SPARSE OPTIMAL CONTROL OF THE KORTEWEG-DE VRIES-BURGERS EQUATION ON A BOUNDED DOMAIN

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Abstract. In this work we consider measure-valued optimal control problems involving the KdV-Burgers equation as state equation. These optimal control problems are motivated by an inverse problem and a control problem involving the flow of water in a channel over topography. Well-posedness of the optimal control problem is established which involves the investigation of the KdV-Burgers equation for irregular source terms. Moreover optimality conditions for the control problem are derived. Efficient numerical schemes based on spectral methods are proposed for the state and adjoint equation, as well as adequate optimization methods. The work is illustrated by several numerical examples.

Key words. Sparse optimal control, Korteweg-de Vries equation, Inverse problem, Spectral methods

AMS subject classifications. 35Q93,49J20, 65M70, 74J30

1. Introduction. The Korteweg-de Vries equation first appeared in 1895 in the context of water waves [27]. It was designed to model the evolution of long water waves in a channel of rectangular cross section when the effects of nonlinearity and dispersion balance. This phenomenon gives rise to the so-called soliton, a wave traveling at constant speed without losing its shape. This equation has been theoretically widely studied: much work has been devoted to the derivation of the equation from Euler equations [43, 17, 47], but also to the proof of their well-posedness in various contexts [37, 26, 6] - periodic domain, on the real line, bounded domain -, to their controllability [41, 23, 18, 13]. One application of the KdV equation that is of particular interest for us is the modeling of a flow in a narrow channel over an obstacle [36, 43, 44]. In that case, a source term is added on the right-hand side of the KdV equation, that represents the derivative of the topography under the flow, and the resulting equation is called the forced Korteweg-de Vries equation. More general we consider the forced KdVB -equation which describes the viscous flow over a topography. This is done on the one hand to include viscosity and on the other hand for mathematical reasons. The idea of this paper is to provide a framework to tackle two kinds of problems regarding the Korteweg-de Vries equation: an inverse problem - are we able to reconstruct a time varying topography, e.g., the locations of jumps with varying height at the bottom of the channel, from the noisy observations of wave patterns ? - and a control problem - is it possible to create a certain desired wave while acting on the topography ? As pointed out earlier, the considered topography ia assumed to consist of jumps with time varying heights. Thus its derivative is a linear combination of Dirac measures with time-independent positions x_i and time-dependent magnitudes u_i, i.e.,

$$\sum_{i=1}^{N} u_i(t)\delta_{x_i}(x). \tag{1.1}$$

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Then the control problem consist of finding the optimal positions and optimal time-depending heights of the jumps given noisy measurements of the wave patterns or a desired wave. As pointed out in [28] and [29] the space of vector measures $\mathcal{M}(\Omega_c, L^2(I))$ with values in $L^2(I)$ contains sources of the form (1.1) and the use of its norm $\|\cdot\|_{\mathcal{M}(\Omega_c,L^2(I))}$ as control cost functional enhances optimal controls of the form (1.1). In this perspective, we follow the path introduced in [28, 29] and focus on the optimal control problem

$$\min_{u\in\mathcal{M}(\Omega_c,L^2(I)),y\in Y} J(y,u) = \frac{1}{2}\left(\|\chi_{\Omega_o}y - z_1\|^2_{L^2(I\times\Omega_o)} + \|\chi_{\Omega_o}y(T) - z_2\|^2_{L^2(\Omega_o)}\right) + \alpha\|u\|_{\mathcal{M}(\Omega_c,L^2(I))} \tag{1.2}$$

with $(z_1, z_2) \in L^2(I \times \Omega_o) \times L^2(\Omega_o)$ where $y \in Y$ is the solution of the Korteweg-de Vries-Burgers equation with Dirichlet and Neumann boundary conditions on $\Omega = (0, L)$ (the space Y will be defined later on)

$$\begin{cases} \partial_t y + \partial_x y + \partial_{xxx} y + y\partial_x y - \gamma\partial_{xx} y = u \text{ in } I\times\Omega, & \text{(1.3a)}\\ y(\cdot,0) = y(\cdot,L) = \partial_x y(\cdot,L) = 0 \text{ in } I, & \text{(1.3b)}\\ y(0,\cdot) = y_0(\cdot) \text{ in } \Omega. & \text{(1.3c)} \end{cases}$$

The control acts on the control domain $\Omega_c \subseteq \Omega$. The state variable y is tracked on the observation domain $\Omega_o \subseteq \Omega$. The parameter $\alpha > 0$ is called control cost parameter and $\gamma \geq 0$ viscosity parameter. Similar types of measure-valued control problems have already been studied in the case of linear elliptic and parabolic equations [40, 15, 8, 16] and [9, 12, 11, 10]. Our approach is connected to [25] where the authors use the control cost functional

$$u \mapsto \alpha\|u\|_{L^1(\Omega_c,L^2(I))} + \frac{\varepsilon}{2}\|u\|^2_{L^2(I\times\Omega_c)}$$

in connection with parabolic optimal control problems. This control cost functional promotes sparsity patterns of the optimal control which are constant in time (directional sparsity, joint sparsity). Our problem setting is equivalent to theirs for $\varepsilon = 0$. Moreover we solve (1.2) using a continuation strategy involving the the regularization term $(\varepsilon/2)\|\cdot\|^2_{L^2(I\times\Omega_c)}$ for $\varepsilon \to 0$. The mathematical challenges of this work are twofold: on the one hand we shall prove well-posedness of the state equation in the presence of an irregular source term while on the other hand sparse optimal control of a nonlinear dispersive partial differential equation is also a novel question.

This paper is organized as follows. In Section 2 we discuss properties of the control space $\mathcal{M}(\Omega_c, L^2(I))$. Section 3, Section 4 and Section 5 deal with the well-posedness of the optimal control problem with measure-valued controls. This includes a study of the forward PDE problem with irregular data. Section 6 displays the optimality conditions of (1.2). Afterwards, we expose in Section 7 the various numerical strategies we adopt for the solution of the state equation and of the optimization problem. We conclude with some numerical examples on the Korteweg-de Vries equation, which include: an inverse problem and a control problem.

2. Control space. In this section we introduce the control space $\mathcal{M}(\Omega_c, L^2(I))$ and its properties. The control set Ω_c is any relatively closed subset of Ω. Let $u\colon \mathcal{B}(\Omega_c) \to L^2(I)$ be a countably additive mapping on the Borel sets $\mathcal{B}(\Omega_c)$ of Ω_c with values in $L^2(I)$. For u we denote by $|u| \in \mathcal{M}^+(\Omega_c)$ (positive regular Borel measure) the total variation measure defined by

$$|u|(B) = \sup_{\pi} \sum_{E\in\pi} \|u(E)\|_{L^2(I)}$$

where π is the set of all disjoint partitions of $B \in \mathcal{B}(\Omega_c)$. The space

$$\mathcal{M}(\Omega_c, L^2(I)) = \{u\colon \mathcal{B}(\Omega_c) \to L^2(I)\colon \quad u \text{ countably additive}, \quad |u|(\Omega_c) < \infty\}$$

is the space of vector measures with values in $L^2(I)$. Equipped with the norm

$$\|u\|_{\mathcal{M}(\Omega_c,L^2(I))} = |u|(\Omega_c)$$

it is a Banach space. The support of u, respectively of its total variation measure $|u|$, is defined by

$$\operatorname{supp} u = \operatorname{supp} |u| = \Omega \setminus \left(\bigcup\{B \text{ open in } \Omega_c|\ |u|(B) = 0\}\right).$$

The vector measure u possesses a Radon-Nikodym derivative, see [30],

$$u' \in L^\infty((\Omega_c, |u|), L^2(I)) \text{ with } \|u'(\cdot)\|_{L^2(I)} \equiv 1 \tag{2.1}$$

with respect to its total variation measure $|u|$. So u can be represented in the following way

$$\mathrm{d}u = u'\ \mathrm{d}|u|.$$

Next we introduce the space $\mathcal{C}(\Omega_c, L^2(I))$ of vector-valued continuous functions $p\colon \bar{\Omega}_c \to L^2(I)$ with $p|_{\partial\Omega\cap\bar{\Omega}_c} = 0$. Equipped with the norm

$$\|p\|_{\mathcal{C}(\Omega_c,L^2(I))} = \max_{x\in\Omega_c} \|p(x,\cdot)\|_{L^2(I)}$$

it is a separable Banach space. The dual space of $\mathcal{C}(\Omega_c, L^2(I))$ can be characterized by $\mathcal{M}(\Omega_c, L^2(I))$, i.e.,

$$\mathcal{C}(\Omega_c, L^2(I))^* \cong \mathcal{M}(\Omega_c, L^2(I)).$$

A proof is given in [24]. The duality pairing between $\mathcal{C}(\Omega_c, L^2(I))$ and $\mathcal{M}(\Omega_c, L^2(I))$ takes the form

$$\langle u, p\rangle_{\mathcal{M}(\Omega_c,L^2(I)),\mathcal{C}(\Omega_c,L^2(I))} = \int_\Omega (u', p)_{L^2(I)}\ \mathrm{d}|u|.$$

Next we introduce the space $L^2(I, \mathcal{M}(\Omega_c))$. It is the space of weakly-$*$ measurable functions $u\colon I \to \mathcal{M}(\Omega_c)$ which satisfy

$$\int_0^T \|u(t)\|^2_{\mathcal{M}(\Omega_c)}\ \mathrm{d}t < \infty$$

where $\mathcal{M}(\Omega_c)$ is the space of Borel measures on Ω_c and $\|\cdot\|_{\mathcal{M}(\Omega_c)}$ is the total variation norm in $\mathcal{M}(\Omega_c)$. Furthermore it can be identified with the dual space of $\mathcal{C}(\Omega_c)$ which is the space of continuous functions on $\bar{\Omega}_c$ with $p|_{\partial\Omega\cap\bar{\Omega}_c} = 0$. There holds

$$\mathcal{M}(\Omega_c, L^2(I)) \hookrightarrow L^2(I, \mathcal{M}(\Omega_c)). \tag{2.2}$$

Since $d = 1$ we have the embedding $H^1_0(\Omega) \hookrightarrow \mathcal{C}(\Omega_c)$. Thus there holds $L^2(I, H^1_0(\Omega)) \hookrightarrow L^2(I, C(\Omega_c))$ and by duality $L^2(I, \mathcal{M}(\Omega_c)) \hookrightarrow L^2(I, H^{-1}(\Omega))$.

3. Well-posedness of the state equation. In this section we discuss the well-posedness of the state equation for irregular sources from $L^2(I, H^{-1}(\Omega))$ which includes $\mathcal{M}(\Omega_c, L^2(I))$ according to the last section. For the proof of the well-posedness we need the following results for the linear KdVB-equation.

3.1. Well-posedness of the linear Korteweg-de Vries Burgers. First we consider

$$\begin{cases} \partial_t y + \partial_x y + \partial_{xxx} y - \gamma \partial_{xx} y = f \text{ in } I \times \Omega, & (3.1a) \\ y(\cdot, 0) = y(\cdot, L) = \partial_x y(\cdot, L) = 0 \text{ in } I, & (3.1b) \\ y(0, \cdot) = y_0(\cdot) \text{ in } \Omega, & (3.1c) \end{cases}$$

and its dual counter part

$$\begin{cases} -\partial_t p - \partial_x p - \partial_{xxx} p - \gamma \partial_{xx} p = \phi \text{ in } I \times \Omega, & (3.2a) \\ p(\cdot, 0) = p(\cdot, L) = \partial_x p(\cdot, 0) = 0 \text{ in } I, & (3.2b) \\ p(T, \cdot) = p_T(\cdot) \text{ in } \Omega. & (3.2c) \end{cases}$$

We denote by $A \colon L^2(\Omega) \to L^2(\Omega)$ the linear differential operator

$$Aw = -\partial_{xxx} w - \partial_x w + \gamma \partial_{xx} w$$

with the dense domain $\mathcal{D}(A) \subset L^2(\Omega)$ defined by

$$\mathcal{D}(A) = \left\{ w \in H^3(\Omega) \text{ s.t. } w(0) = w(L) = \partial_x w(L) = 0 \right\}.$$

The adjoint operator $A^* \colon L^2(\Omega) \to L^2(\Omega)$ is given by

$$A^* w = \partial_{xxx} w + \partial_x w + \gamma \partial_{xx} w$$

with the domain

$$\mathcal{D}(A^*) = \left\{ w \in H^3(\Omega) \text{ s.t. } w(0) = w(L) = \partial_x w(0) = 0 \right\}.$$

The operators A and A^* generate strongly continuous semigroups of contractions on $L^2(\Omega)$ denoted by $W(\cdot) \colon L^2(\Omega) \to L^2(\Omega)$ and $W^*(\cdot) \colon L^2(\Omega) \to L^2(\Omega)$. The reader is referred to [41] for a proof. In the sequel, we will denote by $\mathcal{B}$ the Banach space $C(I, L^2(\Omega)) \cap L^2(I, H_0^1(\Omega))$ endowed with the norm

$$\|y\|_{\mathcal{B}} = \|y\|_{\mathcal{C}(\bar{I}, L^2(\Omega))} + \|y\|_{L^2(I, H_0^1(\Omega))}$$

Furthermore we introduce the space

$$\mathcal{V} = \{ v \in H^2(\Omega) \cap H_0^1(\Omega) \colon \partial_x v(0) = 0 \}$$

endowed with the norm $\| \cdot \|_{\mathcal{V}} = \| \partial_{xx} \cdot \|_{L^2(\Omega)}$. The following existence and uniqueness result can be found in [5, Section 2].

Proposition 3.1. *Let $(f, y_0) \in L^1(I, L^2(\Omega)) \times L^2(\Omega)$ and $(\phi, p_T) \in L^1(I, L^2(\Omega)) \times L^2(\Omega)$. Then equations* (3.1a)-(3.1c) *have a unique (mild) solution $y \in \mathcal{B}$ which is given by*

$$y(t) = W(t) y_0 + \int_0^t W(t-s) f(s) \, \mathrm{d}s \quad \forall t \in I$$

and there exists a constant $c > 0$ independent of y_0, f and y such that

$$\|y\|_{\mathcal{B}} \le c\left(\|f\|_{L^1(I,L^2(\Omega))} + \|y_0\|_{L^2(\Omega)}\right)$$

holds. Furthermore equations (3.2a)-(3.2c) *have a unique solution $p \in \mathcal{B}$ given by*

$$p(t) = W^*(T-t)p_T + \int_t^T W^*(s-t)\phi(s)\,\mathrm{d}s \quad \forall t \in I \tag{3.3}$$

and there exits a constant $c > 0$ such that

$$\|p\|_{\mathcal{B}} \le c\left(\|\phi\|_{L^1(I,L^2(\Omega))} + \|p_T\|_{L^2(\Omega)}\right)$$

holds.

Next we introduce a weak formulation of (3.1) for sources $f \in L^2(I, H^{-1}(\Omega))$.

Definition 3.2. *For $(f, y_0) \in L^2(I, H^{-1}(\Omega)) \times L^2(\Omega)$ a function $y \in C(\bar{I}, L^2(\Omega))$ is called a weak solution of* (3.1a)-(3.1c) *if it satisfies the following equation*

$$\int_0^T (y,\phi)_{L^2(\Omega)}\,\mathrm{d}t + (y(T),p_T)_{L^2(\Omega)} = \int_0^T \langle f,p\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,\mathrm{d}t + (y_0,p(0))_{L^2(\Omega)} \tag{3.4}$$

for all $(\phi, p_T) \in L^1(I, L^2(\Omega)) \times L^2(\Omega)$, where $p = p(\phi, p_T) \in \mathcal{B}$ is the mild solution of (3.2a)-(3.2c)*.*

In order to show existence of a weak solution we utilize a strategy based on the approximation of the data.

Proposition 3.3. *Let $(f, y_0) \in L^2(I, H^{-1}(\Omega)) \times L^2(\Omega)$. Then, there exists a unique weak solution $y \in \mathcal{B} \cap H^1(I, \mathcal{V}^*)$ of* (3.1a)-(3.1c)*. Furthermore there exists a constant $C(T, L) > 0$ such that the following estimate holds*

$$\|y\|_{\mathcal{B}} + \|\partial_t y\|_{L^2(I,\mathcal{V}^*)} \le C(T,L)\left(\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2(I,H^{-1}(\Omega))}\right). \tag{3.5}$$

Proof. We choose sequences

- $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{C}^1(\bar{I}, L^2(\Omega))$ with $f_n \to f$ in $L^2(I, H^{-1}(\Omega))$
- $\{y_{0,n}\}_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ with $y_{0,n} \to y_0$ in $L^2(\Omega)$

which exist due to density. According to [3, Part 2, Proposition 3.3] it exists a unique classical solution

$$y_n \in \mathcal{C}(\bar{I}, \mathcal{D}(A)) \cap \mathcal{C}^1(\bar{I}, L^2(\Omega))$$

of (3.1) for data f_n and $y_{0,n}$ which satisfies the weak form (3.4). Furthermore it can be shown that y_n satisfy the following estimate

$$\|y_n\|_{\mathcal{B}} \le C(T,L)\left(\|y_{n,0}\|_{L^2(\Omega)} + \|f_n\|_{L^2(I,H^{-1}(\Omega))}\right). \tag{3.6}$$

For a proof see Appendix A.1. This estimate implies that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}$ and therefore there exists a $y \in \mathcal{B}$ which satisfies (3.4) with the data (f, y_0). This means that y is a weak solution of (3.1). Its uniqueness can be shown using standard arguments. Furthermore we can pass to the limit in (3.6) and get the first part of (3.5). Next we choose any $\psi \in \mathcal{C}_c^\infty(I, \mathcal{D}(A^*))$ and set $\phi = \partial_t\psi - A^*\psi$ in (3.2). Therefore ψ is the solution of (3.2a)-(3.2c) and it holds

$$\begin{aligned}\int_0^T (y,\partial_t\psi)_{L^2(\Omega)}\,\mathrm{d}t &= \int_0^T (y,A^*\psi)_{L^2(\Omega)} + \langle f,\psi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\mathrm{d}t \\ &\le C(T,L)\left(\|y\|_{L^2(I,H_0^1(\Omega))} + \|f\|_{L^2(I,H^{-1}(\Omega))}\right)\|\psi\|_{L^2(I,\mathcal{V})}.\end{aligned}$$

Due to the density of $\mathcal{D}(A^*)$ in $\mathcal{V}$, it holds $y \in H^1(I, \mathcal{V}^*)$ and

$$\|\partial_t y\|_{L^2(I,V^*)} \leq C(T, L) \left(\|f\|_{L^2(I,H^{-1}(\Omega))} + \|y_0\|_{L^2(\Omega)}\right).$$

□

REMARK 3.4. *Proposition 3.3 allows us to define the linear solution operator*

$$\mathcal{L} : L^2(I, H^{-1}(\Omega)) \times L^2(\Omega) \to \mathcal{B}, (f, y_0) \mapsto y, \tag{3.7}$$

where y is the weak solution of (3.1a) - (3.1c)*.*

3.2. Well-posedness of the KdVB equation. We consider in this section the nonlinear KdVB equation (1.3a) - (1.3c) with sources from $f \in L^2(I, H^{-1}(\Omega))$. In particular we show existence for short time and small data for $\gamma = 0$ and time-global existence for $\gamma > 0$. First of all we introduce a suitable solution concept for the KdVB equation

$$\begin{cases} \partial_t y + \partial_x y + \partial_{xxx} y + y\partial_x y - \gamma\partial_{xx} y = f \text{ in } I \times \Omega, & (3.8a) \\ y(.,0) = y(.,L) = \partial_x y(.,L) = 0, \text{ in } I, & (3.8b) \\ y(0,.) = 0 \text{ in } \Omega, & (3.8c) \end{cases}$$

for sources from $L^2(I, H^{-1}(\Omega))$.

DEFINITION 3.5. *For $(y_0, f) \in L^2(\Omega) \times L^2(I, H^{-1}(\Omega))$ a function $y \in \mathcal{B}$ is called a weak solution of* (3.8a) - (3.8c) *if it satisfies the following fixed point equation*

$$y = \mathcal{L}(f - y\partial_x y, y_0),$$

or in other words

$$\int_0^T (y, \phi)_{L^2(\Omega)} \,\mathrm{d}t + (y(T), p_T)_{L^2(\Omega)} = \int_0^T \langle f - y\partial_x y, p\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \,\mathrm{d}t + (y_0, p(0))_{L^2(\Omega)} \tag{3.9}$$

for all $(\phi, p_T) \in L^1(I, L^2(\Omega)) \times L^2(\Omega)$, where $p(\phi, p_T) \in \mathcal{B}$ is the solution of (3.2a)-(3.2c)*.*

The last definition makes sense considering the next Lemma which is also needed for the proof of existence of a solution to (3.8a) - (3.8c).

LEMMA 3.6. *Let $T > 0$, $y \in \mathcal{B}$ and $z \in \mathcal{B}$, then it exists a $c > 0$ such that*

$$\|y\partial_x y - z\partial_x z\|_{L^2(I,H^{-1}(\Omega))} \leq c\,T^{1/4}\|y + z\|_{\mathcal{B}}\|y - z\|_{\mathcal{B}}.$$

Proof. The proof is provided in Appendix A.2 and is largely inspired from [22].

□

Let us define for an arbitrary $\theta \leq T$ the space

$$\mathcal{B}_\theta = C([0,\theta], L^2(\Omega)) \cap L^2((0,\theta), H_0^1(\Omega)), \tag{3.10}$$

endowed with the norm

$$\|y\|_{\mathcal{B}_\theta} = \|y\|_{C([0,\theta],L^2(\Omega))} + \|y\|_{L^2([0,\theta],H_0^1(\Omega))}. \tag{3.11}$$

PROPOSITION 3.7. *For any $f \in L^2(I, H^{-1}(\Omega))$ and $y_0 \in L^2(\Omega)$, there exists a $T^* \in [0,T]$ depending on $\|f\|_{L^2(I,H^{-1}(\Omega))}$ and $\|y_0\|_{L^2(\Omega)}$ such that the system* (3.8a) - (3.8c) *admits a unique weak solution $y \in \mathcal{B}_{T^*} \cap H^1((0,T^*), \mathcal{V}^*)$ which satisfies* (3.9) *with $T = T^*$. Moreover there exists a constant $C > 0$ such that*

$$\|y\|_{\mathcal{B}_{T^*}} + \|\partial_t y\|_{L^2((0,T^*),\mathcal{V}^*)} \leq C\left(\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2(I,H^{-1}(\Omega))}\right) \tag{3.12}$$

holds.

Proof. For $\theta \in (0,T]$, we define the operator $\Psi^\theta_{f,y_0} : \mathcal{B}_\theta \mapsto \mathcal{B}_\theta$ as

$$\Psi_{f,y_0}(z) = \mathcal{L}(f - z\partial_x z, y_0). \tag{3.13}$$

which is the weak solution of (3.1a)-(3.1c) with $T = \theta$ for the data $(f - z\partial_x z, y_0)$. Estimate (3.5) and Lemma 3.6 imply

$$\|\Psi_{f,y_0}(y)\|_{\mathcal{B}_\theta} \leq C_1\left(\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2(I,H^{-1}(\Omega))}\right) + C_2\theta^{1/4}\|y\|^2_{\mathcal{B}_\theta} \tag{3.14}$$

and

$$\|\Psi_{f,y_0}(y) - \Psi_{f,y_0}(z)\|_{\mathcal{B}_\theta} \leq C_2\theta^{1/4}\|y+z\|_{\mathcal{B}_\theta}\|y-z\|_{\mathcal{B}_\theta}.$$

We choose $\theta > 0$ such that

$$\begin{cases} r = 3C_1\left(\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2(0,T,H^{-1}(\Omega))}\right) & \text{(3.15a)} \\ C_2\theta^{1/4} r \leq \dfrac{1}{3} & \text{(3.15b)} \end{cases}$$

holds. Therefore, by considering the ball $B = \{y \in \mathcal{B}_\theta; \|y\|_{\mathcal{B}_\theta} \leq r\}$ we have

$$\Psi_{f,y_0}(B) \subset B$$

and for all $(y,z) \in B$

$$\|\Psi_{f,y_0}(y) - \Psi_{f,y_0}(z)\|_{\mathcal{B}_\theta} \leq \frac{2}{3}\|y-z\|_{\mathcal{B}_\theta}.$$

As a consequence, we can apply the Banach fixed point theorem which implies the existence of a unique fix point y of $\Psi_{f,y_0}(\cdot)$. The estimate for $\|y\|_{\mathcal{B}_\theta}$ follows by construction. The estimate of the time-derivative follows from (3.5) and Lemma 3.6. □

REMARK 3.8. *According to the proof of Proposition 3.7, an upper bound for T^* is given by*

$$T^* \leq \frac{C(T,L)}{\left(\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2(I,H^{-1}(\Omega))}\right)^4}.$$

The bigger $\|f\|_{L^2(I,H^{-1}(\Omega))}$, the shorter we can ensure the existence of the solution. But one can also consider T fixed and shift the constraint on the data

$$\|f\|_{L^2(I,H^{-1}(\Omega))} + \|y_0\|_{L^2(\Omega)} \leq \frac{C(T,L)}{T^{1/4}}. \tag{3.16}$$

3.2.1. Global well-posedness of the nonlinear KdVB equation. Proposition 3.7 guarantees local well-posedness in time. Therefore, global well-posedness will follow from a priori estimates for the nonlinear problem on $[0,T]$. We will show that such estimates exist in the case $\gamma > 0$.

THEOREM 3.9. *Let $\gamma > 0$. For any $(f, y_0) \in L^2(I, H^{-1}(\Omega)) \times L^2(\Omega)$ let $y \in \mathcal{B}_\theta$, $\theta \in (0,T]$, be a time-local solution of* (3.8a)-(3.8c)*, then y satisfies*

$$\|y\|_{\mathcal{B}_t} \leq c \left(\frac{\sqrt{\gamma}+1}{\sqrt{\gamma}} \right) \left(\|y_0\|_{L^2(\Omega)} + \frac{1}{\sqrt{\gamma}} \|f\|_{L^2(I,H^{-1}(\Omega))} \right) \tag{3.17}$$

for any $t \in (0,T]$ and some $c > 0$ independent of y and the data. Moreover there holds

$$\|\partial_t y\|_{L^2((0,t),\mathcal{V}^*)} \leq C(T, L, y_0, f, \gamma).$$

Proof. We consider first that y is a classical solution $y \in \mathcal{C}(\bar{I}, \mathcal{D}(A)) \cap C^1(\bar{I}, L^2(\Omega))$ for smooth data. Then equation (3.8a) holds in $L^2(\Omega)$ and we can multiply it with y which yields

$$\frac{1}{2}\frac{d}{dt}\int_0^L y^2 dx + |\partial_x y(t,0)|^2 + \gamma \int_0^L (\partial_x y)^2 dx = \langle f, y \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

Applying Cauchy-Schwarz followed by Young's inequality to the right-hand side leads to

$$\frac{1}{2}\frac{d}{dt}\int_0^L y^2 dx + |\partial_x y(t,0)|^2 + \gamma \int_0^L (\partial_x y)^2 dx \leq \frac{1}{2\gamma}\|f\|^2_{H^{-1}(\Omega)} + \frac{\gamma}{2}\|y\|^2_{H_0^1(\Omega)}.$$

And eventually

$$\frac{1}{2}\frac{d}{dt}\int_0^L y^2 dx + \frac{\gamma}{2} \int_0^L (\partial_x y)^2 dx \leq \frac{1}{2\gamma}\|f\|^2_{H^{-1}(\Omega)}.$$

Integration between 0 and t yields

$$\|y(t,\cdot)\|^2_{L^2(\Omega)} \leq \|y_0\|^2_{L^2(\Omega)} + \frac{1}{\gamma}\|f\|^2_{L^2(I,H^{-1}(\Omega))}. \tag{3.18}$$

And integration between 0 and T gives

$$\|y(T,.)\|^2_{L^2(\Omega)} + \gamma\|y\|^2_{L^2(I,H_0^1(\Omega))} \leq \|y_0\|^2_{L^2(\Omega)} + \frac{1}{\gamma}\|f\|^2_{L^2(I,H^{-1}(\Omega))},$$

which results in

$$\|y\|^2_{L^2(I,H_0^1(\Omega))} \leq \frac{1}{\gamma}\|y_0\|^2_{L^2(\Omega)} + \frac{1}{\gamma^2}\|f\|^2_{L^2(I,H^{-1}(\Omega))}. \tag{3.19}$$

Adding (3.18) and (3.19) gives the global estimate (3.17) for smooth data which can be extended by density to $L^2(I, H^{-1}(\Omega)) \times L^2(\Omega)$. The estimate for $\|y\|_{L^2((0,t),\mathcal{V}^*)}$ follows from (3.5), Lemma 3.6 and the global estimate for $\|y\|_{\mathcal{B}}$. □

REMARK 3.10. *The constant in the estimate* (3.17) *explodes for $\gamma \to 0$.*

4. Well-posedness of the optimization problem. Existence of an optimum. We start by defining the admissible set of controls

$$U_{ad} = \begin{cases} \{u \in \mathcal{M}(\Omega_c, L^2(I)) \colon \|u\|_{\mathcal{M}(\Omega_c,L^2(I))} \leq \hat{c}\} & \text{if } \gamma = 0 \\ \mathcal{M}(\Omega_c, L^2(I)) & \text{otherwise} \end{cases}$$

where $\hat{c} < \frac{C(T,L)}{T^{1/4}} - \|y_0\|_{L^2(\Omega)}$. The constant $C(T, L)$ comes from (3.16) and we assume that

$$\|y_0\|_{L^2(\Omega)} < \frac{C(T,L)}{T^{1/4}}. \tag{4.1}$$

Moreover we introduce the non-linear control-to-state operator

$$S \colon U_{ad} \to \mathcal{B} \cap H^1(I, \mathcal{V}^*), \quad u \mapsto y \tag{4.2}$$

where y is a weak solution of (1.3a)-(1.3c) for a given $y_0 \in L^2(\Omega)$ which satisfies (4.1) in the case $\gamma = 0$. According to Remark 3.8 the mapping S is well-defined. In [14] the authors used a similar strategy for the definition of the control-to-state mapping. The control-to-observation operator is denoted by

$$S_{obs} \colon U_{ad} \to L^2(I \times \Omega_o) \times L^2(\Omega_o), \quad u \mapsto (\chi_{\Omega_o} y, \chi_{\Omega_o} y(T)).$$

Thus we can rewrite problem (1.2) in its reduced form given by

$$\min_{u \in U_{ad}} \frac{1}{2}\|S_{obs}(u) - z\|^2_{L^2(I\times\Omega_o)\times L^2(\Omega_o)} + \alpha\|u\|_{\mathcal{M}(\Omega_c,L^2(I))}. \tag{4.3}$$

with $z \in L^2(I \times \Omega_o) \times L^2(\Omega_o)$.

PROPOSITION 4.1. *There exists a solution* $(\bar{y}, \bar{u}) \in \mathcal{B} \times U_{ad}$ *to the optimal control problem* (1.2).

Proof. Case $\gamma > 0$. The cost function J is a positive function. Thus its infimum $\bar{J}$ exists and there exists a minimizing sequence $(u_n, y_n) \in U_{ad} \times \mathcal{B}$ such that $J(u_n, y_n) \to \bar{J}$ as $n \to \infty$. Furthermore there exists an $\varepsilon > 0$ such that

$$\varepsilon + J(0, 0) \geq \alpha\|u_n\|_{\mathcal{M}(\Omega_c,L^2(I))} \tag{4.4}$$

for n large enough. Therefore u_n is bounded, which implies the existence of an element $\bar{u} \in \mathcal{M}(\Omega_c, L^2(I))$ and a subsequence u_{n_k} converging in the weak-$*$ topology of $\mathcal{M}(\Omega_c, L^2(I))$ towards $\bar{u}$. For each u_{n_k}, we define $y_{n_k} = Su_{n_k} \in \mathcal{B} \cap H^1(I, \mathcal{V}^*)$, cf., [7, Corollary 3.30]. Thanks to the global estimate (3.17), there exists a exists a $\bar{y} \in L^2(I, H^1_0(\Omega)) \cap L^\infty(I, L^2(\Omega)) \cap H^1(I, \mathcal{V}^*)$ such that

$$y_{n_k} \rightharpoonup^* \bar{y} \quad \text{in} \quad L^2(I, H^1_0(\Omega)) \cap L^\infty(I, L^2(\Omega)) \cap H^1(I, \mathcal{V}^*)$$

and a $\hat{y} \in L^2(\Omega)$ such that $y_{n_k}(T) \rightharpoonup \hat{y}$ in $L^2(\Omega)$. Moreover the Aubin-Lions-Lemma, c.f. [45, Chapter 3, Proposition 1.3], implies

$$y_{n_k} \to \bar{y} \quad \text{in} \quad L^2(I, \mathcal{C}_0(\Omega)).$$

It remains to show that the limit $\bar{y}$ is indeed a weak solution of (1.3a) - (1.3c) for the control $\bar{u}$. Due to the weak-to-weak continuity of time-point evaluation operator $y \mapsto y(T)$ from $H^1(I, \mathcal{V}^*)$ to $\mathcal{V}^*$ there holds $\hat{y} = y(T)$. The convergence of the linear terms in (3.9) is obvious. The nonlinear term converges due to the strong convergence of y_{n_k}, since

$$\int_0^T \langle y_{n_k}\partial_x y_{n_k} - \bar{y}\partial_x\bar{y}, p\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\ \mathrm{d}t = -\int_0^T (y_{n_k}^2 - \bar{y}^2, \partial_x p)_{L^2(\Omega)}\ \mathrm{d}t$$
$$\leq \|y_{n_k} - \bar{y}\|_{L^2(I,\mathcal{C}_0(\Omega))}\|y_{n_k} + \bar{y}\|_{\mathcal{C}(\bar{I},L^2(\Omega))}\|\partial_x p\|_{L^2(I\times\Omega)}.$$

Therefore $\bar{y}$ is a weak solution of (1.3a)-(1.3c) for the control $\bar{u}$ and it holds $\bar{y} \in \mathcal{B} \cap H^1(I,\mathcal{V}^*)$. Moreover there holds

$$S_{obs}(u_{n_k}) \rightharpoonup S_{obs}(\bar{u}) \quad \text{in } L^2(I\times\Omega_o)\times L^2(\Omega_o).$$

The tracking functional is weak lower semi-continuous in $L^2(I\times\Omega_o)\times L^2(\Omega_o)$ and the control cost term is weak-$*$ lower semi continuous in $\mathcal{M}(\Omega_c, L^2(I))$ therefore $(\bar{y},\bar{u})$ is a solution of (1.2).

Case $\gamma = 0$. In this case we rely on the estimate (3.12) and use that the set U_{ad} is weak-$*$ closed. □

REMARK 4.2. *We know for a minimizing sequence* $(u_n, y_n) \in \mathcal{M}(\Omega_c, L^2(I))\times\mathcal{B}$ *that there exists an* $\varepsilon > 0$ *and an* $N \in \mathbb{N}$ *such that for any* $n > N$

$$\|u_n\|_{L^2(I,H^{-1}(\Omega))} \leq c\|u_n\|_{\mathcal{M}(\Omega_c,L^2(I))} \leq \frac{c}{\alpha}\left(J(0,0)+\varepsilon\right).$$

Therefore, we can always find a regularization parameter $\alpha > 0$ *such that the condition* (3.16) *is satisfied.*

5. Well-posedness of the optimal control problem for $\gamma = 0$ without norm-constraint. This section is based on ideas from [32]. There the state equation is considered as an equality constraint in the pair (y,u) in contrast to the reduced approach from the previous sections. Moreover we add the term $\frac{\eta}{4}\|y\|^4_{L^4(I\times\Omega)}$ for $\eta > 0$ to the cost functional. In the following we will see that the boundedness of the state in the $L^4(I,L^4(\Omega))$-norm makes the state equation a well-defined equality constraint and ensures the time-global existence of y in $\mathcal{B}$. Thus we consider an optimal control problem of the form

(5.1)
$$\min_{u\in\mathcal{M}(\Omega_c,L^2(I)),\, y\in\tilde{Y}} J(y,u) = \frac{1}{2}\left(\|\chi_{\Omega_o}y - z_1\|^2_{L^2(I\times\Omega_o)} + \|\chi_{\Omega_o}y(T) - z_2\|^2_{L^2(\Omega_o)}\right)$$
$$+\alpha\|u\|_{\mathcal{M}(\Omega,L^2(I))} + \frac{\eta}{4}\|y\|^4_{L^4(I\times\Omega)}.$$

subject to (1.3a)-(1.3c). The state space is chosen as $\tilde{Y} = L^4(I,L^4(\Omega))\cap H^1(I,\mathcal{V}^*)$. The $L^4(I,L^4(\Omega))$-norm is chosen since it guarantees the following regularity of the non-linearity.

LEMMA 5.1. *Let* $y \in L^4(I,L^4(\Omega))$. *Then* $y\partial_x y \in L^2(I,H^{-1}(\Omega))$.

Therefore the following definition is reasonable.

DEFINITION 5.2. *Let* $(y,u) \in \tilde{Y}\times\mathcal{M}(\Omega_c,L^2(I))$ *such that* $y(T,\cdot) \in L^2(\Omega)$. *They are called a state-control pair if they satisfy the equation*

(5.2)
$$\int_0^T (y,\phi)_{L^2(\Omega)}\ \mathrm{d}t + (y(T),p_T)_{L^2(\Omega)} = \int_0^T \langle f - y\partial_x y, p\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\ \mathrm{d}t + (y_0,p(0))_{L^2(\Omega)}$$

for all $(\phi, p_T) \in L^{4/3}(I, L^2(\Omega)) \times L^2(\Omega)$*, where* $p = p(\phi, p_T) \in \mathcal{B}$ *is the mild solution of* (3.2a)-(3.2c).

Next we show that the existence of a control-state pair implies the time-global existence of y in $\mathcal{B}$.

PROPOSITION 5.3. *Let* $(y, u) \in \tilde{Y} \times \mathcal{M}(\Omega_c, L^2(I))$ *be a state-control pair. Then* $y \in \mathcal{B} \cap H^1(I, \mathcal{V}^*)$ *is a weak solution of* (1.3a) - (1.3c) *in the sense of Definition 3.5 for* $\gamma = 0$ *and satisfies the global estimate*

$$\|y\|_{\mathcal{B}} \leq C(T, L) \left(\|f\|_{L^2(I, H^{-1}(\Omega))} + \|y_0\|_{L^2(\Omega)} + \|y\|^2_{L^4(I, L^4(\Omega))} \right) \tag{5.3}$$

and

$$\|\partial_t y\|_{L^2(I, \mathcal{V}^*)} \leq C(T, L, f, y_0).$$

Proof. For the first part of the estimate, one proceed in a similar manner as in Appendix A.1 but this time for the non-linear KdV equation. The critical term is then the non-linearity, the rest is identical. Let $y \in \mathcal{C}(\bar{I}, \mathcal{D}(A)) \cap \mathcal{C}^1(\bar{I}, L^2(\Omega))$ be the classical solution of (1.3a) - (1.3c) for smooth data ($\mathcal{C}^\infty$). We multiply (3.8a) which holds in $L^2(\Omega)$ for a.e. $t \in I$ with y and get for the nonlinear term

$$\int_0^L y^2 \partial_x y \, \mathrm{d}x = -2 \int_0^L y^2 \partial_x y \, \mathrm{d}x.$$

Due to the boundary conditions, this term vanishes. Next, we test with xy and see

$$\int_0^L xy^2 \partial_x y \, \mathrm{d}x = - \int_0^L y^3 + 2xy^2 \partial_x y \, \mathrm{d}x.$$

This yields

$$\int_0^L xy^2 \partial_x y \, \mathrm{d}x = -\frac{1}{3} \int_0^L y^3 \, \mathrm{d}x.$$

This additional term is then treated in the same manner as the source term f in Appendix A.1. This explains the necessity of the boundedness of $\|y\|_{L^3(I \times \Omega)}$, which is guaranteed by the fact that $y \in L^4(I, L^4(\Omega))$. The estimate for the time derivative $\|\partial_t y\|_{L^2(I, \mathcal{V}^*)}$ follows from (5.2), Lemma 5.1 and the global estimate for $\|y\|_{\mathcal{B}}$. ∎

Now we are in the position to show the existence of an optimal control.

PROPOSITION 5.4. *There exists a solution* $(\bar{y}, \bar{u}) \in \tilde{Y} \times \mathcal{M}(\Omega_c, L^2(I))$ *to the optimal control problem* (5.1).

Proof. Since the cost functional J is positive it has an infimum $\bar{J}$. Moreover we already have shown in Proposition 3.7 existence of a control-state pair $(y, u) \in \tilde{Y} \times \mathcal{M}(\Omega_c, L^2(I))$ (for small data, see Remark 3.8). Thus there exists a minimizing sequence of state-control pairs $(y_n, u_n) \in \tilde{Y} \times \mathcal{M}(\Omega_c, L^2(I))$ such that $J(y_n, u_n) \to \bar{J}$ as $n \to \infty$. Furthermore there is a $\varepsilon > 0$ such that

$$\varepsilon + J(0, 0) \geq \alpha \|u_n\|_{\mathcal{M}(\Omega_c, L^2(I))} + \frac{\eta}{4} \|y\|^4_{L^4(I \times \Omega)} \tag{5.4}$$

for n large enough. Therefore the sequence (y_n, u_n) is bounded in $L^4(I, L^4(\Omega)) \times \mathcal{M}(\Omega_c, L^2(I))$. Thanks to Proposition 5.3, we also have (y_n, u_n) is bounded in $\mathcal{B} \cap$

$H^1(I,\mathcal{V}^*)\times\mathcal{M}(\Omega_c,L^2(I))$. This implies the existence of an element $(\bar{y},\bar{u})\in\mathcal{B}\cap H^1(I,\mathcal{V}^*)\times\mathcal{M}(\Omega_c,L^2(I))$ and a state-control subsequence (y_{n_k},u_{n_k}) such that u_{n_k} converges in the weak-$*$ topology of $\mathcal{M}(\Omega_c,L^2(I))$ towards $\bar{u}$ and y_{n_k} converges in the weak-$*$ topology of $\mathcal{B}\cap H^1(I,\mathcal{V}^*)$ towards $\bar{y}$. Moreover existence of $\hat{y}\in L^2(\Omega)$ such that $y_{n_k}(T)\rightharpoonup y(T)$ in $L^2(\Omega)$ is ensured. With the Aubin-Lions lemma, we get also

$$y_{n_k}\to\bar{y}\quad\text{in}\quad L^2(I,\mathcal{C}_0(\Omega)).$$

Next we check that $(\bar{y},\bar{u})$ is a state-control pair according to Definition 5.2. The convergence of the linear terms in (5.2) is obvious. The convergence of the nonlinear terms is due to strong convergence of y_{n_k} towards $\bar{y}$. Therefore $(\bar{y},\bar{u})$ is a state-control pair and satisfies the weak formulation (5.2). The tracking functional (in which we include the L^4 part) is weakly lower semi-continuous in $L^4(I\times\Omega)\times L^2(\Omega)$, the control term is weakly-$*$ lower semi-continuous in $\mathcal{M}(\Omega_c,L^2(I))$. Therefore the state-control pair $(\bar{y},\bar{u})$ is a minimizer of J. ∎

In the next section we proceed with the derivation of the optimality conditions for problem (1.2) posed on U_{ad}.

6. First order optimality conditions. Next we discuss the differentiability of the control-to-state operator S defined in (4.2).

PROPOSITION 6.1. *The control to state operator $S\colon U_{ad}\to\mathcal{B}$ is continuously Fréchet-differentiable. Its derivative*

$$S'(u)\colon\mathcal{M}(\Omega_c,L^2(I))\to\mathcal{B},\ \delta u\mapsto\delta y$$

at $u\in U_{ad}$ is given by the solution operator of the linear tangent equation

$$\begin{cases}\partial_t\delta y+\partial_x\delta y-\gamma\partial_{xx}\delta y+\partial_{xxx}\delta y+\partial_x(y\delta y)=\delta u\ \text{in}\ I\times\Omega, & \text{(6.1a)}\\ \delta y(\cdot,0)=\delta y(\cdot,L)=\partial_x\delta y(\cdot,L)=0\ \text{on}\ I, & \text{(6.1b)}\\ \delta y(0,\cdot)=0\ \text{on}\ \Omega. & \text{(6.1c)}\end{cases}$$

Proof. First we mention that the non-linearity $F:\mathcal{B}\to L^2(I,H^{-1}(\Omega)),\ y\mapsto y\partial_x y$ is Fréchet differentiable since there holds

$$\|F(y+\delta y)-F(y)-F'(y)\delta y\|_{L^2(I,H^{-1}(\Omega))}\le\frac{1}{2}\|\delta y\|^2_{L^4(I\times\Omega)}\le c\,\|\delta y\|^2_{\mathcal{B}}$$

with $F'(y)\delta y=\partial_x(y\delta y)$ for any $\delta y\in\mathcal{B}\hookrightarrow L^4(I\times\Omega)$. Then we differentiate the fixed point equation $y=\mathcal{L}(u-y\partial_x y,y_0)$ with respect to (y,u) in direction $(\delta y,\delta u)\in\mathcal{B}\times\mathcal{M}(\Omega_c,L^2(I))$ and get

$$\delta y=\mathcal{L}'(\delta u-\partial_x(y\delta y))\tag{6.2}$$

where $\mathcal{L}'(\cdot)=\mathcal{L}(\cdot,0)$. In the Appendix A.3 it is shown that (6.2) has a unique solution $\delta y\in\mathcal{B}$. Actually δy is the weak solution of (6.1a)-(6.1c) in the sense of (3.9). Next we show that $S'(u)\delta u:=\delta y$ is the Fréchet derivative of S. This will result from the study of

$$\frac{1}{\|\delta u\|_{\mathcal{M}(\Omega_c,L^2(I))}}\|S(u+\delta u)-S(u)-S'(u)\delta u\|_{\mathcal{B}}=\frac{1}{\|\delta u\|_{\mathcal{M}(\Omega_c,L^2(I))}}\|\tilde{y}-y-\delta y\|_{\mathcal{B}}.$$

Calling $w=\tilde{y}-y-\delta y\in\mathcal{B}$ the function w then satisfies

$$\begin{cases}\partial_t w+\partial_x w+\partial_{xxx}w-\gamma\partial_{xx}w+\tilde{y}\partial_x\tilde{y}-y\partial_x y-\partial_x(y\delta y)=0\text{ in }I\times\Omega,\\ w(\cdot,0)=w(\cdot,L)=\partial_x w(\cdot,L)=0\text{ in }I,\\ w(0,\cdot)=0\text{ in }\Omega.\end{cases}$$

in the weak sense of (3.9). After rearranging the terms we end up with

$$\begin{cases} \partial_t w + \partial_x w + \partial_{xxx} w - \gamma \partial_{xx} w + \partial_x (yw) = -(\tilde{y} - y)\partial_x(\tilde{y} - y) \text{ in } I \times \Omega, \\ w(\cdot, 0) = w(\cdot, L) = \partial_x w(\cdot, L) = 0 \text{ in } I, \\ w(0, \cdot) = 0 \text{ in } \Omega. \end{cases}$$

According to Appendix A.3 and Lemma 3.6 it holds

$$\|w\|_{\mathcal{B}} \leq C(T, L, u) \|(\tilde{y} - y)\partial_x(\tilde{y} - y)\|_{L^2(I, H^{-1}(\Omega))} \leq C(T, L, u)\, T^{1/4} \|\tilde{y} - y\|_{\mathcal{B}}.$$

Therefore the conclusion follows from local Lipschitz continuity of S. Hence we provide the following lemma which concludes the proof. ∎

LEMMA 6.2. *The control-to-state operator $S\colon U_{ad} \to \mathcal{B}$ is locally Lipschitz continuous, i.e., for every $u \in U_{ad}$ there exists a neighbourhood $V \subset U_{ad}$ and a constant $C(T, L, u, \tilde{u}) > 0$ such that*

$$\|S(u) - S(\tilde{u})\|_{\mathcal{B}} \leq C(T, L, u, \tilde{u})\, \|u - \tilde{u}\|_{\mathcal{M}(\Omega_c, L^2(I))} \quad \forall \tilde{u} \in V.$$

Proof. We define $y = S(u) \in \mathcal{B}$ and $\tilde{y} = S(\tilde{u}) \in \mathcal{B}$ for $\tilde{u} \in V$. Therefore, the difference $w = \tilde{y} - y$ satisfies the equation

$$\begin{cases} \partial_t w + \partial_x w + \partial_{xxx} w - \gamma \partial_{xx} w + \dfrac{1}{2}\partial_x((y + \tilde{y})w) = \tilde{u} - u \text{ in } I \times \Omega, \\ w\cdot, 0) = w(\cdot, L) = \partial_x w(\cdot, L) = 0 \text{ in } I, \\ w(0, x) = 0 \text{ in } \Omega, \end{cases}$$

in the weak sense of (3.9). According to Appendix A.3 $w \in \mathcal{B}$ satisfies the a priori estimates

$$\|w\|_{\mathcal{B}} \leq C(T, L, u, \tilde{u}) \|u - \tilde{u}\|_{\mathcal{M}(\Omega_c, L^2(I))}.$$

Thus we can conclude that the solution operator S is locally Lipschitz continuous. ∎

Therefore the control-to-observation operator S_{obs} is also Fréchet differentiable and its derivative is given by

$$S'_{obs}(u)\colon \delta u \mapsto (\chi_{\Omega_o} \delta y, \chi_{\Omega_o} \delta y(T))$$

where $\delta y \in \mathcal{B}$ is the weak solution of (6.1). Next we introduce the adjoint control to observation operator $S'^{\star}$.

PROPOSITION 6.3. *Let $u \in U_{ad}$. There exists a bounded linear operator*

$$S'^{\star}_{obs}(u)\colon L^2(I \times \Omega_o) \times L^2(\Omega_o) \to \mathcal{C}(\Omega_c, L^2(I)), (\phi, p_T) \mapsto p$$

which fulfills

$$\begin{aligned} (6.3) \quad (S'_{obs}(u)\delta u, \phi)_{L^2(I \times \Omega_o)} &+ ((S'_{obs}(u)\delta u)(T), p_T)_{L^2(\Omega_o)} \\ &= \langle \delta u, S'^{\star}_{obs}(u)(\phi, p_T) \rangle_{\mathcal{M}(\Omega_c, L^2(I)), \mathcal{C}(\Omega_c, L^2(I))} \\ &\qquad \forall \delta u \in \mathcal{M}(\Omega_c, L^2(I)),\ (\phi, p_T) \in L^2(I \times \Omega_o) \times L^2(\Omega_o). \end{aligned}$$

Moreover it is the solution operator of

$$\begin{cases} -\partial_t p - \partial_x p - \gamma\partial_{xx} p - \partial_{xxx} p - y\partial_x p = \phi \text{ in } I\times\Omega, & \text{(6.4a)}\\ p(\cdot,0) = p(\cdot,L) = \partial_x p(\cdot,0) = 0 \text{ in } I, & \text{(6.4b)}\\ p(T,\cdot) = p_T \text{ in } \Omega & \text{(6.4c)}\end{cases}$$

with $y = S(u)$ *for* $(\phi, p_T) \in L^2(I\times\Omega_o)\times L^2(\Omega_o)$.

Proof. First of all we mention that $y\partial_x p \in L^1(I, L^2(\Omega))$ holds for $y \in \mathcal{B}$ and $p \in \mathcal{B}$, cf., Lemma A.5. We use the weak formulation of the tangent equation and get

$$\begin{aligned}(6.5)\quad &(S'_{obs}(u)\delta u, \phi)_{L^2(I\times\Omega_o)} + ((S'_{obs}(u)\delta u)(T), p_T)_{L^2(\Omega_o)}\\ &\quad= (\delta y, \chi^*_{\Omega_o}\phi)_{L^2(I\times\Omega)} + (\delta y(T), \chi^*_{\Omega_o} p_T)_{L^2(\Omega)} - (\delta y, y\partial_x p)_{L^2(I\times\Omega)}\\ &\quad= \langle \delta u, p\rangle_{\mathcal{M}(\Omega_c,L^2(I)),\mathcal{C}(\Omega_c,L^2(I))}\end{aligned}$$

for $\delta u \in \mathcal{M}(\Omega_c, L^2(I))$ and $y = S(u) \in \mathcal{B}$ where $\chi^*_{\Omega_o}$ is the extension operator to Ω. By comparing with (6.3), we set $S'^{\star}(u)(\phi, p_T) := \chi_{\Omega_c} p$ where p solves the fixed point equation

$$p(t) = W^*(T-t)p_T + \int_t^T W^*(s-t)(\phi(s) - y(s)\partial_x p(s))\,\mathrm{d}s, \quad t \in I. \tag{6.6}$$

In Appendix A.4 we show that the fixed point equation (6.6) has a unique solution $p \in \mathcal{B} \hookrightarrow \mathcal{C}(\Omega_c, L^2(I))$ which depends continuously on (ϕ, p_T). ∎

Next we derive first order optimality conditions using tools from convex analysis.

PROPOSITION 6.4. *Let* $(\bar{y}, \bar{u}) \in \mathcal{B}\times U_{ad}$ *be a solution of* (1.2)*. Then* $\bar{u}$ *satisfies the following variational variational inequality*

$$\langle -\chi_{\Omega_c}\bar{p}, u - \bar{u}\rangle_{\mathcal{C}(\Omega_c,L^2(I)),\mathcal{M}(\Omega_c,L^2(I))} + \|\bar{u}\|_{\mathcal{M}(\Omega_c,L^2(I))} \le \|u\|_{\mathcal{M}(\Omega_c,L^2(I))} \quad \forall u \in U_{ad}, \tag{6.7}$$

where $\bar{p}$ *is the solution of the adjoint state equation*

$$\begin{cases} -\partial_t\bar{p} - \partial_x\bar{p} - \gamma\partial_{xx}\bar{p} - \partial_{xxx}\bar{p} - \bar{y}\partial_x\bar{p} = \chi_{\Omega_o}\bar{y} - z_1 \text{ in } \Omega,\\ \bar{p}(\cdot,0) = \bar{p}(\cdot,L) = \partial_x\bar{p}(\cdot,0) = 0 \text{ in } I,\\ \bar{p}(T,\cdot) = \chi_{\Omega_o} y(T) - z_2 \text{ in } \Omega.\end{cases}$$

Proof. We define

$$F(u_1,u_2) = \frac{1}{2}\left(\|y_1 - z_1\|^2_{L^2(I\times\Omega_o)} + \|y_2 - z_2\|^2_{L^2(\Omega_o)}\right)$$

for $(y_1, y_2) \in L^2(I\times\Omega_o)\times L^2(\Omega_o)$ and $\psi(u) = \|u\|_{\mathcal{M}(\Omega_c,L^2(I))}$. Since F and S_{obs} are Fréchet differentiable the directional derivative of $F\circ S_{obs}$ at $\bar{u}$ has the form

$$D(F\circ S_{obs}, \bar{u}, \delta u) = \langle S'^{\star}_{obs}(\bar{u})(S_{obs}(\bar{u}) - z), \delta u\rangle_{\mathcal{C}(\Omega_c,L^2(I)),\mathcal{M}(\Omega_c,L^2(I))}, \quad \delta u \in \mathcal{C}(\Omega_c, L^2(I)).$$

Then we set $\chi_{\Omega_c}\bar{p} := S'^{\star}_{obs}(\bar{u})(S_{obs}(\bar{u}) - z)$. An element $\bar{u} \in U_{ad}$ is optimal if and only if

$$F\circ S_{obs}(\bar{u}) + \psi(\bar{u}) \le F\circ S_{obs}(u) + \psi(u) \quad \forall u \in U_{ad}$$

and in particular

$$F \circ S_{obs}(\bar{u}) + \psi(\bar{u}) \leq F \circ S_{obs}(\bar{u} + \varepsilon(u - \bar{u})) + \psi(\bar{u} + \varepsilon(u - \bar{u}))$$

for some $0 < \varepsilon$ small enough such that $\bar{u} + \varepsilon(u - \bar{u}) \in U_{ad}$ holds. Using the convexity of ψ we get

$$\frac{F \circ S_{obs}(\bar{u}) - F \circ S_{obs}(\bar{u} + \varepsilon(u - \bar{u}))}{\varepsilon} + \psi(\bar{u}) \leq \psi(u)$$

which implies

$$\langle -\chi_{\Omega_c}\bar{p}, u - \bar{u}\rangle_{\mathcal{C}(\Omega_c,L^2(I)),\mathcal{M}(\Omega_c,L^2(I))} + \psi(\bar{u}) \leq \psi(u) \quad \forall u \in U_{ad}.$$

□

The subgradient conditions can be equivalently reformulated in the following form.

PROPOSITION 6.5. *The subgradient condition* (6.7) *is equivalent to*

$$\alpha\|\bar{u}\|_{\mathcal{M}(\Omega_c,L^2(I))} + \phi^\star(-\chi_{\Omega_c}\bar{p}) = \langle -\chi_{\Omega_c}\bar{p}, \bar{u}\rangle_{\mathcal{C}(\Omega_c,L^2(I)),\mathcal{M}(\Omega_c,L^2(I))}, \quad \bar{u} \in U_{ad} \tag{6.8}$$

with

$$\phi^\star(p) = \sup_{u \in U_{ad}} [\langle u, p\rangle_{\mathcal{C}(\Omega_c,L^2(I)),\mathcal{M}(\Omega_c,L^2(I))} - \alpha\|u\|_{\mathcal{M}(\Omega_c,L^2(I))}]$$

for $p \in \mathcal{C}(\Omega_c, L^2(I))$.

Proof. This a well known characterization of the subdifferential of a convex function, cf., [21]. However the assertion can be easily derived from the definition of the of $\phi^\star$ and (6.7). □

Next we characterize $\phi^\star$.

LEMMA 6.6. *The functional* $\phi^\star(p)\colon \mathcal{C}(\Omega_c, L^2(I)) \to \mathbb{R}$ *has the form*

$$\phi^\star(p) = \hat{c}\left(\|p\|_{\mathcal{C}(\Omega_c,L^2(I))} - \alpha\right)^+$$

where $(\cdot)^+ = \max(0, \cdot)$ *is the positive part of a function.*

Proof.

$$\begin{aligned}\phi^\star(p) &= \sup_{u \in U_{ad}} [\langle u, p\rangle_{\mathcal{C}(\Omega_c,L^2(I)),\mathcal{M}(\Omega_c,L^2(I))} - \alpha\|u\|_{\mathcal{M}(\Omega_c,L^2(I))}] \\ &= \sup_{\lambda \in [0,\hat{c}]} \sup_{\|u\|_{\mathcal{M}(\Omega_c,L^2(I))}=1,} \lambda[\langle u, p\rangle_{\mathcal{C}(\Omega_c,L^2(I)),\mathcal{M}(\Omega_c,L^2(I))} - \alpha].\end{aligned}$$

This implies the assertion since

$$\|p\|_{\mathcal{C}(\Omega_c,L^2(I))} = \sup_{\|u\|_{\mathcal{M}(\Omega_c,L^2(I))}=1} \langle u, p\rangle_{\mathcal{C}(\Omega_c,L^2(I)),\mathcal{M}(\Omega_c,L^2(I))}.$$

□

Using Proposition 6.5 and Lemma 6.6 we can derive the following structural properties of the optimal control.

PROPOSITION 6.7. *Let* $\bar{u} \in U_{ad}$ *be an optimal control of* (1.2). *Moreover let* $|\bar{u}| \in \mathcal{M}(\Omega)$ *be its total-variation measure and* $\bar{u}'$ *its the Radon-Nikodym-derivative. Furthermore let* $\bar{p}$ *be the corresponding optimal adjoint state. Then there holds*

$$0 = \left(\|\bar{u}\|_{\mathcal{M}(\Omega_c,L^2(I))} - \hat{c}\right)\bar{\lambda}, \tag{6.9}$$

$$\operatorname{supp}|\bar{u}| \subseteq \{x \in \Omega\colon \|\chi_{\Omega_c}\bar{p}\|_{L^2(I)} = \alpha + \bar{\lambda}\} \tag{6.10}$$

with $\bar\lambda = (\|\chi_{\Omega_c}\bar p\|_{\mathcal{C}(\Omega_c,L^2(I))} - \alpha)^+$. *Moreover we have*

$$\bar u'(x) = -\frac{\chi_{\Omega_c}\bar p(x)}{\alpha+\bar\lambda} \quad \text{in } L^1((\mathring{\Omega}_c,|\bar u|),L^2(I)). \tag{6.11}$$

Proof. Using $\bar u \in U_{ad}$, $\|\chi_{\Omega_c}\bar p\|_{\mathcal{C}(\Omega_c,L^2(I))} - \bar\lambda \le \alpha$ and (6.8) we can estimate

$$\begin{aligned}
\alpha\|\bar u\|_{\mathcal{M}(\Omega_c,L^2(I))} &= \int_{\Omega_c} (\bar u', \chi_{\Omega_c}\bar p)_{L^2(I)} \,\mathrm{d}|\bar u| - \hat c\bar\lambda \\
&\le \int_{\Omega_c} (\bar u', \chi_{\Omega_c}\bar p)_{L^2(I)} \,\mathrm{d}|\bar u| - \bar\lambda\|\bar u\|_{\mathcal{M}(\Omega_c,L^2(I))} \\
&\le \int_{\Omega_c} \|\bar u'\|_{L^2(I)}\|\chi_{\Omega_c}\bar p\|_{L^2(I)} \,\mathrm{d}|\bar u| - \bar\lambda\|\bar u\|_{\mathcal{M}(\Omega_c,L^2(I))} \\
&\le \|\bar u\|_{\mathcal{M}(\Omega_c,L^2(I))}(\|\chi_{\Omega_c}\bar p\|_{\mathcal{C}(\Omega_c,L^2(I))} - \bar\lambda) \le \alpha\|\bar u\|_{\mathcal{M}(\Omega_c,L^2(I))}.
\end{aligned} \tag{6.12}$$

Thus the last chain of inequalities holds with equality and therefore we have

$$\int_{\Omega_c} (\bar u', \chi_{\Omega_c}\bar p)_{L^2(I)} - \|\bar u'\|_{L^2(I)}\|\chi_{\Omega_c}\bar p\|_{L^2(I)} \,\mathrm{d}|\bar u| = 0$$

which implies

$$\bar u' = -\frac{1}{\|\chi_{\Omega_c}\bar p\|_{L^2(I)}}\,\chi_{\Omega_c}\bar p \quad \text{in } L^1((\Omega_c,|\bar u|),L^2(I)).$$

Moreover we have

$$\int_{\Omega_c} \|\chi_{\Omega_c}\bar p\|_{L^2(I)} - \bar\lambda - \alpha \,\mathrm{d}|\bar u| = 0$$

which then implies (6.10) and (6.11). Equality in (6.12) also implies

$$(\hat c - \alpha\|\bar u\|_{\mathcal{M}(\Omega_c,L^2(I))})\bar\lambda = 0.$$

□

REMARK 6.8. *If the constraint* $\|\bar u\|_{\mathcal{M}(\Omega_c,L^2(I))} \le \hat c$ *is not active there holds* $\bar\lambda = 0$ *and we recover the usual optimality conditions; c.f. [28]. This can be guaranteed for a large enough parameter* α.

7. Discretization and numerical results. This section is devoted to the discretization of the optimization problem (1.2) - (1.3c), as well as some insights into the numerical optimization process we use.

7.1. Discretization of state and adjoint equation. In the case of bounded domains numerous schemes are available, be it finite differences [19, 51], finite elements [49, 1], finite volumes [20], discontinuous Galerkin schemes [4, 50], or polynomial spectral methods [33, 34, 42]. One of the most recent and efficient method for the Korteweg-de Vries equation is proposed in [33]. The linear term is treated by a Petrov-Galerkin method on the Legendre polynomials, while the nonlinear term is treated pseudospectrally on the Chebyschev collocation points. Shortly after, Shen [42] proposed an improvement of this Petrov-Galerkin method with nearly optimal computational complexity. This will be our method of choice and we recall it here briefly. Since it has never been used on optimal control problem, we also discuss an appropriate time discretization which allows the derivation of a consistent adjoint time stepping.

7.1.1. The dual Petrov-Galerkin method. The interesting feature of this method lies in the choice of the test and trial functions bases. They are chosen as a compact combination of Legendre polynomials in such a way that the trial functions satisfy the underlying boundary conditions of the equation and the test functions satisfy the dual boundary conditions (3.2b). Therefore, most matrices involved in the resolution of the problem are sparse or well-conditioned [42]. We present the method for some reference domain $[-1,1]$, but it can be extended to any other domain of type $[a,b]$ by scaling the Legendre polynomials and the integrals. We denote by P_N the space of polynomials of degree $\leq N$ and set

$$V_N = \{y \in P_N : y(1) = y(-1) = \partial_x y(1) = 0\}\,, \tag{7.1}$$

$$V_N^* = \{y \in P_N : y(1) = y(-1) = \partial_x y(-1) = 0\}\,. \tag{7.2}$$

We consider then the semi-discrete problem: find

$$y_N : \quad I \to V_N, \quad t \mapsto y_N(t,\cdot),$$

such that for almost every $t \in I$

$$\begin{aligned} &\langle \partial_t y_N, \varphi_N\rangle - (y_N, \partial_x \varphi_N) + (\partial_x y_N, \partial_{xx}\varphi_N) + \\ &\qquad \gamma\,(\partial_x y_N, \partial_x \varphi_N) - \left(\frac{y_N^2}{2}, \partial_x \varphi_N\right) = \langle u, \varphi_N\rangle, \quad \forall v_N \in V_N^*, \end{aligned} \tag{7.3}$$

where $(\cdot,\cdot)$ denotes the usual $L^2(\Omega)$ spatial inner product and $\langle\cdot,\cdot\rangle$ is the spatial duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, as in the continuous case. Let us point out that the weak solution of (3.1a) - (3.1c) has $\mathcal{B}\cap H^1(I,\mathcal{V}^*)$ regularity, and therfore also satisfies (7.3) for all test functions in $\mathcal{V}$.

Denoting by L_k the kth Legendre polynomial, we can define the basis functions as follows (see Figure 1)

$$\phi_k(x) = L_k(x) - \frac{2k+3}{2k+5}L_{k+1}(x) - L_{k+2}(x) + \frac{2k+3}{2k+5}L_{k+3}(x), \tag{7.4}$$

$$\psi_k(x) = L_k(x) + \frac{2k+3}{2k+5}L_{k+1}(x) - L_{k+2}(x) - \frac{2k+3}{2k+5}L_{k+3}(x). \tag{7.5}$$

Thus for $N \geq 3$, we have

$$\begin{aligned} V_N &= \{\phi_0, \phi_1, ..., \phi_{N-3}\}\,,\\ V_N^* &= \{\psi_0, \psi_1, ..., \psi_{N-3}\}\,. \end{aligned} \tag{7.6}$$

Then, we represent the semi-discrete state variable $y_N(t,\cdot)$ on the spectral space and give its vector representation:

$$y_N(t,\cdot) = \sum_{k=0}^{N-3} \hat{y}_k(t)\phi_k(\cdot), \quad \mathbf{y}(t) = (\hat{y}_0(t), \hat{y}_1(t), ..., \hat{y}_{N-3}(t))^T\,. \tag{7.7}$$

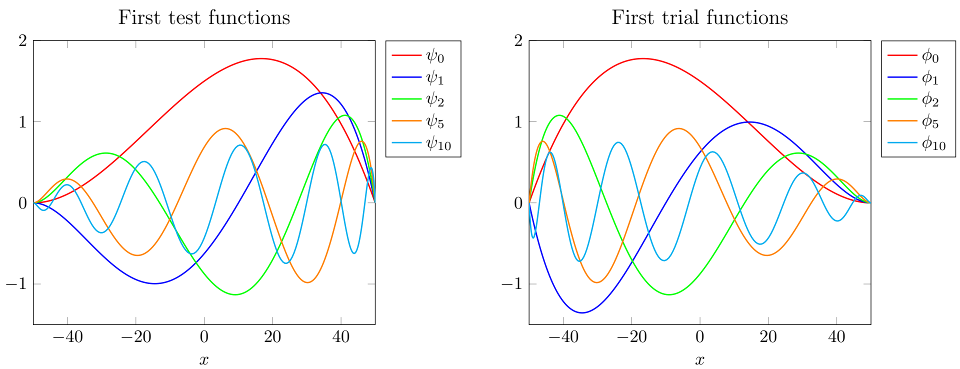

FIG. 1. *First test and trial functions.*

The vector representation of the control is given by:

$$\mathbf{u}(t) = (\langle u_N(t,\cdot), \psi_0(\cdot)\rangle, \langle u_N(t,\cdot), \psi_1(\cdot)\rangle, ..., \langle u_N(t,\cdot), \psi_{N-3}(\cdot)\rangle)^T \tag{7.8}$$

where the expression for the semi-discrete control $u_N(t,\cdot)$ is given in Section 7.1.2. Afterwards, one builds the matrices M, P, Q and S of size $(N-2)\times(N-2)$ with coefficients m_{ij}, p_{ij}, q_{ij} and s_{ij} defined as follows:

$$m_{ij} = (\phi_j, \psi_i), \quad p_{ij} = -(\partial_x \phi_j, \psi_i), \quad q_{ij} = (\partial_x \phi_j, \partial_x \psi_i), \quad s_{ij} = (\partial_x \phi_j, \partial_{xx} \psi_i). \tag{7.9}$$

The variational formulation (7.3) thus yields

$$M\frac{d\mathbf{y}}{dt} + (-P + \gamma Q + S)\,\mathbf{y} + F(\mathbf{y}) = \mathbf{u}, \tag{7.10}$$

where $F(\mathbf{y})$ represents the nonlinear term and is treated as suggested in [42] i.e. using the pseudospectral approach. It means that the nonlinearity is evaluated in the spatial domain, that we choose to be the Chebyshev-Gauss-Lobatto (CGL) points, and transfered back in the Legendre spectral space. We therefore need to be able to transform back and forth from the spectral space of Legendre coefficients to the values on the CGL points. This can be done using the fast Fourier transform (FFT) and the Chebyshev-Legendre transform. However, for the polynomial degrees we consider here (between 160 and 512), we rather use the direct method which is faster and easier to handle, especially when it comes to finding a discrete adjoint (see Section 7.1.3). We build beforehand the matrices $L_1 = (\phi_j(x_i))$ and $L_2 = (\psi_j(x_i))$, $i = 0\ldots N$, $j = 0\ldots N-3$, where the points x_i are the CGL points such that

$$L_1\mathbf{y}(t) = \tilde{\mathbf{y}}(t) = (y_N(t,x_0), y_N(t,x_2), ...y_N(t,x_N))^T, \tag{7.11}$$

and the same holds with L_2 for a variable in the dual space. Note that the spatial space, in which $\tilde{\mathbf{y}}(t)$ lies and the spectral space, in which $\mathbf{y}(t)$ lies do not apparently have the same dimensions ($N+1$ versus $N-2$). This is only due to the fact that the three boundary conditions are already included in the spectral space and not in the spatial space.

7.1.2. Semi-discretization of the control u**.** By $\{x_i\}, i = 0, 1, \ldots, N$, we denote the grid points of the spatial mesh (i.e. in our case the CGL points). The control is then discretized as follows

$$u_N(t,\cdot) = \sum_{j=0}^{N} \hat{u}_j(t)\delta_{x_j}(\cdot) \tag{7.12}$$

where the functions $\hat{u}_j(t)$ are the time-dependent coefficients of the control at the grid points and δ_{x_j} are Dirac functionals located at the grid points. Hence, we clearly have $u_N \in \mathcal{M}(\Omega_c, L^2(I))$ and can easily define any inner product required in the resolution of the state equation (7.8)

$$\langle u_N(t,\cdot), \psi(\cdot)\rangle = \sum_{j=0}^{N} \hat{u}_j(t)\psi(x_j). \tag{7.13}$$

Beside $\mathbf{u}(t)$ defined in (7.8) we introduce another vectorized representation of the variable $u_N(t,\cdot)$ in the space spanned by the Dirac delta functions previously mentioned

$$\hat{\mathbf{u}}(t) = (\hat{u}_0(t), \ldots, \hat{u}_N(t))^T \,. \tag{7.14}$$

This one will be useful in the definition of the various norms involved in the optimization process and described in the forthcoming sections.

7.1.3. Time stepping scheme and adjoint - Crank-Nicolson-Leap-Frog. We have to deal with a problem of high order derivative. Therefore an explicit temporal discretization would lead to excessively small time steps in order to get stability. An implicit method should rather be considered. In [31], the authors prove convergence of a pseudospectral method with backward Euler scheme for the KdV equation. However in practice, the first order accuracy in time authorizes only very small time steps. A second order implicit scheme like the Crank-Nicolson scheme should be preferable. This scheme has the advantage of being a method of choice in optimal control: using the representation of the Crank-Nicolson scheme as a continuous Galerkin method of degree one (continuous linear trial functions and discontinuous piecewise constant test functions) allows us to give directly the concrete form of the adjoint, tangent and additional adjoint equations leading to the exact computation of the discrete gradient and Hessian (see, e.g. [35]). Note that the use of the correct discrete derivatives is important for the convergence of our optimization algorithm. However, the Crank-Nicolson scheme is computationally demanding. An alternative is then the two-step Crank-Nicolson Leap Frog method. In this setting, the third derivative is treated implicitely and the nonlinear term is treated explicitely. This method has already been extensively used for the KdV equation [42, 33, 34]. Commonly, it is initialized by a semi-implicit step. We suggest also a slight modification of the last step of this two-step method in order to get a discrete adjoint that is consistent with the continuous adjoint in both the distributed control problem or the terminal observations problem.

The time domain is devided into N_T intervals $0 = t_0 < t_1 .... < t_{N_T} = T$. While the first and last one have length $\frac{\Delta t}{2}$, the other steps are of equal length Δt. We denote hereafter the discrete state variable representation

$$\mathbf{y}_n = \mathbf{y}(t_n) = (\hat{y}_0(t_n), \hat{y}_1(t_n), ..., \hat{y}_{N-3}(t_n))^T \,, \quad n = 0 \ldots N_T$$

and for the control

$$
\begin{aligned}
\mathbf{u}_n &= \mathbf{u}(t_n) = \left(\langle u_N(t_n,\cdot),\psi_0(\cdot)\rangle, \langle u_N(t_n,\cdot),\psi_1(\cdot)\rangle, ..., \langle u_N(t_n,\cdot),\psi_{N-3}(\cdot)\rangle\right)^T, \\
\hat{\mathbf{u}}_n &= \hat{\mathbf{u}}(t_n) = \left(\hat{u}_0(t_n),\dots,\hat{u}_N(t_n)\right)^T, \quad n = 0\dots N_T
\end{aligned} \tag{7.15}
$$

where the inner products are evaluated according to (7.13). With the help of the matrices defined in (7.9), the forward scheme is as follows

$$
\begin{cases}
\frac{1}{2}\left(M+\Delta t S\right)\mathbf{y}_1 = \frac{1}{2}\left(M+\Delta t\left(P+\gamma Q\right)\right)\mathbf{y}_0 + \frac{1}{2}\Delta t F(\mathbf{y}_0) + \frac{1}{2}\Delta t \mathbf{u}_0 \\
\frac{1}{2}\left(M+\Delta t S\right)\mathbf{y}_{n+1} = \frac{1}{2}\left(M-\Delta t S\right)\mathbf{y}_{n-1} + \Delta t\left(P+\gamma Q\right)\mathbf{y}_n \\
\qquad\qquad + \Delta t F(\mathbf{y}_n) + \Delta t \mathbf{u}_n,\ n = 1\dots N_T-1 \\
M\mathbf{y}_{N_T+1} = \frac{1}{2}\left(M\mathbf{y}_{N_T} + M\mathbf{y}_{N_T-1}\right) + \frac{1}{2}\Delta t\left(P+\gamma Q\right)\mathbf{y}_{N_T} \\
\qquad\qquad + \frac{1}{2}\Delta t F(\mathbf{y}_{N_T}) - \frac{1}{2}\Delta t S \mathbf{y}_{N_T-1}
\end{cases} \tag{7.16}
$$

The first step is semi-implicit. The interior steps are classical CNLF steps. The last step - due to our modification - is not a regular, well-known step, but it is designed in such a way that the discrete adjoint shall be consistent with a CNLF discretization of the continuous adjoint. Notice that as Δt vanishes, it becomes just a mean between the two preceding steps.

As for the resolution of the optimization problem, the objective functional at the discrete level takes the form

$$
\begin{aligned}
J_{N,N_T}(\mathbf{y}_0,\dots,\mathbf{y}_{N_T},\mathbf{u}_0,\dots,\mathbf{u}_{N_T}) &= \frac{1}{2}\sum_{i=1}^{N_T}\Delta t\left(\mathbf{y}_i-\mathbf{z}_{1,i}\right)^t A\left(\mathbf{y}_i-\mathbf{z}_{1,i}\right) + \\
&\frac{1}{2}\left(\mathbf{y}_{N_T}-\mathbf{z}_2\right)^t A\left(\mathbf{y}_{N_T}-\mathbf{z}_2\right) + \alpha\sum_{j=1}^{N}\left(\sum_{i=1}^{N_T}\Delta t\hat{\mathbf{u}}_{ij}^2\right)^{1/2}
\end{aligned} \tag{7.17}
$$

This allows also to define a discrete Lagrangian:

$$
\begin{aligned}
&\mathcal{L}_{N,N_T}(\mathbf{y}_0,\dots,\mathbf{y}_{N_T},\hat{\mathbf{u}}_0,\dots,\hat{\mathbf{u}}_{N_T},\mathbf{u}_0,\dots,\mathbf{u}_{N_T},\mathbf{p}_0,\dots,\mathbf{p}_{N_T-1}) = \\
&\qquad J_{N,N_T}(\mathbf{y}_0,\dots,\mathbf{y}_{N_T},\hat{\mathbf{u}}_0,\dots,\hat{\mathbf{u}}_{N_T}) + \\
&\mathbf{p}_0^T\left[\frac{1}{2}\left(M+\Delta t S\right)\mathbf{y}_1 - \frac{1}{2}\left(M+\Delta t P+\gamma\Delta t Q\right)\mathbf{y}_0 - \frac{1}{2}\Delta t F(\mathbf{y}_0) - \frac{1}{2}\Delta t\mathbf{u}_0\right] + \\
&\sum_{k=1}^{N_T-2}\mathbf{p}_k^T\left[\frac{1}{2}\left(M+\Delta t S\right)\mathbf{y}_{k+1} - \frac{1}{2}\left(M-\Delta t S\right)\mathbf{y}_{k-1} - \Delta t\left(P+\gamma Q\right)y_k\right. \\
&\qquad \left. -\Delta t F(y_k) - \Delta t\mathbf{u}_k\right] + \\
&\mathbf{p}_{N_T-1}^T\left[M\mathbf{y}_{N_T} - \frac{1}{2}\left(M\mathbf{y}_{N_T-1}+M\mathbf{y}_{N_T-2}\right) - \frac{1}{2}\Delta t\left(P+\gamma Q\right)\mathbf{y}_{N_T-1}\right. \\
&\qquad \left. -\frac{1}{2}\Delta t F(\mathbf{y}_{N_T-1}) + \frac{1}{2}\Delta t S\mathbf{y}_{N_T-2}\right].
\end{aligned} \tag{7.18}
$$

From (7.18), one can classically derive the discrete adjoint, hereafter written in the case of terminal observations (i.e. we consider only the tracking part depending on z_2 in (7.17)), which will be of particular interest in the numerical examples:

$$\begin{cases} M^T \mathbf{p}_{N_T-1} = -A(\mathbf{y}_{N_T} - \mathbf{z}_2) \\ \frac{1}{2}\left(M^T + \Delta t S^T\right)\mathbf{p}_{N_T-2} = \frac{1}{2}\left(M^T + \Delta t P^T + \gamma \Delta t Q^T\right)\mathbf{p}_{N_T-1} \\ \qquad\qquad + \frac{1}{2}\Delta t F^{'}(\mathbf{y}_{N_T-1})^T \mathbf{p}_{N_T-1} \\ \frac{1}{2}\left(M^T + \Delta t S^T\right)\mathbf{p}_{n-2} = \frac{1}{2}\left(M^T - \Delta t S^T\right)\mathbf{p}_n + \Delta t\left(P^T + \gamma Q^T\right)\mathbf{p}_{n-1} \\ \qquad\qquad + \Delta t F^{'}(\mathbf{y}_{n-1})^T \mathbf{p}_{n-1}, \quad n = 2\ldots N_T - 1. \end{cases} \tag{7.19}$$

It starts with a projection of the tracking term in the state space into the adjoint space. This is followed by a single semi-implicit step and regular CNLF steps.

REMARK 7.1. *Without this modification of the last forward step, the discrete adjoint obtained in the case of terminal observation would be*

$$\begin{cases} \frac{1}{2}(M^T + \Delta t S^T)\mathbf{p}_{N_T-1} = -A(\mathbf{y}_{N_T} - \mathbf{z}_2) \\ \frac{1}{2}\left(M^T + \Delta t S^T\right)\mathbf{p}_{N_T-2} = \left(\Delta t P^T + \gamma \Delta t Q^T\right)\mathbf{p}_{N_T-1} + \Delta t F^{'}(\mathbf{y}_{N_T-1})^T \mathbf{p}_{N_T-1} \\ \frac{1}{2}\left(M^T + \Delta t S^T\right)\mathbf{p}_{n-2} = \frac{1}{2}\left(M^T - \Delta t S^T\right)\mathbf{p}_n + \Delta t\left(P^T + \gamma Q^T\right)\mathbf{p}_{n-1} \\ \qquad\qquad + \Delta t F^{'}(\mathbf{y}_{n-1})^T \mathbf{p}_{n-1}, \quad n = 2\ldots N_T - 1. \end{cases} \tag{7.20}$$

The major difference lies in the second step. It does not correspond to a consistent discretization of the continuous adjoint equation: the time derivative is not reconstructed. It results in practice in huge oscillations in time of the adjoint variable.

REMARK 7.2. *Unlike the other steps, the modification of the last forward step implies the inversion of the matrix M, which is typically ill-conditionned. However, this drawback is easily overcome by the use of a regularization term of type εI_d with epsilon of order h^2, where h is the average space step.*

The advantage of computing the adjoint lies in the following proposition

PROPOSITION 7.3. *Following the definition of the discrete objective functional J_{N,N_T} (7.17) and discrete lagrangian $\mathcal{L}_{N,N_T}$ (7.18), the discrete derivative of the reduced tracking cost term is given by $(-\mathbf{p}_0, \ldots, -\mathbf{p}_{N_T-1})$, where $\mathbf{p}$ is the discrete adjoint state defined by the numerical scheme (7.19) in the case of terminal observations.*

Proof. The proof is done by standard differentiation of the discrete Lagrangian. ∎

7.2. Numerical treatment of the optimization problem.

7.2.1. The regularized problem. Numerically, the optimization problem is solved via a Newton type method. This requires an additional regularization term which we introduce as follows

$$\begin{aligned} \min_{u\in L^2(\Omega_c\times I), y\in Y} J(y,u) &= \frac{1}{2}\left(\|\chi_{\Omega_o} y - z_1\|^2_{L^2(I\times\Omega_o)} + \|\chi_{\Omega_o} y(T) - z_2\|^2_{L^2(\Omega_o)}\right) \\ &\quad + \alpha\|u\|_{L^1(\Omega_c, L^2(I))} + \frac{\varepsilon}{2}\|u\|^2_{L^2(\Omega_c\times I)} \\ &= f(y) + \alpha\|u\|_{L^1(\Omega_c, L^2(I))} + \frac{\varepsilon}{2}\|u\|^2_{L^2(\Omega_c\times I)}, \end{aligned} \tag{7.21}$$

where u and y are subject to Korteweg-de Vries Burgers equation. This enables us to look for controls in the space $L^2(I \times \Omega)$. Since the embedding $L^1(\Omega, L^2(I)) \hookrightarrow \mathcal{M}(\Omega_c, L^2(I))$ is isometric, then any $y \in L^1(\Omega, L^2(I))$ will satisfy

$$\psi(u) = \|u\|_{\mathcal{M}(\Omega_c, L^2(I))} = \|u\|_{L^1(I, L^2(\Omega))} = \int_0^L \|u\|_{L^2(I)} dx. \tag{7.22}$$

In [25], the authors study this problem in the linear case, and prove that this setting promotes a *striped sparsity pattern*. In the nonlinear case, the development is analogous. To obtain the optimality conditions, one introduces the proximal map (see, e.g. [2]) of the $L^1(\Omega, L^2(I))$ norm:

$$\mathrm{Prox}_{\psi/\varepsilon}(q)(x) = \frac{1}{\varepsilon}\left(\varepsilon - \frac{\alpha}{\|q(x)\|_{L^2(I)}}\right)^+ q(x) \quad \text{for } q \in L^2(\Omega, L^2(I)). \tag{7.23}$$

We can then express the optimality condition for (7.21) by the pointwise projection formula

$$\bar{u}_\varepsilon = -\frac{1}{\varepsilon}\left(1 - \frac{\alpha}{\|p_\varepsilon\|_{L^2(I)}}\right)^+ p_\varepsilon. \tag{7.24}$$

It is then directly possible to solve via a Newton type method the equation

$$F(u_\varepsilon) = u_\varepsilon + \frac{1}{\varepsilon}\left(1 - \frac{\alpha}{\|p_\varepsilon\|_{L^2(I)}}\right)^+ p_\varepsilon = 0. \tag{7.25}$$

Following [39], one can also reformulate this optimality condition thanks to the "normal map", due to Robinson:

$$G(q_\varepsilon) = \varepsilon q_\varepsilon + \nabla f(\mathrm{Prox}_{\psi/\varepsilon}(q_\varepsilon)) = 0, \tag{7.26}$$

for an auxiliary variable q_ε such that $u_\varepsilon = \mathrm{Prox}_{\psi/\varepsilon}(q_\varepsilon)$. One easily prove in that case that $q_\varepsilon = -\frac{1}{\varepsilon} p_\varepsilon$ and equivalence between (7.25) and (7.26).

To solve problem (7.26), we apply a Newton-type method, and since the proximal operator is not differentiable, we use the concept of semismoothness to derive a general derivative for G [48]. An iterative method (conjugate gradient) is used to solve the Newton system. If a simple continuation strategy is often suggested to achieve global convergence in that case [25], this is not sufficient for our problem, due to the nonlinearity of the state equation. A globalization strategy based on the decrease of the objective functional is required. We implement it based on the truncated conjugate gradient algorithm for trust region problems by Steihaug [39, 46].

7.2.2. Discretization of the control and mass lumping. In the unregularized problem, the controls are discretized as nodal Dirac measures. In the regularized case, this is not possible anymore as we need a discretization that is compatible with $L^2(\Omega_c \times I)$ functionals. Thus, for this regularized case, one need to find a discretization that should lead to an equivalent problem to (1.2) in the limiting case $\varepsilon \to 0$ (i.e. to obtain a Dirac measure discretization as in Section 7.1.2). We suggest to follow the strategy in [28]. The control is discretized in space with piecewise linear, continuous finite elements on a grid whose nodes are the Chebyschev-Gauss-Lobatto points $(x_n), n = 0 \dots N$ previsously mentioned.

Moreover, the various norms involved in the optimization problem shall be computed via mass lumping, i.e, the trapezoidal rule is used for the evaluation of the spatial integrals for each cell. Therefore, for $u_h = \sum_{j=1}^{N_T} \chi_{I_j} \sum_{n=0}^{N} \hat{\mathbf{u}}_{jn} e_n$, where $I_j = (t_{j-1}, t_j]$ is the j^{th} time interval considered and e_n is the basis vector for piecewise linear, continuous finite elements centered at the grid point x_n, it holds
(7.27)

$$\|u_h\|_{L^1(\Omega,L^2(I))} = \sum_{n=0}^{N} d_n \left(\sum_{j=1}^{N_T} \Delta t \hat{\mathbf{u}}_{nj}^2 \right)^{1/2}, \quad \|u_h\|^2_{L^2(\Omega,L^2(I))} = \sum_{n=0}^{N} d_n \left(\sum_{j=1}^{N_T} \Delta t \hat{\mathbf{u}}_{nj}^2 \right),$$

$$\langle u_h, \psi \rangle = \sum_{j=1}^{N_T} \chi_{I_j} \sum_{n=0}^{N} d_n \hat{\mathbf{u}}_{jn} \psi_n$$

for all spectral basis test functions ψ where we have denoted $\psi_n = \psi(x_n)$, where $d_n = \int_{\Omega_c} e_n \, \mathrm{d}x$.

REMARK 7.4. *With* (7.27) *as expressions for the norms and inner products, all projections formulae involved at the continuous level in the optimization process can be transfered one to one to the (spatial) discrete vector. Therefore,* (7.25) *or* (7.26) *hold at the discrete level.*

7.3. Numerical examples. The numerical examples illustrate the problem of wave generation. We want to investigate the optimal position and vertical movement of a wavemaker. A similar study has been carried out in a slighty different framework in [38], where one possible application was to design artificial surfing facilities. For that purpose, we use a version of the forced Korteweg-de Vries equation with physical parameters, as the one derived in [36]

$$\partial_t y + f \partial_x y - \frac{1}{6} \partial_{xxx} y - \frac{3}{2} y \partial_x y = u \tag{7.28}$$

where the parameter f is proportional to $F - 1$ (F is the Froude number). Its value determines if a flow is subcritical ($f \leq 0$) or supercritical ($f > 0$). The forcing u can be interpreted as the spatial derivative of the bottom topography. In both examples, we consider the minimization problem

$$\min_{u \in \mathcal{M}(\Omega_c, L^2(I)), y \in Y} J(y, u) = \frac{1}{2} \|\chi_{\Omega_o} y(T) - y_d\|^2_{L^2(\Omega_o)} + \alpha \|u\|_{\mathcal{M}(\Omega_c, L^2(I))} \tag{7.29}$$

for different Ω_c, Ω_o and y_d, where y is the solution of (7.28). In particular, we emphasize that $U_{ad} = \mathcal{M}(\Omega_c, L^2(I))$.

Inverse problem. In the first example, we want to reconstruct the time varying bottom topography which creates waves. These waves are observed at final time T. We generate beforehand a wave with the following source term

$$u(t, x) = \begin{cases} 10\delta_{\{x=1.5\}}, \text{ when } 0 < t \leq 2.5 \\ 0, \text{ when } 2.5 \leq t \leq 5 \end{cases} \tag{7.30}$$

in a subcritical configuration ($f = -0.5$). The time horizon is $T = 5$. The space-time grid is parametrized with $N = 256$ and $\Delta t = 0.01$. The generated wave is displayed in Fig 2. One can see a series of downstream waves - note that the flow enters the domain from the right - generated by the bottom topography induced by u (recall that u is proportional to the derivative of the topography), while a solitary wave is going upstream, at constant speed and constant height, which is typical for the Korteweg-de Vries equation.. Afterwards, a white Gaussian noise is added to the y-profile at final time T such that the magnitude of the noise is in average 5 percent of the magnitude of the original signal $\left(\frac{\|\hat{y}-\hat{y}_{noisy}\|_2}{\|\hat{y}\|_2} \approx 0.05\right)$.

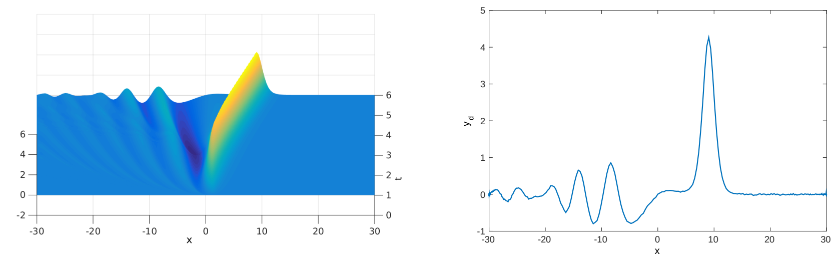

Fig. 2. *Left: wave generated by the forcing term u, for $f = -0.5$ (i.e. subcritical flow). Right: y_d-profile at final time with Gaussian white noise.*

At the end of the continuation strategy, the support of the control is, according to Proposition 6.7, included in the domains where $\|p\|_{L^2(I)} = \alpha$. One can see on Fig. 3 (left) that the final control is a point source, located close to $x = 1.5$. The time profile of the control is depicted on Fig. 3 (right) and it follows the original one quite well. We point out that the oscillations are caused by the noise. The noisy observation enters the adjoint equation as initial condition and the time profile of $\bar{u}$ is proportional to $t \mapsto \bar{p}(t, \hat{x})$, where $\hat{x}$ is the reconstructed position of the point source. The results of the optimization process are shown on Fig.4, for $\alpha = 0.1$.

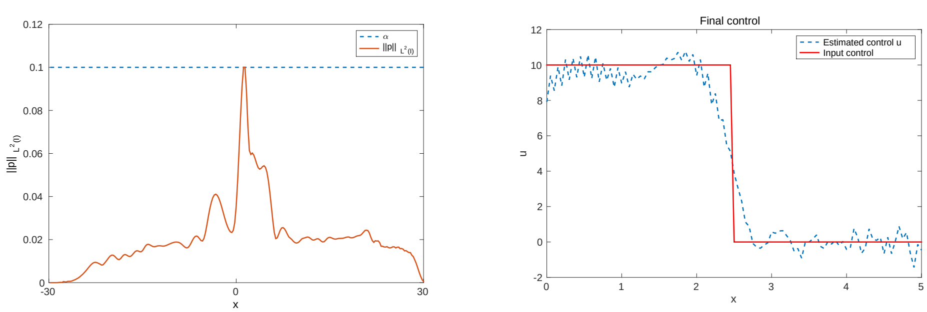

Fig. 3. *Left: spatial support of the control, determined by $\|p_\varepsilon\|_{L^2(0,T)} = \alpha$. Here $\alpha = 0.1$ and $\varepsilon = 10^{-6}$. Right: recovered control for $\varepsilon = 10^{-6}$*

<u>*Control example.*</u> The second example is a control example. We divide the domain $\Omega = [-L, L]$ into two subdomains. The domain $\Omega_c = [-L, 0]$ is the control domain, while $\Omega_o = [0, L]$ is the observation domain. See Fig. 5 (left) for the description of the setup. The flow enters the domain from the right and we want to create the wave profile shown on Fig.5 (right) on Ω_o at final time T, acting only on Ω_c. The $L^2(I)$ norm

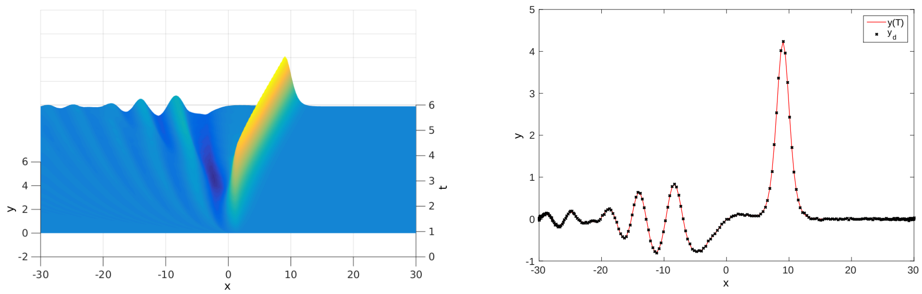

FIG. 4. *Recovered state y and final observation $y(T)$ for $\varepsilon = 10^{-6}$*

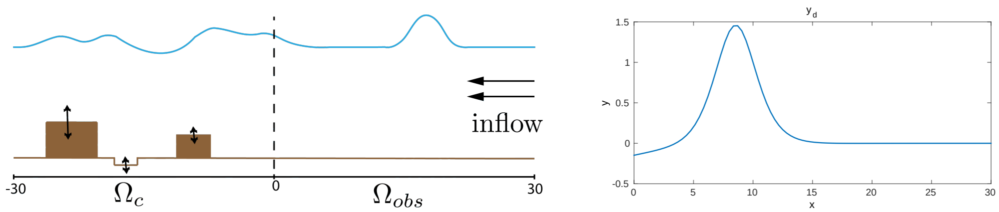

FIG. 5. *Left: Setup of the control problem. Right: Objective wave at final time T.*

of the adjoint state is depicted on Fig. 6. As in the previous example, we infer from that the support of the control. At the end of the continuation strategy, it boils down to two point sources, the points where $\|p\|_{L^2(I)} = \alpha$. One can see clearly on Fig. 7 that it has smooth variations in time, as induced by the L^2 norm in $\|\cdot\|_{\mathcal{M}(\Omega_c, L^2(I))}$. The optimized state is shown on Fig. 8 for $\varepsilon = 10^{-5}$. An upstream going soliton is created that matches accurately at final time the objective wave. Naturally, spurious waves are created on the control domain that which have no contribution to the cost functional.

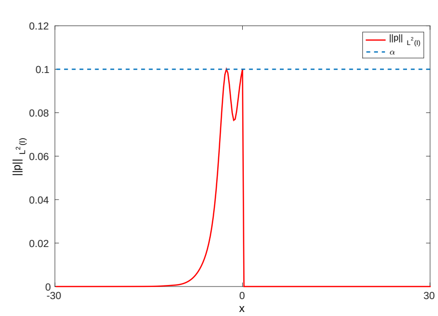

FIG. 6. *Spatial support of the control, determined by $\|p_\varepsilon\|_{L^2(0,T)} = \alpha$, for $\varepsilon = 10^{-5}$.*

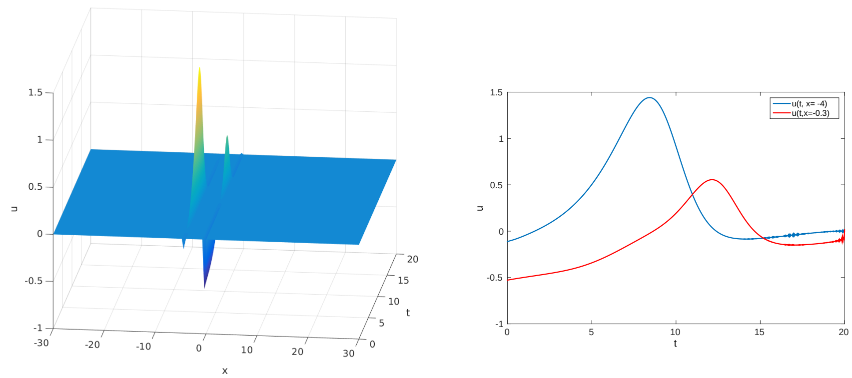

FIG. 7. *Optimal control u, $\varepsilon = 10^{-5}$*

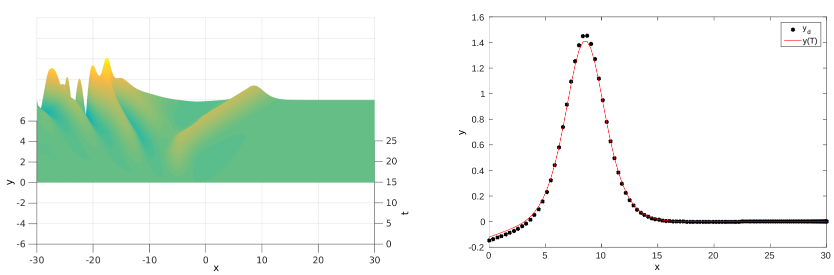

FIG. 8. *Left: evolution of the optimized state y for $\varepsilon = 10^{-5}$. Right: final optimized state $y(T)$, compared with the objective shape for $\varepsilon = 10^{-5}$.*

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Appendix A. Well-posedness of the state equation, tangent equation and adjoint equation.

A.1. Linear estimates. *Proof.* [Proof linear estimate (3.6)] The proof is largely inspired from [41, 23]. Let $y \in \mathcal{C}(\bar{I}, \mathcal{D}(A)) \cap \mathcal{C}^1(\bar{I}, L^2(\Omega))$ be the classical solution of (3.1) for smooth versions of the data f and y_0. We multiply (3.1a) which holds in $L^2(\Omega)$ for a.e. $t \in I$ with y and get

$$\frac{1}{2}\frac{d}{dt}\int_0^L y^2 \,\mathrm{d}x + |\partial_x y(t,0)|^2 + \gamma \int_0^L (\partial_x y)^2 \,\mathrm{d}x = \langle f, y\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}. \tag{A.1}$$

Applying the Cauchy-Schwarz followed by Young's inequality to the right-hand side leads to

$$\frac{1}{2}\frac{d}{dt}\int_0^L y^2 \,\mathrm{d}x + |\partial_x y(t,0)|^2 + \gamma \int_0^L (\partial_x y)^2 \,\mathrm{d}x \le \frac{1}{2}\|f\|^2_{H^{-1}(\Omega)} + \frac{1}{2}\|y\|^2_{H_0^1(\Omega)}. \tag{A.2}$$

We proceed in the same manner testing with xy

$$\frac{1}{2}\frac{d}{dt}\int_0^L xy^2 \,\mathrm{d}x - \frac{1}{2}\int_0^L y^2 \,\mathrm{d}x + \frac{3}{2}\int_0^L (\partial_x y)^2 \,\mathrm{d}x + \gamma\int_0^L x(\partial_x y)^2 \,\mathrm{d}x = \langle f, xy\rangle_{H_0^1(\Omega)}. \tag{A.3}$$

The right-hand side is treated using the Cauchy-Schwarz and Young's inequality

$$\begin{aligned}\langle f, xy\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} &\le \|q\|_{H^{-1}(\Omega)}\|xy\|_{H_0^1(\Omega)} \\ &\le \|f\|_{H^{-1}(\Omega)}\|y + x\partial_x y\|_{L^2(\Omega)} \\ &\le \frac{1+L^2}{2}\|f\|^2_{H^{-1}(\Omega)} + \frac{1}{2}\|y\|^2_{L^2(\Omega)} + \frac{1}{2}\|\partial_x y\|^2_{L^2(\Omega)}.\end{aligned} \tag{A.4}$$

Adding (A.2), (A.3) (with the upper bound (A.4)) and omitting $|\partial_x y(t,0)|^2$ on the left-hand side yields

$$\frac{1}{2}\frac{d}{dt}\int_0^L (1+x)y^2 \,\mathrm{d}x + \left(\frac{1}{2}+\gamma\right)\|y\|^2_{H_0^1(\Omega)} \le \left(1+\frac{L^2}{2}\right)\|f\|^2_{H^{-1}(\Omega)} + \|y\|^2_{L^2(\Omega)}.$$

After integration between 0 and $0 < t < T$ we have

$$\begin{aligned}\|y(t)\|^2_{L^2(\Omega)} + (1+2\gamma)\int_0^t \|y\|^2_{H_0^1(\Omega)} \,\mathrm{d}t \le \left(2+L^2\right)\int_0^t \|f\|^2_{H^{-1}(\Omega)} \,\mathrm{d}t \\ + \|y_0\|^2_{L^2(\Omega)} + 2\int_0^t \|y\|^2_{L^2(\Omega)} \,\mathrm{d}t.\end{aligned}$$

Then the standard Gronwall inequality gives us

$$\|y(t)\|^2_{L^2(\Omega)} \le e^{2t}\left(\left(2+L^2\right)\|f\|^2_{L^2(I,H^{-1}(\Omega))} + \|y_0\|_{L^2(\Omega)}\right).$$

This yields

$$\|y\|_{\mathcal{C}(\bar{I},L^2(\Omega))} + \|y\|_{L^2(I,H_0^1(\Omega))} \le c(T,L)\left(\|f\|_{L^2(I,H^{-1}(\Omega))} + \|y_0\|_{L^2(\Omega)}\right)$$

for some $c(T,L) > 0$ independent of y, f and y_o. ◻

A.2. Nonlinear state equation. *Proof.* [Proof of Lemma 3.6] The proof is inspired from [22, Theorem 2.8]. Let us consider $y \in \mathcal{B}$ and $z \in \mathcal{B}$. There holds $\mathcal{B} \hookrightarrow L^4(I \times \Omega)$ and therefore we can estimate using $\|y\|^2_{L^\infty(\Omega)} \leq c\|y\|_{L^2(\Omega)}\|y\|_{H^1_0(\Omega)}$

$$\|y\partial_x y - z\partial_x z\|_{L^2(I,H^{-1}(\Omega))} = \frac{1}{2}\left(\int_0^T \left(\sup_{\|\varphi\|_{H^1_0(\Omega)}=1} (y^2 - z^2, \partial_x\varphi)_{L^2(\Omega)}\right)^2 \mathrm{d}t\right)^{1/2}$$
$$\leq \frac{1}{2}\|z-y\|_{L^4(I\times\Omega)}\|z+y\|_{L^4(I\times\Omega)}$$
$$\leq \frac{1}{2}\|z-y\|^{1/2}_{C(\bar I,L^2(\Omega))}\|z-y\|^{1/2}_{L^2(I,L^\infty(\Omega))}\|z+y\|^{1/2}_{C(\bar I,L^2(\Omega))}\|z+y\|^{1/2}_{L^2(I,L^\infty(\Omega))}$$
$$\leq c\,\|z-y\|^{1/2}_{C(\bar I,L^2(\Omega))}\|z+y\|^{1/2}_{C(\bar I,L^2(\Omega))}$$
$$\left(\int_0^T \|z-y\|_{L^2(\Omega)}\|z-y\|_{H^1_0(\Omega)}\ \mathrm{d}t\right)^{1/4}\left(\int_0^T \|z+y\|_{L^2(\Omega)}\|z+y\|_{H^1_0(\Omega)}\ \mathrm{d}t\right)^{1/4}$$
$$\leq c\,T^{1/4}\|z-y\|^{3/4}_{C(\bar I,L^2(\Omega))}\|z-y\|^{1/4}_{L^2(I,H^1_0(\Omega))}\|z+y\|^{3/4}_{C(\bar I,L^2(\Omega))}\|z+y\|^{1/4}_{L^2(I,H^1_0(\Omega))}$$
$$\leq c\,T^{1/4}\|y-z\|_{\mathcal{B}}\|y+z\|_{\mathcal{B}}$$

∎

A.3. The tangent equation. Next we analyze the well-posedness of the tangent equation.

$$\begin{cases} \partial_t\delta y + \partial_x\delta y + \partial_{xxx}\delta y - \gamma\delta\partial_{xx}y + \partial_x(y\delta y) = \delta u \text{ in } I\times\Omega, & \text{(A.5a)}\\ \delta y(\cdot,0) = \delta y(\cdot,L) = \partial_x\delta y(\cdot,L) = 0 \text{ in } I, & \text{(A.5b)}\\ \delta y(0,x) = \delta y_0 \text{ in } \Omega. & \text{(A.5c)}\end{cases}$$

Definition A.1. *Let* $(\delta u, \delta y_0) \in L^2(I,H^{-1}(\Omega)) \times L^2(\Omega)$ *and* $y \in \mathcal{B}$. *A function* $\delta y \in \mathcal{B}$ *is called a solution of* (A.5a)-(A.5c) *if it satisfies the fixed point equation*

$$\delta y = \mathcal{L}(\delta u - \partial_x(y\delta y), \delta y_0)$$

where $\mathcal{L}$ *is the solution operator from Remark 3.4.*

Proposition A.2. *Let* $(\delta u, \delta y_0) \in L^2(I,H^{-1}(\Omega)) \times L^2(\Omega)$ *and* $y \in \mathcal{B}$. *Then, there exists a unique solution* $\delta y \in \mathcal{B}$ *of* (A.5a)-(A.5c). *Furthermore, there exists a constant* $\widetilde{C}(T,L,\|y\|_{\mathcal{B}})$ *such that the following estimate holds*

$$\|\delta y\|_{\mathcal{B}} + \|\partial_t\delta y\|_{L^2(I,\mathcal{V}^*)} \leq \widetilde{C}\left(\|\delta y_0\|_{L^2(\Omega)} + \|\delta u\|_{L^2(I,H^{-1}(\Omega))}\right). \tag{A.6}$$

Proof. We define the linear mapping

$$\Psi_{\delta u,\delta y_0,y}\colon \mathcal{B}_\theta \to \mathcal{B}_\theta, \quad \Psi_{\delta u,\delta y_0,y}(\delta y) = \mathcal{L}(\delta u - \partial_x(y\delta y), \delta y_0)$$

with $\mathcal{B}_\theta$ defined as in (3.10) and (3.11) and $\mathcal{L}$ being the linear KdV operator described in Remark 3.4. Our goal is to show that under some constraints on θ, $\Psi_{\delta u,\delta y_0,y}$ is a contraction mapping, such that the Banach fixed point theorem can be applied. First we estimate $\partial_x(y\delta y)$ in the $L^2((0,\theta),H^{-1}(\Omega))$-norm

$$\|\partial_x(y\delta y)\|_{L^2((0,\theta),H^{-1}(\Omega))} = \left(\int_0^\theta \left(\sup_{\|v\|_{H_0^1(\Omega)}=1} (y\delta y, \partial_x v)_{L^2(\Omega)}\right)^2 \mathrm{d}t\right)^{1/2} \tag{A.7}$$

$$\leq \left(\int_0^\theta \|\delta y\|^2_{L^2(\Omega)} \|y\|^2_{L^\infty(\Omega)} \mathrm{d}t\right)^{1/2} \leq c\,\|y\|_{\mathcal{C}([0,\theta],L^2(\Omega))} \left(\int_0^\theta \|\delta y\|_{H_0^1(\Omega)} \|\delta y\|_{L^2(\Omega)} \mathrm{d}t\right)^{1/2}$$
$$\leq c\,\theta^{1/4}\,\|y\|_{\mathcal{C}([0,\theta],L^2(\Omega))} \|\delta y\|^{1/2}_{\mathcal{C}([0,\theta],L^2(\Omega))} \|\delta y\|^{1/2}_{L^2((0,\theta),H_0^1(\Omega))}$$
$$\leq c\,\theta^{1/4}\,\|y\|_{\mathcal{B}_\theta} \|\delta y\|_{\mathcal{B}_\theta}$$

Therefore we can estimate

$$\begin{aligned}\|\Psi(\delta y)_{\delta u,\delta y_0,y}\|_{\mathcal{B}_\theta} &\leq \widetilde{C}\left(\|\delta y_0\|_{L^2(\Omega)} + \|\delta u\|_{L^2(I,H^{-1}(\Omega))} + \|\partial_x(y\delta y)\|_{L^2((0,\theta),H^{-1}(\Omega))}\right)\\ &\leq \widetilde{C}\left(\|\delta y_0\|_{L^2(\Omega)} + \|\delta u\|_{L^2(I,H^{-1}(\Omega))}\right) + \hat{C}\theta^{1/4}\|y\|_{\mathcal{B}_\theta}\|\delta y\|_{\mathcal{B}_\theta}\end{aligned}$$

and

$$\|\Psi_{\delta u,\delta y_0,y}(\delta y_1) - \Psi_{\delta u,\delta y_0,y}(\delta y_2)\|_{\mathcal{B}_\theta} \leq \hat{C}\theta^{1/4}\|y\|_{\mathcal{B}_\theta}\|\delta y_1 - \delta y_2\|_{\mathcal{B}_\theta}.$$

Now we set $r = 3\widetilde{C}\left(\|\delta y_0\|_{L^2(\Omega)} + \|\delta u\|_{L^2(I,H^{-1}(\Omega))}\right)$ and introduce the ball

$$B = \{\delta y \in \mathcal{B}_\theta \colon \|\delta y\|_{\mathcal{B}_\theta} \leq r\}.$$

Next we choose θ small enough such that

$$\hat{C}\theta^{1/4}\|y\|_{\mathcal{B}_\theta} \leq \frac{1}{3}$$

holds. Then the following inequalities hold

$$\|\Psi_{\delta u,\delta y_0,y}(\delta y)\|_{\mathcal{B}_\theta} \leq \frac{2}{3}r, \quad \|\Psi_{\delta u,\delta y_0,y}(\delta y_1) - \Psi_{\delta u,\delta y_0,y}(\delta y_2)\|_{\mathcal{B}_\theta} \leq \frac{1}{3}\|\delta y_1 - \delta y_2\|_{\mathcal{B}_\theta}$$

which imply that $\Psi_{\delta u,\delta y_0,y}$ is a contraction mapping on B. So we can apply the Banach fixed point theorem which guarantees the existence of a unique fixed point δy of $\Psi_{\delta u,\delta y_0,y}$ which is a solution of (A.5a)-(A.5c) in $(0,\theta)$ with initial value δy_0. Since θ is independent of δy_0 we can apply this strategy successively starting at $t=0$ with δy_0 and using $\delta y(k\theta)$ as initial points for $k = 1,2,3,\ldots,N$ until T is reached. The concatenation of all $\delta y(k\theta)$ for $k = 1,2,3,\ldots,N$ is a solution of (A.5a) - (A.5c). Existence of a unique solution is thus proven. Concerning the estimates (A.6), the proof is very similar to the non-variable coefficients case (see Appendix A.1). We just mention the main differences. In the case of a smooth solution $\delta y \in \mathcal{C}(\bar{I},\mathcal{D}(A)) \cap \mathcal{C}^1(I,L^2(\Omega))$ we multiply (A.5a) by δy and estimate the term involving y in the following way

$$\begin{aligned}\left|\int_0^L \delta y \partial_x (y\delta y)\,\mathrm{d}x\right| = \left|-\int_0^L y\delta y \partial_x \delta y\,\mathrm{d}x\right| &\leq \frac{1}{2\varepsilon}\int_0^L y^2\delta y^2\,\mathrm{d}x + \frac{\varepsilon}{2}\|\delta y\|^2_{H_0^1(\Omega)}\\ &\leq \frac{1}{2\varepsilon}\|y\|^2_{L^\infty(\Omega)}\|\delta y\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\|\delta y\|^2_{H_0^1(\Omega)}\end{aligned}$$

for any $\varepsilon > 0$. In the same manner, multiplying (A.5a) by $x\delta y$ leads to

$$\left|\int_0^L x\delta y\partial_x(y\delta y)\right| \,\mathrm{d}x = \left|-\int_0^L y\delta y^2 \,\mathrm{d}x - \int_0^L xy\delta y\partial_x\delta y \,\mathrm{d}x\right| \\ \leq \left(\|y\|_{L^\infty(\Omega)} + \frac{L^2}{2\varepsilon}\|y\|^2_{L^\infty(\Omega)}\right)\|\delta y\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\|\delta y\|^2_{H^1_0(\Omega)}$$

for any $\varepsilon > 0$. Based on these estimates the a priori estimate (A.6) can be shown. The estimate for $\|\partial_t y\|_{L^2(I,\mathcal{V}^*)}$ can be shown based on (A.7) and (A.6). ◻

A.4. The adjoint equation. Next we study the following equation

$$\begin{cases} -\partial_t p - \partial_x p - \partial_{xxx} p - \gamma\partial_{xx} p - y\partial_x p = \phi \text{ in } I\times\Omega, & \text{(A.8a)} \\ p(\cdot,0) = p(\cdot,L) = \partial_x p(\cdot,0) = 0 \text{ on } I, & \text{(A.8b)} \\ p(T) = p_T \text{ in } \Omega. & \text{(A.8c)} \end{cases}$$

for any $y \in \mathcal{B}$.

Definition A.3. *A function $p \in \mathcal{B}$ is called a solution of* (A.8a)-(A.8c) *if it solves the fixed point equation*

$$p(t) = W^*(t)p_T + \int_t^T W^*(s-t)(\phi(s) + y(s)\partial_x p(s)) \,\mathrm{d}s.$$

Proposition A.4. *Let $(\phi, p_T) \in L^1(I, L^2(\Omega)) \times L^2(\Omega)$. Then the equation* (A.8a)-(A.8c) *has a unique solution $p \in \mathcal{B}$. Furthermore there exists a constant $c\,(\|y\|_{\mathcal{B}}) > 0$ such that*

$$\|p\|_{\mathcal{B}} \leq c\,(\|p_T\|_{L^2(\Omega)} + \|\phi\|_{L^1(I,L^2(\Omega))}) \tag{A.9}$$

holds.

Proof. The proof uses similar arguments as the proof of Proposition A.2. In particular it is based on the Banach fixed theorem and Lemma (A.5). The estimate (A.9) follows from

$$\left|\int_0^L y\partial_x pp \,\mathrm{d}x\right| = \left|\frac{1}{2}\int_0^L p^2\partial_x y \,\mathrm{d}x\right| \leq c\,\|p\|_{H^1_0(\Omega)}\|p\|_{L^2(\Omega)}\|y\|_{H^1_0(\Omega)} \\ \leq \frac{c}{2\varepsilon}\|p\|^2_{L^2(\Omega)}\|y\|^2_{H^1_0(\Omega)} + \frac{\varepsilon}{2}\|p\|^2_{H^1_0(\Omega)}$$

for any $\varepsilon > 0$ and

$$\left|\int_0^L y\partial_x p(L-x)p \,\mathrm{d}x\right| = \left|\frac{1}{2}\int_0^L (L-x)p^2\partial_x y - yp^2 \,\mathrm{d}x\right| \\ \leq c\left(\|p\|_{H^1_0(\Omega)}\|p\|_{L^2(\Omega)}\|y\|_{H^1_0(\Omega)} + \|p\|^2_{L^2(\Omega)}\|y\|_{H^1_0(\Omega)}\right) \\ \leq \|p\|^2_{L^2(\Omega)}\left(\frac{c}{2\varepsilon}\|y\|^2_{H^1_0(\Omega)} + \|y\|_{H^1_0(\Omega)}\right) + \frac{\varepsilon}{2}\|p\|^2_{H^1_0(\Omega)}.$$

for any $\varepsilon > 0$. ∎

LEMMA A.5. *Let $y \in \mathcal{B}$, $p \in \mathcal{B}$. Then it holds*

$$\|y\partial_x p\|_{L^1(0,T,L^2(\Omega))} \leq c\,T^{1/4}\|y\|_{\mathcal{B}}\|p\|_{\mathcal{B}}.$$

Proof. [Proof of Lemma A.5]

$$\begin{aligned}
\|y\partial_x p\|_{L^1(I,L^2(\Omega))} &\leq \int_0^T \|y\|_{L^\infty(\Omega)}\|p\|_{H_0^1(\Omega)}\,\mathrm{d}t \\
&\leq c\int_0^T \|y\|_{L^2(\Omega)}^{1/2}\|y\|_{H_0^1(\Omega)}^{1/2}\|p\|_{H_0^1(\Omega)}\,\mathrm{d}t \\
&\leq c\,\|y\|_{C(\bar{I},L^2(\Omega))}^{1/2}\left(\int_0^T \|y\|_{H_0^1(\Omega)}\,\mathrm{d}t\right)^{1/2}\|p\|_{L^2(I,H_0^1(\Omega))} \\
&\leq c\,T^{1/4}\|y\|_{C(\bar{I},L^2(\Omega))}^{1/2}\|y\|_{L^2(I,H_0^1(\Omega))}^{1/2}\|p\|_{L^2(I,H_0^1(\Omega))},
\end{aligned}$$

which implies the assertion. ∎

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