boldsymbol{c}}$, where $\boldsymbol{c} = \boldsymbol{c}_j$ for some $j \in \mathcal{C}$. The number of elements intersecting $\Omega_i^{\boldsymbol{c}}$ is bounded by

$$\sum_{T \cap \Omega_i^{\boldsymbol{c}} \ne \emptyset} 1 \le c d_i^3 |T_{min}|^{-1} \le c c_I^3, \tag{25}$$

as $d_i^3 \sim c_I^3 |T_{min}|$. Again, we insert the local estimates (17) which depend on the position of the patch of elements S_T. If $r_{S_T} > 0$ we get with $|T| \le c h^3 r_T^{3(1-\mu)}$ and

$$r_T^t \le c d_I^t \le c c_I^t h^{t/\mu} \quad \text{if} \quad t \ge 0, \qquad r_T^t \le c h^{t/\mu} \quad \text{if} \quad t < 0, \tag{26}$$

as well as the choice $t := (7/2 - \ell - 3/p)(1-\mu) - \kappa_j$ and $|T_{min}| = h^{3/\mu}$

$$|u - Z_h u|_{H^\ell(T)} \le c c_I^{[\Theta_\ell - \kappa_j]_+} h^{(2-\ell-\kappa_j)/\mu} |T_{min}|^{1/2-1/p} |u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(S_T)}. \tag{27}$$

The same estimate holds for $r_{S_T} = 0$ even without the factor $c_I^{[\Theta_\ell - \kappa_j]_+}$ due to (16) and $c \ge \varrho_{j,T} \ge h^{1/\mu}$ in case of $\varrho_{j,T} > 0$. From (18) we conclude with (25)

$$|u - Z_h u|_{H^\ell(\Omega_i^{\boldsymbol{c}})} \le c c_I^{[\Theta_\ell - \kappa_j]_+ + 3/2 - 3/p} h^{(7/2-\ell-3/p-\kappa_j)/\mu} |u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(\Omega_i')}. \tag{28}$$

With a similar technique we can show an estimate on $\Omega_{i,j}^{\boldsymbol{e},\pm}$ for $i = I-1, I$ and $j = 0, \ldots, i$. For all $T \cap \Omega_{i,j}^{\boldsymbol{e},\pm} \ne \emptyset$ with $r_{S_T} > 0$ we conclude from (16) the estimate

$$|u - Z_h u|_{H^\ell(T)} \le c c_I^{[\Theta_\ell - \delta_k]_+} h^{(2-\ell-\delta_k)/\mu} |T_{min}|^{1/2-1/p} d_{i,j}^{\delta_k - \tilde{\alpha}_k} |u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(S_T)}. \tag{29}$$

We easily confirm that this estimate holds also in case of $r_{S_T} = 0$. The number of elements which intersect $\Omega^{\boldsymbol{e},\pm}_{i,j}$ is of order

$$\sum_{T\cap\Omega^{\boldsymbol{e}}_{i,j}\neq\emptyset} 1 \le c d_i^2 d_{i,j} |T_{min}|^{-1} \le c_I^2 h^{2/\mu} d_{i,j} |T_{min}|^{-1}.$$

Consequently, we get from (29) and the Hölder inequality as in (18)

$$|u - Z_h u|_{H^\ell(\Omega^{\boldsymbol{e},\pm}_{i,j})} \le c c_I^{[\Theta_\ell-\delta_k]_+ + 1 - 2/p} h^{(3-2/p-\ell-\delta_k)/\mu} d_{i,j}^{1/2-1/p+\delta_k-\tilde\alpha_k} |u|_{W^{2,p}_{\vec\alpha,\vec\delta}(\Omega^{\boldsymbol{e},\pm}_{i,j}{}')}.$$

Summing up over all $\Omega^{\boldsymbol{e},\pm}_{i,j}$ for $j = 0, \ldots, i$ yields

$$\begin{aligned}
&\left(\sum_{j=0}^{i} |u - Z_h u|^2_{H^\ell(\Omega^{\boldsymbol{e},\pm}_{i,j})}\right)^{1/2} \\
&\quad \le c c_I^{[\Theta_\ell-\delta_k]_+ + 1 - 2/p} h^{(3-2/p-\ell-\delta_k)/\mu} \left(\sum_{j=0}^{i} d_{i,j}^{s_k p'}\right)^{1/p'} |u|_{W^{2,p}_{\vec\alpha,\vec\delta}(\Omega_i')} \\
&\quad \le c c_I^{[\Theta_\ell-\delta_k]_+ + 1 - 2/p} h^{(3-2/p-\ell-\delta_k+[s_k]_-)/\mu} |u|_{W^{2,p}_{\vec\alpha,\vec\delta}(\Omega_i')}, \qquad (30)
\end{aligned}$$

where we used the estimate (24) and the fact that $c_I^{[s_k]_-} \le 1$ ($c_I \ge 1$) in the last step.

For all $T \cap \tilde\Omega^{\boldsymbol{e}}_i \neq \emptyset$ there holds $\rho_{j,S_T} \sim 1$ and as the number of these elements is of order

$$\sum_{T\cap\tilde\Omega^{\boldsymbol{e}}_i\neq\emptyset} 1 \le c d_i^2 |T_{min}|^{-1} \le c_I^2 h^{2/\mu} |T_{min}|^{-1},$$

we get

$$|u - Z_h u|_{H^\ell(\tilde\Omega^{\boldsymbol{e}}_i)} \le c c_I^{[\Theta_\ell-\delta_k]_+ + 1 - 2/p} h^{(3-2/p-\ell-\delta_k)/\mu} |u|_{W^{2,p}_{\vec\alpha,\vec\delta}(\Omega^{\boldsymbol{e}}_i{}')}. \qquad (31)$$

Finally, from the decomposition (15) and the estimates (30) and (31) we conclude the estimate on $\Omega^{\boldsymbol{e}}_i$ for $i = I-1, I$. □

Furthermore, we need some interpolation error estimates in the $L^\infty(\Omega)$-norm on the subsets Ω_i, and here, we use the nodal interpolant $I_h\colon C(\overline\Omega) \to Y_h$ due to its stability in the $L^\infty(\Omega)$-norm. In the following result we will hide the parameter c_I in the generic constant c as it is not needed for the terms to which we apply these estimates.

Lemma 4. *Let some function $u \in L^\infty(\Omega_i^{(m+1)})$, $m \in \{0,1\}$, be given satisfying the following properties:*

- $D^{\boldsymbol\alpha} u \in W^{1,\infty}_{\vec\beta,\vec\varrho}(\Omega_i^{(m+1)})$ *for all $|\boldsymbol\alpha| = 1$ with $\vec\beta \in [0,2)^{d'}$ and $\vec\varrho \in [0,5/3)^d$,*
- $u \equiv 0$ *on $\Omega \setminus \Omega_R$.*

Define $\kappa_j = \max\{\beta_j, \max_{k\in X_j} \varrho_k\}$ and $\tilde\beta_k := \max\{\beta_j : j \in \mathcal{C}$ such that $k \in X_j\}$. Then, for all corners $\boldsymbol{c} := \boldsymbol{c}_j$, $j \in \mathcal{C}$, and edges $\boldsymbol{e} := \boldsymbol{e}_k$, $k \in \mathcal{E}$, the following estimates hold:

a) For $i = 0, 1, \ldots, I-2-m$ there hold the estimates

$$\begin{aligned}
\|u - I_h u\|_{L^\infty(\Omega_i^{\boldsymbol{c},(m)})} &\le c h^2 d_i^{2(1-\mu)-\kappa_j} |u|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\Omega_i^{\boldsymbol{c},(m+1)})}, \\
\|u - I_h u\|_{L^\infty(\Omega_{i,j}^{\boldsymbol{e},\pm,(m)})} &\le c h^2 d_i^{2(1-\mu)-\varrho_k} d_{i,j}^{\varrho_k-\tilde\beta_k} |u|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\Omega_{i,j}^{\boldsymbol{e},\pm,(m+1)})}, \quad j = 0, \ldots, i, \\
\|u - I_h u\|_{L^\infty(\tilde\Omega_i^{\boldsymbol{e},(m)})} &\le c h^2 d_i^{2(1-\mu)-\varrho_k} |u|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\tilde\Omega_i^{\boldsymbol{e},(m+1)})}.
\end{aligned}$$

b) For $i = I-1-m, \dots, I$ there hold the estimates

$$\|u - I_h u\|_{L^\infty(\Omega_i^{\mathbf{c},(m)})} \le c h^{(2-\kappa_j)/\mu} |u|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\Omega_i^{\mathbf{c},(m+1)})},$$
$$\|u - I_h u\|_{L^\infty(\Omega_{i,j}^{\mathbf{e},\pm,(m)})} \le c h^{(2-\varrho_k)/\mu} d_{i,j}^{\varrho_k - \tilde\beta_k} |u|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\Omega_{i,j}^{\mathbf{e},\pm,(m+1)})}, \quad j = 0,\dots,i$$
$$\|u - I_h u\|_{L^\infty(\tilde\Omega_i^{\mathbf{e},(m)})} \le c h^{(2-\varrho_k)/\mu} |u|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\tilde\Omega_i^{\mathbf{e},(m+1)})}.$$

Proof. We prove the assertion merely for $m = 0$ since the extension to $m = 1$ is simple. Let $T \in \mathcal{T}_h$ be an arbitrary element. The index j is chosen such that $T \subset U_j$, where $\{U_j\}$ is the covering used in definition (8). The result then follows from the local estimates

$$\|u - I_h u\|_{L^\infty(T)} \le c h_T^2 \rho_{j,T}^{-\beta_k} \prod_{k\in X_j} \left(\frac{r_{k,T}}{\rho_{j,T}}\right)^{-\varrho_k} |u|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(T)}, \qquad \text{if } r_T > 0, \tag{32}$$
$$\|u - I_h u\|_{L^\infty(T)} \le c h_T^{2-\varrho_k} \rho_{j,T}^{\varrho_k-\beta_j} |u|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(T)} \qquad \text{if } r_{k,T} = 0, \rho_{j,T} > 0 \tag{33}$$
$$\|u - I_h u\|_{L^\infty(T)} \le c h_T^{2-\beta_j} |u|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(T)}, \qquad \text{if } \rho_{j,T} = 0, \tag{34}$$

which have been derived in the proof of [3, Lemma 4.8]. We have to distinguish among certain situations of how T is located such that the distances $r_{k,T}$ and $\rho_{j,T}$ can be estimated against the constants d_i and $d_{i,j}$.

We start with an estimate on $\Omega_i^{\mathbf{c}}$ for $i = 0,\dots,I-2$. Let $T \cap \Omega_i^{\mathbf{c}} \ne \emptyset$ be the element where the maximum of $|u(x) - I_h u(x)|$ is attained. We apply (32) and the simplification

$$\rho_{j,T}^{-\beta_k} \prod_{k\in X_j} \left(\frac{r_{k,T}}{\rho_{j,T}}\right)^{-\varrho_k} \le r_T^{-\kappa_j} \tag{35}$$

shown in the proof of [3, Lemma 4.4] to arrive at

$$\|u - I_h u\|_{L^\infty(T)} \le c h^2 r_T^{2(1-\mu)-\kappa_j} |u|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(T)} \tag{36}$$

and conclude the result using $r_T \sim d_i$.

To obtain the desired estimates for $i = I-1, I$ we distinguish among the cases that T touches the singular points or not. For elements with $r_T = 0$ we take (33) or (34) and insert $h_T^{-\varrho_k} \rho_{j,T}^{\varrho_k - \beta_j} \le h_T^{-\kappa_j}$ (this follows from $\rho_{j,T} > 0 \Rightarrow \rho_{j,T} \ge c h_T$). For elements with $r_T > 0$, we use (36) as well as $r_T \le c d_I \sim c h^{1/\mu}$ instead. Both arguments lead to the estimate

$$\|u - I_h u\|_{L^\infty(T)} \le c h^{(2-\kappa_j)/\mu} |u|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(T)}. \tag{37}$$

This implies the assertion for the domains $\Omega_i^{\mathbf{c}}$.

Next, we show the estimate on $\Omega_{i,j}^{\mathbf{e},\pm}$ in case of $i = 0,\dots,I-2$. Let $\mathbf{c}_{j_1}$ and $\mathbf{c}_{j_2}$, $j_1, j_2 \in \mathcal{C}$, denote the endpoints of the edge $\mathbf{e}$. For $T \subset U_{j_p}$, $p \in \{1,2\}$ we apply the local estimate

$$\|u - I_h u\|_{L^\infty(T)} \le c h^2 r_{k,T}^{2(1-\mu)-\varrho_k} \rho_{j_p,T}^{\varrho_k - \beta_{j_p}} |u|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(T)},$$

that we conclude from (32), see also [3, Equation (4.34)], and exploit that $r_{k,T} \sim d_i$ for $T \cap \Omega_{i,j}^{\mathbf{e},\pm} \ne \emptyset$, and $\rho_{j_1,T} \sim d_{i,j}$ if $T \cap \Omega_{i,j}^{\mathbf{e},+} \ne \emptyset$, and $\rho_{j_2,T} \sim d_{i,j}$ if $T \cap \Omega_{i,j}^{\mathbf{e},-} \ne \emptyset$. This leads to the local estimate

$$\|u - I_h u\|_{L^\infty(T)} \le c h^2 d_i^{2(1-\mu)-\varrho_k} d_{i,j}^{\varrho_k - \tilde\beta_k} |u|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(T)} \tag{38}$$

from which we conclude the assertion for $i = 0,\dots,I-2$. For $i = I-1, I$ we distinguish among the cases $r_T > 0$ and $r_T = 0$. To show an estimate for $r_T > 0$ we insert the property $d_i \sim h^{1/\mu}$ into (38).

In case of $r_T = 0$ and $T \subset U_{j_p}$, $p \in \{1,2\}$, we insert $\rho_{j_p,T} \sim d_{i,j}$ into the local estimate (33). In both cases we obtain

$$\|u - I_h u\|_{L^\infty(T)} \le c h^{(2-\varrho_k)/\mu} d_{i,j}^{\varrho_k - \beta_{j_p}} |u|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(T)},$$

which yields the assertion as $T \subset \Omega^{\boldsymbol{e},\pm\prime}_{i,j}$. The estimates on $\tilde\Omega^{\boldsymbol{e}}_i$ follow from the same strategy exploiting that $\rho_{j_p,T} \sim 1$, $p \in \{1,2\}$, for all $T \cap \tilde\Omega^{\boldsymbol{e}}_i \neq \emptyset$. □

Next, we define the function $\tilde y := \omega y$, where $\omega \in C^\infty(\Omega)$ is a smooth cut-off function satisfying

$$\omega|_{\Omega_{R/2}} \equiv 1 \qquad \text{and} \qquad \operatorname{supp}\omega \subset \Omega_R. \tag{39}$$

Note that this function coincides with y near the singular points. In the next steps we show some error estimates for a certain Ritz-projection of this local solution that we denote by

$$\tilde y_h \in Y_h(\Omega_R) := \{v_h \in C(\overline\Omega_R) \colon v_h = w_h|_{\Omega_R} \text{ for some } w_h \in Y_h\},$$

and this function is defined by

$$a_{\Omega_R}(\tilde y - \tilde y_h, v_h) := \int_{\Omega_R} (\nabla(\tilde y - \tilde y_h)\cdot\nabla v_h + (\tilde y - \tilde y_h)v_h) = 0 \qquad \forall v_h \in Y_h(\Omega_R). \tag{40}$$

First, we show error estimates for this this solution in the norms $H^1(\Omega_R)$ and $L^2(\Omega_R)$.

Lemma 5. *Assume that $D^{\boldsymbol\alpha} y \in W^{1,2}_{\vec\alpha,\vec\delta}(\Omega)$ for $|\boldsymbol\alpha| = 1$ with weights $\vec\alpha \in [0,1)^{d'}$ and $\vec\delta \in [0,2/3)^d$ that fulfill*

$$\frac12 - \lambda^{\boldsymbol{c}_j} < \alpha_j \le 1 - \mu, \quad j \in \mathcal{C}, \qquad 1 - \lambda^{\boldsymbol{e}_k} < \delta_k \le 1 - \mu, \quad k \in \mathcal{E}.$$

For the functions $\tilde y := \omega y$ with ω from (39) and $\tilde y_h$ from (40) the error estimates

$$\|\tilde y - \tilde y_h\|_{H^\ell(\Omega_R)} \le c h^{2-\ell}\left(|\tilde y|_{W^{2,2}_{\vec\alpha,\vec\delta}(\Omega_R)} + \|\tilde y\|_{H^1(\Omega_0)}\right)$$

hold for $\ell \in \{0,1\}$.

Proof. We denote by $\Omega_{R,h} := \cup\{\overline T \colon T \in \mathcal{T}_h,\ T \cap \Omega_R \neq \emptyset\}$ the union of all elements that intersect Ω_R. Next, we introduce the Calderon extension which extends $\tilde y \colon \Omega_R \to \mathbb{R}$ smoothly to some function $\breve y \colon \Omega_{R,h} \to \mathbb{R}$ that coincides with $\tilde y$ on Ω_R.

The continuity of this extension operator in classical Sobolev spaces is proved in [15, §2.2] from which we deduce $\|\breve y\|_{H^2(\Omega_{R,h}\setminus\Omega_{R/2})} \le c\|\tilde y\|_{H^2(\Omega_0)}$. As the weights are bounded by a constant within $\Omega_{R,h}\setminus\Omega_{R/2}$ we conclude $D^{\boldsymbol\alpha}\breve y \in W^{1,2}_{\vec\alpha,\vec\delta}(\Omega_{R,h})$ for $|\boldsymbol\alpha| = 1$. For $\breve y$ we can define the Scott-Zhang interpolant in $Y_h(\Omega_{R,h})$ (which is not possible on Ω_R as the mesh does not resolve the boundary of Ω_R). From the Céa-Lemma and the local interpolation error estimates from (17) we conclude using the assumptions on μ

$$\begin{aligned}
\|\tilde y - \tilde y_h\|_{H^1(\Omega_R)} &\le c \inf_{\chi \in Y_h(\Omega_R)} \|\tilde y - \chi\|_{H^1(\Omega_R)} \le c\|\breve y - Z_h\breve y\|_{H^1(\Omega_{R,h})} \\
&\le c h |\breve y|_{W^{2,2}_{\vec\alpha,\vec\delta}(\Omega_{R,h})} \le c h \left(|\breve y|_{W^{2,2}_{\vec\alpha,\vec\delta}(\Omega_{R/2})} + |\breve y|_{H^2(\Omega_{R,h}\setminus\Omega_{R/2})}\right) \\
&\le c h \left(|\tilde y|_{W^{2,2}_{\vec\alpha,\vec\delta}(\Omega_R)} + \|\tilde y\|_{H^2(\Omega_0)}\right).
\end{aligned} \tag{41}$$

Here, we exploited the fact that the weights are of order one within $\Omega \setminus \Omega_{R/2}$ and the continuity of the Calderon extension. Moreover, we confirm the estimate

$$\|\tilde y\|_{H^2(\Omega_0)} \le c\left(\|\tilde y\|_{H^1(\Omega_0)} + |\tilde y|_{W^{2,2}_{\vec\alpha,\vec\delta}(\Omega_0)}\right)$$

which implies together with (41) the assertion for $\ell = 1$.

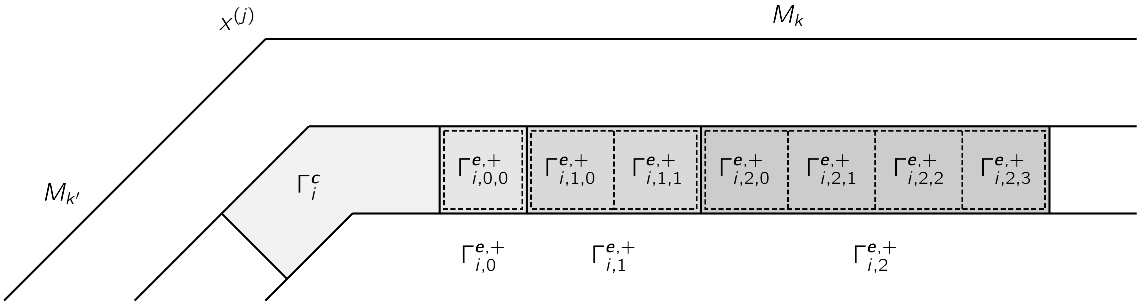


Figure 2: Illustration of the sets introduced in (47).

The estimate in $L^2(\Omega_R)$ is a consequence of the Aubin-Nitsche method using the dual problem

$$-\Delta w + w = \tilde y - \tilde y_h \text{ in } \Omega_R, \qquad \partial_n w = 0 \text{ on } \partial\Omega_R.$$

The estimate (41) is applicable for the error $w - w_h$, with the Ritz-projection $w_h \in Y_h(\Omega_R)$ of w, as well, and the weighted regularity result from Theorem 1 provides the estimate

$$|w|_{W^{2,2}_{\vec\alpha,\vec\delta}(\Omega_R)} \le c\|\tilde y - \tilde y_h\|_{L^2(\Omega_R)}.$$

□

The next step is to show an initial error estimate on a single boundary strip Γ_i. Afterwards we will combine this result to a global estimate.

Lemma 6. *Let $y \in H^1(\Omega) \cap L^\infty(\Omega)$, $\tilde y := \omega y$ with ω from (39), and $\tilde y_h \in Y_h(\Omega_R)$ as in (40). Then, for arbitrary $i \in \{1, \ldots, I\}$ there holds the local estimate*

$$\begin{aligned}
&\|\tilde y - \tilde y_h\|_{L^2(\Gamma_i)} \\
&\quad \le c\Bigg(|\ln h|^2 \sum_{\substack{\boldsymbol{e}:=\boldsymbol{e}_k\\ k\in\mathcal{E}}} \left(\sum_{j=0}^{i} d_i d_{i,j} \|\tilde y - I_h \tilde y\|^2_{L^\infty(\Omega_{i,j}^{\boldsymbol{e},\pm\prime})} + d_i \|\tilde y - I_h\tilde y\|^2_{L^\infty(\tilde\Omega_i^{\boldsymbol{e}\prime})} \right) \\
&\qquad + |\ln h|^2 \sum_{\substack{\boldsymbol{c}:=\boldsymbol{c}_j\\ j\in\mathcal{C}}} d_i^2 \|\tilde y - I_h\tilde y\|^2_{L^\infty(\Omega_i^{\boldsymbol{c}\prime})} + d_i^{-1}\|\tilde y - \tilde y_h\|^2_{L^2(\Omega_i')} \Bigg)^{1/2}.
\end{aligned} \tag{42}$$

Proof. Let us first discuss the estimate in case of $i = 1, \ldots, I-2$. To obtain the desired result on the boundary part $\Gamma_i^{\boldsymbol{c}}$ we apply the Hölder inequality with $|\Gamma_i^{\boldsymbol{c}}| \sim d_i^2$, and a trace theorem (note that $\tilde y - \tilde y_h \in C(\overline\Omega)$). This leads to

$$\|\tilde y - \tilde y_h\|_{L^2(\Gamma_i^{\boldsymbol{c}})} \le d_i \|\tilde y - \tilde y_h\|_{L^\infty(\Gamma_i^{\boldsymbol{c}})} \le d_i \|\tilde y - \tilde y_h\|_{L^\infty(\Omega_i^{\boldsymbol{c}})}. \tag{43}$$

Now we can apply the local maximum norm estimate from Theorem 10.1 and Example 10.1 in [22], which reads in our situation

$$\|\tilde y - \tilde y_h\|_{L^\infty(\Omega_i^{\boldsymbol{c}})} \le c\left(|\ln h| \|\tilde y - I_h\tilde y\|_{L^\infty(\Omega_i^{\boldsymbol{c}\prime})} + d^{-3/2}\|\tilde y - \tilde y_h\|_{L^2(\Omega_i')}\right), \tag{44}$$

with $d := \operatorname{dist}(\partial\Omega_i^{\boldsymbol{c}\prime} \setminus \Gamma,\ \partial\Omega_i^{\boldsymbol{c}} \setminus \Gamma)$. Due to our construction we find that $d \sim d_i$. Inserting (44) into (43) yields (42) for $i = 1, \ldots, I-2$ with $\Gamma_i^{\boldsymbol{c}}$ instead of Γ_i on the left-hand side.

To show the estimate on the part $\Gamma_i^{\boldsymbol{e}}$ we cannot apply this technique directly as the measure of $\Gamma_i^{\boldsymbol{e}}$ is only of order d_i. We would then obtain a worse estimate. One can apply a coordinate transformation with the aim that the edge $\boldsymbol{e}$ coincides with the z-axis, and that $z = 0$ and $z = L$ correspond to the endpoints of $\boldsymbol{e}$. We introduce a further decomposition, namely

$$\begin{aligned}\Omega_{i,j,k}^{\boldsymbol{e},+,(m)} &:= \Big\{x\in\Omega_{i,j}^{\boldsymbol{e},+,(m)}\colon z(x)\in\big((1+A+2^j+k-m)d_i,\\ &\qquad\qquad (2+A+2^j+k+m)d_i\big)\Big\},\\ \Omega_{i,j,k}^{\boldsymbol{e},-,(m)} &:= \Big\{x\in\Omega_{i,j}^{\boldsymbol{e},-,(m)}\colon z(x)\in\big(L-(2+A+2^j+k+m)d_i,\\ &\qquad\qquad L-(1+A+2^j+k-m)d_i\big)\Big\},\end{aligned}\tag{45}$$

for $k=0,\dots,2^j-1$ and $m\in\{0,1\}$. To shorten the notation we write

$$\Omega_{i,j,k}^{\boldsymbol{e},\pm} := \Omega_{i,j,k}^{\boldsymbol{e},\pm,(0)} \qquad\text{and}\qquad \Omega_{i,j,k}^{\boldsymbol{e},\pm\,\prime} := \Omega_{i,j,k}^{\boldsymbol{e},\pm,(1)}.$$

The sets $\{\Omega_{i,j,k}^{\boldsymbol{e},\pm,(m)}\}_{k=0}^{2j-1}$ form a decomposition of $\Omega_{i,j}^{\boldsymbol{e},\pm,(m)}$. Analogously we introduce a decomposition of $\tilde\Omega_i^{\boldsymbol{e},(m)}$, namely

$$\tilde\Omega_{i,k}^{\boldsymbol{e},(m)} := \Big\{x\in\tilde\Omega_i^{\boldsymbol{e},(m)}\colon z(x)\in\big((1+A+2^{i+1}+k-m)d_i,\\ (2+A+2^{i+1}+k+m)d_i\big)\Big\}\tag{46}$$

for $k=0,\dots,K$ with some $K\sim d_i^{-1}$ and $m\in\{0,1\}$. Again, we denote the boundary parts by

$$\Gamma_{i,j,k}^{\boldsymbol{e},\pm} := \partial\Omega_{i,j,k}^{\boldsymbol{e},\pm}\cap\Gamma,\qquad \tilde\Gamma_{i,k}^{\boldsymbol{e}} := \partial\tilde\Omega_{i,k}^{\boldsymbol{e}}\cap\Gamma,\tag{47}$$

which are illustrated in Figure 2, and confirm the desired properties

$$|\Gamma_{i,j,k}^{\boldsymbol{e},\pm}|\sim d_i^2,\qquad |\tilde\Gamma_{i,k}^{\boldsymbol{e}}|\sim d_i^2.\tag{48}$$

Due to this construction we moreover have the properties

$$\operatorname{dist}(\partial\Omega_{i,j,k}^{\boldsymbol{e},\pm\,\prime}\setminus\Gamma,\ \partial\Omega_{i,j,k}^{\boldsymbol{e},\pm}\setminus\Gamma)\sim d_i \quad\text{and}\quad \operatorname{dist}(\partial\tilde\Omega_{i,k}^{\boldsymbol{e}\,\prime}\setminus\Gamma,\ \partial\tilde\Omega_{i,k}^{\boldsymbol{e}}\setminus\Gamma)\sim d_i,\tag{49}$$

which play a role in the local maximum norm estimate (44). Exploiting the decompositions (45) and (46), the Hölder inequality with (48) and a trace theorem leads to

$$\begin{aligned}\|\tilde y-\tilde y_h\|_{L^2(\Gamma_i^{\boldsymbol{e}})}^2 &= \sum_{j=0}^{i}\sum_{k=0}^{2^j-1}\|\tilde y-\tilde y_h\|_{L^2(\Gamma_{i,j,k}^{\boldsymbol{e},\pm})}^2+\sum_{k=0}^{K}\|\tilde y-\tilde y_h\|_{L^2(\tilde\Gamma_{i,k}^{\boldsymbol{e}})}^2\\ &\le cd_i^2\left(\sum_{j=0}^{i}\sum_{k=0}^{2^j-1}\|\tilde y-\tilde y_h\|_{L^\infty(\Omega_{i,j,k}^{\boldsymbol{e},\pm})}^2+\sum_{k=0}^{K}\|\tilde y-\tilde y_h\|_{L^\infty(\tilde\Omega_{i,k}^{\boldsymbol{e}})}^2\right).\end{aligned}$$

Several applications of the local maximum norm estimate (44) with the properties (49) yield

$$\begin{aligned}\|\tilde y-\tilde y_h\|_{L^2(\Gamma_i^{\boldsymbol{e}})}^2 &\le cd_i^2\left(\sum_{j=0}^{i}\sum_{k=0}^{2^j-1}\left(|\ln h|^2\|\tilde y-I_h\tilde y\|_{L^\infty(\Omega_{i,j,k}^{\boldsymbol{e},\pm\,\prime})}^2+d_i^{-3}\|\tilde y-\tilde y_h\|_{L^2(\Omega_{i,j,k}^{\boldsymbol{e},\pm\,\prime})}^2\right)\right.\\ &\qquad\left.+\sum_{k=0}^{K}\left(|\ln h|^2\|\tilde y-I_h\tilde y\|_{L^\infty(\tilde\Omega_{i,k}^{\boldsymbol{e}\,\prime})}^2+d_i^{-3}\|\tilde y-\tilde y_h\|_{L^2(\tilde\Omega_{i,k}^{\boldsymbol{e}\,\prime})}^2\right)\right)\\ &\le c\left(\sum_{j=0}^{i}d_id_{i,j}|\ln h|^2\|\tilde y-I_h\tilde y\|_{L^\infty(\Omega_{i,j}^{\boldsymbol{e},\pm\,\prime})}^2\right.\\ &\qquad\left.+d_i|\ln h|^2\|\tilde y-I_h\tilde y\|_{L^\infty(\tilde\Omega_i^{\boldsymbol{e}\,\prime})}^2+d_i^{-1}\|\tilde y-\tilde y_h\|_{L^2(\Omega_i')}^2\right).\end{aligned}$$

In the last step we exploited that $K \sim d_i^{-1}$ and that $d_i 2^j = d_{i,j}$. From this we obtain the estimate (42) on the subset $\Gamma_i^{\boldsymbol{e}}$.

It remains to show the desired estimates also for $i = I-1, I$ which cannot be shown with the same technique, since the local maximum norm estimate (44) is not applicable if $\Omega_i^{\boldsymbol{c}\prime}$ and $\Omega_i^{\boldsymbol{e}\prime}$ contain the singular points. Therefore, we insert $I_h\tilde y$ as intermediate function and apply the triangle inequality which leads to

$$\|\tilde y - \tilde y_h\|_{L^2(\Gamma_i^{\boldsymbol{c}})} \le c\left(\|\tilde y - I_h\tilde y\|_{L^2(\Gamma_i^{\boldsymbol{c}})} + \|I_h\tilde y - \tilde y_h\|_{L^2(\Gamma_i^{\boldsymbol{c}})}\right). \tag{50}$$

The Hölder inequality with $|\Gamma_i^{\boldsymbol{c}}| \sim d_i^2$, and a trace theorem imply

$$\|\tilde y - I_h\tilde y\|_{L^2(\Gamma_i^{\boldsymbol{c}})} \le c d_i \|\tilde y - I_h\tilde y\|_{L^\infty(\Omega_i^{\boldsymbol{c}})}. \tag{51}$$

To estimate the second part of (50) we consider an arbitrary boundary element $E \in \partial\mathcal{T}_h$ intersecting $\Omega_i^{\boldsymbol{c}}$ and its corresponding tetrahedron $T \in \mathcal{T}_h$, and apply a trace theorem as well as norm equivalences on a reference setting. Thus,

$$\|I_h\tilde y - \tilde y_h\|_{L^2(E)} \le c h_T^{-1/2}\|I_h\tilde y - \tilde y_h\|_{L^2(T)}. \tag{52}$$

and due to $h_T^{-1} \le h^{-1/\mu} \sim d_i^{-1}$ for all $T \cap \Omega_i^{\boldsymbol{c}\prime} \ne \emptyset$, as well as $|\Omega_i^{\boldsymbol{c}}| \sim d_i^3$, we get

$$\begin{aligned}\|I_h\tilde y - \tilde y_h\|_{L^2(\Gamma_i^{\boldsymbol{c}})} &\le c d_i^{-1/2}\|I_h\tilde y - \tilde y_h\|_{L^2(\Omega_i^{\boldsymbol{c}\prime})}\\ &\le c\left(d_i\|\tilde y - I_h\tilde y\|_{L^\infty(\Omega_i^{\boldsymbol{c}\prime})} + d_i^{-1/2}\|\tilde y - \tilde y_h\|_{L^2(\Omega_i^{\boldsymbol{c}\prime})}\right).\end{aligned}$$

This estimate together with (51) and (50) yields (42) on $\Gamma_i^{\boldsymbol{c}}$ for $i = I-1, I$.

On $\Gamma_i^{\boldsymbol{e}}$ we use again the decomposition (15), the triangle inequality, and the Hölder inequality with (15) to arrive at

$$\begin{aligned}\|\tilde y - \tilde y_h\|^2_{L^2(\Gamma_i^{\boldsymbol{e}})} &\le \sum_{j=0}^{i}\left(\|\tilde y - I_h\tilde y\|^2_{L^2(\Gamma_{i,j}^{\boldsymbol{e},\pm})} + \|I_h\tilde y - \tilde y_h\|^2_{L^2(\Gamma_{i,j}^{\boldsymbol{e},\pm})}\right)\\ &\quad + \|\tilde y - I_h\tilde y\|^2_{L^2(\tilde\Gamma_i^{\boldsymbol{e}})} + \|I_h\tilde y - \tilde y_h\|^2_{L^2(\tilde\Gamma_i^{\boldsymbol{e}})}\\ &\le \sum_{j=0}^{i}\left(d_i d_{i,j}\|\tilde y - I_h\tilde y\|^2_{L^\infty(\Omega_{i,j}^{\boldsymbol{e},\pm\prime})} + \|I_h\tilde y - \tilde y_h\|^2_{L^2(\Gamma_{i,j}^{\boldsymbol{e},\pm})}\right)\\ &\quad + d_i\|\tilde y - I_h\tilde y\|^2_{L^\infty(\tilde\Omega_i^{\boldsymbol{e}\prime})} + \|I_h\tilde y - \tilde y_h\|^2_{L^2(\tilde\Gamma_i^{\boldsymbol{e}})}.\end{aligned} \tag{53}$$

From (52) and $|\Omega_{i,j}^{\boldsymbol{e},\pm\prime}| \sim d_i^2 d_{i,j}$ we obtain

$$\begin{aligned}\|I_h\tilde y - \tilde y_h\|_{L^2(\Gamma_{i,j}^{\boldsymbol{e},\pm})} &\le d_i^{-1/2}\|I_h\tilde y - \tilde y_h\|_{L^2(\Omega_{i,j}^{\boldsymbol{e},\pm\prime})}\\ &\le d_i^{1/2} d_{i,j}^{1/2}\|\tilde y - I_h\tilde y\|_{L^\infty(\Omega_{i,j}^{\boldsymbol{e},\pm\prime})} + d_i^{-1/2}\|\tilde y - \tilde y_h\|_{L^2(\Omega_{i,j}^{\boldsymbol{e},\pm\prime})},\end{aligned}$$

and with the same arguments using $|\tilde\Omega_i^{\boldsymbol{e}\prime}| \sim d_i^2$

$$\|I_h\tilde y - \tilde y_h\|_{L^2(\tilde\Gamma_i^{\boldsymbol{e}})} \le d_i^{1/2}\|\tilde y - I_h\tilde y\|_{L^\infty(\tilde\Omega_i^{\boldsymbol{e}\prime})} + d_i^{-1/2}\|\tilde y - \tilde y_h\|_{L^2(\tilde\Omega_i^{\boldsymbol{e}\prime})}.$$

From these estimates and (53) we finally conclude (42) in case of $i = I-1, I$. □

The next step of the proof is to derive a finite element error estimate on the boundary part $\Gamma_{R/2}$ defined in (12) which is under influence of corner and edge singularities.

Theorem 7. *Let $\tilde y := \omega y \in H^1(\Omega_R)$ with ω defined as in (39). Assume that $D^{\boldsymbol{\alpha}} y \in W^{1,2}_{\vec\alpha,\vec\delta}(\Omega) \cap W^{1,\infty}_{\vec\beta,\vec\varrho}(\Omega)$ for $|\boldsymbol{\alpha}| = 1$ with weight vectors $\vec\alpha \in [0,1)^{d'}$, $\vec\beta \in [0,2)$, $\vec\delta \in [0,2/3)^d$, $\vec\varrho \in [0,5/3)^d$. The refinement parameter μ satisfies the inequalities*

$$\begin{aligned}\alpha_j &\le 1-\mu, &\quad \beta_j &\le 3-2\mu, &\quad &\forall j \in \mathcal{C},\\ \delta_k &\le 1-\mu, &\quad \varrho_k &\le \frac{5}{2}-2\mu, &\quad &\forall k \in \mathcal{E}.\end{aligned} \tag{54}$$

Then, the Ritz projection $\tilde y_h \in Y_h(\Omega_R)$ from (40) fulfills the estimate

$$\|\tilde y - \tilde y_h\|_{L^2(\Gamma_{R/2})} \le ch^2 |\ln h|^{3/2} \left(|\tilde y|_{W^{2,2}_{\vec\alpha,\vec\delta}(\Omega_R)} + \|\tilde y\|_{H^1(\Omega_0)} + |\tilde y|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\Omega_R)} \right).$$

Proof. We consider the decomposition of the boundary $\overline\Gamma_{R/2} = \overline\Gamma_1 \cup \ldots \cup \overline\Gamma_I$ into the segments introduced in (13) and estimate the terms on the right-hand side of the estimate from Lemma 6. First, the terms involving the interpolation error, more precisely

$$E_i := \sum_{\substack{\boldsymbol{e}:=\boldsymbol{e}_k\\ k\in\mathcal{E}}} \left(\sum_{j=0}^{i} d_i d_{i,j} \|\tilde y - I_h \tilde y\|^2_{L^\infty(\Omega^{\boldsymbol{e},\pm\prime}_{i,j})} + d_i \|\tilde y - I_h\tilde y\|^2_{L^\infty(\tilde\Omega^{\boldsymbol{e}\prime}_i)} \right) + \sum_{\substack{\boldsymbol{c}:=\boldsymbol{c}_j\\ j\in\mathcal{C}}} d_i^2 \|\tilde y - I_h\tilde y\|^2_{L^\infty(\Omega^{\boldsymbol{c}\prime}_i)}$$

are discussed. Inserting the local estimates from Lemma 4 yields for $i = 1, \ldots, I-3$

$$E_i \le ch^4 \Bigg(\sum_{\substack{\boldsymbol{e}:=\boldsymbol{e}_k\\ k\in\mathcal{E}}} d_i^{2(5/2-2\mu-\varrho_k)} \left(\sum_{j=0}^{i} d_{i,j}^{2(1/2+\varrho_k-\tilde\beta_k)} |\tilde y|^2_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\Omega^{\boldsymbol{e},\pm\prime\prime}_{i,j})} + |\tilde y|^2_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\tilde\Omega^{\boldsymbol{e}\prime\prime}_i)} \right) + \sum_{\substack{\boldsymbol{c}:=\boldsymbol{c}_j\\ j\in\mathcal{C}}} d_i^{2(3-2\mu-\kappa_j)} |\tilde y|^2_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\Omega^{\boldsymbol{c}\prime\prime}_i)} \Bigg) \le ch^4 |\tilde y|^2_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\Omega''_i)}, \tag{55}$$

where we used the refinement condition (54) as well as (24) with $s_k = 1/2 + \varrho_k - \tilde\beta_k$ in the last step. In case of $i = I-2, \ldots, I$ we obtain with Lemma 4

$$E_i \le c \left(\sum_{\substack{\boldsymbol{e}:=\boldsymbol{e}_k\\ k\in\mathcal{E}}} h^{2(5/2-\varrho_k+[1/2+\varrho_k-\tilde\beta_k]_-)/\mu} |\tilde y|^2_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\Omega^{\boldsymbol{e}\prime\prime}_i)} + \sum_{\substack{\boldsymbol{c}:=\boldsymbol{c}_j\\ j\in\mathcal{C}}} h^{2(3-\kappa_j)/\mu} |\tilde y|^2_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\Omega^{\boldsymbol{c}\prime\prime}_i)} \right) \le ch^4 |\tilde y|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\Omega''_i)}. \tag{56}$$

Inserting the estimates (55) and (56) into (42) and summing up over all Γ_i for $i = 1, \ldots, I$ yields with $I \sim |\ln h|$ the estimate

$$\|\tilde y - \tilde y_h\|^2_{L^2(\Gamma_{R/2})} \le c \left(|\ln h|^3 h^4 |\tilde y|^2_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\Omega_R)} + \|\gamma^{-1/2}(\tilde y - \tilde y_h)\|^2_{L^2(\Omega_R)} \right), \tag{57}$$

where $\gamma(x) := d_I + r(x)$. Note, that there holds $\gamma(x) \ge d_i = 2d_{i-1}$ if $x \in \Omega_i$.

In the remainder of the proof we will discuss the second term on the right-hand side of (57) which requires an estimate for a weighted $L^2(\Omega_R)$-error. Therefore, we adopt the technique that was applied in the proof of Lemma 6.2 in [19] where a duality argument was used. First we decompose the error into

$$\|\gamma^{-1/2}(\tilde y - \tilde y_h)\|_{L^2(\Omega_R)} \le \|\gamma^{-1/2}(\tilde y - \tilde y_h)\|_{L^2(\Omega_{R/4})} + \|\gamma^{-1/2}(\tilde y - \tilde y_h)\|_{L^2(\Omega_0 \cup \Omega_1)}. \tag{58}$$

On the outermost subdomain $\Omega_0 \cup \Omega_1$ we exploit that $\gamma \sim 1$ and can directly use the global finite element error estimate from Lemma 5. As a consequence we get

$$\|\gamma^{-1/2}(\tilde y - \tilde y_h)\|_{L^2(\Omega_0\cup\Omega_1)} \le c\|\tilde y - \tilde y_h\|_{L^2(\Omega_R)} \le ch^2 \left(|\tilde y|_{W^{2,2}_{\vec\alpha,\vec\delta}(\Omega_R)} + \|\tilde y\|_{H^1(\Omega_0)} \right). \tag{59}$$

For the innermost rings we apply the representation

$$\|\gamma^{-1/2}(\tilde y - \tilde y_h)\|_{L^2(\Omega_{R/4})} = \sup_{\substack{g\in C_0^\infty(\Omega_{R/4})\\ \|g\|_{L^2(\Omega_{R/4})}=1}} (\gamma^{-1/2}(\tilde y - \tilde y_h), g) \tag{60}$$

and consider the auxiliary problem

$$-\Delta w + w = \gamma^{-1/2} g \quad \text{in} \quad \Omega_R, \qquad \partial_n w = 0 \quad \text{on} \quad \partial\Omega_R. \tag{61}$$

From the weak formulation of (61) we can deduce

$$(\gamma^{-1/2}(\tilde y - \tilde y_h), g) = (\tilde y - \tilde y_h, \gamma^{-1/2} g) = a_{\Omega_R}(\tilde y - \tilde y_h, w), \tag{62}$$

Analogous to Lemma 5 we define the Scott-Zhang interpolant of the Calerdon extension of w, namely $[Z_h\breve w]|_{\Omega_R} \in Y_h(\Omega_R)$ and obtain

$$\begin{aligned} a_{\Omega_R}(\tilde y - \tilde y_h, w) &= a_{\Omega_R}(\tilde y - \tilde y_h, w - Z_h\breve w) \\ &\le c \sum_{i=0}^{I} \Bigg(\sum_{\substack{\boldsymbol{c}:=\boldsymbol{c}_j \\ j\in\mathcal{C}}} \|\tilde y - \tilde y_h\|_{H^1(\Omega_i^{\boldsymbol{c}})} \|w - Z_h\breve w\|_{H^1(\Omega_i^{\boldsymbol{c}})} \\ &\qquad + \sum_{\substack{\boldsymbol{e}:=\boldsymbol{e}_k \\ k\in\mathcal{E}}} \|\tilde y - \tilde y_h\|_{H^1(\Omega_i^{\boldsymbol{e}})} \|w - Z_h\breve w\|_{H^1(\Omega_i^{\boldsymbol{e}})} \Bigg). \end{aligned} \tag{63}$$

First, we insert the local finite element error estimate from Corollary 9.1 in [22], which reads in our situation

$$\|\tilde y - \tilde y_h\|_{H^1(\Omega_i^{\boldsymbol{c}})} \le c \left(|\tilde y - Z_h \tilde y|_{H^1(\Omega_i^{\boldsymbol{c}\prime})} + d_i^{-1} \|\tilde y - Z_h\tilde y\|_{L^2(\Omega_i^{\boldsymbol{c}\prime})} + d_i^{-1} \|\tilde y - \tilde y_h\|_{L^2(\Omega_i^{\boldsymbol{c}\prime})} \right). \tag{64}$$

The estimate remains true when replacing $\boldsymbol{c}$ by $\boldsymbol{e}$.

In order to derive estimates for the terms on the right-hand side of (63) we consider the cases $i = 3, \dots, I-3$ and $i = I-2, \dots, I$ as well as $i = 0, 1, 2$ separately.

In case of $i = 3, \dots, I-3$, we obtain with the local estimates from Lemma 3 and (64)

$$\begin{aligned} \|\tilde y - \tilde y_h\|_{H^1(\Omega_i^{\boldsymbol{c}})} &\le c \left(h d_i^{5/2-\mu-\kappa_j} |\tilde y|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\Omega_i'')} + d_i^{-1} \|\tilde y - \tilde y_h\|_{L^2(\Omega_i')} \right), \\ \|w - Z_h\breve w\|_{H^1(\Omega_i^{\boldsymbol{c}})} &\le c h d_i^{1/2-\mu} |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')}, \end{aligned}$$

where we also exploited $hd_i^{-\mu} \le hd_I^{-\mu} = c_I^{-\mu} \le 1$ to simplify the interpolation error estimate in $L^2(\Omega_i^{\boldsymbol{c}})$. Combining both estimates yields for $i = 3, \dots, I-3$

$$\begin{aligned} &\|\tilde y - \tilde y_h\|_{H^1(\Omega_i^{\boldsymbol{c}})} \|w - Z_h\breve w\|_{H^1(\Omega_i^{\boldsymbol{c}})} \\ &\le c \left(h^2 d_i^{3-2\mu-\kappa_j} |\tilde y|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\Omega_i'')} + h d_i^{-1/2-\mu} \|\tilde y - \tilde y_h\|_{L^2(\Omega_i')} \right) |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')} \\ &\le c \left(h^2 |\tilde y|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\Omega_i'')} + c_I^{-\mu} \|\gamma^{-1/2}(\tilde y - \tilde y_h)\|_{L^2(\Omega_i')} \right) |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')}. \end{aligned} \tag{65}$$

The last step is a consequence of the assumption upon μ and the definition of the domains Ω_i, more precisely we exploited $d_i^{-\mu} \le d_I^{-\mu} \le c_I^{-\mu} h^{-1}$.

In case of $i = I-2, \dots, I$ we get analogously

$$\begin{aligned} \|\tilde y - \tilde y_h\|_{H^1(\Omega_i^{\boldsymbol{c}})} &\le c \left(h^{(5/2-\kappa_j)/\mu} |\tilde y|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\Omega_i'')} + d_i^{-1} \|\tilde y - \tilde y_h\|_{L^2(\Omega_i')} \right), \\ \|w - Z_h\breve w\|_{H^1(\Omega_i^{\boldsymbol{c}})} &\le c c_I^{[1/2-\mu]_+} h^{1/(2\mu)} |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')}. \end{aligned}$$

Combining both estimates leads to

$$\begin{aligned} &\|\tilde y - \tilde y_h\|_{H^1(\Omega_i^{\boldsymbol{c}})} \|w - Z_h\breve w\|_{H^1(\Omega_i^{\boldsymbol{c}})} \\ &\le c \left(h^{(3-\kappa_j)/\mu} |\tilde y|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\Omega_i'')} + c_I^{[1/2-\mu]_+} h^{1/(2\mu)} d_I^{-1} \|\tilde y - \tilde y_h\|_{L^2(\Omega_i')} \right) |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')} \\ &\le c \left(h^2 |\tilde y|_{W^{2,\infty}_{\vec\beta,\vec\varrho}(\Omega_i'')} + c_I^{\max\{-1/2,-\mu\}} \|\gamma^{-1/2}(\tilde y - \tilde y_h)\|_{L^2(\Omega_i')} \right) |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')}. \end{aligned} \tag{66}$$

The last step follows from the assumption upon μ and the fact that $d_I = c_I h^{1/\mu}$. For $i = 0, 1, 2$ we can insert the global finite element error estimate from Lemma 5 and the interpolation error estimate from Lemma 3 taking into account that the factors d_0, d_1 and d_2 are of order one. With the continuity property of the Calderon extension $\breve{w}$ we get

$$\begin{aligned}\|\tilde{y} - \tilde{y}_h\|_{H^1(\Omega_1^{\boldsymbol{c}\prime})} \|w - Z_h \breve{w}\|_{H^1(\Omega_1^{\boldsymbol{c}\prime})} &\le ch^2 \left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} + \|\tilde{y}\|_{H^1(\Omega_0)} \right) |\breve{w}|_{H^2(\Omega_1'' \cup (\Omega_{R,h} \setminus \Omega_{R/2}))} \\ &\le ch^2 \left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} + \|\tilde{y}\|_{H^1(\Omega_0)} \right) \left(|w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_R)} + \|w\|_{H^1(\Omega_0)} \right). \end{aligned} \tag{67}$$

We can repeat the same strategy to show the appropriate estimates on $\Omega_i^{\boldsymbol{e}}$, and apply Lemma 3 with $s_k = 1/2 + \varrho_k - \tilde{\beta}_k$, as well as (64) with $\boldsymbol{c}$ replaced by $\boldsymbol{e}$. Moreover, we have to exploit the refinement condition

$$2\mu \le 5/2 - \varrho_k + [s_k]_- = \begin{cases} 5/2 - \varrho_k, & \text{if } s_k \ge 0, \\ 3 - \tilde{\beta}_k, & \text{if } s_k < 0, \end{cases}$$

which follows from (54). Consequently, we arrive at

$$\begin{aligned} &\|\tilde{y} - \tilde{y}_h\|_{H^1(\Omega_i^{\boldsymbol{e}})} \|w - Z_h \breve{w}\|_{H^1(\Omega_i^{\boldsymbol{e}})} \\ &\le c \left(h^2 |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_i'')} + c_I^{\max\{-1/2,-\mu\}} \|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\Omega_i')} \right) |w|_{W^{2,2}_{\boldsymbol{1/2},\boldsymbol{1/2}}(\Omega_i')}, \end{aligned} \tag{68}$$

for $i = 3, \ldots, I$. Finally, we easily confirm that the estimate (67) remains true when replacing $\boldsymbol{c}$ by $\boldsymbol{e}$, and we have covered also the cases $i = 0, 1, 2$.

We may now insert the estimates (65), (66), (67) and (68) into (63) which implies

$$\begin{aligned} &a_{\Omega_R}(\tilde{y} - \tilde{y}_h, w) \\ &\le c \sum_{i=3}^{I} \left(h^2 |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_i'')} + c_I^{\max\{-1/2,-\mu\}} \|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\Omega_i')} \right) |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')} \\ &\quad + ch^2 \left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} + \|\tilde{y}\|_{H^1(\Omega_0)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_R)} \right) \left(|w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_R)} + \|w\|_{H^1(\Omega_0)} \right) \\ &\le ch^2 |\ln h|^{1/2} \left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} + \|\tilde{y}\|_{H^1(\Omega_0)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_R)} \right) \\ &\quad + cc_I^{\max\{-1/2,-\mu\}} \|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\Omega_{R/4})}. \end{aligned} \tag{69}$$

In the last step we used $I \sim |\ln h|$ und inserted the a-priori estimate

$$|w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_R)} + \|w\|_{H^1(\Omega_R)} \le c\|g\|_{L^2(\Omega_R)} = c$$

shown already in [3, Theorem 4.8].

Inserting now (69) into (62) yields together with (60)

$$\begin{aligned} \|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\Omega_{R/4})} &\le ch^2 |\ln h|^{1/2} \left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} + \|\tilde{y}\|_{H^1(\Omega_0)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_R)} \right) \\ &\quad + cc_I^{\max\{-1/2,-\mu\}} \|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\Omega_{R/4})}. \end{aligned} \tag{70}$$

The desired result follows from a kick-back argument. Therefore, we choose c_I sufficiently large such that $cc_I^{\max\{-1/2,-\mu\}} \le 1/2$, and hence, the second term on the right-hand side of (70) can be neglected. Finally, we insert (70) together with (59) into (58), insert the resulting estimate into (57), and arrive at the assertion. □

Now we are able to prove the main result of this section.

Theorem 8. *Let y denote the weak solution of (6) and y_h its finite element approximation (10), with input data satisfying $f \in C^{0,\sigma}(\overline{\Omega})$ for some $\sigma \in (0,1)$, and $g \equiv 0$. Assume that $\{\mathcal{T}_h\}_{h>0}$ is a family of locally refined triangulations according to condition (11). Moreover, let be given weights $\vec{\alpha}, \vec{\beta} \in \mathbb{R}^{d'}_+$ and $\vec{\delta}, \vec{\varrho} \in \mathbb{R}^d_+$ satisfying*

$$\begin{aligned} &\frac{1}{2} - \lambda^{\boldsymbol{c}_j} < \alpha_j \le 1 - \mu, && 2 - \lambda^{\boldsymbol{c}_j} < \beta_j \le 3 - 2\mu, && \forall j \in \mathcal{C}, \\ &1 - \lambda^{\boldsymbol{e}_k} < \delta_k \le 1 - \mu, && 2 - \lambda^{\boldsymbol{e}_k} < \varrho_k \le \frac{5}{2} - 2\mu, && \forall k \in \mathcal{E}. \end{aligned} \tag{71}$$

Then, some $c > 0$ exists such that

$$\|y - y_h\|_{L^2(\Gamma)} \le ch^2 |\ln h|^{3/2} \left(\sum_{|\boldsymbol{\alpha}|=1} \left(\|D^{\boldsymbol{\alpha}} y\|_{W^{1,2}_{\vec{\alpha},\vec{\delta}}(\Omega)} + \|D^{\boldsymbol{\alpha}} y\|_{W^{1,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega)} \right) + \|y\|_{L^\infty(\Omega)} \right). \tag{72}$$

Proof. Let ω be the cut-off function defined in (39). In order to apply Theorem 7 we insert the intermediate function $\tilde{y}_h$ from (40) and exploit that $\tilde{y} := \omega y$ coincides with y in $\Omega_{R/2}$. This leads to

$$\|y - y_h\|_{L^2(\Gamma_{R/4})} \le c \left(\|\tilde{y} - \tilde{y}_h\|_{L^2(\Gamma_{R/2})} + \|\tilde{y}_h - y_h\|_{L^2(\Gamma_{R/4})} \right). \tag{73}$$

For the first part we may now apply the result of Theorem 7 and obtain

$$\|\tilde{y} - \tilde{y}_h\|_{L^2(\Gamma_{R/2})} \le ch^2 |\ln h|^{3/2} \left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} + \|\tilde{y}\|_{H^1(\Omega_0)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_R)} \right). \tag{74}$$

Note that it is possible to construct a cut-off function ω satisfying (39) and

$$\|D^{\boldsymbol{\alpha}} \omega\|_{L^\infty(\Omega_R)} \le 2^{|\boldsymbol{\alpha}|} \le c = c(|\boldsymbol{\alpha}|) \qquad \forall \boldsymbol{\alpha} \in \mathbb{N}_0^3.$$

Using the Leibniz rule we then get

$$\begin{aligned} |\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} = |\omega y|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} &\le c \left(|y|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} + \|y\|_{W^{1,2}(\Omega \setminus \Omega_{R/2})} \right) \\ &\le c \left(\sum_{|\boldsymbol{\alpha}|=1} \|D^{\boldsymbol{\alpha}} y\|_{W^{1,2}_{\vec{\alpha},\vec{\delta}}(\Omega)} + \|y\|_{L^2(\Omega)} \right), \end{aligned} \tag{75}$$

and analogously

$$|\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_R)} \le c \left(\sum_{|\boldsymbol{\alpha}|=1} \|D^{\boldsymbol{\alpha}} y\|_{W^{1,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega)} + \|y\|_{L^\infty(\Omega)} \right). \tag{76}$$

Let us discuss the second part of (73). The function $\tilde{y}_h - y_h$ is discrete harmonic on $\Omega_{R/2}$. Hence, the discrete Caccioppoli estimate from Lemma 3.3 in [8] yields

$$\|\tilde{y}_h - y_h\|_{H^1(\Omega_{R/4})} \le cd^{-1} \|\tilde{y}_h - y_h\|_{L^2(\Omega_{R/2})}, \qquad d := \operatorname{dist}(\partial\Omega_{R/2} \setminus \Gamma, \partial\Omega_{R/4} \setminus \Gamma), \tag{77}$$

and with our construction we have $d = 1/4$. With a trace theorem and (77) we then obtain

$$\begin{aligned} \|\tilde{y}_h - y_h\|_{L^2(\Gamma_{R/4})} &\le c\|\tilde{y}_h - y_h\|_{H^1(\Omega_{R/4})} \le c\|\tilde{y}_h - y_h\|_{L^2(\Omega_{R/2})} \\ &\le c \left(\|\tilde{y} - \tilde{y}_h\|_{L^2(\Omega_R)} + \|y - y_h\|_{L^2(\Omega)} \right), \end{aligned}$$

where the last step holds due to $y = \tilde{y}$ on $\Omega_{R/2}$. Then, Lemma 5 and Theorem 2 imply

$$\|\tilde{y}_h - y_h\|_{L^2(\Gamma_{R/4})} \le ch^2 \left(\sum_{|\boldsymbol{\alpha}|=1} \|D^{\boldsymbol{\alpha}} y\|_{W^{1,2}_{\vec{\alpha},\vec{\delta}}(\Omega)} + \|y\|_{L^2(\Omega)} \right), \tag{78}$$

where we also applied the estimate (75). Consequently we get from (73) the estimate

$$\begin{aligned}&\|y-y_h\|_{L^2(\Gamma_{R/4})}\\&\quad\le ch^2|\ln h|^{3/2}\left(\sum_{|\boldsymbol{\alpha}|=1}\left(\|D^{\boldsymbol{\alpha}}y\|_{W^{1,2}_{\vec\alpha,\vec\delta}(\Omega)}+\|D^{\boldsymbol{\alpha}}y\|_{W^{1,\infty}_{\vec\beta,\vec\varrho}(\Omega)}\right)+\|y\|_{L^\infty(\Omega)}\right).\end{aligned}\tag{79}$$

In the interior of the boundary we directly apply the trace theorem in the L^∞-norm and use the local estimate (44) to arrive at

$$\begin{aligned}\|y-y_h\|_{L^2(\Gamma\setminus\Gamma_{R/4})}&\le c\|y-y_h\|_{L^\infty(\Gamma\setminus\Gamma_{R/4})}\le c\|y-y_h\|_{L^\infty(\Omega\setminus\Omega_{R/4})}\\&\le c\left(|\ln h|\|y-I_hy\|_{L^\infty(\Omega\setminus\Omega_{R/8})}+\|y-y_h\|_{L^2(\Omega\setminus\Omega_{R/8})}\right)\\&\le c\left(|\ln h|h^2|y|_{W^{2,\infty}(\Omega\setminus\Omega_{R/16})}+h^2|y|_{W^{2,2}_{\vec\alpha,\vec\delta}(\Omega)}\right),\end{aligned}\tag{80}$$

where the last step is a consequence of a standard interpolation error estimate and the global finite element error estimate from Theorem 2. From (79) and (80) we finally conclude the desired estimate. □

Remark 9. *The assumption* (71) *is always true for the choice* $1/3<\mu<1/4+\lambda/2$ *with* λ *defined in* (7)*, as it is always possible to find weights satisfying the inequalities. A possible choice would be*

$$\begin{aligned}\alpha_j&=\max\{0,1/2-\lambda^{\boldsymbol{c}_j}+\varepsilon\}, & \beta_j&=\max\{0,2-\lambda^{\boldsymbol{c}_j}+\varepsilon\},\\ \delta_k&=\max\{0,1-\lambda^{\boldsymbol{e}_k}+\varepsilon\}, & \varrho_k&=\max\{0,2-\lambda^{\boldsymbol{e}_k}+\varepsilon\},\end{aligned}$$

with sufficiently small $\varepsilon>0$.

4 Error estimates for the optimal control problem

4.1 Optimality conditions and regularity results

Let us recall the optimal control problem (1)–(3). The state equation is linear and uniquely solvable which allows us to introduce the linear and bounded operator $S\colon L^2(\Gamma)\to L^2(\Omega)$ as the mapping $u\mapsto Su:=y$, where y is the solution of (2). The optimization problem is then equivalent to

$$j(u):=J(Su,u)\to\min!\quad\text{s.t.}\quad u\in U_{ad}.\tag{81}$$

It is already well-known [21] that this problem possesses a unique solution $u\in U_{ad}$ which satisfies the following optimality system:

Lemma 10. *Let* $(y,u)\in H^1(\Omega)\times L^2(\Gamma)$ *denote the unique solution of the optimal control problem* (1)–(3)*. Then there exists a function* $p\in H^1(\Omega)$ *which fulfils the system*

$$\begin{aligned}-\Delta y+y&=0 & -\Delta p+p&=y-y_d &&\text{in }\Omega,\\ \partial_n y&=u & \partial_n p&=0 &&\text{on }\Gamma,\end{aligned}$$

$$(p+\alpha u,w-u)_\Gamma\ge0\qquad\forall w\in U_{ad}.\tag{82}$$

The variational inequality is equivalent to the projection formula

$$u=\Pi_{ad}\left(-\frac{1}{\alpha}p|_\Gamma\right),\tag{83}$$

where the operator $\Pi_{ad}\colon L^2(\Gamma)\to U_{ad}$ *denotes the* $L^2(\Gamma)$*-projection onto* U_{ad}.

Due to the convexity of the optimization problem this is also a sufficient optimality condition. Using the solution operator $P\colon L^2(\Omega) \to H^1(\Omega)$ of the adjoint equation we may write $p = P(y - y_d)$. It is easy to confirm that the adjoint of the control-to-state operator can be represented as $S^* := \tau \circ P$ (where τ is the trace operator), which implies $p|_\Gamma = S^*(y - y_d)$.

The optimality system presented in Lemma 10 can be solved by a finite element approximation. While the state and the adjoint state are discretized by piecewise linear finite elements, see (9), the control is sought in the space

$$U_h := \{w_h \in L^\infty(\Gamma)\colon w_h|_E \in \mathcal{P}_0 \quad \forall E \in \partial\mathcal{T}_h\}. \tag{84}$$

The fully discrete optimality system reads
Find $(y_h, u_h, p_h) \in Y_h \times (U_h \cap U_{ad}) \times Y_h$*:*

$$\begin{cases} a(y_h, v_h) &= (u_h, v_h)_\Gamma & \forall v_h \in Y_h, \\ a(v_h, p_h) &= (y_h - y_d, v_h)_\Omega & \forall v_h \in Y_h, \\ (p_h + \alpha u_h, w_h - u_h)_\Gamma &\geq 0 & \forall w_h \in U_h \cap U_{ad}. \end{cases} \tag{85}$$

The discrete control-to-state operator $S_h\colon L^2(\Gamma) \to Y_h$ is the solution operator of the first equation in (85).

Due to the polynomial degree used for the control approximation the convergence rate is limited by one [9], i. e., with some constant $c > 0$ there holds

$$\|u - u_h\|_{L^2(\Gamma)} \leq ch.$$

In [24, Theorem 4.2.1] it has been shown that this convergence rate is achieved for arbitrary polyhedral domains as $u \in H^1(\Gamma)$. However, we will see later that the control is even more regular, meaning in some weighted $H^2(\Gamma)$ space, except in the vicinity of those points where the control transitions into the active set. This motivates the use of a linear control approximation which can be simply realized in a postprocessing step without additional computational effort by an application of the projection formula

$$u_h^* := \Pi_{ad}\left(-\frac{1}{\alpha} p_h\right). \tag{86}$$

Note that u_h^* is piecewise linear but in general not in the trace space of Y_h. In the remainder of this section we show that u_h^* is an approximation of the optimal control which converges with rate 2 (up to logarithmic factors) if either the singularities are weak enough or the sequence of meshes is refined appropriately.

The challenging part is the proof of an error estimate for the discrete state in the $L^2(\Omega)$-norm. Once such a result is established an estimate for the control follows from boundedness properties of the solution operators for the state and adjoint equation, Lipschitz properties of the projection formula, and the finite element error estimates shown in the previous section. We basically follow the idea of Meyer/Rösch [14] who propose a decomposition of the discretization error of the state variable by means of

$$\|Su - S_h u_h\|_{L^2(\Omega)} \leq \|(S - S_h)u\|_{L^2(\Omega)} + \|S_h(u - R_h u)\|_{L^2(\Omega)} + \|S_h(R_h u - u_h)\|_{L^2(\Omega)}, \tag{87}$$

where $R_h\colon C(\Gamma) \to U_h$ denotes the midpoint interpolant defined by $[R_h u]|_E = u(x_E)$ for all $E \in \partial\mathcal{T}_h$, when $x_E \in E$ is the barycenter of $E \in \partial\mathcal{T}_h$. The first term can be bounded by using Theorem 2. The latter two terms on the right-hand side are discussed in the following. However, in order to obtain optimal error estimates for these terms a structural assumption upon the active set is necessary:

Assumption 1. *Let* $\mathcal{A}^- := \{x \in \Gamma\colon u(x) = u_a\}$*,* $\mathcal{A}^+ := \{x \in \Gamma\colon u(x) = u_b\}$*, and* $\mathcal{I} := \{x \in \Gamma\colon u(x) \in (u_a, u_b)\}$*. It is assumed that the set* $g := (\overline{\mathcal{A}^+} \cup \overline{\mathcal{A}^-}) \cap \overline{\mathcal{I}}$ *consists of a finite number of curves having finite length.*

In all contributions of which we are aware about estimates for the state in $L^2(\Omega)$ when a full discretization is used, similar assumptions are demanded. To achieve the convergence rate two in the second term of (87) $H^2(\Gamma)$-regularity of the control is required (in the sense of weighted spaces). In the vicinity of g this is, as a rule, not the case as the control could have a kink along g. Hence, only linear convergence can be shown at elements intersecting these lines, but Assumption 1 allows us to retain global quadratic convergence as well. Therefore, in [2, 12, 14] the assumption $|\cup\{E \in \partial\mathcal{T}_h\colon E\cap g\}| \le ch$ is demanded which would directly follow from our assumption. However, our assumption allows us to conclude even a sharper relation in the subsets where the mesh is refined locally.

In the following we decompose the boundary triangulation $\partial\mathcal{T}_h$ into two sets

$$\mathcal{K}_1 := \cup\overline{\{E \in \partial\mathcal{T}_h\colon E \cap g \neq \emptyset\}}, \qquad \mathcal{K}_2 := \Gamma \setminus \mathcal{K}_1.$$

Finally, we can show the following regularity result as consequence of some applications of Theorem 1 in a bootstrapping fashion.

Theorem 11. *Assume that $y_d \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$. Let $\varepsilon > 0$ be a sufficiently small real number, and let $\vec\alpha, \vec\beta, \vec\gamma \in \mathbb{R}^{d'}$ and $\vec\delta, \vec\varrho, \vec\tau \in \mathbb{R}^d$ be weight vectors defined by*

$$\begin{aligned} \alpha_j &:= \max\{0, \frac{1}{2} - \lambda^{\mathbf{c}_j} + \varepsilon\}, & \delta_k &:= \max\{0, 1 - \lambda^{\mathbf{e}_k} + \varepsilon\}, \\ \beta_j &:= \max\{0, 2 - \lambda^{\mathbf{c}_j} + \varepsilon\}, & \varrho_k &:= \max\{0, 2 - \lambda^{\mathbf{e}_k} + \varepsilon\}, \\ \gamma_j &:= \max\{0, 1 - \lambda^{\mathbf{c}_j} + \varepsilon\}, & \tau_k &:= \max\{0, \frac{3}{2} - \lambda^{\mathbf{e}_k} + \varepsilon\}, \end{aligned}$$

for all $j \in \mathcal{C}$ and $k \in \mathcal{E}$. Then, the solution (y,u,p) of the optimality system from Lemma 10 satisfy

$$\begin{aligned} &D^{\boldsymbol\alpha} y \in W^{1,2}_{\vec\alpha,\vec\delta}(\Omega), \\ &D^{\boldsymbol\alpha} p \in W^{1,2}_{\vec\alpha,\vec\delta}(\Omega) \cap W^{1,\infty}_{\vec\beta,\vec\varrho}(\Omega) \cap W^{1,2}_{\vec\gamma,\vec\tau}(\Gamma), \\ &D^{\boldsymbol\alpha} u \in W^{0,\infty}_{\vec\gamma,\vec\delta}(\mathcal{K}_1) \cap W^{1,2}_{\vec\gamma,\vec\tau}(\mathcal{K}_2), \end{aligned}$$

for all $|\boldsymbol\alpha| = 1$.

Proof. With bootstrapping arguments taking regularity results in classical function spaces as well as trace and embedding theorems into account we obtain

$$p \in H^{3/2+\varepsilon}(\Omega) \Rightarrow p \in H^1(\Gamma) \Rightarrow u \in H^1(\Gamma) \Rightarrow y \in H^{3/2+\varepsilon}(\Omega) \hookrightarrow C^{0,\sigma}(\overline{\Omega}),$$

with some $\sigma \in (0,\varepsilon)$. From Theorem 1 we then conclude

$$D^{\boldsymbol\alpha} y \in W^{1,2}_{\vec\alpha,\vec\delta}(\Omega), \quad D^{\boldsymbol\alpha} p \in W^{1,2}_{\vec\alpha,\vec\delta}(\Omega) \cap W^{1,\infty}_{\vec\beta,\vec\varrho}(\Omega), \qquad \forall |\boldsymbol\alpha| = 1.$$

A trace theorem and the embeddings from [13, Lemma 8.1.1] imply

$$D^{\boldsymbol\alpha} p \in W^{1,\infty}_{\vec\beta,\vec\varrho}(\Gamma) \hookrightarrow W^{1,2}_{\vec\gamma,\vec\tau}(\Gamma) \cap W^{0,\infty}_{\vec\gamma,\vec\delta}(\Gamma).$$

Note, that in order to get the validity of the embeddings one has to take into account that $\varepsilon > 0$ can be chosen arbitrarily but small. Due to (83) we moreover have

$$u = \begin{cases} -\alpha^{-1} p, & \text{on } \mathcal{I}, \\ u_a, & \text{on } \mathcal{A}^-, \\ u_b, & \text{on } \mathcal{A}^+, \end{cases}$$

Consequently, away from the set g the control u inherits the regularity of the adjoint state p and the control bounds u_a and u_b. □

4.2 Error estimates for the midpoint interpolant

First we derive some local estimates for the midpoint interpolant exploiting regularity in weighted Sobolev spaces.

Lemma 12. *Let $E \in \partial\mathcal{T}_h$ be an arbitrary boundary element with $E \subset U_j \cap \Gamma$ for some $j \in \mathcal{C}$ (recall the covering $\{U_j\}$ used in (8)). We define the number $\kappa_j := \max\{\beta_j, \max_{k\in X_j} \delta_k\}$. The following assertions hold:*

a) If $|u|_{W^{2,2}_{\vec\beta,\vec\delta}(E)} \le c$ with $\vec\beta \in [0,3/2)^{d'}$ and $\vec\delta \in [0,1)^d$, there holds

$$\left|\int_E (u(x) - R_h u)\mathrm{d}s_x\right| \le c h_E^2 |E|^{1/2} |u|_{W^{2,2}_{\vec\beta,\vec\delta}(E)} \cdot \begin{cases} r_E^{-\kappa_j}, & \text{if } r_E > 0, \\ h_E^{-\kappa_j}, & \text{if } r_E = 0. \end{cases} \tag{88}$$

b) If $|u|_{W^{1,\infty}_{\vec\beta,\vec\delta}(E)} \le c$ with $\vec\beta \in [0,1)^{d'}$ and $\vec\delta \in [0,1/2)^d$, there holds

$$\|u - R_h u\|_{L^\infty(E)} \le c h_E |u|_{W^{1,\infty}_{\vec\beta,\vec\delta}(E)} \cdot \begin{cases} r_E^{-\kappa_j}, & \text{if } r_E > 0, \\ h_E^{-\kappa_j}, & \text{if } r_E = 0. \end{cases} \tag{89}$$

Proof. We adapt the proof of similar results from [12, 2] for the two-dimensional case to the three-dimensional one. Our technique differs slightly as regularity results in weighted Sobolev spaces for polyhedral domains have to be exploited.
a) First, we apply the transformation to the reference triangle $\hat E$ and introduce a polynomial $\hat w$. Note that the property $\int_{\hat E} \hat w = \int_{\hat E} \hat R_h \hat w$ holds for arbitrary first-order polynomials $\hat w \in \mathcal{P}_1$. Together with a stability estimate for the midpoint interpolant, the embedding $W^{2,1+\varepsilon}(\hat E) \hookrightarrow L^\infty(\hat E)$ which holds for arbitrary $\varepsilon > 0$, and the Bramble-Hilbert Lemma we arrive at

$$\begin{aligned} \left|\int_E (u(x) - R_h u)\mathrm{d}s_x\right| &\le c|E| \left|\int_{\hat E} (\hat u(\hat x) - \hat R_h \hat u)\mathrm{d}s_{\hat x}\right| \\ &\le c|E| \left(\left|\int_{\hat E} (\hat u(\hat x) - \hat w(\hat x))\mathrm{d}s_{\hat x}\right| + \left|\int_{\hat E} \hat R_h(\hat u - \hat w)\mathrm{d}s_{\hat x}\right|\right) \\ &\le c|E| \|\hat u - \hat w\|_{L^\infty(\hat E)} \le c|E| \|\hat u - \hat w\|_{W^{2,1+\varepsilon}(\hat E)} \\ &\le c|E| |\hat u|_{W^{2,1+\varepsilon}(\hat E)}. \end{aligned} \tag{90}$$

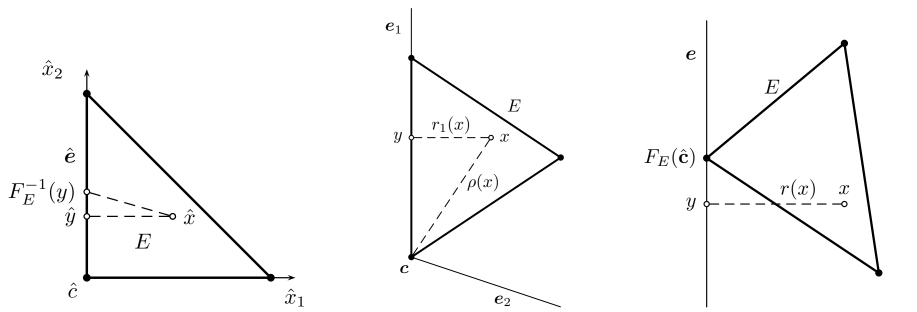
If $r_E > 0$ we use the trivial embedding $L^2(\hat E) \hookrightarrow L^{1+\varepsilon}(\hat E)$ (note that we can chose $\varepsilon \in (0,1)$), apply the transformation back to E and introduce the weights which yields

$$\begin{aligned} |\hat u|_{W^{2,1+\varepsilon}(\hat E)} &\le c h_E^2 |E|^{-1/2} |u|_{H^2(E)} \\ &\le c h_E^2 |E|^{-1/2} \rho_{j,E}^{-\beta_j} \prod_{k\in X_j} \left(\frac{r_{k,E}}{\rho_{j,E}}\right)^{-\delta_k} |u|_{W^{2,2}_{\vec\beta,\vec\delta}(E)}. \end{aligned} \tag{91}$$

Using this and property (35) we conclude the desired estimate for the case $r_E > 0$ from (90) and (91).

If $r_E = 0$ we have reduced regularity and apply embeddings into appropriate weighted Sobolev spaces. The weighted Sobolev spaces used on the reference element are defined analogous to (8), with the modification that $\hat\rho(\hat x) = |\hat x|$ and $\hat r(\hat x) = \hat x_1$ are the corner and edge weight, respectively. Here we assume without loss of generality that elements in $\partial\mathcal{T}_h$ have at most one edge which is contained in an edge of Γ, and we define the reference transformation $F_E \colon \hat E \to E$ in such a way that the edge $\hat{\boldsymbol{e}}$ of $\hat E$ having endpoints $\hat{\boldsymbol{c}} := (0,0)$ and $(0,1)$ is mapped to the singular edge of E. The extension to the case that two edges of E are contained in edges of Γ is obvious and is hence not explained further. Let us derive some relations between the weights in $\hat E$ and E. One quickly realizes that the way in which the element E touches an edge $\boldsymbol{e}_k$ of Γ has effects on the role the weight functions play. Note that the following results hold due to the assumed shape-regularity of $\mathcal{T}_h$. Consider the case illustrated in Figure 3b where an edge of E is completely contained in the edge $\boldsymbol{e}_k$. We define the quantities $y := \arg\min_{v\in\boldsymbol{e}_k} |v - x|$ and $\hat y = \arg\min_{\hat v\in\hat e} |\hat v - \hat x|$. From the assumed shape-regularity we get the relation

$$r_k(x) = |x - y| \sim h_E |\hat x - F_E^{-1}(y)| \sim h_E |\hat x - \hat y| = h_E \hat r(\hat x). \tag{92}$$



(a) Reference element

(b) Element touching an edge and a corner

(c) Element touching an edge in a single point

Figure 3: The reference element $\hat{E}$ and the different positions of the original element E.

In contrast to this, if E touches the edge $\boldsymbol{e}_k$ only in a single point, see Figure 3c, we get

$$r_k(x) = |x - y| \sim |x - F_E(\hat{\boldsymbol{c}})| \sim h_E |\hat{x} - \hat{\boldsymbol{c}}| = h_E \hat{\rho}(\hat{x}). \tag{93}$$

Moreover, if E touches a corner $\boldsymbol{c}$ of Γ there holds

$$\rho(x) = |x - \boldsymbol{c}| \sim h_E |\hat{x} - \hat{\boldsymbol{c}}| = h_E \hat{\rho}(\hat{x}). \tag{94}$$

In the following we will make use of the embedding $W^{2,2}_{\beta_j,\delta_k}(\hat{E}) \hookrightarrow W^{2,1+\varepsilon}(\hat{E})$ [13, Lemma 8.1.1] which holds if $\beta_j < 3/2$, $\delta_k < 1$, provided that $\varepsilon > 0$ is sufficiently small.

We continue estimating the right-hand side of (90) and discuss four possible situations separately. If one edge of E is contained in the edge $\boldsymbol{e}_k$ and E is away from the corners we use the property (92), and the fact that $\rho_{j,E} > 0$, to estimate

$$|\hat{u}|_{W^{2,1+\varepsilon}(\hat{E})} \leq c|\hat{u}|_{W^{2,2}_{\delta_k,\delta_k}(\hat{E})} \leq c h_E^{2-\delta_k} |E|^{-1/2} \rho_{j,E}^{\delta_k - \beta_j} |u|_{W^{2,2}_{\vec{\beta},\vec{\delta}}(E)}.$$

If E touches the edge only in a single point we apply (93) instead of (92) and get

$$|\hat{u}|_{W^{2,1+\varepsilon}(\hat{E})} \leq c|\hat{u}|_{W^{2,2}_{\delta_k,0}(\hat{E})} \leq c h_E^{2-\delta_k} |E|^{-1/2} \rho_{j,E}^{\delta_k - \beta_j} |u|_{W^{2,2}_{\vec{\beta},\vec{\delta}}(E)}.$$

If E touches additionally the corner $\boldsymbol{c}_j$ and has an edge contained in $\boldsymbol{e}_k$, we get with (92) and (94)

$$|\hat{u}|_{W^{2,1+\varepsilon}(\hat{E})} \leq c|\hat{u}|_{W^{2,2}_{\beta_j,\delta_k}(\hat{E})} \leq c h_E^{2-\beta_j} |E|^{-1/2} |u|_{W^{2,2}_{\vec{\beta},\vec{\delta}}(E)}.$$

If E touches the corner $\boldsymbol{c}_j$, but the edges $\overline{\boldsymbol{e}}_k$, $k \in X_j$, only in $\boldsymbol{c}_j$, the property (94) yields

$$|\hat{u}|_{W^{2,1+\varepsilon}(\hat{E})} \leq c|\hat{u}|_{W^{2,2}_{\beta_j,0}(\hat{E})} \leq c h_E^{2-\beta_j} |E|^{-1/2} |u|_{W^{2,2}_{\vec{\beta},\vec{\delta}}(E)}.$$

Moreover, as $\rho_{j,E} \geq c h_E$ if $\varrho_{j,E} > 0$ (neighboring elements have equivalent diameter), we conclude the simplifications

$$h_E^{-\delta_k} \rho_{j,E}^{\delta_k - \beta_j} \leq h_E^{-\max\{\delta_k,\beta_j\}} \leq h_E^{-\kappa}, \qquad h_E^{-\beta_j} \leq h_E^{-\kappa_j}, \tag{95}$$

and get from all four cases discussed above that

$$|\hat{u}|_{W^{2,1+\varepsilon}(\hat{E})} \leq c h_E^{2-\kappa_j} |E|^{-1/2} |u|_{W^{2,2}_{\vec{\beta},\vec{\delta}}(E)}.$$

Together with (90) the estimate (88) follows for $r_E = 0$.

b) To show the estimate in the $L^\infty(E)$-norm we use again the transformation to a reference element, insert a polynomial $\hat{w} \in \mathcal{P}_0$, and apply an embedding as well as the Bramble-Hilbert Lemma to obtain

$$\|u - R_h u\|_{L^\infty(E)} \le c\|\hat{u} - \hat{w}\|_{L^\infty(\hat{E})} \le c|\hat{u}|_{W^{1,2+\varepsilon}(E)}. \tag{96}$$

The case $r_E > 0$ is easy since $u \in W^{1,\infty}(E)$. Transforming back to E and inserting the weights yields

$$\begin{aligned} |\hat{u}|_{W^{1,2+\varepsilon}(\hat{E})} \le c h_E |u|_{W^{1,\infty}(E)} &\le c h_E \rho_{j,E}^{-\beta_j} \prod_{k\in X_j} \left(\frac{r_{k,E}}{\rho_{j,E}}\right)^{-\delta_k} |u|_{W^{1,\infty}_{\vec{\beta},\vec{\delta}}(E)} \\ &\le c h_E r_E^{-\kappa_j} |u|_{W^{1,\infty}_{\vec{\beta},\vec{\delta}}(E)}, \end{aligned} \tag{97}$$

where the latter step is an application of (35).

If $r_E = 0$ we proceed as in the proof of part a), and derive the estimate

$$|\hat{u}|_{W^{1,2+\varepsilon}(\hat{E})} \le c h_E |u|_{W^{1,\infty}_{\vec{\beta},\vec{\delta}}(E)} \cdot \begin{cases} h_E^{-\delta_k} \rho_{j,E}^{\delta_k - \beta_j}, & \text{if } \rho_{j,E} > 0, \\ h_E^{-\beta_j}, & \text{if } \rho_{j,E} = 0. \end{cases} \tag{98}$$

where we used, depending on the way in which E touches the edge, one of the embeddings

$$\begin{aligned} &W^{0,\infty}_{\delta_k,\delta_k}(\hat{E}) \hookrightarrow W^{0,2+\varepsilon}(\hat{E}), && W^{0,\infty}_{\delta_k,0}(\hat{E}) \hookrightarrow W^{0,2+\varepsilon}(\hat{E}), \\ &W^{0,\infty}_{\beta_j,\delta_k}(\hat{E}) \hookrightarrow W^{0,2+\varepsilon}(\hat{E}), && W^{0,\infty}_{\beta_j,0}(\hat{E}) \hookrightarrow W^{0,2+\varepsilon}(\hat{E}) \end{aligned}$$

which hold under the assumptions $\vec{\beta} \in [0,1)^{d'}$ and $\vec{\delta} \in [0,1/2)^d$. Inserting (98) into (96) and applying the simplification (95) leads to the desired estimate in case of $r_E = 0$. □

These local estimates allow us to prove an estimate for the second term on the right-hand side of (87).

Lemma 13. *Let $y_d \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$, and let Assumption 1 be satisfied. The refinement parameter is chosen such that $\mu < \frac{1}{4} + \frac{\lambda}{2}$ holds. Then the estimate*

$$\|S_h(u - R_h u)\|_{L^2(\Omega)} \le c h^2 |\ln h| \eta, \tag{99}$$

holds with

$$\eta := |u|_{H^1(\Gamma)} + \|u\|_{L^\infty(\Gamma)} + |u|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(\mathcal{K}_2)} + |u|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(\mathcal{K}_1)}$$

and weight vectors defined as in Theorem 11 with $\varepsilon > 0$ sufficiently small.

Proof. We introduce the functions $z_h := S_h(u - R_h u)$ and $v = P z_h$ which implies $v|_\Gamma = S^* z_h$. Then the term under consideration can be written as

$$\|z_h\|^2_{L^2(\Omega)} = \|S_h(u - R_h u)\|^2_{L^2(\Omega)} = (u - R_h u, (S_h^* - S^*) z_h)_\Gamma + (u - R_h u, v)_\Gamma. \tag{100}$$

With the trace theorem from [6, Theorem 1.6.6] an the finite element error estimates from Theorem 2 (note that $\mu < 1/4 + \lambda/2 < \lambda$) we conclude for the first term

$$\begin{aligned} (u - R_h u, (S_h^* - S^*) z_h)_\Gamma &\le \|u - R_h u\|_{L^2(\Gamma)} \|(S_h^* - S^*) z_h\|^{1/2}_{L^2(\Omega)} \|(S_h^* - S^*) z_h\|^{1/2}_{H^1(\Omega)} \\ &\le c h^{3/2} \|u - R_h u\|_{L^2(\Gamma)} \|z_h\|_{L^2(\Omega)}. \end{aligned} \tag{101}$$

On $\mathcal{K}_1$ we get an estimate for the midpoint interpolant using Assumption 1 and stability of R_h, hence

$$\|u - R_h u\|_{L^2(\mathcal{K}_1)} \le c\|u - R_h u\|_{L^\infty(\mathcal{K}_1)} |\mathcal{K}_1|^{1/2} \le c h^{1/2} \|u\|_{L^\infty(\Gamma)}. \tag{102}$$

On the remaining set $\mathcal{K}_2$ we apply a standard estimate for the $L^2(\Gamma)$-projection and obtain

$$\begin{aligned} \|u - R_h u\|_{L^2(\mathcal{K}_2)} &\le \|u - Q_h u\|_{L^2(\mathcal{K}_2)} + \|Q_h u - R_h u\|_{L^2(\mathcal{K}_2)} \\ &\le c h^{1/2} \left(|u|_{H^{1/2}(\Gamma)} + |u|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(\mathcal{K}_2)} \right). \end{aligned} \tag{103}$$

The estimate used for the term $\|Q_h u - R_h\|_{L^2(\mathcal{K}_2)}$ will be shown later in (113) where even a higher approximation order is proved. Inserting (102) and (103) into (101) leads to an estimate for the first term in (100), namely

$$(u - R_h u, (S_h^* - S^*)z_h)_\Gamma \le ch^2\eta\|z_h\|_{L^2(\Omega)}. \tag{104}$$

For the second term on the right-hand side of (100) we introduce the $L^2(\Gamma)$-projection onto U_h as intermediate function and obtain with orthogonality properties and standard estimates for Q_h

$$\begin{aligned}(u - R_h u, v)_\Gamma &= (u - Q_h u, v - Q_h v)_\Gamma + (Q_h u - R_h u, v)_\Gamma \\ &\le ch^2|u|_{H^1(\Gamma)}\|z_h\|_{L^2(\Omega)} + (Q_h u - R_h u, v)_\Gamma,\end{aligned} \tag{105}$$

where we applied the a-priori estimate

$$\|v\|_{H^1(\Gamma)} + \|v\|_{L^\infty(\Gamma)} \le c\|v\|_{H^{3/2+\varepsilon}(\Omega)} \le c\|z_h\|_{L^2(\Omega)}, \tag{106}$$

which follows for some sufficiently small $\varepsilon > 0$ from trace and embedding theorems, and elliptic regularity results. The estimate (106) for the $L^\infty(\Gamma)$- and $L^2(\Gamma)$-norm of v will be used later.

For the second term in (105) we distinguish between boundary elements $E \subset \mathcal{K}_1$ and $E \subset \mathcal{K}_2$. On $\mathcal{K}_2$ the solution possesses the regularity $D^{\boldsymbol{\alpha}} u \in W^{1,2}_{\vec{\gamma},\vec{\tau}}(\mathcal{K}_2)$ for all $|\boldsymbol{\alpha}| = 1$, as stated in Theorem 11, where the largest weight is defined by

$$\kappa := \max_{j\in\mathcal{C},k\in\mathcal{E}}\{\gamma_j, \tau_k\} = \max_{j\in\mathcal{C},k\in\mathcal{E}}\{0, 1 - \lambda^{\boldsymbol{c}_j} + \varepsilon, 3/2 - \lambda^{\boldsymbol{e}_k} + \varepsilon\} = \max\{0, 3/2 - \lambda + \varepsilon\}. \tag{107}$$

Using the element-wise definition of the $L^2(\Gamma)$-projection and the fact that $R_h u$ is constant on each element we get

$$\begin{aligned}\|Q_h u - R_h u\|^2_{L^2(\mathcal{K}_2)} &= \sum_{E\subset\mathcal{K}_2}\int_E\left(|E|^{-1}\int_E u(y)\mathrm{d}s_y - [R_h u]|_E\right)^2\mathrm{d}s_x \\ &= \sum_{E\subset\mathcal{K}_2}|E|^{-1}\left(\int_E(u(y) - [R_h u]|_E)\mathrm{d}s_y\right)^2.\end{aligned} \tag{108}$$

Now the local estimates from Lemma 12 can be inserted. In case of $r_E > 0$ the estimate (88) yields together with the refinement condition

$$|E|^{-1}\left(\int_E(u(y) - [R_h u]|_E)\mathrm{d}s_y\right)^2 \le c\left(h^2 r_E^{2(1-\mu)-\kappa}|u|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(E)}\right)^2, \tag{109}$$

and in case of $r_E = 0$ we get with $h_E = h^{1/\mu}$

$$|E|^{-1}\left(\int_E(u(y) - [R_h u]|_E)\mathrm{d}s_y\right)^2 \le c\left(h^{(2-\kappa)/\mu}|u|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(E)}\right)^2. \tag{110}$$

Moreover, the assumption $\mu < 1/4 + \lambda/2$ implies $\mu \le 1 - \kappa/2$, since

$$1 - \kappa/2 = 1 - \frac{1}{2}\max\{0, 3/2 - \lambda + \varepsilon\} = \min\{1, 1/4 + \lambda/2 - \varepsilon\} \ge \mu, \tag{111}$$

where the last step is valid when ε is chosen sufficiently small. Hence, (109) and (110) become

$$|E|^{-1}\left(\int_E(u(y) - [R_h u]|_E)\mathrm{d}s_y\right)^2 \le c\left(h^2|u|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(E)}\right)^2 \tag{112}$$

for arbitrary $E \in \mathcal{E}_h$, $E \subset \mathcal{K}_2$. Inserting this into (108) yields

$$\|Q_h u - R_h u\|_{L^2(\mathcal{K}_2)} \le ch^2|u|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(\mathcal{K}_2)} \tag{113}$$

and with the Cauchy-Schwarz inequality and (106) we finally arrive at

$$(Q_h u - R_h u, v)_{L^2(\mathcal{K}_2)} \le ch^2 |u|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(\mathcal{K}_2)} \|z_h\|_{L^2(\Omega)}. \tag{114}$$

On the set $\mathcal{K}_1$ the solution satisfies only $D^{\boldsymbol{\alpha}} u \in W^{0,\infty}_{\vec{\gamma},\vec{\delta}}(\mathcal{K}_1)$ for all $|\boldsymbol{\alpha}| = 1$. We denote the largest weight by

$$\kappa_\infty := \max_{j\in\mathcal{C},k\in\mathcal{E}} \{\gamma_j, \delta_k\} = \max_{j\in\mathcal{C},k\in\mathcal{E}} \{0, 1-\lambda^{\mathbf{c}_j} + \varepsilon, 1-\lambda^{\mathbf{e}_k} + \varepsilon\}. \tag{115}$$

With the element-wise definition of Q_h we obtain

$$\begin{aligned}
(Q_h u - R_h u, v)_{L^2(\mathcal{K}_1)} &= \sum_{E\subset\mathcal{K}_1} \int_E (Q_h u - R_h u)|_E v(x) \mathrm{d}s_x \\
&\le \|v\|_{L^\infty(\Gamma)} \sum_{E\subset\mathcal{K}_1} \int_E \left| |E|^{-1} \int_E u(y)\mathrm{d}s_y - [R_h u]|_E \right| \mathrm{d}s_x \\
&\le \|v\|_{L^\infty(\Gamma)} \sum_{E\subset\mathcal{K}_1} \left| \int_E (u(y) - [R_h u]|_E) \,\mathrm{d}s_y \right| \\
&\le \|v\|_{L^\infty(\Gamma)} \sum_{E\subset\mathcal{K}_1} \|u - R_h u\|_{L^\infty(E)} |E|.
\end{aligned} \tag{116}$$

To obtain a sharp error estimate, we recall the decomposition (13)

$$\Gamma_{R/n} := \{x \in \Gamma \colon r(x) < R/n\}, \qquad \tilde{\Gamma}_{R/n} := \Gamma \setminus \Gamma_{R/n},$$

with sufficiently small $R > 0$ that we set without loss of generality equal to one, and use the dyadic decomposition

$$\Gamma_i := \begin{cases} \{x \in \Gamma \colon d_{i+1} < r(x) < d_i\}, & \text{for } i = 0, \ldots, I-1, \\ \{x \in \Gamma \colon \quad 0 < r(x) < d_I\}, & \text{for } i = I, \end{cases} \quad \text{with} \quad d_i = 2^{-i}. \tag{117}$$

The inner-most domain has radius $d_I = ch^{1/\mu}$ with a mesh-independent constant $c > 1$ which results in $I \sim |\ln h|$. The patch with the neighboring sets is denoted by

$$\Gamma_i' := \operatorname{int}\left(\overline{\Gamma_{\max\{0,i-1\}}} \cup \overline{\Gamma_i} \cup \overline{\Gamma_{\min\{I,i+1\}}}\right).$$

Within the set Γ_i, $i = 0, \ldots, I$, all elements E have diameter $h_E \sim h d_i^{1-\mu}$. Assumption 1 then implies that

$$\sum_{\substack{E\subset\mathcal{K}_1\\ E\cap\tilde{\Gamma}_{R/2}\neq\emptyset}} |E| \le ch, \qquad \sum_{\substack{E\subset\mathcal{K}_1\\ E\cap\Gamma_i\neq\emptyset}} |E| \le chd_i^{1-\mu}, \quad i = 0, \ldots, I. \tag{118}$$

With (117) we obtain

$$\sum_{\substack{E\subset\mathcal{K}_1\\ E\cap\Gamma_{R/2}\neq\emptyset}} \|u - R_h u\|_{L^\infty(E)} |E| \le \sum_{i=1}^{I} \sum_{\substack{E\subset\mathcal{K}_1\\ E\cap\Gamma_i\neq\emptyset}} \|u - R_h u\|_{L^\infty(E)} |E|. \tag{119}$$

From Lemma 12 we conclude the local estimate

$$\|u - R_h u\|_{L^\infty(E)} |E| \le chd_i^{1-\mu-\kappa_\infty} |E| |u|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(E)} \qquad \forall E \subset \mathcal{K}_1, E \cap \Gamma_i \neq \emptyset, \tag{120}$$

for all $i = 1, \ldots, I$, where we used the properties $h_E \sim hd_i^{1-\mu}$, and in particular if $r_E = 0$

$$h_E^{1-\kappa_\infty} = h^{1+(1-\mu-\kappa_\infty)/\mu} \le chd_I^{1-\mu-\kappa_\infty}.$$

Inserting (118) and (120) into (119) yields

$$\sum_{\substack{E\subset\mathcal{K}_1\\ E\cap\Gamma_{R/2}\neq\emptyset}} \|u-R_hu\|_{L^\infty(E)}|E| \le ch^2\sum_{i=1}^{I} d_i^{2(1-\mu)-\kappa_\infty}|u|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(\Gamma_i'\cap\mathcal{K}_1)}. \tag{121}$$

Next, we confirm that the condition $\mu \le 1-\kappa_\infty/2$ holds. Taking (115) and the assumption upon μ into account yields for sufficiently small $\varepsilon>0$

$$1-\frac{\kappa_\infty}{2} = 1-\frac{1}{2}\max_{j\in\mathcal{C},k\in\mathcal{E}}\{0,1-\lambda^{\boldsymbol{e}_k}+\varepsilon,1-\lambda^{\boldsymbol{c}_j}+\varepsilon\} \ge \min\{1,1/4+\lambda/2-\varepsilon\}\ge\mu.$$

As a consequence, (121) leads together with $I\sim|\ln h|$ to

$$\sum_{\substack{E\subset\mathcal{K}_1\\ E\cap\Gamma_{R/2}\neq\emptyset}} \|u-R_hu\|_{L^\infty(E)}|E| \le ch^2|\ln h||u|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(\mathcal{K}_1)}. \tag{122}$$

The extension to elements contained in or intersecting $\tilde{\Gamma}_{R/2}$ is easy as these elements satisfy $r_E\sim c$ and $h_E\sim h$. Exploiting also (118) yields

$$\sum_{\substack{E\subset K_1\\ E\cap\tilde{\Gamma}_{R/2}\neq\emptyset}} \|u-R_hu\|_{L^\infty(E)}|E| \le ch|u|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(\mathcal{K}_1)}\sum_{\substack{E\subset K_1\\ E\cap\tilde{\Gamma}_{R/2}\neq\emptyset}}|E| \le ch^2|u|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(\mathcal{K}_1)}. \tag{123}$$

Consequently, we deduce from (122) and (123) that

$$\sum_{E\subset\mathcal{K}_1} \|u-R_hu\|_{L^\infty(E)}|E| \le ch^2|\ln h||u|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(\mathcal{K}_1)}. \tag{124}$$

Inserting (124) into (116) yields together with (106)

$$(Q_hu-R_hu,v)_{L^2(\mathcal{K}_1)} \le ch^2|\ln h||u|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(\mathcal{K}_1)}\|z_h\|_{L^2(\Omega)}. \tag{125}$$

Combining the estimates (100), (104), (105), (114) and (125), and dividing by the term $\|z_h\|_{L^2(\Omega)}$ leads to the desired result (99). □

4.3 Supercloseness of the midpoint interpolant

It remains to derive an estimate for the third term on the right-hand side of (87), and we exploit a principle which is called *supercloseness* in the literature. This principle relies on the fact that the interpolant of the continuous solution u is closer to the discrete solution u_h than u itself.

Lemma 14. *Assume that $y_d\in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma\in(0,1)$, and let Assumption 1 be satisfied. If $\mu<\frac{1}{4}+\frac{\lambda}{2}$, then there holds*

$$\|S_h(R_hu-u_h)\|_{L^2(\Omega)} \le ch^2|\ln h|^{3/2}\eta, \tag{126}$$

where

$$\begin{aligned}\eta := {}& |u|_{H^1(\Gamma)}+|u|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(\mathcal{K}_2)}+|u|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(\mathcal{K}_1)}+|y|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega)}\\ &+|p|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(E)}+\sum_{|\boldsymbol{\alpha}|=1}\|D^{\boldsymbol{\alpha}}p\|_{W^{1,2}_{\vec{\alpha},\vec{\delta}}(\Omega)}+\sum_{|\boldsymbol{\alpha}|=1}\|D^{\boldsymbol{\alpha}}p\|_{W^{1,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega)}+\|p\|_{L^\infty(\Omega)}\end{aligned}$$

with the weight vectors defined in Theorem 11 and $\varepsilon>0$ chosen sufficiently small.

Proof. Firstly, one confirms that the variational inequality (82) holds also pointwise and hence

$$(\alpha R_h u + R_h p, u_h - R_h u)_\Gamma \geq 0,$$

where we used u_h as test function. Secondly, if we test the discrete variational inequality (85) with $R_h u$ we get

$$(\alpha u_h + p_h, R_h u - u_h)_\Gamma \geq 0.$$

Summing up both inequalities yields

$$\alpha \|u_h - R_h u\|^2_{L^2(\Gamma)} \leq (R_h p - p_h, u_h - R_h u)_\Gamma.$$

Once we have shown an estimate for the right-hand side the assertion follows as S_h is bounded, i. e. $\|S_h v\|_{L^2(\Omega)} \leq c\|v\|_{L^2(\Gamma)}$ for all $v \in L^2(\Gamma)$. Introducing the intermediate functions p and $S_h^*(S_h R_h u - y_d)$ leads to

$$\begin{aligned}\alpha \|u_h - R_h u\|^2_{L^2(\Gamma)} \leq\ & (R_h p - p, u_h - R_h u)_\Gamma \\ & + (p - S_h^*(S_h R_h u - y_d), u_h - R_h u)_\Gamma \\ & + (S_h^*(S_h R_h u - y_d) - p_h, u_h - R_h u)_\Gamma, \end{aligned} \tag{127}$$

and it remains to discuss the three terms on the right-hand side. Up to here, the proof coincides with the proof of [12, Proposition 4.5].

Taking into account the decomposition $\mathcal{E}_h$ of Γ and exploiting that u_h and $R_h u$ are constant on each boundary element $E \in \mathcal{E}_h$ leads to

$$\begin{aligned}(R_h p - p, u_h - R_h u)_\Gamma &= \sum_{E\in\mathcal{E}_h} \int_E ([R_h p]|_E - p(x))(u_h - R_h u)|_E \mathrm{d}s_x \\ &= \sum_{E\in\mathcal{E}_h} (u_h - R_h u)|_E \int_E ([R_h p]|_E - p(x))\mathrm{d}s_x. \end{aligned} \tag{128}$$

For the adjoint state we have shown in Theorem 11 that $D^{\boldsymbol{\alpha}} p \in W^{1,2}_{\vec{\gamma},\vec{\tau}}(\Gamma)$ for all $|\boldsymbol{\alpha}| = 1$. We insert the local estimate (88) from Lemma 12 to arrive at

$$\int_E ([R_h p]|_E - p(x))\mathrm{d}s_x \leq c|E|^{1/2} |p|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(E)} \begin{cases} h^2 r_E^{2(1-\mu)-\kappa}, & \text{if } r_E > 0, \\ h^{(2-\kappa)/\mu}, & \text{if } r_E = 0, \end{cases} \tag{129}$$

with κ from (107). Inserting the assumption $\mu \leq 1 - \kappa/2$, see (111), yields

$$\int_E ([R_h p]_E - p(x))\mathrm{d}s_x \leq ch^2 |E|^{1/2} |p|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(E)} \qquad \forall E \in \mathcal{E}_h.$$

The estimate (128) then becomes

$$\begin{aligned}(R_h p - p, u_h - R_h u)_\Gamma &\leq c \sum_{E\in\mathcal{E}_h} |(u_h - R_h u)|_E|\, h^2 |E|^{1/2} |p|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(E)} \\ &\leq c \sum_{E\in\mathcal{E}_h} h^2 |p|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(E)} \|u_h - R_h u\|_{L^2(E)} \\ &\leq ch^2 |p|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(\Gamma)} \|u_h - R_h u\|_{L^2(\Gamma)}. \end{aligned} \tag{130}$$

For the second term in (127) we insert the representation $p|_\Gamma = S^*(Su - y_d)$ and with appropriate intermediate functions we get

$$\begin{aligned}\|p - S_h^*(S_h R_h u - y_d)\|_{L^2(\Gamma)} &= \|(S^* - S_h^*)(y - y_d)\|_{L^2(\Gamma)} + \|S_h^*(S - S_h)u\|_{L^2(\Gamma)} \\ &\quad + \|S_h^* S_h(u - R_h u)\|_{L^2(\Gamma)} \leq ch^2 |\ln h|^{3/2} \eta. \end{aligned}$$

In the last step we inserted the finite element error estimate from Theorem 8 for the first term, the stability of S_h^* as operator from $L^2(\Omega)$ to $L^2(\Gamma)$ and the estimate of Theorem 2 for the second term, and the result of Lemma 13 for the third term. With an application of the Cauchy-Schwarz inequality we then obtain

$$(p - S_h^*(S_h R_h u - y_d), u_h - R_h u)_\Gamma \le ch^2 |\ln h|^{3/2} \eta \|u_h - R_h u\|_{L^2(\Gamma)}. \tag{131}$$

For the third term in (127) we insert the representation of the discrete adjoint state, namely $p_h|_\Gamma = S_h^*(S_h u_h - y_d)$, and observe that it is non-positive by

$$(S_h^*(S_h R_h u - y_d) - p_h, u_h - R_h u)_\Gamma = (S_h(R_h u - u_h), S_h(u_h - R_h u)) \le 0.$$

Hence, we can neglect this term. From the estimates (127), (130) and (131) we conclude the estimate (126). □

4.4 Error estimates for the postprocessing approach

Inserting now the results of the Lemmas 13 and 14 into (87) yields an estimate for the state. From this we can conclude an estimate for the adjoint state and the control as well.

Theorem 15. *Let Assumption 1 be satisfied, and assume that $y_d \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$. Moreover, the refinement parameter is chosen such that $\frac{1}{3} < \mu < \frac{1}{4} + \frac{\lambda}{2}$ holds. Then, the estimate*

$$\|u - u_h^*\|_{L^2(\Gamma)} + \|y - y_h\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Gamma)} \le ch^2 |\ln h|^{3/2} \eta, \tag{132}$$

is fulfilled, where

$$\begin{aligned} \eta := {} & |u|_{H^1(\Gamma)} + \|u\|_{L^\infty(\Gamma)} + |u|_{W^{2,2}_{\vec\gamma,\vec\tau}(\mathcal{K}_2)} + |u|_{W^{1,\infty}_{\vec\gamma,\vec\delta}(\mathcal{K}_1)} + |y|_{W^{2,2}_{\vec\alpha,\vec\delta}(\Omega)} \\ & + |p|_{W^{2,2}_{\vec\gamma,\vec\tau}(\Gamma)} + \sum_{|\boldsymbol{\alpha}|=1} \|D^{\boldsymbol{\alpha}} p\|_{W^{1,2}_{\vec\alpha,\vec\delta}(\Omega)} + \sum_{|\boldsymbol{\alpha}|=1} \|D^{\boldsymbol{\alpha}} p\|_{W^{1,\infty}_{\vec\beta,\vec\varrho}(\Omega)} + \|p\|_{L^\infty(\Omega)}, \end{aligned}$$

with the weight vectors defined in Theorem 11 and $\varepsilon > 0$ chosen sufficiently small.

Proof. The estimate for the state variable follows from the decomposition (87), Theorem 2 and the Lemmata 13 and 14.

From the representations $p|_\Gamma = S^*(y - y_d)$ and $p_h|_\Gamma = S_h^*(y_h - y_d)$, as well as the triangle inequality we get an estimate for the adjoint state

$$\|p - p_h\|_{L^2(\Gamma)} \le \|(S^* - S_h^*)(y - y_d)\|_{L^2(\Gamma)} + \|S_h^*(y - y_h)\|_{L^2(\Gamma)}.$$

It remains to insert the error estimate on the boundary from Theorem 8, the stability of S_h^* from $L^2(\Omega)$ to $L^2(\Gamma)$, and the estimate already derived for the state.

Inserting the projection formula (86) and exploiting the non-expansivity of the projection operator Π_{ad}, see e. g. [26, Proposition 46.5], leads to

$$\|u - u_h^*\|_{L^2(\Gamma)} = \|\Pi_{ad}\left(-\frac{1}{\alpha}p\right) - \Pi_{ad}\left(-\frac{1}{\alpha}p_h\right)\|_{L^2(\Gamma)} \le c\alpha^{-1}\|p - p_h\|_{L^2(\Gamma)}.$$

The assertion then directly follows from the error estimate for the adjoint state. □

5 Numerical experiments

In order to confirm the convergence rate predicted in Theorem 15 we computed the experimental convergence rates for the numerical approximation of the slightly modified problem

$$J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Gamma)} + (g_2, y)_\Gamma \to \min!$$

subject to

$$-\Delta y + y = f \qquad \text{in } \Omega,$$
$$\partial_n y = u + g_1 \qquad \text{on } \Gamma,$$

$$u \in U_{ad} := \{v \in L^2(\Gamma)\colon u_a < v \text{ a.e. on } \Gamma\},$$

where $g_1, g_2 \in L^2(\Gamma)$ are correction terms that are used to construct an exact solution for this problem. The corresponding adjoint equation then reads

$$-\Delta p + p = y - y_d \qquad \text{in } \Omega,$$
$$\partial_n p = g_2 \qquad \text{on } \Gamma.$$

The projection formula (83) holds as usual.

The problem is solved in a Fichera domain $\Omega := (-1,1)^3 \setminus [0,1]^3$ and the control bound is set to $u_a := -120$. Moreover, the regularization parameter $\alpha = 10^{-2}$ is chosen. The exact solution is given by

$$\bar{y} = \bar{p} := \begin{cases} \rho^{\lambda^c}\left(\frac{r}{\rho}\right)^{\lambda^e}, & \text{if } x_3 > 0, \\ \rho^{\lambda^c}, & \text{if } x_3 \leq 0, \end{cases}$$

where $\rho(x) := |x|$ and $r(x) := \sqrt{x_1^2 + x_2^2}$. Moreover, we choose $\lambda^c = 0.84$ and $\lambda^e = 2/3$ so that this solution possesses the regularity one would expect in general cases for the domain Ω. To be more specifically, the solution is the singular function at the corner $(0,0,0)$ and the edge $(0,0,x_3)$, $x_3 > 0$. The input data f, y_d, g_1 and g_2 can be computed by means of the optimality system. Note that the integration of the force vectors involving f and y_d requires special caution. The source terms are in this example very irregular as we omitted the terms depending on the angels. This makes the construction of a benchmark problem easier but the solution is not harmonic. To achieve an appropriate accuracy for the force vector one must use adaptive integration schemes (up to 6 recursive steps). The discretized optimality system is then solved with a primal-dual active set strategy and a GMRES method is applied to the unconstrained auxiliary problems. We refined the mesh locally with a red-green-blue refinement strategy proposed by Bey [5] until the refinement criterion (11) is satisfied. To show that the refinement criterion and the convergence rates are sharp we computed the numerical solution on a sequence of locally refined meshes with refinement parameters $\mu \in \{1, 0.777, 0.666, 0.5\}$.

In Figure 4 it can be seen that the refinement parameter $\mu = 0.5$ which satisfies our refinement criterion used in Theorem 15 ($\mu = 0.5 < 1/4 + \lambda/2 = 7/12$) guarantees quadratic convergence (up to logarithmic influences). On quasi-uniform meshes we observe the convergence rate $1/2 + \lambda \approx 1.1667$ and this is exactly the rate which is proved in [24, Theorem 4.2.6]. The choice $\mu = 0.6666$ which would guarantee optimal convergence of a finite element approximation in $H^1(\Omega)$ and $L^2(\Omega)$ (see Theorem 2) is obviously not sufficient for optimal convergence of the discrete control variable.

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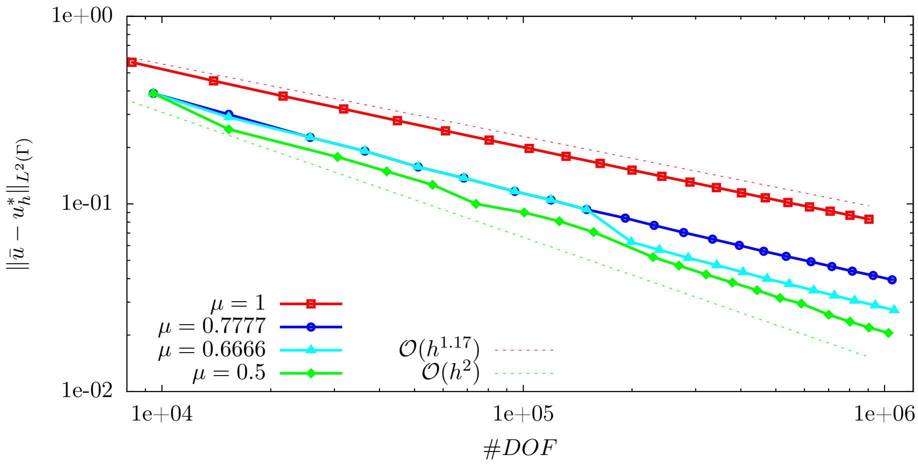

Figure 4: Error for the discrete control variable u_h^* in the $L^2(\Gamma)$ norm for refinement parameters $\mu = 1, 0.777, 0.666, 0.5$ plotted against the number of degrees of freedom N of the computational mesh $\mathcal{T}_h$. Dashed lines indicate the behavior predicted by our theory.

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