WEAK DIFFERENTIABILITY OF THE CONTROL-TO-STATE MAPPING IN A PARABOLIC CONTROL PROBLEM WITH HYSTERESIS

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Abstract. We consider the heat equation on a bounded domain subject to an inhomogeneous forcing in terms of a rate-independent (hysteresis) operator and a control variable.

The aim of the paper is to establish a functional analytical setting which allows to prove weak differentiability properties of the control-to-state mapping. Using results of [BK] on the weak differentiability of scalar rate-independent operators, we prove Bouligand differentiability in suitable Bochner spaces of the control-to-state mapping in a parabolic problem.

1. Introduction and Problem formulation

The aim of this article is to study weak differentiability properties of a parabolic control problem with a rate-independent hysteresis operator. More precisely, we consider the following problem.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with sufficiently smooth boundary $\Gamma := \partial\Omega \in C^{2+\alpha}$ for $\alpha > 0$ and denote $\Omega_T := \Omega \times (0,T)$ and $\Gamma_T := \Gamma \times (0,T)$. Given a control $u \in L^2(\Omega_T)$, we shall consider the following control problem for the heat equation coupled to a rate-independent operator $\mathcal{W}$:

$$y_t - \Delta y = u + w, \qquad \text{in} \quad \Omega_T, \tag{1a}$$

$$w = \mathcal{W}[y], \qquad \text{in} \quad \Omega_T, \tag{1b}$$

$$\mathcal{B}[y] = 0, \qquad \text{on} \quad \Gamma_T, \tag{1c}$$

$$y(\cdot, 0) = y_0, \qquad \text{on} \quad \Omega. \tag{1d}$$

Here, $\mathcal{W}$ denotes a large class of operators, in particular rate-independent operators, which shall be define precisely in the following. Moreover, $\mathcal{B}$ specifies a linear boundary operator corresponding to homogeneous Dirichlet data $\mathcal{B}[y] = y|_{\Gamma_D} = 0$ on a subpart of the boundary $\Gamma_D \subset \Gamma$ with non-zero measure $|\Gamma_0| > 0$ and homogeneous Neumann boundary data on the remaining part of the boundary $\Gamma_N := \Gamma \setminus \Gamma_D$.

The operator $\mathcal{W}$. The operator $\mathcal{W}$ is constructed as a space-dependent version of an operator $\mathcal{V}$,

$$\mathcal{W}[y](x,t) = \mathcal{V}[y(x,\cdot)](t), \quad (x,t) \in \Omega \times [0,T]. \tag{2}$$

2010 *Mathematics Subject Classification.* 47J40, 35K10, 34K35.

Key words and phrases. Heat equation, rate independence, hysteresis operator, optimal control, weak differentiability.

Thus, $\mathcal{W}$ represents a family of operators acting on $y(x,\cdot)$, viewed as a function of time, at every $x \in \Omega$.

We remark that if one wants to include a space-dependent initial condition for $\mathcal{V}$, one would write $\mathcal{W}[y](x,t) = \mathcal{V}[y(x,\cdot),x](t)$ instead of (2); we will not do this in this paper.

Concerning the operator $\mathcal{V}$, we assume that

$$\mathcal{V} : C[0,T] \times \Omega \to C[0,T] \tag{3}$$

is Lipschitz continuous; more precisely, we require that there exists an $L > 0$ such that

$$|\mathcal{V}[v](t) - \mathcal{V}[\tilde{v}](t)| \le L \sup_{0\le s\le t} |v(s) - \tilde{v}(s)| \tag{4}$$

holds for every $v, \tilde{v} \in C[0,T]$, every $t \in [0,T]$ and every $x \in \Omega$. Condition (4) also implies causality. We moreover assume linear growth

$$|\mathcal{V}[v](t)| \le L \sup_{0\le s\le t} |v(s)| + c_0 \tag{5}$$

for arguments as above, and some $c_0 > 0$.

The properties (4) and (5) carry over to the operator $\mathcal{W}$ defined in (3). Denoting

$$\|y(x,\cdot)\|_{\infty,t} = \sup_{0\le s\le t} |y(x,s)|, \tag{6}$$

we immediately obtain that

$$\|\mathcal{W}[y](x,\cdot) - \mathcal{W}[\tilde{y}](x,\cdot)\|_{\infty,t} \le L\|y(x,\cdot) - \tilde{y}(x,\cdot)\|_{\infty,t}, \tag{7}$$

$$\|\mathcal{W}[y](x,\cdot)\|_{\infty,t} \le L\|y(x,\cdot)\|_{\infty,t} + c_0, \tag{8}$$

holds for every $y, \tilde{y} \in L^2(\Omega; C[0,T])$, for a.e. $x \in \Omega$ and every $t \in [0,T]$. Thus,

$$\mathcal{W} : L^2(\Omega; C[0,T]) \to L^2(\Omega; C[0,T]) \tag{9}$$

is well-defined.

Under the assumptions above, the following existence and uniqueness result is a consequence of Theorems X.1.1 and X.1.2 of [Vis]. In the following, we shall either use the space

$$V = H^1_{\Gamma_D} = \{v \in H^1_0 : v|_{\Gamma_D} = 0\},$$

in case $|\Gamma_N| > 0$ or

$$V = H^1_0,$$

in case $|\Gamma_N| = 0$.

Theorem 1 (Existence and Uniqueness, see [Vis]). *For every $u \in L^2(\Omega_T)$ and every $y_0 \in V$, the initial-boundary value problem given by* (1) *has a unique solution*

$$y \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V), \qquad \mathcal{W}[y] \in L^2(\Omega; C[0,T]).$$

Proof. The existence proof is based on the compactness of the embeddding of $H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)$ into $L^2(\Omega; C[0,T])$, see [Vis]. □

Theorem 1 guarantees that the control-to-state operator

$$y = Su\,, \qquad S : L^2(\Omega_T) \to H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)\,,$$

is well-defined. Assume for a moment that the control-to-state operator S is differentiable w.r.t. some suitable norms, then for an increment $h \in L^2(\Omega_T)$ we would have

$$S(u+h) = Su + S'(u)h + o(\|h\|)\,, \tag{10}$$

where the first order approximation $d = S'(u)h$ to the difference $S(u+h) - Su$ depends linearly upon h and is expected to solve a linear problem, obtained from linearising the original problem.

When $\mathcal{W}$ is a hysteresis operator, $\mathcal{W}$ (and thus S) are not differentiable in the classical sense. Nevertheless, let us consider the formal linearisation of (1). Given functions $y = Su$ and h, we want to determine functions d and p as solutions of

$$d_t - \Delta d = h + p, \qquad \text{in} \quad \Omega_T, \tag{11a}$$

$$p = \mathcal{W}'[y;d], \qquad \text{in} \quad \Omega_T, \tag{11b}$$

$$\mathcal{B}[d] = 0, \qquad \text{on} \quad \Gamma_T, \tag{11c}$$

$$d(\cdot,0) = 0, \qquad \text{on} \quad \Omega. \tag{11d}$$

Here, $\mathcal{W}'[y;d]$ denotes the directional derivative of $\mathcal{W}$ at y in the direction d. We do not assume that the mapping $d \mapsto \mathcal{W}'[y;d]$ is linear; indeed, hysteresis operators do not satisfy this property, but they possess directional derivatives, see [BK]. Thus, we term the system (11) the **first order problem**; it is nonlinear whenever the mapping $d \mapsto \mathcal{W}'[y;d]$ is not linear.

Our aim is to derive weak differentiability properties of the control-to-state operator S from corresponding properties of the operator $\mathcal{W}$. More precisely, out main results shows the following Theorem, which will be proved in Section 3 below:

Theorem 2 (Weak differentiability of the state-to-control map S)**.**
The control-to-state mapping $u \mapsto Sy$ has a Bouligand derivative when considered as an operator

$$S : L^2(0,T;L^\infty(\Omega)) \to H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)\,. \tag{12}$$

The derivative $d = S'(u;h)$ is given by the solution of the first order problem (11).

Weak differentiability of $\mathcal{V}$ and $\mathcal{W}$. We assume that the operator $\mathcal{V} : C[0,T] \to C[0,T]$ has the following weak differentiability properties.

(i) For every $v, \eta \in C[0,T]$, the pointwise derivative $\mathcal{V}^{PD}[v;\eta] : [0,T] \to \mathbb{R}$ defined by

$$\mathcal{V}^{PD}[v;\eta](t) = \lim_{\lambda \downarrow 0} \frac{\mathcal{V}[v+\lambda\eta](t) - \mathcal{V}[v](t)}{\lambda} \tag{13}$$

exists for all $t \in [0,T]$ and is a regulated function. Linearity of the mapping $\eta \to \mathcal{V}^{PD}[v;\eta]$ is not assumed.

(ii) Let $p \in (1,\infty)$ and $r \in [1,\infty)$ be given. There exists a non-negative function $\rho : (0,\varepsilon_0) \to \mathbb{R}_+$ with $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$\|\mathcal{V}[v+\eta] - \mathcal{V}[v] - \mathcal{V}^{PD}[v;\eta]\|_{L^r(0,t)} \le \rho(\|\eta\|_{\infty,t})\|\eta'\|_{L^p(0,t)} \tag{14}$$

holds for all $v \in C[0,T]$, $\eta \in W^{1,p}(0,T)$, $t \in [0,T]$.

Remark 3. *The foregoing estimate implies that the operator $\mathcal{V}$ is Bouligand differentiable when considered as an operator*

$$\mathcal{V} : W^{1,p}(0,T) \to L^r(0,T)\,.$$

Accordingly, the pointwise derivative $\mathcal{V}^{PD}[v;\eta]$ is called the Bouligand derivative of $\mathcal{V}$, and we shall denote it by $\mathcal{V}'[v;\eta]$ instead.

The estimate

$$\|\mathcal{V}'[v;\eta] - \mathcal{V}'[v;\zeta]\|_{\infty,t} \le L\|\eta-\zeta\|_{\infty,t}\,, \quad \text{for all } \eta,\zeta \in C[0,T], \tag{15}$$

follows immediately from (13) and (4). In particular, since $\mathcal{V}'[v;0]=0$,

$$\|\mathcal{V}'[v;\eta]\|_{\infty,t} \le L\|\eta\|_{\infty,t}\,, \quad \text{for all } \eta \in C[0,T], \tag{16}$$

We now define pointwise in $x \in \Omega$

$$\mathcal{W}'[y;d](x,t) = \mathcal{V}'[y(x,\cdot);d(x,\cdot)](t). \tag{17}$$

Due to (15) and (16), the operator

$$d \mapsto \mathcal{W}'[y;d]\,, \quad L^2(\Omega;C[0,T]) \to L^2(\Omega;G[0,T])$$

is well-defined; here, $G[0,T]$ denotes the space of regulated functions on $[0,T]$.

Due to (16) and (17), the operator $\eta \mapsto \mathcal{V}'[v;\eta]$ satisfies the assumptions of Theorems X.1.1 and X.1.2 in [Vis], which can be extended to cover the range space $G[0,T]$ instead of $C[0,T]$ for the hysteresis operator. This yields the following

Theorem 4. *The first order problem given by* (11) *has a unique solution*

$$d \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V), \qquad p \in L^2(\Omega;G[0,T]).$$

We remark that the function p has less regularity than the corresponding function w in the original problem (1).

This regularity is consistent with standard parabolic regularity; given $(h+p) \in L^2(\Omega_T)$, then parabolic regularity yields $d \in L^2(\Omega;W^{1,2}(0,T)) \cap L^\infty((0,T);H^1(\Omega))$.

We also have the following estimate.

Theorem 5. *The solution d of the first order problem (11a – 11d) satisfies*

$$\int_0^T\!\!\int_\Omega d_t^2\,dx\,dt + \sup_{t\in[0,T]}\int_\Omega |\nabla d|^2\,dx \le C_1(T)\int_0^T\!\!\int_\Omega h^2\,dx\,dt \tag{18}$$

as well as

$$\|d\|_{L^\infty(\Omega_T)} \le C_2(T)\int_0^T \|h(\cdot,t)\|_\infty\,dt\,. \tag{19}$$

The constants $C_1(T)$ and $C_2(T)$ do not depend on h.

Proof. The proof of (18) follows from estimate (22) in Lemma 6 by setting $z := d$, $f := |h|$ and $g := h + p$ as well as by noting that

$$|p + h|(x, t) \leq L \sup_{s \leq t} |d(x, s)| + |h(x, t)| \,,$$

Moreover, (19) follows from estimate (23) in Lemma 6 in Section 2. □

2. Regularity estimates

The following Lemma 6 provides a parabolic regularity statement for the heat equation subject to rate-independent operator satisfying the Lipschitz continuity (8).

Lemma 6 (Parabolic regularity). *Consider the parabolic problem:*

$$z_t - \Delta z = g, \qquad in \quad \Omega_T, \tag{20a}$$

$$\mathcal{B}[z] = 0, \qquad on \quad \Gamma_T, \tag{20b}$$

$$z(\cdot, 0) = 0, \qquad on \quad \Omega, \tag{20c}$$

where $g \in L^2(\Omega; G[0, T])$ satisfies the following estimate

$$|g|(x, t) \leq L \sup_{s \leq t} |z(x, s)| + f(x, t). \tag{21}$$

for a non-negative function $f(x, t) \geq 0$.

(1) *Assume that $f \in L^2(\Omega_T)$. Then, for all $T > 0$*

$$\int_0^T \int_\Omega (z_t)^2 \, dxdt + \sup_{t \in [0,T]} \int_\Omega |\nabla z(t)|^2 \, dx \leq C_1(T) \int_0^T \int_\Omega f^2 \, dxdt, \tag{22}$$

where the constant $C_1(T)$ grows at most linearly in T.

(2) *Assume that $f \in L^1([0, T]; L^\infty(\Omega))$. Then, for all $T > 0$*

$$\sup_{t \in [0,T]} \|z\|_{L^\infty_x}(t) \leq C_2(T) \int_0^T \|f\|_{L^\infty_x}(s) \, ds, \tag{23}$$

where the constant $C_2(T)$ grows at most exponentially in T.

Remark 7. *The estimate (22) implies the continuity at zero of the mapping*

$$f \in L^2(\Omega_T) \mapsto z \in L^2(\Omega; W^{1,2}(0, T)) \cap L^\infty((0, T); H^1(\Omega))$$

with a bound which grows at most linearly in T. In fact, the estimates (29) and (30) below imply even the continuity at zero of the mapping

$$(f, z_0) \in L^2(\Omega_T) \times H^1(\Omega) \mapsto z \in L^2(\Omega; W^{1,2}(0, T)) \cap L^\infty((0, T); H^1(\Omega))$$

regardless of $z(0) = 0$ as considered in (20c). We remark that these estimates do not imply linearity away from zero, which can not be expected in general for rate-independent evolutions. However, we are only interested in continuity at zero.

Moreover, the estimate (23) implies the continuity at zero of the mapping

$$f \in L^1([0, T]; L^\infty(\Omega)) \mapsto z \in L^\infty(\Omega_T)$$

with a constant, which grows at most exponentially in T. Again, the solution operator is nonlinear in general and continuity is only proved at zero.

Proof. We prove first estimate (22). Our goal is to obtain an a priori estimate for z in terms of f. To this end, we test (20a) formally with z_t and integrate over Ω_T. We point out that the formal integration with z_t can be made rigorous by a suitable approximation procedure whenever solutions to (20) have full parabolic regularity, in particular, $z_t \in L^2(\Omega_T)$. This is standard in the case $V = H_0^1$ since we have $g \in L^2(\Omega_T)$ and otherwise smooth data, boundary and coefficients, see e.g. [QS]. For $V = H^1_{\Gamma_D}$, the required regularity $z_t \in L^2(\Omega_T)$ follows from $\Delta z \in L^2(\Omega_T)$, which can be shown following [Chi][Chapter 11] by using that problem (20) is linear with constant coefficients and homogeneous boundary and initial data and $\nabla g \in L^2((0,T);V')$.

By integrating by parts and using (21), we obtain

$$\begin{aligned}\int_0^T\!\!\int_\Omega (z_t)^2\,dxds' + \int_0^T\!\!\int_\Omega \partial_t\left(\frac{|\nabla z|}{2}\right)\,dxds' &\le \int_0^T\!\!\int_\Omega |g||z_t|\,dxds' \\ \le L\int_0^T\!\!\int_\Omega \sup_{s\le t}|z(x,s)||z_t(x,s')|\,dxds' &+ \int_0^T\!\!\int_\Omega f|z_t|\,dxds',\end{aligned} \tag{24}$$

where we remark that all boundary terms vanish for the considered homogeneous boundary operator $\mathcal{B}$ in (20b). Moreover, we may replace the second term in the first line by $\frac{1}{2}\int_0^T \frac{d}{dt}\int_\Omega |\nabla z|^2 dxdt$.

In order to handle the first term on the right hand side of (24), we use that

$$\sup_{0\le s\le t}|z(x,s)| \le \int_0^T |z_t(x,s)|ds$$

and estimate with Young's inequality

$$\begin{aligned}\int_0^T\!\!\int_\Omega \sup_{s\le t}|z(x,s)||z_t(x,s')|\,dxds' &\le \int_0^T\!\!\int_\Omega\int_0^T |z_t(x,s)||z_t(x,s')|\,dsdxds' \\ &\le \int_0^T\!\!\int_\Omega\int_0^T \frac{|z_t(x,s)|^2}{2}\,dsdxds' + \int_0^T\!\!\int_\Omega\int_0^T \frac{|z_t(x,s')|^2}{2}\,dsdxds' \\ &\le T\int_0^T\!\!\int_\Omega |z_t(x,s')|^2\,dxds'.\end{aligned}$$

Coming back to (24), we obtain by using Young's inequality with a constant $\theta > 0$,

$$\begin{aligned}\int_0^T\!\!\int_\Omega (z_t)^2\,dxds' &+ \int_\Omega \frac{|\nabla z(T)|^2}{2}\,dx \\ &\le \int_\Omega \frac{|\nabla z(0)|^2}{2}\,dx + L\,T\int_0^T\!\!\int_\Omega |z_t|^2 dxds' \\ &\quad + \frac{\theta}{2}\int_0^T\!\!\int_\Omega |z_t|^2\,dxds' + \frac{1}{2\theta}\int_0^T\!\!\int_\Omega f^2\,dxds',\end{aligned} \tag{25}$$

where we ignore for the moment that we actually have $z(0) = 0$.

The estimate (25) has the following two consequences: First, choosing $T_1 := \frac{1}{3L}$ and setting $\theta = \frac{2}{3}$, we obtain

$$\frac{1}{3}\int_0^T\int_\Omega (z_t)^2\,dxds' \le \int_\Omega \frac{|\nabla z(0)|^2}{2}\,dx + \frac{3}{4}\int_0^T\int_\Omega f^2\,dxds', \qquad \text{for all} \quad T \le T_1. \tag{26}$$

Secondly, choosing θ such that $1 = LT_1 + \frac{\theta}{2}$, that is, $\theta = \frac{4}{3}$, we conclude that

$$\int_\Omega \frac{|\nabla z(T)|^2}{2}\,dx \le \int_\Omega \frac{|\nabla z(0)|^2}{2}\,dx + \frac{3}{8}\int_0^T\int_\Omega f^2\,dxds', \qquad \text{for all} \quad T \le T_1. \tag{27}$$

Next, we shall iterate the estimates (26) and (27) and consider subsequent time intervals (T_n, T_{n+1}), where $T_n := nT_1$ for $n = 0, 1, \ldots$. For any cylinder $\Omega \times (T_n, T_{n+1})$, we can perform the same arguments as above for (24) and obtain analog estimates for (26) and (27), i.e.

$$\int_{T_n}^{T_{n+1}}\int_\Omega (z_t)^2\,dxds' \le \frac{3}{2}\int_\Omega |\nabla z(T_n)|^2\,dx + \frac{9}{4}\int_{T_n}^{T_{n+1}}\int_\Omega f^2\,dxds', \tag{28}$$

$$\int_\Omega |\nabla z(T_{n+1})|^2\,dx \le \int_\Omega |\nabla z(T_n)|^2\,dx + \frac{3}{4}\int_{T_n}^{T_{n+1}}\int_\Omega f^2\,dxds', \tag{29}$$

Then, iterating (29) yields directly

$$\int_\Omega |\nabla z(T_{n+1})|^2\,dx \le \int_\Omega |\nabla z(0)|^2\,dx + \frac{3}{4}\int_0^{T_{n+1}}\int_\Omega f^2\,dxds'. \tag{30}$$

Moreover, by repeatedly using (29) and summing the intervals (T_{n-1}, T_n), we obtain

$$\begin{aligned}\int_0^{T_n}\int_\Omega (z_t)^2\,dxds' &\le n\frac{3}{2}\int_\Omega |\nabla z(0)|^2\,dx + \sum_{k=0}^{n-1}(n+1-k)\frac{9}{8}\int_{T_k}^{T_{k+1}}\int_\Omega f^2\,dxds',\\ &\le \frac{3n}{2}\int_\Omega |\nabla z(0)|^2\,dx + \frac{9(n+1)}{8}\int_0^{T_n}\int_\Omega f^2\,dxds',\end{aligned} \tag{31}$$

where we remark that $\frac{3n}{2} = O(T) = \frac{9(n+1)}{8}$.

By recalling that the remainder system (20) is in fact subject to zero initial data (20c), i.e. $z(0) = 0$, we obtain (22) for all $T > 0$.

We shall now prove (23). More precisely, we will show that the solution of the remainder problem (20) can be estimated in $L^\infty(\Omega_T)$ via a maximum principle argument. We refer, for instance, to [Chi] for weak maximum principles for the heat equation with the homogeneous boundary operator $\mathcal{B}$ as given in (20b) and inhomogeneity $g \in L^2(\Omega_T)$.

Then, by applying the weak maximum principle for parabolic equations to (20), we are able to estimate the growth of the L^∞-norm of z as

$$\partial_t \|z\|_{L^\infty_x}(t) \le L \sup_{s\le t} \|z\|_{L^\infty_x} + \|f\|_{L^\infty_x}(t), \tag{32}$$

where we have used $\|\sup_{s\le t}|z|\|_{L^\infty_x} = \sup_{s\le t}\|z\|_{L^\infty_x}$. Moreover, we recall that $z(0) = 0$.

Therefore, a natural guess for an upper solution is the monotone increasing solution of the initial-value problem

$$\begin{cases} \frac{d}{dt}M(t) = L\,M(t) + \|f\|_{L^\infty_x}(t), \\ M(0) = \|z\|_{L^\infty_x} = 0. \end{cases} \tag{33}$$

Indeed, by denoting $m(t) := \|z\|_{L^\infty_x}(t)$, we have

$$\begin{cases} \frac{d}{dt}(m-M) \le L\left[\sup_{s\le t} m - M\right] \le L\left[\sup_{s\le t}(m-M)\right] \\ (m-M)(0) = 0, \end{cases} \tag{34}$$

since $\sup_{s\le t}(m - M + M) \le \sup_{s\le t}(m-M) + M(t)$ as $M(t)$ is monotone increasing. As a consequence we can estimate the growth behaviour of (34) by

$$(m-M)(t) \le (m-M)(0)\, e^{L\,t} = 0,$$

which implies with the solution of (33)

$$\|z\|_{L^\infty_x}(t) \le \int_0^t e^{L\,(t-s)} \|f\|_{L^\infty_x}(s)\, ds, \qquad \text{for all} \quad t\in[0,T], \tag{35}$$

which yields (23) for all $T > 0$. □

3. Weak differentiability of S

Proof of Theorem 2. We consider an increment $h \in L^2(\Omega_T)$ of a given nominal control $u \in L^2(\Omega_T)$. We denote by

$$y := Su, \qquad \text{and} \qquad y_h := S[u+h]$$

the corresponding states, and by

$$w_h := \mathcal{W}[y_h]$$

the corresponding output of the hysteresis operator. Then, y_h and w_h solve the system

$$\begin{aligned} (y_h)_t - \Delta y_h &= (u+h) + w_h, && \text{in} \quad \Omega_T, \\ w_h &= \mathcal{W}[y_h], && \text{in} \quad \Omega_T, \\ \mathcal{B}[y_h] &= y_\Gamma, && \text{on} \quad \Gamma_T, \\ y_h(\cdot,0) &= y_0, && \text{on} \quad \Omega, \end{aligned}$$

Moreover, we define the differences

$$d_h := y_h - y, \qquad p_h := w_h - w. \tag{36}$$

Our goal is to show that the solutions d and p of the first order problem are indeed first order approximations of d_h and p_h.

The differences $d_h - d$ and $p_h - p$ satisfy the following system of equations.

$$(d_h - d)_t - \Delta(d_h - d) = p_h - p, \qquad \text{in} \quad \Omega_T, \tag{37a}$$
$$p_h - p = \mathcal{W}[y_h] - \mathcal{W}[y] - \mathcal{W}'[y;d], \qquad \text{in} \quad \Omega_T, \tag{37b}$$
$$\mathcal{B}[d_h - d] = 0, \qquad \text{on} \quad \Gamma_T, \tag{37c}$$
$$(d_h - d)(\cdot,0) = 0, \qquad \text{on} \quad \Omega, \tag{37d}$$

We now want to estimate $|p_h - p|$. From (7) we get

$$\begin{aligned} |\mathcal{W}[y_h(x,\cdot)] - \mathcal{W}[(y+d)(x,\cdot)]|\,(t) &\le L \sup_{s\le t} |y_h(x,s) - y(x,s) - d(x,s)| \\ &= L \sup_{s\le t} |d_h(x,s) - d(x,s)| \end{aligned}$$

for a.e. $x \in \Omega$ and all $t \in [0,T]$.

From (14) and (17), we obtain for a.e. $x \in \Omega$ and all $t \in [0,T]$

$$\begin{aligned} &\big|\mathcal{W}[(y+d)(x,\cdot)] - \mathcal{W}[y(x,\cdot)] - \mathcal{W}'[y(x,\cdot); d(x,\cdot)]\big|\,(t) \\ &\quad \le \|d_t(x,\cdot)\|_{L^p_t(0,t)}\, \rho(\|d(x,\cdot)\|_{\infty,t}). \end{aligned} \tag{38}$$

Therefore, we can estimate $|p_h - p|$ against $|d_h - d|$ as

$$\begin{aligned} |p_h - p|(x,t) &= |w_h - w - p| = \big|\mathcal{W}[y_h] - \mathcal{W}[y] - \mathcal{W}'[y;d]\big| \\ &\le |\mathcal{W}[y_h] - \mathcal{W}[y+d]| + \big|\mathcal{W}[y+d] - \mathcal{W}[y] - \mathcal{W}'[y;d]\big| \\ &\le L \sup_{s\le t} |d_h(x,s) - d(x,s)| + \|d_t(x,\cdot)\|_{L^p_t(0,t)}\, \rho(\|d(x,\cdot)\|_{\infty,t}), \end{aligned} \tag{39}$$

for a.e. $x \in \Omega$ and all $t \in [0,T]$.

In the next step, we shall apply the remainder estimate (39) to the remainder problem (37). Let us introduce the notation

$$z(x,t) := d_h - d, \quad f(x,t) := \|d_t(x,\cdot)\|_{L^p_t(0,t)}\, \rho(\|d(x,\cdot)\|_{\infty,t}) \ge 0. \tag{40}$$

Then, the system (37) satisfies the assumptions of Lemma 6. In particular, we have

$$\int_0^T\!\!\int_\Omega (d_h - d)_t^2\, dxdt + \sup_{t\in[0,T]} \int_\Omega |\nabla(d_h - d)|^2\, dx \le C_1(T) \int_0^T\!\!\int_\Omega f^2\, dxdt, \tag{41}$$

We shall now estimate f. Recalling (40), we have

$$\begin{aligned} \int_0^T\!\!\int_\Omega f^2\, dx\, dt &= \int_0^T\!\!\int_\Omega \|d_t(x,\cdot)\|^2_{L^2(0,t)} \rho(\|d(x,\cdot)\|_{\infty,t})^2\, dx\, dt \\ &\le T \int_0^T\!\!\int_\Omega d_t(x,t)^2\, dx\, dt \cdot \rho(\|d\|_{L^\infty(\Omega_T)})^2 . \end{aligned}$$

Using the bounds from Theorem 5, we get from (18) and (19) that

$$\begin{aligned} &\int_0^T\!\!\int_\Omega \|(d_h - d)_t\|^2\, dx\, dt + \sup_{t\in[0,T]} \int_\Omega |\nabla(d_h - d)|^2\, dx \\ &\quad \le C_1(T) \int_0^T\!\!\int_\Omega h^2\, dx\, dt \cdot \rho\Big(C_2(T) \int_0^T \|h(\cdot,t)\|_\infty\, dt\Big)^2 . \end{aligned} \tag{42}$$

This ends the proof of Theorem 2. □

References

[BK] M. Brokate, P. Krejci, *Weak Differentiability of Scalar Hysteresis Operators*, DCDS **35**, no.6, (2015) pp. 2405-2421.

[Chi] Michel Chipot, Elements of Nonlinear Analysis, Birkháuser Advanced Texts, Basel, 2000.

[QS] P. Quittner, Ph. Souplet, Superlinear parabolic problems. Blow-up, global existence and steady states, Birkhauser Advanced Texts, 2007.

[Vis] A. Visintin, Differential Models of Hysteresis, Applied Mathematical Sciences 111, Springer 1994.

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