A pointwise characterization of the subdifferential of the total variation functional*

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Abstract

We derive a new pointwise characterization of the subdifferential of the total variation (TV) functional. It involves a full trace operator which maps certain L^q - vectorfields to integrable functions with respect to the total variation measure of the derivative of a bounded variation function. This full trace operator extents a notion of normal trace, frequently used, for example, to characterize the total variation flow.

Keywords. Total variation, subdifferential characterization, normal trace.
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1 Introduction

The aim of this paper is to derive a new, pointwise characterization of the subdifferential of the TV functional in Lebesgue spaces. This characterization bases on a trace operator, which extends the normal trace of [7]: There, Anzellotti introduces a normal trace $\theta(g, \mathrm{D}u) \in L^1(\Omega; |\mathrm{D}u|)$ for vector fields $g \in W^q(\mathrm{div};\Omega) \cap L^\infty(\Omega, \mathbb{R}^d)$ (see Section 2) that allows the following characterization: $u^* \in \partial \operatorname{TV}(u)$ if and only if, there exists $g \in W_0^q(\mathrm{div};\Omega)$ with $\|g\|_\infty \leq 1$ such that $u^* = -\operatorname{div} g$ and

$$\theta(g, \mathrm{D}u) = 1 \quad \text{in } L^1(\Omega; |\mathrm{D}u|).$$

This approach is commonly used to characterize the total variation flow, as for example in [3, 4, 5, 6, 8, 9, 11].

Introducing a "full" trace operator $T : D \subset W^q(\mathrm{div};\Omega) \cap L^\infty(\Omega, \mathbb{R}^d) \to L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u|)$, we sharpen this result by showing that the set $\partial \operatorname{TV}(u)$ can be described as: $u^* \in \partial \operatorname{TV}(u)$ if and only if, there exists $g \in D \cap W_0^q(\mathrm{div};\Omega)$ with $\|g\|_\infty \leq 1$ such that $u^* = -\operatorname{div} g$ and

$$Tg = \sigma_u \quad \text{in } L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u|),$$

where $\sigma_u \in L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u|)$ is the density function such that $\mathrm{D}u = \sigma_u |\mathrm{D}u|$.

The outline of the paper is as follows: In the second section we give some preliminary results about functions of bounded variation, introduce a straightforward generalization of the space $H(\mathrm{div})$ and state an approximation result. The third section is the main section, where we first repeat the term of normal trace introduced in [7], then introduce the notion of full trace, and, using this notion, show a characterization of the subdifferential of the total variation (TV) functional. In the fourth section we address some topics where the full trace characterization of the TV subdifferential can be applied: We use it to reformulate well known results, such as a characterization of the total variation flow, a characterization of Cheeger sets and optimality conditions for mathematical imaging problems, in terms of the full trace operator. In the last section we give a conclusion.

2 Preliminaries

This section is devoted to introduce notation and basic results. After some preliminary definitions, we start with a short introduction to functions of bounded variation. For further information and proofs we refer to to [2, 21, 15]. For convenience, we always assume $\Omega \subset \mathbb{R}^d$ to be a bounded Lipschitz domain. Further, throughout this work, we often denote $\int_\Omega \phi$ or $\int_\Omega \phi \mathrm{d}x$ instead of $\int_\Omega \phi(x) \mathrm{d}x$ for the Lebesgue integral of a measureable function ϕ, when the usage of the Lebesgue measure and the integration variable are clear from the context.

We use a standard notation for continuously differentiable-, compactly supported- or integrable functions. However, in order to avoid ambiguity, we define the space of continuously differentiable functions on a closed set:

Definition 1 (Continuous functions on a closed set)**.** *Given a domain* $A \subsetneq \mathbb{R}^d$ *and* $m \in \mathbb{N}$*, we define*

$$C(\overline{A}, \mathbb{R}^m) = \{\phi : \overline{A} \to \mathbb{R}^m \mid \phi \text{ is uniformly continuous on } A\},$$

$$C^k(\overline{A}, \mathbb{R}^m) = \{\phi : \overline{A} \to \mathbb{R}^m \mid \mathrm{D}^\alpha \phi \in C(\overline{A}, \mathbb{R}^m) \text{ for all } |\alpha| \leq k\}$$

and

$$C^\infty(\overline{A}, \mathbb{R}^m) = \bigcap_{k \in \mathbb{N}} C^k(\overline{A}, \mathbb{R}^m).$$

Note that for bounded domains, $\phi \in C(\overline{A}, \mathbb{R}^m)$ is equivalent to ϕ being the restriction of a function in $C_c(\mathbb{R}^d, \mathbb{R}^m)$. This also applies to $C^k(\overline{A}, \mathbb{R}^m)$ and $C^\infty(\overline{A}, \mathbb{R}^m)$ with $C_c^k(\mathbb{R}^d, \mathbb{R}^m)$ and $C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$, respectively, by virtue of Whitney's Extension Theorem [20, Theorem 1]. For unbounded domains, however, this is generally not true.

Definition 2 (Finite Radon measure)**.** *Let* $\mathcal{B}(\Omega)$ *be the Borel* σ*-algebra generated by the open subsets of* Ω*. We say that a function* $\mu : \mathcal{B}(\Omega) \to \mathbb{R}^m$*, for* $m \in \mathbb{N}$*, is a finite* $\mathbb{R}^m$*-valued Radon measure if* $\mu(\emptyset) = 0$ *and* μ *is* σ*-additive. We denote by* $\mathcal{M}(\Omega)$ *the space of all finite Radon measures on* Ω*. Further we denote by* $|\mu|$ *the variation of* $\mu \in \mathcal{M}(\Omega)$*, defined by*

$$|\mu|(E) = \sup \left\{ \sum_{i=0}^{\infty} |\mu(E_i)| \,\middle|\, E_i \in \mathcal{B}(\Omega),\ i \geq 0,\ \text{pairwise disjoint}, E = \bigcup_{i=0}^{\infty} E_i \right\},$$

for $E \in \mathcal{B}(\Omega)$*. Note that* $|\mu(E_i)|$ *denotes the Euclidean norm of* $\mu(E_i) \in \mathbb{R}^m$*.*

Definition 3 (Functions of bounded variation)**.** *We say that a function* $u \in L^1(\Omega)$ *is of bounded variation, if there exists a finite* $\mathbb{R}^d$*-valued Radon measure, denoted by* $Du = (D_1 u, ..., D_d u)$*, such that for all* $i \in \{1, ..., d\}$*,* $D_i u$ *represents the distributional derivative of* u *with respect to the* i*th coordinate, i.e., we have*

$$\int_\Omega u \partial_i \phi = - \int_\Omega \phi \, \mathrm{d} D_i u \quad \text{for all } \phi \in C_c^\infty(\Omega).$$

By $\mathrm{BV}(\Omega)$ *we denote the space of all functions* $u \in L^1(\Omega)$ *of bounded variation.*

Definition 4 (Total variation)**.** *For* $u \in L^1(\Omega)$*, we define the functional* $\mathrm{TV} : L^1(\Omega) \to \overline{\mathbb{R}}$ *as*

$$\mathrm{TV}(u) = \sup \left\{ \int_\Omega u \operatorname{div} \phi \,\middle|\, \phi \in C_c^\infty(\Omega, \mathbb{R}^d),\ \|\phi\|_\infty \leq 1 \right\}$$

where we set $\mathrm{TV}(u) = \infty$ *if the set is unbounded from above. We call* $\mathrm{TV}(u)$ *the total variation of* u*.*

Proposition 1. *The functional* $\mathrm{TV} : L^1(\Omega) \to \overline{\mathbb{R}}$ *is convex and lower semi-continuous with respect to* L^1*-convergence. For* $u \in L^1(\Omega)$ *we have that*

$$u \in \mathrm{BV}(\Omega) \text{ if and only if } \mathrm{TV}(u) < \infty.$$

In addition, the total variation of u *coincides with the variation of the measure* Du*, i.e.,* $\mathrm{TV}(u) = |Du|(\Omega)$*. Further,*

$$\|u\|_{\mathrm{BV}} := \|u\|_{L^1} + \mathrm{TV}(u)$$

defines a norm on $\mathrm{BV}(\Omega)$ *and endowed with this norm,* $\mathrm{BV}(\Omega)$ *is a Banach space.*

Definition 5 (Strict Convergence)**.** *For* $(u_n)_{n\in\mathbb{N}}$ *with* $u_n \in \mathrm{BV}(\Omega)$*,* $n \in \mathbb{N}$*, and* $u \in \mathrm{BV}(\Omega)$ *we say that* $(u_n)_{n\in\mathbb{N}}$ *strictly converges to* u *if*

$$\|u_n - u\|_{L^1} \to 0 \text{ and } \mathrm{TV}(u_n) \to \mathrm{TV}(u)$$

as $n \to \infty$*.*

Definition 6 (Lebesgue Point)**.** *Let* $f \in L^p(\Omega)$*,* $1 \le p < \infty$*. We say that* $x \in \Omega$ *is a Lebesgue point of* f *if*

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y) - f(x)| \,\mathrm{d}y \to 0$$

as $n \to \infty$*. Note that here,* $|B(x,r)|$ *denotes the Lebesgue measure of the ball with radius* r *around* $x \in \Omega$*.*

Remark 1. *Remember that for any* $f \in L^p(\Omega)$*,* $1 \le p < \infty$*, almost every* $x \in \Omega$ *is a Lebesgue point of* f *(see [15, Corollary 1.7.1]).*

Next we recall some standard notations and facts from convex analysis. For proofs and further introduction we refer to [14].

Definition 7 (Convex conjugate and subdifferential)**.** *For a normed vector space* V *and a function* $F : V \to \overline{\mathbb{R}}$ *we define its convex conjugate, or Legendre-Fenchel transform, denoted by* $F^* : V^* \to \overline{\mathbb{R}}$*, as*

$$F^*(u^*) = \sup_{v\in V} \langle v, u^* \rangle_{V,V^*} - F(v).$$

Further F *is said to be subdifferentiable at* $u \in V$ *if* $F(u)$ *is finite and there exists* $u^* \in V^*$ *such that*

$$\langle v - u, u^* \rangle_{V,V^*} + F(u) \le F(v)$$

for all $v \in V$*. The element* $u^* \in V^*$ *is then called a subgradient of* F *at* u *and the set of all subgradients at* u *is denoted by* $\partial F(u)$*.*

Definition 8 (Convex indicator functional)**.** *For a normed vector space V and $U \subset V$ a convex set, we denote by $\mathcal{I}_U : V \to \overline{\mathbb{R}}$ the convex indicator functional of U, defined by*

$$\mathcal{I}_U(u) = \begin{cases} 0 & \text{if } u \in U, \\ \infty & \text{else.} \end{cases}$$

Next we define the space $W^q(\operatorname{div};\Omega)$, which is fundamental for the characterization of the TV subdifferential.

Definition 9 (The space $W^q(\operatorname{div};\Omega)$)**.** *Let $1 \leq q < \infty$ and $g \in L^q(\Omega, \mathbb{R}^d)$. We say that $\operatorname{div} g \in L^q(\Omega)$ if there exists $w \in L^q(\Omega)$ such that for all $v \in C_c^\infty(\Omega)$*

$$\int_\Omega \nabla v \cdot g = - \int_\Omega v w.$$

Furthermore we define

$$W^q(\operatorname{div};\Omega) = \left\{ g \in L^q(\Omega, \mathbb{R}^d) \mid \operatorname{div} g \in L^q(\Omega) \right\}$$

with the norm $\|g\|_{W^q(\operatorname{div})}^q := \|g\|_{L^q}^q + \|\operatorname{div} g\|_{L^q}^q$.

Remark 2. *Density of $C_c^\infty(\Omega)$ in $L^p(\Omega)$ implies that, if there exists $w \in L^q(\Omega)$ as above, it is unique. Hence it makes sense to write $\operatorname{div} g = w$. By completeness of $L^q(\Omega)$ and $L^q(\Omega, \mathbb{R}^d)$ it follows that $W^q(\operatorname{div};\Omega)$ is a Banach space when equipped with $\| \cdot \|_{W^q(\operatorname{div})}$.*

Remark 3. *Note that $W^q(\operatorname{div};\Omega)$ is just a straightforward generalization of the well known space $H(\operatorname{div};\Omega)$. Also classical results like density of $C^\infty(\overline{\Omega}, \mathbb{R}^d)$ and existence of a normal trace on $\partial\Omega$ can be derived for $W^q(\operatorname{div};\Omega)$ as straightforward generalizations of the proofs given for example in [16, Chapter 1].*

Definition 10. *For $1 \leq q < \infty$, we define*

$$W_0^q(\operatorname{div};\Omega) = \overline{C_c^\infty(\Omega, \mathbb{R}^d)}^{\|\cdot\|_{W^q(\operatorname{div})}}.$$

Remark 4. *By density it follows that, for $g \in W_0^q \operatorname{div};\Omega)$, we have*

$$\int_\Omega \nabla v g = - \int_\Omega v \operatorname{div} g$$

for all $v \in C^\infty(\overline{\Omega})$.

The following approximation result will be needed in the context of the full trace.

Proposition 2. *If Ω is a bounded Lipschitz domain, $1 \leq q < \infty$ and $g \in W^q(\operatorname{div};\Omega)$, there exists a sequence of vector fields $(g_n)_{n \geq 0} \subset C^\infty(\overline{\Omega}, \mathbb{R}^d)$ such that*

1. $\|g_n - g\|_{W^q(\mathrm{div})} \to 0$ *as* $n \to \infty$,

2. $\|g_n\|_\infty \le \|g\|_\infty$ *for each* $n \in \mathbb{N}$, *if* $\|g\|_\infty < \infty$,

3. $g_n(x) \to g(x)$ *for every Lebesgue point* $x \in \Omega$ *of* g.

4. $\|g_n - g\|_{\infty,\overline{\Omega}} \to 0$ *as* $n \to \infty$, *if, additionally,* $g \in C(\overline{\Omega}, \mathbb{R}^d)$.

A proof can be found in the Appendix.

3 Subdifferential of TV

In order to describe the subdifferential of the TV functional, for $u \in \mathrm{BV}(\Omega)$, we need a notion of trace for $W^q(\mathrm{div};\Omega)$ vector fields in $L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u|)$.

3.1 The normal trace

We first revisit the normal trace introduced in [7]. We do so by defining it for $W^q(\mathrm{div};\Omega)$ vector fields as a closed operator. In this subsection, if not restricted further, let always be $1 \le q < \infty$, $p = \frac{q}{q-1}$ if $q \ne 1$ or $p = \infty$ else, and Ω a bounded Lipschitz domain.

Proposition 3. *Set* $\tilde{D}_N := W^q(\mathrm{div};\Omega) \cap L^\infty(\Omega, \mathbb{R}^d)$. *Then, with* $u \in \mathrm{BV}(\Omega) \cap L^p(\Omega)$ *fixed, for any* $z \in \tilde{D}_N$ *there exists a function* $\theta(z, \mathrm{D}u) \in L^1(\Omega; |\mathrm{D}u|)$ *such that*

$$\int_\Omega \theta(z, \mathrm{D}u)\psi \,\mathrm{d}|\mathrm{D}u| = -\int_\Omega u \,\mathrm{div}(z\psi)\,\mathrm{d}x$$

for all $\psi \in C_c^\infty(\Omega)$.

Proof. For $z \in \tilde{D}_N$ we define

$$\begin{aligned} L_z : C_c^\infty(\Omega) &\to \mathbb{R} \\ \psi &\mapsto -\int_\Omega u\,\mathrm{div}(z\psi)\,\mathrm{d}x \end{aligned}$$

and show that L_z can be extended to a linear, continuous operator from $C_0(\Omega)$ to $\mathbb{R}$.

It is clear that L_z is well-defined and linear, hence by definition of $C_0(\Omega)$ as closure of $C_c^\infty(\Omega)$ with respect to $\|\cdot\|_\infty$, it suffices to show that L_z is continuous with respect to $\|\cdot\|_\infty$. With $\psi \in C_c^\infty(\Omega)$ and $(z_n)_{n\ge 0} \subset C^\infty(\overline{\Omega}, \mathbb{R}^d)$ converging to z as in Proposition 2, we estimate

$$\begin{aligned} |L_z(\psi)| &= \lim_{n\to\infty} \left| -\int_\Omega u\,\mathrm{div}(z_n\psi)\,\mathrm{d}x \right| = \lim_{n\to\infty} \left| \int_\Omega z_n\psi \,\mathrm{dD}u \right| \\ &\le \|z\|_\infty \int_\Omega |\psi|\,\mathrm{d}|\mathrm{D}u| \le \|z\|_\infty \|\psi\|_\infty |\mathrm{D}u|(\Omega), \end{aligned}$$

where we used that $\|z_n - z\|_{W^q(\text{div})} \to 0$ as $n \to \infty$ and that $\|z_n\|_\infty \leq \|z\|_\infty$ for each $n \in \mathbb{N}$.

Thus, for any $z \in W^q(\text{div};\Omega) \cap L^\infty(\Omega, \mathbb{R}^d)$, we have that $L_z \in C_0(\Omega)^* = \mathcal{M}(\Omega)$ and we can write $(z, \mathrm{D}u)$ for the Radon measure associated with L_z. Performing the above calculations for $\psi \in C_c^\infty(A)$ with any open $A \subset \Omega$ yields $|L_z(\psi)| \leq \|z\|_\infty \|\psi\|_\infty |\mathrm{D}u|(A)$. Thus it follows that $(z, \mathrm{D}u) \ll |\mathrm{D}u|$ and hence by the Radon-Nikodym theorem there exists $\theta(z, \mathrm{D}u) \in L^1(\Omega; |\mathrm{D}u|)$ such that $(z, \mathrm{D}u) = \theta(z, \mathrm{D}u)|\mathrm{D}u|$. □

With that we can define the normal trace operator and prove additional properties:

Proposition 4 (Normal trace operator)**.** *With $\tilde{D}_N$ as in Proposition 3 and $u \in \mathrm{BV}(\Omega) \cap L^p(\Omega)$ fixed, the operator*

$$\begin{array}{rcl} \widetilde{T_N} : \tilde{D}_N \subset W^q(\text{div};\Omega) & \to & L^1(\Omega; |\mathrm{D}u|) \\ z & \mapsto & \theta(z, \mathrm{D}u) \end{array}$$

with $\theta(z, \mathrm{D}u)$ the density function of the measure $(z, \mathrm{D}u)$ with respect to $|\mathrm{D}u|$ as above, is well-defined and closeable. Further, with $T_N : D_N \to L^1(\Omega; |\mathrm{D}u|)$ denoting the closure of $\widetilde{T_N}$ defined on $D_N \subset W^q(\text{div};\Omega)$, we have that, for $z \in D_N$,

$$\|T_N z\|_\infty \leq \|z\|_\infty$$

whenever $z \in L^\infty(\Omega, \mathbb{R}^d)$ and, for $\phi \in C(\overline{\Omega}, \mathbb{R}^d) \cap W^q(\text{div};\Omega)$, that

$$T_N \phi = \phi \cdot \sigma_u \in L^1(\Omega; |\mathrm{D}u|)$$

where σ_u is the density function of $\mathrm{D}u$ w.r.t. $|\mathrm{D}u|$.

Proof. Well-definition is clear since the representation of L_z as a measure and also its density function with respect to $|\mathrm{D}u|$ is unique. Let now $(z_n)_{n\geq 0}, (\tilde{z}_n)_{n \geq 0} \subset \tilde{D}_N$ be two sequences converging to z in $W^q(\text{div};\Omega)$ and suppose that $\widetilde{T_N} z_n \to h$ and $\widetilde{T_N} \tilde{z}_n \to \tilde{h}$ with $h, \tilde{h} \in L^1(\Omega; |\mathrm{D}u|)$. With $\psi \in C_c^\infty(\Omega)$ we can write, using $\lim\limits_{n\to\infty} \text{div}(z_n \psi) = \text{div}(z\psi) = \lim\limits_{n\to\infty} \text{div}(\tilde{z}_n\psi)$ in $L^q(\Omega)$,

$$\begin{aligned} \int_\Omega h\psi \, \mathrm{d}|\mathrm{D}u| &= \lim_{n\to\infty} \int_\Omega (\widetilde{T_N} z_n)\psi \, \mathrm{d}|\mathrm{D}u| = \lim_{n\to\infty} - \int_\Omega u \, \text{div}(z_n\psi) \, \mathrm{d}x \\ &= \lim_{n\to\infty} - \int_\Omega u \, \text{div}(\tilde{z}_n\psi) \, \mathrm{d}x = \lim_{n\to\infty} \int_\Omega (\widetilde{T_N} \tilde{z}_n)\psi \, \mathrm{d}|\mathrm{D}u| \\ &= \int_\Omega \tilde{h}\psi \, \mathrm{d}|\mathrm{D}u| \end{aligned}$$

and thus, by density, $h = \tilde{h}$ and, consequently, $\widetilde{T_N}$ is closeable. The assertion $\|T_N z\|_\infty \leq \|z\|_\infty$ for $z \in D_N$ follows from $\left|\int_A \theta(z, \mathrm{D}u) \, \mathrm{d}|\mathrm{D}u|\right| \leq \|z\|_\infty |\mathrm{D}u|(A)$, for all $A \subset \Omega$ measurable, in the case that $\|z\|_\infty < \infty$, since then $z \in \tilde{D}_N$. If $\|z\|_\infty = \infty$, the inequality is trivially satisfied.

In order to show that $T_N\phi = \phi \cdot \sigma_u$ for $\phi \in C(\overline{\Omega}, \mathbb{R}^d) \cap W^q(\mathrm{div};\Omega)$ first note that $\phi \in \tilde{D}_N$. Thus, $T_N\phi$ is defined and we can use that, due to continuity of ϕ, the approximating vector fields $(\phi_n)_{n\geq 0}$ as in Proposition 2 converge uniformly to ϕ and write, again for $\psi \in C_c^\infty(\Omega)$,

$$\begin{aligned}\int_\Omega (T_N\phi)\psi \,\mathrm{d}|\mathrm{D}u| &= -\int_\Omega u \operatorname{div}(\phi\psi)\,\mathrm{d}x = \lim_{n\to\infty} -\int_\Omega u \operatorname{div}(\phi_n\psi)\,\mathrm{d}x \\ &= \lim_{n\to\infty} \int_\Omega \phi_n\psi \,\mathrm{dD}u = \int_\Omega (\phi\cdot\sigma_u)\psi\,\mathrm{d}|\mathrm{D}u|. \qquad \square\end{aligned}$$

Remark 5. *Note that by similar arguments one could also show that* $\widehat{T_N} : X(\Omega) := W^q(\mathrm{div};\Omega) \cap L^\infty(\Omega,\mathbb{R}^d) \to L^1(\Omega;|\mathrm{D}u|)$ *is continuous, when* X *is equipped with the norm* $\|z\|_X := \|z\|_\infty + \|\operatorname{div} z\|_{L^q}$.

We therefore have a suitable notion of normal trace for a dense subset of $W^q(\mathrm{div};\Omega)$. The closedness of the operator T_N can be interpreted as follows: If $z \in W^q(\mathrm{div};\Omega) \cap L^\infty(\Omega,\mathbb{R}^d)$ is sufficiently regular in the sense that the normal trace of its approximating vector fields as in Proposition 2 converges to some $h \in L^1(\Omega;|\mathrm{D}u|)$ with respect to $\|\cdot\|_{L^1}$ (which is satisfied for example if z_n converges pointwise $|\mathrm{D}u|$-a.e.), then $T_N z = h = \lim_{n\to\infty}(z_n \cdot \sigma_u)$ with σ_u again the density function of $\mathrm{D}u$ with respect to $|\mathrm{D}u|$.

3.2 The full trace

As we can see in Proposition 4 the normal trace only provides information about the vector field g in the direction σ_u. In the following we introduce a notion of trace which gives full vector information $|\mathrm{D}u|$-a.e. As for the normal trace, we also define the full trace for a dense subset of $W^q(\mathrm{div};\Omega)$-vector fields, where again, throughout this subsection, we assume that $1 \leq q < \infty$. As we will see, existence of a full trace is a stronger condition than existence of a normal trace as above. Moreover, the full trace extends the notion of normal trace in the following sense: If for $g \in W^q(\mathrm{div};\Omega) \cap L^\infty(\Omega,\mathbb{R}^d)$ there exists a full trace $h \in L^1(\Omega,\mathbb{R}^d;|\mathrm{D}u|)$, this implies that the normal trace $T_N g$ can be written as $T_N g = h \cdot \sigma_u$. First we need to define a notion of convergence:

Definition 11. *Let* $g \in W^q(\mathrm{div};\Omega) \cap L^\infty(\Omega,\mathbb{R}^d)$. *For* $(g_n)_{n\geq 0} \subset C(\overline{\Omega},\mathbb{R}^d) \cap W^q(\mathrm{div};\Omega)$ *we say that* $(g_n)_{n\geq 0} \tilde{\to} g$ *if*

1. $\|g_n - g\|_{W^q(\mathrm{div})} \to 0$,
2. $\|g_n\|_\infty \leq \|g\|_\infty$,
3. $g_n(x) \to g(x)$ *for every Lebesgue point* x *of* g.

Note that by Proposition 2, for every $g \in W^q(\mathrm{div}, \Omega)$ there exists a sequence $(g_n)_{n\geq 0} \subset C^\infty(\overline{\Omega}, \mathbb{R}^d)$ converging to g in the above sense.

Definition 12 (Full trace operator)**.** *With $u \in \mathrm{BV}(\Omega)$, define*

$$T : D \subset W^q(\mathrm{div}; \Omega) \cap L^\infty(\Omega, \mathbb{R}^d) \to L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u|)$$

by

$$v = Tg$$

whenever

$$\begin{cases} \textit{for all } (g_n)_{n\geq 0} \subset C^\infty(\overline{\Omega}, \mathbb{R}^d) \textit{ such that } g_n \overset{\sim}{\to} g, \\ \quad \textit{it follows that } \|g_n - v\|_{L^1(\Omega,\mathbb{R}^d;|\mathrm{D}u|)} \to 0, \end{cases} \tag{1}$$

where

$$D = \big\{ g \in W^q(\mathrm{div}; \Omega) \cap L^\infty(\Omega, \mathbb{R}^d) \mid \\ \textit{there exists } v \in L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u|) \textit{ satisfying } (1) \big\}.$$

Clearly, such $v = Tg$ is unique in $L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u|)$ and hence T is well-defined. The next two propositions give some basic properties of the trace operator. It is shown that T is consistent with the normal trace operator and, as one would expect, is the identity for continuous vector fields. In the following we denote by $|\mathrm{D}^a u|$ the absolute continuous part of the measure $|\mathrm{D}u|$ with respect to $\mathcal{L}^d$.

Proposition 5. *For $u \in \mathrm{BV}(\Omega)$ and $g \in D$ with D as in Definition 12, we have that*

$$Tg = g \quad |\mathrm{D}^a u| - a.e.,$$

$$\|Tg\|_\infty \leq \|g\|_\infty.$$

Proof. Take $(g_n)_{n\geq 0} \overset{\sim}{\to} g$ as in Definition 11. By L^q-convergence of $(g_n)_{n\geq 0}$ to g, there exists a subsequence of $(g_n)_{n\geq 0}$, denoted by $(g_{n_i})_{i\geq 0}$ converging pointwise $\mathcal{L}^d$-almost everywhere – and thus $|\mathrm{D}^a u|$-a.e. – to g. Now by convergence of $(g_{n_i})_{i\geq 0}$ to Tg in $L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u|)$ there exists a subsequence, again denoted by $(g_{n_i})_{i\geq 0}$, converging to Tg $|\mathrm{D}u|$-a.e. Since we can write $|\mathrm{D}u| = |\mathrm{D}^a u| + |\mathrm{D}^s u|$ where $|\mathrm{D}^s u|$ denotes the singular part of $|\mathrm{D}u|$ with respect to $\mathcal{L}^d$, this implies convergence of $(g_{n_i})_{i\geq 0}$ to Tg $|\mathrm{D}^a u|$ -a.e. Together, by uniqueness of the pointwise limit, it follows $Tg = g$ $|\mathrm{D}^a u|$-a.e.

Since

$$|Tg| = |\lim_{i\to\infty} g_{n_i}| \leq \|g\|_\infty \quad |\mathrm{D}u|\text{-a.e.},$$

also the second assertion follows. □

Proposition 6. *For $u \in \mathrm{BV}(\Omega)$ and for any $\phi \in C(\overline{\Omega}, \mathbb{R}^d) \cap W^q(\mathrm{div};\Omega)$, it follows that $\phi \in D$ and*

$$T\phi = \phi$$

as a function in $L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u|)$. If, in addition, $u \in L^p(\Omega)$ with $p = \frac{q}{q-1}$ for $1 < q < \infty$ and $p = \infty$ for $q = 1$ such that the normal trace operator, mapping to $L^1(\Omega; |\mathrm{D}u|)$, is defined on D, then for any $g \in D$ we have that

$$T_N g = Tg \cdot \sigma_u.$$

Proof. For the first assertion, we need to show that for any $(\phi_n)_{n\geq 0} \overset{\sim}{\to} \phi$,

$$\int_\Omega |\phi_n - \phi| \,\mathrm{d}|\mathrm{D}u| \to 0 \text{ as } n \to \infty.$$

But this follows from Lebesgue's dominated convergence theorem, using that $|\phi_n - \phi| \leq 2\|\phi\|_\infty$ and that for continuous functions every point is a Lebesgue point. Now take $g \in D$ and assume $u \in L^p(\Omega)$. Since $D \subset L^\infty(\Omega, \mathbb{R}^d)$, the normal trace $T_N g$ is defined and, with $(g_n)_{n\geq 0}$ as in Proposition 2, we have

$$\int_\Omega |Tg \cdot \sigma_u - T_N g_n| \,\mathrm{d}|\mathrm{D}u| \leq \int_\Omega |Tg - g_n| \,\mathrm{d}|\mathrm{D}u| \to 0.$$

where we used that, by Proposition 4, $T_N g_n = g_n \cdot \sigma_u$ and that $|\sigma_u| = 1$. By closedness of T_N the second assertion follows. □

Note that, by density of $C(\overline{\Omega}, \mathbb{R}^d)$ in $W^q(\mathrm{div};\Omega)$, Proposition 6 in particular implies that the full trace operator is densely defined.

In [7, Theorem 1.9] it was shown that, for $u \in \mathrm{BV}(\Omega) \cap L^p(\Omega)$ and $g \in W^q(\mathrm{div};\Omega) \cap L^\infty(\Omega, \mathbb{R}^d)$, with $p = \frac{q}{q-1}$ for $1 < q < \infty$ and $p = \infty$ for $q = 1$, denoting by $\theta(g, \mathrm{D}u)$ the normal trace of g as in Proposition 4, the following Gauss-Green formula holds:

$$\int_\Omega u \,\mathrm{div}\, g \,\mathrm{d}x + \int_\Omega \theta(g, \mathrm{D}u)\, |\mathrm{D}u| = \int_{\partial\Omega} [g \cdot \nu] u^\Omega \,\mathrm{d}\mathcal{H}^{d-1},$$

where $[g \cdot \nu] \in L^\infty(\partial\Omega; \mathcal{H}^{d-1})$ and $u^\Omega \in L^1(\partial\Omega; \mathcal{H}^{d-1})$ denote the boundary trace functions of g and u, respectively. As an immediate consequence of this and Proposition 6, we can present a Gauss-Green formula for the full trace:

Corollary 1. *For $g \in D$, $u \in \mathrm{BV}(\Omega) \cap L^p(\Omega)$ and $[g \cdot \nu]$ as in [7, Theorem 1.2], with $p = \frac{q}{q-1}$ for $1 < q < \infty$ and $p = \infty$ for $q = 1$, we have*

$$\int_\Omega u \,\mathrm{div}\, g \,\mathrm{d}x + \int_\Omega Tg \,\mathrm{D}u = \int_{\partial\Omega} [g \cdot \nu] u^\Omega \,\mathrm{d}\mathcal{H}^{d-1}.$$

3.3 Subdifferential characterization

We will now use the notion of full trace to describe the subdifferential of the TV functional. In order to do so, we first remember a well known result, which provides a characterization by using an integral equation. Note that here we define

$$\mathrm{TV} : L^p(\Omega) \to \overline{\mathbb{R}}, \quad 1 < p \leq \frac{d}{d-1},$$

as

$$\mathrm{TV}(u) = \sup \left\{ \int_\Omega u \operatorname{div} \phi \,\middle|\, \phi \in C_c^\infty(\Omega, \mathbb{R}^d),\ \|\phi\|_\infty \leq 1 \right\}$$

where TV may also attain the value ∞.

Proposition 7 (Integral characterization)**.** *Let $\Omega \subset \mathbb{R}^d$ with $d \geq 2$, $1 < p \leq \frac{d}{d-1}$, $q = \frac{p}{p-1}$ and $u \in L^p(\Omega)$, $u^* \in L^q(\Omega)$. Then $u^* \in \partial \mathrm{TV}(u)$ if and only if*

$$\begin{cases} u \in \mathrm{BV}(\Omega) \text{ and there exists } g \in W_0^q(\operatorname{div}; \Omega) \\ \text{with } \|g\|_\infty \leq 1 \text{ such that } u^* = -\operatorname{div} g \text{ and} \\ \displaystyle\int_\Omega \mathbf{1} \, \mathrm{d}|\mathrm{D}u| = -\int_\Omega u \operatorname{div} g. \end{cases}$$

Proof. For the sake of completeness, we elaborate on the proof: Denoting by $C = \left\{ \operatorname{div} \phi \,|\, \phi \in C_c^\infty(\Omega, \mathbb{R}^d),\ \|\phi\|_\infty \leq 1 \right\}$, we have

$$\mathrm{TV}(u) = \mathcal{I}_C^*(u),$$

where $\mathcal{I}_C^*$ denotes the polar of $\mathcal{I}_C$ [14, Definition I.4.1], and, consequently, see [14, Example I.4.3],

$$\mathrm{TV}^*(u^*) = \mathcal{I}_C^{**}(u^*) = \mathcal{I}_{\overline{C}}(u^*)$$

where the closure of C is taken with respect to the L^q norm. Using the equivalence [14, Proposition I.5.1]

$$u^* \in \partial \mathrm{TV}(u) \quad \Leftrightarrow \quad \mathrm{TV}(u) + \mathrm{TV}^*(u^*) = (u, u^*)_{L^p, L^q},$$

it therefore suffices to show that

$$\overline{C} = \{ \operatorname{div} g \,|\, g \in W_0^q(\operatorname{div}, \Omega),\ \|g\|_\infty \leq 1 \} =: K$$

to obtain the desired assertion. Since clearly $C \subset K$, it is sufficient for $\overline{C} \subset K$ to show that K is closed with respect to the L^q norm. For this purpose take $(g_n)_{n \geq 0} \subset W_0^q(\operatorname{div}; \Omega)$ with $\|g_n\|_\infty \leq 1$ such that

$$\operatorname{div} g_n \to h \text{ in } L^q(\Omega) \text{ as } n \to \infty.$$

By boundedness of $(g_n)_{n \geq 0}$ there exists a subsequence $(g_{n_i})_{i \geq 0}$ weakly converging to some $g \in L^q(\Omega, \mathbb{R}^d)$. Now for any $\phi \in C_c^\infty(\Omega)$,

$$\int_\Omega g \cdot \nabla \phi = \lim_{i \to \infty} \int_\Omega g_{n_i} \cdot \nabla \phi = \lim_{i \to \infty} -\int_\Omega \operatorname{div}(g_{n_i}) \phi = -\int_\Omega h \phi,$$

from which follows that $g \in W^q(\mathrm{div};\Omega)$ and $\mathrm{div}\, g = h$. To show that $\|g\|_\infty \leq 1$ and $g \in W_0^q(\mathrm{div};\Omega)$ note that the set

$$\{(f, \mathrm{div}\, f) \,|\, f \in W_0^q(\mathrm{div};\Omega),\ \|f\|_\infty \leq 1\} \subset L^q(\Omega, \mathbb{R}^{d+1})$$

forms a convex and closed – and therefore weakly closed – subset of $L^q(\Omega, \mathbb{R}^{d+1})$ [14, Section I.1.2]. Since the sequence $((g_{n_i}, \mathrm{div}\, g_{n_i}))_{i\geq 0}$ is contained in this set and converges weakly in $L^q(\Omega, \mathbb{R}^{d+1})$ to $(g, \mathrm{div}\, g)$, we have $g \in W_0^q(\mathrm{div};\Omega)$ and $\|g\|_\infty \leq 1$, hence $\mathrm{div}\, g \in K$. For $K \subset \overline{C}$ it suffices to show that, for any $g \in W_0^q(\mathrm{div};\Omega)$ with $\|g\|_\infty \leq 1$ fixed, we have for all $v \in L^p(\Omega)$ that

$$\int_\Omega v \,\mathrm{div}\, g \leq \mathrm{TV}(v)$$

since this implies $\mathrm{TV}^*(\mathrm{div}\, g) = \mathcal{I}_{\overline{C}}(\mathrm{div}\, g) = 0$. Now for such a $v \in L^p(\Omega)$ we can assume that $v \in \mathrm{BV}(\Omega)$ since in the other case the inequality is trivially satisfied. Thus we can take a sequence $(v_n)_{n\geq 0} \subset C^\infty(\overline{\Omega})$ strictly converging to v [2, Theorem 3.9], for which we can also assume that $v_n \to v$ with respect to $\|\cdot\|_{L^p}$. Using Remark 4 it follows

$$\begin{aligned} \int_\Omega v \,\mathrm{div}\, g &= \lim_{n\to\infty} \int_\Omega v_n \,\mathrm{div}\, g = \lim_{n\to\infty} -\int_\Omega \nabla v_n \cdot g \\ &\leq \lim_{n\to\infty} \int_\Omega |\nabla v_n||g| \leq \lim_{n\to\infty} \mathrm{TV}(v_n) = \mathrm{TV}(v). \square \end{aligned}$$

Remark 6. *Note that in the last part of the proof of Proposition 7 we have in particular shown that for any $g \in W_0^q(\mathrm{div};\Omega)$ with $\|g\|_\infty \leq 1$, where $q = \frac{p}{p-1}$ and $1 < p \leq \frac{d}{d-1}$, and any $v \in L^p(\Omega)$, the inequality*

$$\int_\Omega v \,\mathrm{div}\, g \leq \mathrm{TV}(v)$$

holds.

Using Proposition 7, we can derive the main result of the paper, a characterization of the subdifferential of the TV functional in terms of the full trace operator.

Theorem 1 (Pointwise characterization)**.** *With the assumptions of Proposition 7 we have that $u^* \in \partial\, \mathrm{TV}(u)$ if and only if*

$$\begin{cases} u \in \mathrm{BV}(\Omega) \text{ and there exists } g \in W_0^q(\mathrm{div};\Omega) \\ \text{with } \|g\|_\infty \leq 1 \text{ such that } u^* = -\,\mathrm{div}\, g \text{ and} \\ Tg = \sigma_u \text{ in } L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u|), \end{cases}$$

where σ_u is the density of $\mathrm{D}u$ w.r.t. $|\mathrm{D}u|$.

Proof. Let $u^* \in \partial \operatorname{TV}(u)$: Using Proposition 7, with $g \in W_0^q(\operatorname{div}, \Omega)$ provided there, it suffices to show that, for $(g_n)_{n\geq 0} \subset C^\infty(\overline{\Omega}, \mathbb{R}^d)$ such that $g_n \overset{\sim}{\to} g$ it follows

$$\|\sigma_u - g_n\|_{L^1(\Omega,\mathbb{R}^d;|\mathrm{D}u|)} \to 0.$$

Testing the zero extension of u, denoted by $w \in \operatorname{BV}(\mathbb{R}^d)$, with $(g_n)_{n\geq 0}$ extended to be in $C^1(\mathbb{R}^d, \mathbb{R}^d)$ yields, by virtue of [2, Corollary 3.89],

$$\begin{aligned}
\int_\Omega \mathbf{1}\, \mathrm{d}|\mathrm{D}u| &= -\int_\Omega u \operatorname{div} g \,\mathrm{d}x = \lim_{n\to\infty} -\int_\Omega u \operatorname{div} g_n \,\mathrm{d}x \\
&= \lim_{n\to\infty} -\int_{\mathbb{R}^d} w \operatorname{div} g_n \,\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^d} g_n \,\mathrm{d}\mathrm{D}w \\
&= \lim_{n\to\infty} \left(\int_\Omega g_n \cdot \sigma_u \,\mathrm{d}|\mathrm{D}u| + \int_{\partial\Omega} (g_n \cdot \nu_\Omega) u^\Omega \,\mathrm{d}\mathcal{H}^{d-1} \right) \qquad (2)
\end{aligned}$$

where, $u^\Omega \in L^1(\partial\Omega; \mathcal{H}^{d-1})$ denotes the trace of u on $\partial\Omega$ and ν_Ω is the generalized inner unit normal vector of $\partial\Omega$. Next, we like to show that the boundary term vanishes as $n \to \infty$. By density of $C^\infty(\overline{\Omega})$ in $\operatorname{BV}(\Omega)$ and continuity of the trace operator for BV functions with respect to strict convergence (see [2, Theorem 3.88]), for arbitrary $\epsilon > 0$, there exists $\phi_\epsilon \in C^\infty(\overline{\Omega})$ such that $\|u^\Omega - \phi_\epsilon^\Omega\|_{L^1(\partial\Omega)} < \epsilon$. By the standard Gauss-Green theorem we can write

$$\int_{\partial\Omega} (g_n \cdot \nu_\Omega)\phi_\epsilon \,\mathrm{d}\mathcal{H}^{d-1} = -\int_\Omega \operatorname{div}(g_n)\phi_\epsilon \,\mathrm{d}x - \int_\Omega g_n \cdot \nabla\phi_\epsilon \,\mathrm{d}x$$

and taking the limit as $n \to \infty$ we get, by $g_n \to g$ in $W^q(\operatorname{div}; \Omega)$,

$$\begin{aligned}
\lim_{n\to\infty} \int_{\partial\Omega} (g_n \cdot \nu_\Omega)\phi_\epsilon \,\mathrm{d}\mathcal{H}^{d-1} &= \lim_{n\to\infty} \left(-\int_\Omega \operatorname{div}(g_n)\phi_\epsilon \,\mathrm{d}x - \int_\Omega g_n \cdot \nabla\phi_\epsilon \,\mathrm{d}x \right) \\
&= -\int_\Omega \operatorname{div}(g)\phi_\epsilon \,\mathrm{d}x - \int_\Omega g \cdot \nabla\phi_\epsilon \,\mathrm{d}x = 0.
\end{aligned}$$

For $n \in \mathbb{N}$ we thus have, since $\|g_n\|_\infty \leq \|g\|_\infty$,

$$\begin{aligned}
\left| \int_{\partial\Omega} (g_n \cdot \nu_\Omega) u^\Omega \,\mathrm{d}\mathcal{H}^{d-1} \right| &= \left| \int_{\partial\Omega} (g_n \cdot \nu_\Omega)(u^\Omega - \phi_\epsilon) + (g_n \cdot \nu_\Omega)\phi_\epsilon \,\mathrm{d}\mathcal{H}^{d-1} \right| \\
&\leq \|g_n\|_\infty \|u^\Omega - \phi_\epsilon\|_{L^1(\partial\Omega)} + \left| \int_{\partial\Omega} (g_n \cdot \nu_\Omega)\phi_\epsilon \,\mathrm{d}\mathcal{H}^{d-1} \right| \\
&\leq \epsilon + \left| \int_{\partial\Omega} (g_n \cdot \nu_\Omega)\phi_\epsilon \,\mathrm{d}\mathcal{H}^{d-1} \right|.
\end{aligned}$$

Hence

$$\limsup_n \left| \int_{\partial\Omega} (g_n \cdot \nu_\Omega) u^\Omega \, \mathrm{d}\mathcal{H}^{d-1} \right| \leq \epsilon$$

and, since ϵ was chosen arbitrarily,

$$\lim_{n\to\infty} \int_{\partial\Omega} (g_n \cdot \nu_\Omega) u^\Omega \, \mathrm{d}\mathcal{H}^{d-1} = 0.$$

Together with equation (2) this implies

$$\int_\Omega \mathbf{1} \, \mathrm{d}|\mathrm{D}u| = \lim_{n\to\infty} \int_\Omega g_n \cdot \sigma_u \, \mathrm{d}|\mathrm{D}u|.$$

Using that $|g_n(x)| \leq 1$ for all $x \in \Omega$ and $|\sigma_u(x)| = 1$, $|\mathrm{D}u|$−a.e., we estimate $1 - (g_n \cdot \sigma_u)$:

$$\begin{aligned} 1 - (g_n \cdot \sigma_u) &= \frac{1}{2}|\sigma_u|^2 - (g_n \cdot \sigma_u) + \frac{1}{2}|g_n|^2 + \frac{1}{2}|\sigma_u|^2 - \frac{1}{2}|g_n|^2 \\ &= \frac{1}{2}|\sigma_u - g_n|^2 + \frac{1}{2}|\sigma_u|^2 - \frac{1}{2}|g_n|^2 \\ &\geq \frac{1}{2}|\sigma_u - g_n|^2 \quad |\mathrm{D}u| - \text{a.e.} \end{aligned}$$

Hence we have, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \limsup_{n\to\infty} \int_\Omega |\sigma_u - g_n| \, \mathrm{d}|\mathrm{D}u| &\leq \left(|\mathrm{D}u|(\Omega) \lim_{n\to\infty} \int_\Omega |\sigma_u - g_n|^2 \, \mathrm{d}|\mathrm{D}u| \right)^{\frac{1}{2}} \\ &\leq \left(2|\mathrm{D}u|(\Omega) \lim_{n\to\infty} \int_\Omega 1 - (g_n \cdot \sigma_u) \, \mathrm{d}|\mathrm{D}u| \right)^{\frac{1}{2}} = 0 \end{aligned}$$

from which the assertion follows.

In order to show the converse implication, we assume now that $u \in \mathrm{BV}(\Omega)$ and that there exists $g \in W_0^q(\mathrm{div};\Omega)$ with $\|g\|_{L^\infty} \leq 1$ such that $u^* = -\operatorname{div} g$ and $\sigma_u = Tg$ in $L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u|)$. Using Proposition 7, it is sufficient to show that

$$\int_\Omega \mathbf{1} \, \mathrm{d}|\mathrm{D}u| = -\int_\Omega u \operatorname{div} g \, \mathrm{d}x.$$

Taking $(g_n)_{n\geq 0} \subset C^\infty(\overline{\Omega}, \mathbb{R}^d)$ the approximating sequence as in Proposition 2, we have, analogously to the above, that

$$\int_{\partial\Omega} (g_n \cdot \nu_\Omega) u^\Omega \, \mathrm{d}\mathcal{H}^{d-1} \to 0$$

as $n \to \infty$ and, consequently, as $\lim_{n\to\infty} g_n = \sigma_u$ in $L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u|)$,

$$\begin{aligned}
\int_\Omega \mathbf{1}\,\mathrm{d}|\mathrm{D}u| &= \int_\Omega (\sigma_u \cdot \sigma_u)\,\mathrm{d}|\mathrm{D}u| \\
&= \lim_{n\to\infty} \int_\Omega (g_n \cdot \sigma_u)\,\mathrm{d}|\mathrm{D}u| \\
&= \lim_{n\to\infty} \left(-\int_\Omega \operatorname{div}(g_n) u\,\mathrm{d}x - \int_{\partial\Omega} (g_n \cdot \nu_\Omega) u^\Omega\,\mathrm{d}\mathcal{H}^{d-1} \right) \\
&= -\int_\Omega \operatorname{div}(g) u\,\mathrm{d}x. \qquad \square
\end{aligned}$$

Remark 7. *As one can see, the first two assumptions on the convergence as in Definition 11 indeed are necessary for the techniques applied in the proof of Theorem 1, while the third assumption is only needed to ensure the trace operator to be the identity for continuous vector fields as in Proposition 6.*

Remark 8. *Note that in the proof of Theorem 1 we have in particular shown the following condition for existence of a trace of a $W^q(\operatorname{div};\Omega)$ function g, with $\|g\|_\infty \le 1$, in $L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u|)$, $u \in L^p(\Omega)$, $q = \frac{p}{p-1}$, $1 < p \le \frac{d}{d-1}$:*

$$-\int_\Omega u \operatorname{div} g = \mathrm{TV}(u) \Leftrightarrow u \in \mathrm{BV}(\Omega), g \in D \text{ and } Tg = \sigma_u,$$

where D is the domain of the full trace operator T and σ_u is the density of $\mathrm{D}u$ w.r.t. $|\mathrm{D}u|$.

For the normal trace, a similar well known result follows as a direct consequence of Theorem 1 and Proposition 6:

Corollary 2. *Let the assumptions of Proposition 7 be satisfied. For $u \in L^p(\Omega)$ and $u^* \in L^q(\Omega)$ we have that $u^* \in \partial\,\mathrm{TV}(u)$ if and only if*

$$\begin{cases} u \in \mathrm{BV}(\Omega) \text{ and there exists } g \in W_0^q(\operatorname{div};\Omega) \\ \text{with } \|g\|_\infty \le 1 \text{ such that } u^* = -\operatorname{div} g \text{ and} \\ T_N g = Tg \cdot \sigma_u = 1 \text{ in } L^1(\Omega; |\mathrm{D}u|). \end{cases}$$

At last, let us further specify the expression $Tg = \sigma_u$. This can be done using the decomposition of $\mathrm{D}u$ into an absolute continuous part with respect to the Lebesgue measure, a Cantor part and a jump part, denoted by $\mathrm{D}^a u$, $\mathrm{D}^c u$ and $\mathrm{D}^j u$, respectively [2, Section 3.9]. The absolute continuous part can further be written as $\mathrm{D}^a u = \nabla u\,\mathrm{d}\mathcal{L}^2$ and the jump part as

$$\mathrm{D}^j u = (u^+(x) - u^-(x))\nu_u\,\mathrm{d}\mathcal{H}^1|_{S_u}$$

where $(u^+(x), u^-(x), \nu_u(x))$ represents uniquely, up to a change of sign, the jump at $x \in J_u$, with J_u and S_u denoting the jump set and the discontinuity set, respectively (see [2, Definition 3.67]). Since the measures $\mathrm{D}^a u$, $\mathrm{D}^c u$ and $\mathrm{D}^j u$ are mutually singular and $\mathcal{H}^1(S_u \setminus J_u) = 0$, the following result follows from Theorem 1 and Proposition 5.

Proposition 8. *Let the assumptions of Proposition 7 be satisfied. For $u \in L^p(\Omega)$ and $u^* \in L^q(\Omega)$ we have that $u^* \in \partial \operatorname{TV}(u)$ if and only if $u \in \operatorname{BV}(\Omega)$ and there exists $g \in W_0^q(\operatorname{div};\Omega)$ with $\|g\|_\infty \leq 1$ such that $u^* = -\operatorname{div} g$ and*

$$\begin{aligned} g &= \frac{\nabla u}{|\nabla u|} && \mathcal{L}^d - \text{a.e. on } \Omega \setminus \{x : \nabla u(x) = 0\}, \\ Tg &= \frac{u^+(x) - u^-(x)}{|(u^+(x) - u^-(x))|}\nu_u && \mathcal{H}^1 - \text{a.e. on } S_u, \\ Tg &= \sigma_{C_u} && |\mathrm{D}^c u| - \text{a.e.}, \end{aligned}$$

where σ_{C_u} is the density function of $\mathrm{D}^c u$ with respect to $|\mathrm{D}^c u|$.

4 Applications

In this section we will present some applications where the notation of a full trace together with the subdifferential characterization of the previous section can be used to extend known results involving the subdifferential of the TV functional. Remember that Ω is always assumed to be a bounded Lipschitz domain. For simplicity, we now restrict ourselves to the two dimensional setting, i.e. $\Omega \subset \mathbb{R}^2$, and use the more common notation $H(\operatorname{div};\Omega)$ for the space $W^2(\operatorname{div};\Omega)$.

As already mentioned in the introduction, the term of normal trace for $H(\operatorname{div};\Omega)$ functions is frequently used to describe the total variational flow, i.e. the solution of the formal equation [3, 4]

$$(\mathcal{P}_F) \begin{cases} \frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{\mathrm{D}u}{|\mathrm{D}u|}\right) & \text{in} \quad (0,\infty) \times \Omega \\ u(0,\cdot) = u_0(\cdot) & \text{in} \quad \Omega. \end{cases}$$

Defining the functional $\operatorname{TV} : L^2(\Omega) \to \overline{\mathbb{R}}$, this corresponds to the evolution problem

$$(\mathcal{P}) \begin{cases} \frac{\partial u(t)}{\partial t} + \partial \operatorname{TV}(u(t)) \ni 0 & \text{for} \quad t \in (0,\infty) \\ u(0) = u_0 & \text{in} \quad L^2(\Omega) \end{cases}$$

which appears in the steepest descent method to minimize the TV functional.

A solution to $(\mathcal{P})$ is a continuous function $u : [0,\infty) \to L^2(\Omega)$ with $u(0) = u_0$, which is absolutely continuous on $[a,b]$ for each $0 < a < b$, and hence differentiable almost everywhere, with $\frac{\partial u}{\partial t} \in L^1((a,b), L^2(\Omega))$ and $-\frac{\partial u(t)}{\partial t} \in \partial \operatorname{TV}(u(t))$ for almost every $t \in (0,\infty)$.

Using this notation, one gets the following existence result:

Proposition 9. *Let $u_0 \in L^2(\Omega)$. Then there exists a unique solution to $(\mathcal{P})$.*

Proof. Using [14, Corollary I.6.2] it follows that the closure of the domain of $\partial\,\mathrm{TV}$ is already $L^2(\Omega)$ and thus the result follows from [18, Corollary IV.3.2] $\square$

Using the full trace operator T and Theorem 1 we can now provide an equivalent characterization of a solution to $(\mathcal{P})$. For the proof, we need some properties for the solution which are stated in a lemma.

Lemma 1. *Consider $\partial\,\mathrm{TV}$ as a maximal monotone operator on $L^2(\Omega)$ and denote by*

$$A_0(u) = \operatorname*{arg\,min}_{v \in \partial\,\mathrm{TV}(u)} \|v\|_{L^2}$$

the minimal section *of $\partial\,\mathrm{TV}$.*

If $u_0 \in dom(\partial\,\mathrm{TV})$, then the solution u of $(\mathcal{P})$ satisfies:

(i) $u : [0, \infty) \to L^2(\Omega)$ is right-differentiable with right-derivative D^+u solving

$$D^+u(t) + A_0\big(u(t)\big) = 0 \qquad \text{for all } t \geq 0,$$

(ii) $A_0 \circ u : [0, \infty) \to L^2(\Omega)$, $(A_0 \circ u)(t) = A_0\big(u(t)\big)$ is right-continuous with $t \mapsto \|A_0\big(u(t)\big)\|_{L^2}$ non-increasing,

Proof. The items i and ii follow directly from [18, Proposition IV.3.1] applied to $\partial\,\mathrm{TV}$. $\square$

The characterization of the total variation flow in terms of the full trace then reads as follows.

Proposition 10. *A continuous function $u : [0, \infty) \to L^2(\Omega)$ is a solution to $(\mathcal{P})$ if and only if*

(i) u is absolutely continuous on $[a, b]$ for each $0 < a < b$ with derivative $\frac{\partial u}{\partial t} \in L^1((a, b); L^2(\Omega))$,

(ii) $u(t) \in \mathrm{BV}(\Omega)$ for each $t > 0$, $u(0) = u_0$,

(iii) there exists $g \in L^\infty((0, \infty) \times \Omega, \mathbb{R}^d)$ with $\|g\|_\infty \leq 1$ and

(iv) $g : (0, \infty) \to H_0(\mathrm{div}; \Omega)$ is measurable with $\frac{\partial u(t)}{\partial t} = \mathrm{div}\, g(t)$ as well as

$$Tg(t) = \sigma_u(t) \quad \text{in} \quad L^1(\Omega, \mathbb{R}^2; |\mathrm{D}u(t)|)$$

for almost every $t \in (0, \infty)$.

Proof. First note that without loss of generality, we can assume that $u_0 \in \mathrm{dom}(\partial\,\mathrm{TV})$: From [18, Proposition IV.3.2] follows that for each $t_0 > 0$, the translated solution $t \mapsto u(t+t_0)$ solves $(\mathcal{P})$ with initial value $u(t_0) \in \mathrm{dom}(\partial\,\mathrm{TV})$. Consequently, if the claimed statements are true on each $[t_0, \infty)$, then also on $(0, \infty)$.

Choose $L > 0$. We will now approximate u on $[0, L)$ as well as $\frac{\partial u}{\partial t}$ by piecewise constant functions as follows. Denote by $0 = t_0 < t_1 < \ldots < t_K = L$

a partition of $[0,L)$. For $t \in [0,L)$ denote by $k(t) = \min\,\{k' : t_{k'} > t\}$ as well as $\tau(t) = t_{k(t)} - t_{k(t)-1}$. For each $\varepsilon > 0$ we can now choose, due to the uniform continuity of u on $[0,L]$, a partition which satisfies

$$\|u(t) - u(t_{k(t)})\|_{L^2} < \varepsilon.$$

for all $t \in [0,L)$. It is moreover possible to achieve that these partitions are nested which implies that $t_{k(t)} \to t$, $\tau(t) \to 0$ as $\varepsilon \to 0$, both monotonically decreasing. Then, the function

$$u^\varepsilon : [0,L) \to L^2(\Omega), \qquad u^\varepsilon(t) = u(t_{k(t)})$$

obviously converges to u in $L^\infty((0,L),L^2(\Omega))$. Likewise, the function

$$(u^\varepsilon)' : [0,L) \to L^2(\Omega), \qquad (u^\varepsilon)'(t) = -A_0\big(u(t_{k(t)})\big)$$

satisfies, on the one hand, $-(u^\varepsilon)'(t) \in \partial\,\mathrm{TV}\big(u^\varepsilon(t)\big)$ for $t \in [0,L)$ by definition of A_0, see Lemma 1. On the other hand, for $t \in [0,L)$, we have $t_{k(t)} \to t$ monotonically decreasing, which implies by the right continuity of $t \mapsto A_0\big(u(t)\big)$, see Lemma 1, that

$$\lim_{\varepsilon\to 0}(u^\varepsilon)'(t) = -A_0\big(u(t)\big) \qquad \text{in } L^2(\Omega).$$

Also $\|(u^\varepsilon)'(t)\|_2 \le \|A_0(u_0)\|_2$, again by Lemma 1, so there exists an integrable majorant and by Lebesgue's theorem, $\lim_{\varepsilon\to 0}(u^\varepsilon)' = -A_0 \circ u$ in $L^2((0,L),L^2(\Omega))$. However, Lemma 1 yields $-A_0 \circ u = D^+u$, so $(u^\varepsilon)'$ is indeed approximating $\frac{\partial u}{\partial t}$.

As each u^ε, $(u^\varepsilon)'$ is constant on the finitely many intervals $[t_{k(t)-1}, t_{k(t)})$ and $-(u^\varepsilon)'(t) \in \partial\,\mathrm{TV}\big(u^\varepsilon(t)\big)$, we can choose a vector field g according to Proposition 7 on each of these intervals. Composing these g yields a measurable $g^\varepsilon \in L^2((0,L);H_0(\mathrm{div},\Omega))$, $\|g^\varepsilon\|_\infty \le 1$ in $L^\infty((0,L)\times\Omega,\mathbb{R}^d)$ and such that $(u^\varepsilon)' = \mathrm{div}\,g^\varepsilon$ in the weak sense. Moreover,

$$\int_0^L \int_\Omega \mathbf{1}\,\mathrm{d}|\mathrm{D}u^\varepsilon(t)|\,\mathrm{d}t = -\int_0^L \int_\Omega u^\varepsilon\,\mathrm{div}\,g^\varepsilon\,\mathrm{d}x\,\mathrm{d}t. \tag{3}$$

Now, $\{g^\varepsilon\}$ is bounded in $L^2((0,L),H_0(\mathrm{div},\Omega))$, hence there exists a weakly convergent subsequence (not relabeled) and a limit g with $\|g\|_\infty \le 1$ in $L^\infty((0,L)\times\Omega,\mathbb{R}^d)$. In particular, as $(u^\varepsilon)' = \mathrm{div}\,g^\varepsilon$, we have $\mathrm{div}\,g^\varepsilon \to \frac{\partial u}{\partial t}$ in $L^2((0,L),L^2(\Omega))$. By weak closedness of the divergence operator, also $\mathrm{div}\,g = \frac{\partial u}{\partial t}$.

Finally, taking the limits in (3) yields

$$\int_0^L \int_\Omega \mathbf{1}\,\mathrm{d}|\mathrm{D}u|\,\mathrm{d}t \le \liminf_{\varepsilon\to 0} \int_0^L \int_\Omega \mathbf{1}\,\mathrm{d}|\mathrm{D}u^\varepsilon|\,\mathrm{d}t = -\int_0^L \int_\Omega u\,\mathrm{div}\,g\,\mathrm{d}x\,\mathrm{d}t.$$

On the other hand, as for almost every $t \in (0,L)$, $g \in H_0(\mathrm{div};\Omega)$ and $\|g(t)\|_\infty \le 1$, according to Remark 6 it follows that $-\int_\Omega u(t)\,\mathrm{div}\,g(t) \le \mathrm{TV}\big(u(t)\big)$. Hence, the above is only possible if $-\int_\Omega u(t) \operatorname{div} g(t) = \mathrm{TV}\big(u(t)\big)$ for almost every $t \in (0, L)$. By Remark 8, a full trace then exists, i.e.

$$Tg(t) = \sigma_u(t) \quad \text{in} \quad L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u(t)|) \qquad \text{for a.e. } t \in (0, L).$$

Conversely, if we now assume that $u : [0, \infty) \to L^2(\Omega)$ satisfies i - iv, in order to establish that u is a solution to $(\mathcal{P})$ it is left to show that $-\frac{\partial u(t)}{\partial t} \in \partial\,\mathrm{TV}(u(t))$ for almost every $t \in (0, \infty)$. But since at almost every $t \in (0, \infty)$ we have, for $g \in L^\infty((0, \infty) \times \Omega, \mathbb{R}^d)$ as in iii, that $g(t) \in H_0(\mathrm{div}; \Omega)$, $\|g(t)\|_\infty \leq 1$, $\frac{\partial u(t)}{\partial t} = \operatorname{div} g(t)$ and $Tg(t) = \sigma_u(t)$, this follows as immediate consequence of Theorem 1. □

In a related context, a Cheeger set [12, 17] of a bounded set G of finite perimeter [2, Section 3.3] is defined to be the minimizer of

$$\min_{A \subset \overline{G}} \frac{|\partial A|}{|A|}. \tag{4}$$

Defining the constant

$$\lambda_G = \frac{|\partial G|}{|G|},$$

a sufficient condition for G to be a Cheeger set of itself, or in other words to be calibrable, is that $v := \chi_G$ satisfies the equation [8, Lemma 3]

$$-\operatorname{div}(\sigma_v) = \lambda_G v \quad \text{ on } \mathbb{R}^2, \tag{5}$$

i.e. there exists a vector field $\xi \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$ such that $\|\xi\|_\infty \leq 1$,

$$-\operatorname{div} \xi = \lambda_G v \quad \text{ on } \mathbb{R}^2$$

and

$$\int_{\mathbb{R}^2} \theta(\xi, \mathrm{D}v)\,\mathrm{d}|\mathrm{D}v| = \int_{\mathbb{R}^2} \mathbf{1}\,\mathrm{d}|\mathrm{D}v|.$$

This condition is further equivalent to [8, Theorem 4]:

1. G is convex.

2. ∂G is of class $C^{1,1}$.

3. It holds
$$\operatorname*{ess\,sup}_p \kappa_{\partial G}(p) \leq \frac{P(G)}{|G|},$$

where $\kappa_{\partial G}$ is the curvature of ∂G. Using the full trace operator, we can provide the following sufficient condition for G being calibrable:

Proposition 11. *Let $G \subset \mathbb{R}^2$ be a bounded set of finite perimeter. Then $v = \chi_G \in \mathrm{BV}(\mathbb{R}^2)$ satisfies condition (5) if there exists a bounded Lipschitz domain K such that $\overline{G} \subset K$ and $\xi \in H_0(\mathrm{div}; K)$ with $\|\xi\|_\infty \leq 1$ and $\xi \in D$, where D is the domain of the full trace operator, such that*

$$-\operatorname{div} \xi = \lambda_G v \quad \text{on } K$$

and

$$T\xi = \nu_G \quad \mathcal{H}^1 - \text{ almost everywhere on } \mathcal{F}G,$$

where $\mathcal{F}G$ is the reduced boundary, i.e. the set of all points $x \in \operatorname{supp} |\mathrm{D}\chi_G|$ such that the limit

$$\nu_G(x) := \lim_{\rho \to 0^+} \frac{\mathrm{D}\chi_G(B_\rho(x))}{|\mathrm{D}\chi_G(B_\rho(x))|}$$

exists.

Proof. The proof is straightforward: Using that $|\mathrm{D}\chi_G| = \mathcal{H}^1|_{\mathcal{F}G}$ and that $\mathrm{D}\chi_G = \nu_G |\mathrm{D}\chi_G|$ [2, Section 3.5] it follows that

$$\int_K |\mathrm{D}v| = \int_K T\xi \cdot \nu_G \, \mathrm{d}|\mathrm{D}v| = \int_K \theta(\xi, \mathrm{D}v) \, \mathrm{d}|\mathrm{D}v|.$$

From this and the fact that $\xi \in H_0(\mathrm{div}; K)$ it follows that its extension by 0 to the whole $\mathbb{R}^2$ is contained in $H(G; \mathbb{R}^2)$ and satisfies condition (5). □

The full trace operator can also be used to formulate optimality conditions for optimization problems appearing in mathematical imaging. A typical problem formulation would be

$$\min_{u \in L^2(\Omega)} \mathrm{TV}(u) + F(u), \tag{6}$$

where TV plays the role of a regularization term and $F : L^2(\Omega) \to \overline{\mathbb{R}}$ reflects data fidelity. Under weak assumptions on F we can derive the following general optimality condition:

Proposition 12. *Suppose that $F : L^2(\Omega) \to \overline{\mathbb{R}}$ is such that $\partial(\mathrm{TV} + F) = \partial \mathrm{TV} + \partial F$. Then we have that $u \in L^2(\Omega)$ solves (6) if and only if there exists $g \in H_0(\mathrm{div}; \Omega)$ such that $\|g\|_\infty \leq 1$,*

$$\operatorname{div} g \in \partial F(u)$$

and

$$Tg = \sigma_u \quad \text{in} \quad L^1(\Omega, \mathbb{R}^2; |\mathrm{D}u|)$$

Proof. This follows immediately from $\partial(\mathrm{TV} + F) = \partial \mathrm{TV} + \partial F$ and the characterization of $\partial \mathrm{TV}$ in Theorem 1. □

In [19], a problem of this type, but with a generalized regularization term was considered. Existence and a characterization of solutions to

$$\min_{u\in\mathrm{BV}} \int_\Omega \varphi(|\mathrm{D}u|) + \int_\Omega |Ku - u_0|^2$$

was shown, a problem which appears in denoising, deblurring or zooming of digital images.. For the characterization of optimal solutions, again the term $g \cdot \sigma_u$, with $g \in H(\mathrm{div};\Omega)$, was associated to a measure and then, following [13], it was split into a measure corresponding the absolute continuous part of $\mathrm{D}u$ with respect to the Lebesgue measure and a singular part. By applying Propositions 8 and 12, we can now get a characterization of solutions similar to [19, Propostion 4.1], but in terms of $L^1(\Omega, \mathbb{R}^2; |\mathrm{D}u|)$ functions, for the special case that φ is the identity:

Proposition 13. *Let $u_0 \in L^2(\Omega)$ and $K : L^2(\Omega) \to L^2(\Omega)$ a continuous, linear operator. Then, $u \in L^2(\Omega)$ is a solution to*

$$\min_{u\in\mathrm{BV}} \int_\Omega |\mathrm{D}u| + \int_\Omega |Ku - u_0|^2$$

if and only if $u \in \mathrm{BV}(\Omega)$ and there exists $g \in H_0(\mathrm{div};\Omega)$ with $\|g\|_\infty \le 1$ such that

$$2K^*(Ku - u_0) = \mathrm{div}\, g$$

and

$$\begin{aligned} g &= \frac{\nabla u}{|\nabla u|} && \mathcal{L}^2 - a.e.\ on\ \Omega \setminus \{x : \nabla u(x) = 0\} \\ Tg &= \frac{u^+(x) - u^-(x)}{|(u^+(x) - u^-(x))|}\nu_u && \mathcal{H}^1 - a.e.\ on\ S_u \\ Tg &= \sigma_{C_u} && |\mathrm{D}^c u| - a.e., \end{aligned}$$

where $u^+, u^-, \nu_u, S_u, C_u, \nabla u$ and $|\mathrm{D}^c u|$ are defined as in Proposition 8 and its preceding paragraph.

Proof. By continuity of $F(u) = \int_\Omega |Ku - u_0|^2$ it follows that $\partial(\mathrm{TV} + F) = \partial\,\mathrm{TV} + \partial F$ and we can apply Proposition 12. The characterization follows then by Proposition 8 and the fact that $\partial F(v) = \{2K^*(Ku - u_0)\}$ for any $v \in L^2(\Omega)$. □

The general formulation of an imaging problem as in (6) also applies, for example, to the minimization problem presented in [10]: There, as part of an infinite dimensional modeling of an improved JPEG reconstruction process, one solves

$$\min_{u\in L^2(\Omega)} \mathrm{TV}(u) + \mathcal{I}_U(u) \tag{7}$$

where $U = \{u \in L^2(\Omega) \,|\, Au \in J_n \text{ for all } n \in \mathbb{N}\}$, $A : L^2(\Omega) \to \ell^2$ is a linear basis transformation operator and $(J_n)_{n\in\mathbb{N}} = ([l_n, r_n])_{n\in\mathbb{N}}$ a given data set. Under some additional assumptions, a necessary and sufficient condition for u being a minimizer of (7) is stated in [10, Theorem 5]. Using the full trace operator, this condition can now be extended as follows:

Proposition 14. *With the assumptions of [10, Theorem 5], the function $u \in L^2(\Omega)$ is a minimizer of (7) if and only if $u \in \mathrm{BV}(\Omega) \cap U$ and there exists $g \in H_0(\mathrm{div}; \Omega)$ satisfying*

1. $\|g\|_\infty \leq 1$,
2. $Tg = \sigma_u$, $|\mathrm{D}u|$*-almost everywhere,*
3. $\begin{cases} (\mathrm{div}\, g, a_n)_{L^2} \geq 0 & \text{if} (Au)_n = r_n \neq l_n, \\ (\mathrm{div}\, g, a_n)_{L^2} \leq 0 & \text{if} (Au)_n = l_n \neq r_n, \\ (\mathrm{div}\, g, a_n)_{L^2} = 0 & \text{if} (Au)_n \in \mathring{J}_n, \end{cases} \qquad \forall n \in \mathbb{N}.$

5 Conclusion

We have introduced a trace operator allowing a pointwise evaluation of $W^q(\mathrm{div}; \Omega)$ functions in the space $L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u|)$, for $u \in \mathrm{BV}(\Omega)$. Using this operator, we have derived a subdifferential characterization of the total variation functional when considered as a functional from $L^p(\Omega)$ to the extended reals. This characterization gives an analytical motivation for the notation

$$-\,\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) \in \partial\, \mathrm{TV}(u),$$

frequently used in mathematical imaging problems related to TV minimization. We further have shown that, as on would expect, the concept of full trace extends the normal trace term by Anzellotti [7] and that it can be used in several applications, for example, to characterize the total variational flow.

A An approximation result

Since existence of a suitable approximating sequence for $W^q(\mathrm{div}; \Omega)$-vector fields is frequently used in this work, we give here an example of how to construct such a sequence. For Ω a bounded Lipschitz domain, $1 \leq q < \infty$ and $g \in W^q(\mathrm{div}; \Omega)$, we have to show existence of $(g_n)_{n\geq 0} \subset C^\infty(\overline{\Omega}, \mathbb{R}^d)$ satisfying:

1. $\|g_n - g\|_{W^q(\mathrm{div})} \to 0 \quad \text{as} \quad n \to \infty$,
2. $\|g_n\|_\infty \leq \|g\|_\infty$ for each $n \in \mathbb{N}$ if $g \in L^\infty(\Omega, \mathbb{R}^d) \cap W^q(\mathrm{div}; \Omega)$,
3. $g_n(x) \to g(x)$ for every Lebesgue point $x \in \Omega$ of g,
4. $\|g_n - g\|_{\infty, \overline{\Omega}} \to 0$ as $n \to \infty$, if, additionally, $g \in C(\overline{\Omega}, \mathbb{R}^d)$.

Proof. The proof follows basic ideas presented in [15, Theorem 4.2.3] for a density proof for Sobolev functions. We make use of the Lipschitz property of $\partial\Omega$: For $x \in \partial\Omega$, take $r > 0$ and $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ Lipschitz continuous, such that – upon rotating and relabeling the coordinate axes if necessary – we have

$$\Omega \cap Q_r(x) = \{y \in \mathbb{R}^d \,|\, \gamma(y_1, \dots, y_{d-1}) < y_d\} \cap Q_r(x) \tag{8}$$

where $Q_r(x) = \{y \in \mathbb{R}^d \,|\, |y_i - x_i| < r\,,\, i = 1, .., d\}$. Now for fixed $x \in \partial\Omega$, we define $Q = Q_r(x)$ and $Q' = Q_{\frac{r}{2}}(x)$. In the first step, we suppose that

$$\mathrm{spt}(g) := \overline{\{y \in \Omega : g(y) \neq 0\}} \subset Q'$$

and show that there exist vector fields $g_\epsilon \in C^\infty(\overline{\Omega}, \mathbb{R}^d)$ converging, as $\epsilon \to 0$, to g – in $W^q(\mathrm{div};\Omega)$, pointwise in every Lebesgue-point $y \in \Omega$ and uniformly on $\overline{\Omega}$ if additionally $g \in C(\overline{\Omega}, \mathbb{R}^d)$ – and satisfying the boundedness property 2).

Choose $\alpha = \mathrm{Lip}(\gamma) + 2$ fixed and $0 < \epsilon < \frac{r}{2(\alpha+1)}$ arbitrarily. It follows then by straightforward estimations that, for any $y \in \overline{\Omega \cap Q'}$, with $y^\epsilon = y + \epsilon\alpha e_d$, where e_d is the dth coordinate vector according to (8), we have $\overline{B_\epsilon(y^\epsilon)} \subset \Omega \cap Q$. Now with $\eta : \mathbb{R}^d \to \mathbb{R}$ a standard mollifier kernel supported in the unit ball, we define

$$\eta_\epsilon(y) = \frac{1}{\epsilon^d}\eta\left(\frac{y}{\epsilon}\right).$$

Using that $\overline{B_\epsilon(y^\epsilon)} \subset \Omega \cap Q$, for $y \in \overline{\Omega \cap Q'}$, it follows that the support of the functions

$$x \mapsto \eta_\epsilon(y + \epsilon\alpha e_d - x)$$

is contained in $\Omega \cap Q$. Thus, for $1 \leq j \leq d$, the functions $g^j_\epsilon : \overline{\Omega \cap Q'} \to \mathbb{R}$,

$$\begin{aligned} g^j_\epsilon(y) &= \int\limits_{\mathbb{R}^d} \eta_\epsilon(y + \epsilon\alpha e_d - x) g^j(x)\,\mathrm{d}x \\ &= \int\limits_{\mathbb{R}^d} \eta_\epsilon(y - z) g^j(z + \epsilon\alpha e_d)\,\mathrm{d}z = \left(\eta_\epsilon * g^j_{S_\epsilon}\right)(y), \end{aligned} \tag{9}$$

where

$$g^j_{S_\epsilon}(y) := g^j(y + \epsilon\alpha e_d)$$

denotes the composition of g^j with a translation operator, are well defined. Using standard results, given for example in [1, Section 2.12 and Proposition 2.14], it follows that $g^j_\epsilon \in C^\infty(\overline{\Omega \cap Q'})$ and, extending by 0 outside of $\overline{\Omega \cap Q'}$, that

$$\begin{aligned} \|g^j_\epsilon - g^j\|_{L^q(\Omega\cap Q')} &\leq \|\eta_\epsilon * g^j_{S_\epsilon} - \eta_\epsilon * g^j\|_{L^q(\mathbb{R}^d)} + \|\eta_\epsilon * g^j - g^j\|_{L^q(\mathbb{R}^d)} \\ &\leq \|\eta_\epsilon\|_{L^1(\mathbb{R}^d)}\|g^j_{S_\epsilon} - g^j\|_{L^q(\mathbb{R}^d)} + \|\eta_\epsilon * g^j - g^j\|_{L^q(\mathbb{R}^d)} \to 0 \end{aligned}$$

as $\epsilon \to 0$. By equivalence of norms in $\mathbb{R}^d$ it thus follows that the vector valued functions $g_\epsilon = (g^1_\epsilon, \dots, g^d_\epsilon)$ are contained in $C^\infty(\overline{\Omega \cap Q'})$ and that $\|g_\epsilon - g\|_{L^q(\Omega\cap Q')} \to 0$ as $\epsilon \to 0$. Since, for $i \in \{1 \dots d\}$,

$$\partial_i(\eta_\epsilon * g^j_{S_\epsilon}) = \partial_i\eta_\epsilon * g^j_{S_\epsilon},$$

we have, for $y \in \overline{\Omega \cap Q'}$, that

$$\begin{aligned}
\operatorname{div} g_\epsilon(y) &= \int_{\mathbb{R}^d} \nabla_y(\eta_\epsilon(y-x)) \cdot g_{S_\epsilon}(x)\,\mathrm{d}x \\
&= \int_{\Omega\cap Q} \nabla_y(\eta_\epsilon(y+\epsilon\alpha e_d - z)) \cdot g(z)\,\mathrm{d}z \\
&= -\int_{\Omega\cap Q} \nabla_z(\eta_\epsilon(y+\epsilon\alpha e_d - z)) \cdot g(z)\,\mathrm{d}z \\
&= \int_{\Omega\cap Q} (\eta_\epsilon(y+\epsilon\alpha e_d - z)) \operatorname{div} g(z)\,\mathrm{d}z \\
&= \int_{\mathbb{R}^d} (\eta_\epsilon(y+\epsilon\alpha e_d - z)) \operatorname{div} g(z)\,\mathrm{d}z,
\end{aligned}$$

where we used that $x \mapsto \eta_\epsilon(y+\epsilon\alpha e_d - x) \in C_c^\infty(\Omega \cap Q)$ and the weak definition of div. An argumentation analogous to the above thus yields $\|\operatorname{div} g_\epsilon - \operatorname{div} g\|_{L^q(\Omega\cap Q')} \to 0$ as $\epsilon \to 0$. Now let $y \in \Omega \cap Q'$ be a Lebesgue point of g. Again by equivalence of norms it suffices to show that $g_\epsilon^j(y) \to g^j(y)$ for y being a Lebesgue point of g^j, $1 \le j \le d$. With $\epsilon > 0$ sufficiently small such that, with $t := 1+\alpha$, we have $B_{\epsilon t}(y) \subset \Omega \cap Q$ we can estimate

$$\begin{aligned}
|g_\epsilon^j(y) - g^j(y)| &= \left| \frac{1}{\epsilon^d} \int_{\mathbb{R}^d} \eta\Big(\frac{y-w}{\epsilon}\Big)\big(g^j(w+\epsilon\alpha e_n) - g^j(y)\big)\,\mathrm{d}w \right| \\
&\le C(d)\frac{1}{|B_\epsilon(y)|} \int_{B_\epsilon(y)} |g^j(w+\epsilon\alpha e_n) - g^j(y)|\,\mathrm{d}w \\
&= C(d)\frac{1}{|B_\epsilon(y)|} \int_{B_\epsilon(y+\epsilon\alpha e_n)} |g^j(w) - g^j(y)|\,\mathrm{d}w \\
&\le \tilde{C}(d)\frac{1}{|B_{\epsilon t}(y)|} \int_{B_{\epsilon t}(y)} |g^j(w) - g^j(y)|\,\mathrm{d}w,
\end{aligned}$$

with $C(d), \tilde{C}(d) > 0$ constants depending only on d. Now since y was assumed to be a Lebesgue point of g^j, the desired convergence follows.

Now, additionally suppose that $g \in C(\overline{\Omega}, \mathbb{R}^d)$. Note that $\epsilon > 0$ can also be chosen such that with $\tau = \alpha + 1$, $B_{\epsilon t} \subset \Omega \cap Q$ for all $y \in \overline{\Omega \cap Q'}$, so the above implies

$$\begin{aligned}
|g_\epsilon^j(y) - g^j(y)| &\le \tilde{C}(d)\frac{1}{|B_{\epsilon t}(y)|} \int_{B_{\epsilon t}(y)\cap\Omega\cap Q} |g^j(w) - g^j(y)|\,\mathrm{d}w \\
&\le \tilde{C}(d) \sup_{w \in \overline{B_{\epsilon t}(y)\cap\Omega\cap Q}} \big(|g^j(w) - g^j(y)|\big).
\end{aligned}$$

By uniform continuity of g in the compact set $\overline{\Omega}$ it follows that $\|g_\epsilon^j - g^j\|_{\infty,\overline{\Omega\cap Q}}$ – and thus also $\|g_\epsilon - g\|_{\infty,\overline{\Omega\cap Q'}}$ – converges to zero as $\epsilon \to 0$.

Next we estimate the sup-norm of g_ϵ: Suppose $\|g\|_\infty \leq C$. For $y \in \overline{\Omega \cap Q'}$ we then have:

$$\begin{aligned} |g_\epsilon(y)|^2 &= \frac{1}{\epsilon^{2d}} \sum_{i=1}^{d} \left(\int\limits_{\Omega\cap Q} \sqrt{\eta\Big(\frac{y-w}{\epsilon} + \alpha e_n\Big)} \sqrt{\eta\Big(\frac{y-w}{\epsilon} + \alpha e_n\Big)} g^i(w)\,\mathrm{d}w \right)^2 \\ &\leq \frac{1}{\epsilon^{2d}} \left(\int\limits_{\Omega\cap Q} \eta\Big(\frac{y-w}{\epsilon} + \alpha e_n\Big) \sum_{i=1}^{d} g^i(w)^2\,\mathrm{d}w \right) \cdot \\ &\quad \left(\int\limits_{\Omega\cap Q} \eta\Big(\frac{y-w}{\epsilon} + \alpha e_n\Big)\,\mathrm{d}w \right) \\ &\leq C^2. \end{aligned}$$

At last, since $\mathrm{spt}(g) \subset Q'$ it follows that $\mathrm{spt}(g_\epsilon) \subset Q'$ for sufficiently small ϵ and thus we can extend it by 0 to the rest of $\overline{\Omega}$. Note that the convergence of g_ϵ to g – in $W^q(\Omega, \mathrm{div})$, in every Lebesgue point $y \in \Omega \setminus Q'$ and uniformly on $\overline{\Omega}$ in the case that additionally $g \in C(\overline{\Omega}, \mathbb{R}^d)$ – and also the uniform boundedness on all of $\overline{\Omega}$ are trivially satisfied.

In the second step we make use of the previous calculations to get an approximation to g without additional assumptions: Since $\partial\Omega$ is compact, there exist finitely many cubes $Q_i' = Q_{\frac{r_i}{2}}(x_i)$, $1 \leq i \leq M$ as above, which cover $\partial\Omega$. Let $(\zeta_i)_{0\leq i\leq M}$ be C^∞-functions, such that

$$\begin{cases} 0 \leq \zeta_i \leq 1 \quad \mathrm{spt}(\zeta_i) \subset Q_i' \quad \text{for } 1 \leq i \leq M, \\ 0 \leq \zeta_0 \leq 1 \quad \mathrm{spt}(\zeta_0) \subset \Omega, \\ \displaystyle\sum_{i=0}^{M} \zeta_i \equiv 1 \quad \text{on } \Omega. \end{cases}$$

As shown above, for $g\zeta_i$, $1 \leq i \leq M$ we can construct vector fields $g_{\epsilon,i} \in C^\infty(\overline{\Omega}, \mathbb{R}^d)$ converging to $g\zeta_i$ in the desired sense. By a standard mollifier approximation we can also construct $g_{\epsilon,0}$ converging to $g\zeta_0$ in the desired sense. Setting

$$g_\epsilon = \sum_{i=0}^{M} g_{\epsilon,i}$$

we finally obtain vector fields in $C^\infty(\overline{\Omega}, \mathbb{R}^d)$ converging to g in $W^q(\mathrm{div}; \Omega)$ as $\epsilon \to 0$ and, as one can check easily, satisfying also the additional boundedness and convergence properties 2), 3), 4). □

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