

# SENSITIVITY ANALYSIS AND OPTIMAL CONTROL OF OBSTACLE-TYPE EVOLUTION VARIATIONAL INEQUALITIES\*

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**Abstract.** This paper is concerned with the differential sensitivity analysis and the optimal control of evolution variational inequalities (EVIs) of obstacle type. We demonstrate by means of a counterexample that the solution map  $S$  of an EVI with a unilateral constraint is typically not (weakly) directionally differentiable or Lipschitz continuous in any of the spaces  $H^s(0, T; H)$ ,  $s \geq 1/2$ , where  $(0, T)$  is the time interval and  $H$  is the pivot space of the underlying Gelfand triple  $V \hookrightarrow H \hookrightarrow V^*$ . We further establish that, despite this negative result, the solution operator is always strongly Hadamard directionally differentiable as a function  $S : L^2(0, T; H) \rightarrow L^q(0, T; H)$  for all  $1 \leq q < \infty$ , weakly- $\star$  directionally differentiable as a function  $S : L^2(0, T; H) \rightarrow L^\infty(0, T; H)$ , and weakly directionally differentiable as a function  $S : L^2(0, T; H) \rightarrow L^2(0, T; V)$ . Using the differentiability properties of the map  $S$ , we derive strong stationarity conditions for optimal control problems that are governed by EVIs of obstacle type. The resulting optimality system is compared with that obtained by regularization.

**Key words.** Evolution variational inequality, Parabolic obstacle problem, Hadamard directional differentiability, Sensitivity analysis, Strong stationarity, Optimal control, Signorini problem

**AMS subject classifications.** 35B30 35K85 49J40 49K40 90C31

**1. Introduction.** This paper is concerned with the sensitivity analysis and the optimal control of obstacle-type evolution variational inequalities (EVIs) of the form

$$\left. \begin{aligned} &y \in L^2(0, T; V) \cap H^1(0, T; H), \\ &y \in K \text{ a.e. in } (0, T), \quad y(0) = y_0, \\ &\int_0^T (y', v - y)_H + \langle Ay, v - y \rangle_V - (u, v - y)_H \, dt \geq 0 \\ &\forall v \in L^2(0, T; V), \quad v \in K \text{ a.e. in } (0, T). \end{aligned} \right\} \quad (\text{P})$$

For the precise assumptions on the quantities  $A, H, V, K$  etc., see Section 2 below. We demonstrate that the solution operator  $S : u \mapsto y$  associated with (P) is typically not directionally differentiable as a function from  $L^2(0, T; H)$  to  $H^s(0, T; H)$ ,  $s \geq 1/2$ , (see the counterexample in Section 3), prove the Hadamard directional differentiability of the solution map  $S$  in the spaces  $L^q(0, T; H)$ ,  $1 \leq q < \infty$ , (see Theorem 4.1), and establish strong stationarity conditions for optimal control problems that are governed by EVIs of the type (P) (see Theorem 5.5 and (5.13)). The latter are shown to contain information that is not present in stationarity systems which are obtained by regularization, cf. the results in [5, Section 5.4] and [28, Theorem 6.2].

Let us put our work into perspective: Evolution variational inequalities with unilateral constraints are relevant for numerous applications in engineering, physics, and other disciplines. They emerge, for example, when the parabolic analogues of the classical

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obstacle problem and the Signorini variational inequality are considered, see [17,20,29,35,39,42], in finance, see [27,28,36,37,41], and in ice sheet models, see [10,30,43].

Despite this broad range of applications and the high interest in the optimal control of obstacle-type evolution variational inequalities that it entails, the differentiability properties of solution operators to EVIs of the type (P) are only rarely addressed in the literature. To the author's best knowledge, the only contribution in this field is that of Jarušek et al. in [29], where a parabolic problem with inequality constraints on the boundary is considered. As a consequence, the overwhelming majority of authors resorts to regularization or (semi-)discretization techniques to obtain, e.g., necessary optimality conditions for optimal control problems that are governed by EVIs of the type (P), cf. [1,4,5,9,19,23,25,28]. The aim of this paper is to provide differentiability results for evolution variational inequalities with unilateral constraints that allow to avoid regularization and that may serve as a point of departure for the development of solution algorithms for optimal control problems that take into account the non-smooth behavior of the governing EVI, cf., e.g., the approaches in [11,12,38]. We further demonstrate that our differentiability results give rise to strong stationarity conditions that resemble those derived by Mignot and Puel for the time-independent classical obstacle problem in [32,33] and that contain information which is not obtainable with regularization techniques.

Before we begin with our analysis, we give a short overview of the content and the structure of this paper:

In Section 2, we make precise our assumptions, present a preliminary existence and uniqueness result, and provide some examples of special instances of the problem (P) that illustrate the generality of our approach.

Section 3 contains a counterexample which demonstrates that the solution operator  $S$  to a problem of the type (P) can, in general, not be expected to be directionally differentiable or Lipschitz continuous as a function from  $L^2(0, T; H)$  to  $H^s(0, T; H)$  for any  $s \geq 1/2$ . The results obtained in this section illustrate in particular that the approach of Jarušek et al. in [29] cannot be generalized to EVIs that behave, e.g., like the classical parabolic obstacle problem.

In Section 4, we prove the directional differentiability of the solution map  $S$  to (P) in various  $L^q$ -spaces. See Theorem 4.1 for the main result.

Section 5 is concerned with strong stationarity conditions for optimal control problems that are governed by EVIs of the form (P). Here, we use an auxiliary variational inequality for the directional derivatives of the solution operator  $S$  that is obtained as a byproduct of our sensitivity analysis to derive a stationarity system analogous to that in [32,33]. The resulting system turns out to contain, e.g., an additional sign condition on the adjoint state that is not present in the optimality conditions of [5,28].

Lastly, in Section 6, we give some concluding remarks.

**2. Setting and Preliminaries.** As already mentioned in the introduction, the aim of this paper is to study EVIs of the type

$$\left. \begin{aligned} y &\in L^2(0, T; V) \cap H^1(0, T; H), \\ y &\in K \text{ a.e. in } (0, T), \quad y(0) = y_0, \\ \int_0^T (y', v - y)_H + \langle Ay, v - y \rangle_V - (u, v - y)_H \, dt &\geq 0 \\ \forall v &\in L^2(0, T; V), \quad v \in K \text{ a.e. in } (0, T). \end{aligned} \right\} \quad (\text{P})$$

Our standing assumptions on the quantities in (P) are as follows:

ASSUMPTION 2.1 (Standing Assumptions for the Study of the EVI (P)).

- (i)  $(\Omega, \Sigma, \mu)$  is a complete measure space.
- (ii)  $H := L^2(\Omega, \mu)$ . The norm  $\|\cdot\|_H$  and the product  $(\cdot, \cdot)_H$  are defined as usual.
- (iii)  $V \subset H$  is a separable Hilbert space with dual  $V^*$  such that  $V \hookrightarrow H \hookrightarrow V^*$  is a Gelfand triple, i.e., the embedding  $V \hookrightarrow H$  is continuous and dense and  $H$  is identified with its own dual and subsequently with a subspace of  $V^*$ . By  $\|\cdot\|_V$  and  $\langle \cdot, \cdot \rangle_V$  we denote the norm and the dual pairing in  $V$ .
- (iv) The map

$$V \ni v \mapsto v^+ := \max(0, v) \in V$$

(where  $\max(0, \cdot)$  acts pointwise  $\mu$ -a.e. in  $\Omega$ ) is well-defined and continuous, there exists a constant  $C > 0$  with  $\|v^+\|_V \leq C\|v\|_V$  for all  $v \in V$ , and for every  $z \in L^2(0, T; V) \cap H^1(0, T; H)$  it holds

$$\int_0^T (z', z^+)_H dt = \frac{1}{2} \|z^+(T)\|_H^2 - \frac{1}{2} \|z^+(0)\|_H^2. \quad (2.1)$$

- (v)  $A : V \rightarrow V^*$  is a linear, continuous, symmetric, strongly monotone operator satisfying

$$\langle Av, v^+ \rangle_V \geq \langle Av^+, v^+ \rangle_V \quad \forall v \in V.$$

- (vi)  $K$  is a closed, convex, non-empty subset of  $V$  such that

$$\begin{aligned} v \in K, z \in V &\Rightarrow v + z^+ \in K, \\ v_1 \in K, v_2 \in K &\Rightarrow \min(v_1, v_2) \in K. \end{aligned}$$

- (vii)  $T > 0$  and  $y_0 \in K$  are given and fixed.
- (viii)  $u \in L^2(0, T; H)$  is a given datum (the argument of the solution map).

Some remarks are in order regarding the conditions in Assumption 2.1:

REMARK 2.2.

- (i) The separability of  $V$  and the continuity and density of the embedding  $V \hookrightarrow H$  yield that the space  $H$  is separable as well.
- (ii) Due to the continuity and the boundedness of the maps  $H \ni z \mapsto z^+ \in H$  and  $V \ni v \mapsto v^+ \in V$ , the separability of the spaces  $H$  and  $V$ , and the Pettis measurability theorem, see [24, Corollary 3.1.2], the maps

$$\begin{aligned} L^2(0, T; H) \ni z &\mapsto z^+ \in L^2(0, T; H), \\ L^2(0, T; V) \ni v &\mapsto v^+ \in L^2(0, T; V) \end{aligned}$$

are well-defined and bounded.

- (iii) Following [44], we call a set  $K \subset V$  with the properties in Assumption 2.1(vi) a set with a lower bound or a unilateral constraint set.
- (iv) The sensitivity analysis in Section 4 also works for asymmetric operators  $A$  and time-dependent  $K$ . We consider the situation in Assumption 2.1 here so that the existence of a solution to the problem (P) follows straightforwardly from the results in [5, Chapter 4], see Theorem 2.3 below. If the solvability of the EVI at hand can be established by other means, e.g., by mollification, cf. [27, Section 2.4], then our assumptions can be relaxed accordingly.

From classical results, we obtain:

**THEOREM 2.3.** *In the situation of Assumption 2.1, the EVI (P) admits one and only one solution  $y \in L^2(0, T; V) \cap H^1(0, T; H)$  for every right-hand side  $u \in L^2(0, T; H)$ . Moreover, there exists an absolute constant  $C > 0$  such that the solution operator  $S : u \mapsto y$  associated with (P) satisfies*

$$\begin{aligned} & \|S(u_1) - S(u_2)\|_{L^2(0, T; V)} + \|S(u_1) - S(u_2)\|_{L^\infty(0, T; H)} \\ & \leq C \min(\|u_1 - u_2\|_{L^2(0, T; V^*)}, \|u_1 - u_2\|_{L^1(0, T; H)}) \quad \forall u_1, u_2 \in L^2(0, T; H). \end{aligned} \quad (2.2)$$

*Proof.* The existence of a unique solution  $y \in L^2(0, T; V) \cap H^1(0, T; H)$  to (P) follows from [5, Theorems 4.1, 4.2]. To prove (2.2), let us assume that two right-hand sides  $u_1, u_2 \in L^2(0, T; H)$  with associated solutions  $y_1, y_2$  are given. Then, we may test the EVI for  $y_1$  with the function  $v \in L^2(0, T; V)$  which is equal to  $y_2$  in  $(0, s)$  and equal to  $y_1$  in  $(s, T)$ ,  $s \in (0, T)$ , to arrive at the estimate

$$\int_0^s (y_1', y_2 - y_1)_H + \langle Ay_1, y_2 - y_1 \rangle_V - (u_1, y_2 - y_1)_H dt \geq 0.$$

By exchanging the roles of  $y_1$  and  $y_2$ , by adding the resulting two inequalities and by exploiting the strong monotonicity of  $A$ , we obtain

$$\frac{1}{2} \|y_1(s) - y_2(s)\|_H^2 + c \int_0^s \|y_1 - y_2\|_V^2 dt \leq \int_0^s (u_1 - u_2, y_1 - y_2)_H dt \quad (2.3)$$

for all  $s \in (0, T)$  with some constant  $c > 0$ . Due to the continuity of the embedding  $H^1(0, T; H) \hookrightarrow C([0, T]; H)$  and the structure  $V \hookrightarrow H \hookrightarrow V^*$ , (2.3) implies that there exists an absolute constant  $C > 0$  with

$$\begin{aligned} \|y_1 - y_2\|_{L^2(0, T; V)} & \leq C \|u_1 - u_2\|_{L^2(0, T; V^*)}, \\ \|y_1 - y_2\|_{L^\infty(0, T; H)} & \leq C \|u_1 - u_2\|_{L^1(0, T; H)}. \end{aligned}$$

Using the last two inequalities in (2.3) yields (2.2) as claimed.  $\square$

See also [6, 8, 27] for alternative existence and uniqueness results for EVIs. Let us give some examples of problems that fit into the setting of Assumption 2.1:

**EXAMPLE 2.4** (Primitive Real-Valued EVIs). *Consider the variational inequality*

$$\left. \begin{aligned} & y \in H^1(0, T), \quad y \geq \alpha \text{ a.e. in } (0, T), \quad y(0) = y_0, \\ & \int_0^T y'(v - y) + Ay(v - y) - u(v - y) dt \geq 0 \\ & \forall v \in L^2(0, T), \quad v \geq \alpha \text{ a.e. in } (0, T) \end{aligned} \right\} \quad (2.4)$$

with some  $\alpha \in \mathbb{R}$ ,  $A \in \mathbb{R}^+$ ,  $T \in \mathbb{R}^+$ ,  $y_0 \in [\alpha, \infty)$ ,  $u \in L^2(0, T)$ , where  $L^2(0, T)$  and  $H^1(0, T)$  are the classical  $L^2$ - and  $H^1$ -space, respectively. Then, this EVI is precisely of the form (P) and satisfies all conditions in Assumption 2.1. To see this, define  $\Omega := \{0\}$ , choose  $\mu$  to be the Dirac measure at zero, let  $\Sigma$  be the power set of  $\Omega$  and consider the (trivially complete) measure space  $(\Omega, \Sigma, \mu)$ . For this choice of  $(\Omega, \Sigma, \mu)$ , the space  $H := L^2(\Omega, \mu)$  can obviously be identified with  $\mathbb{R}$ . Doing so and defining  $V := H = \mathbb{R}$ , we obtain a Gelfand triple as in Assumption 2.1(iii) with  $L^2(0, T; V) = L^2(0, T; H) = L^2(0, T)$  and  $H^1(0, T; H) = H^1(0, T)$ . Note that these spaces trivially satisfy the conditions in point (iv) of Assumption 2.1 (see the lemma of Stampacchia, [3, Theorem 5.8.2], for (2.1)). If we combine all of the above and define  $K := [\alpha, \infty)$ , then it follows immediately that (2.4) is covered by our setting.

EXAMPLE 2.5 (Classical Parabolic Obstacle Problems). Let  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 1$ , be a bounded domain endowed with the  $d$ -dimensional Lebesgue measure  $\mathcal{L}^d$ , and define

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad K := \{v \in H^1(\Omega) \mid v \geq \psi \text{ } \mathcal{L}^d\text{-a.e. in } \Omega\},$$

where  $L^2(\Omega)$  and  $H^1(\Omega)$  are defined as usual, see [3], and where  $\psi : \Omega \rightarrow [-\infty, \infty]$  is a Lebesgue measurable function such that the set  $K$  is non-empty. Then,  $\Omega$ ,  $\mu := \mathcal{L}^d$ , the Lebesgue  $\sigma$ -algebra,  $H$  and  $V$  trivially satisfy the conditions in points (i), (ii) and (iii) of Assumption 2.1, and we obtain from [3, Theorem 5.8.2] and [46, Lemma 3.2] that the map  $V \ni v \mapsto v^+ \in V$  is well-defined, continuous and bounded, and that the integration by parts formula (2.1) holds. Using the above and again [3, Theorem 5.8.2], it follows straightforwardly that, e.g., the classical parabolic obstacle problem

$$\left. \begin{aligned} y \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega)), \quad y \in K \text{ a.e. in } (0, T), \quad y(0) = y_0, \\ \int_0^T (y', v - y)_{L^2} + (y, v - y)_{H^1} - (u, v - y)_{L^2} dt \geq 0 \\ \forall v \in L^2(0, T; H^1(\Omega)), \quad v \in K \text{ a.e. in } (0, T) \end{aligned} \right\} \quad (2.5)$$

with a  $y_0 \in K$  is covered by our analysis. Compare also with [27, 28] in this context.

EXAMPLE 2.6 (Signorini-Type Problems). Let  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 1$ , be a bounded Lipschitz domain endowed with the Lebesgue measure  $\mathcal{L}^d$ . Suppose that  $\Gamma_1$  and  $\Gamma_2$  are two disjoint subsets of the boundary  $\Gamma := \partial\Omega$  and define

$$\begin{aligned} H_D^1(\Omega) &:= \{v \in H^1(\Omega) \mid \text{tr}(v) = 0 \text{ } \mathcal{H}^{d-1}\text{-a.e. on } \Gamma_1\}, \\ K &:= \{v \in H_D^1(\Omega) \mid \text{tr}(v) \geq 0 \text{ } \mathcal{H}^{d-1}\text{-a.e. on } \Gamma_2\}, \end{aligned}$$

where  $\mathcal{H}^{d-1}$  denotes the  $(d-1)$ -dimensional Hausdorff measure and where  $\text{tr}$  denotes the trace. Then, it follows completely analogously to Example 2.5 that the EVI

$$\left. \begin{aligned} y \in L^2(0, T; H_D^1(\Omega)) \cap H^1(0, T; L^2(\Omega)), \quad y \in K \text{ a.e. in } (0, T), \quad y(0) = y_0, \\ \int_0^T (y', v - y)_{L^2} + (y, v - y)_{H^1} - (u, v - y)_{L^2} dt \geq 0 \\ \forall v \in L^2(0, T; H_D^1(\Omega)), \quad v \in K \text{ a.e. in } (0, T) \end{aligned} \right\} \quad (2.6)$$

with a  $y_0 \in K$  fits into the setting of Assumption 2.1. The above problem corresponds to that studied by Jarušek et al. in [29]. (Note that, in [29], the time interval is  $\mathbb{R}$ .)

**3. A Counterexample.** To develop intuition for the behavior of the solution operator  $S : u \mapsto y$  associated with the EVI (P) and to get an idea of what to expect when analyzing the differentiability properties of this map, we consider the following special situation:

ASSUMPTION 3.1 (Standing Assumptions for Section 3).

- (i)  $\Omega$ ,  $\Sigma$ ,  $\mu$  are chosen as in Example 2.4, and  $H$ ,  $V$ ,  $V^*$  are identified with  $\mathbb{R}$ ,
- (ii)  $A := 1$ ,  $K := [0, \infty)$ ,  $T := 3$ ,  $y_0 := 0$ ,
- (iii)  $\tilde{u} \in L^2(0, 3)$  is an arbitrary but fixed function satisfying

$$\tilde{u} = \tilde{y}' + \tilde{y} - \varphi \quad (3.1)$$

for some  $\tilde{y}, \varphi \in C^\infty([0, 3])$  with

$$\begin{aligned} \tilde{y} &\equiv 0 \text{ in } [0, 3] \setminus (1, 2), & \tilde{y} &> 0 \text{ in } (1, 2), \\ \varphi &> 0 \text{ in } [0, 3] \setminus [1, 2], & \varphi &\equiv 0 \text{ in } [1, 2]. \end{aligned}$$

For the above choice of  $H$ ,  $V$  etc., (P) is a special instance of the EVI (2.4), namely:

$$\left. \begin{aligned} & y \in H^1(0, 3), \quad y \geq 0 \text{ a.e. in } (0, 3), \quad y(0) = 0, \\ & \int_0^3 y'(v - y) + y(v - y) - u(v - y) dt \geq 0 \\ & \forall v \in L^2(0, 3), \quad v \geq 0 \text{ a.e. in } (0, 3). \end{aligned} \right\} \quad (3.2)$$

We observe:

LEMMA 3.2. *The unique solution of (3.2) with right-hand side  $\tilde{u}$  is precisely the function  $\tilde{y} \in C^\infty([0, 3])$  in (3.1).*

*Proof.* We clearly have  $\tilde{y} \in H^1(0, 3)$ ,  $\tilde{y}(0) = 0$  and  $\tilde{y} \geq 0$  in  $[0, 3]$ , and for every  $0 \leq v \in L^2(0, 3)$ , it holds

$$\int_0^3 \tilde{y}'(v - \tilde{y}) + \tilde{y}(v - \tilde{y}) - \tilde{u}(v - \tilde{y}) dt = \int_0^3 \varphi(v - \tilde{y}) dt = \int_{[0,3] \setminus [1,2]} \varphi v dt \geq 0.$$

This proves the claim.  $\square$

In what follows, we will analyze how the difference quotients

$$\frac{S(\tilde{u} + \tau h) - S(\tilde{u})}{\tau} \in H^1(0, 3), \quad \tau > 0, \quad h \in L^2(0, 3),$$

associated with the solution map  $S : u \mapsto y$  to (3.2) and the right-hand side  $\tilde{u}$  behave when  $\tau$  tends to zero. We begin by proving:

LEMMA 3.3. *Let  $h \in L^2(0, 3)$  be arbitrary but fixed and let  $S$  denote the solution map to (3.2). Then, the family of difference quotients*

$$\left\{ \frac{S(\tilde{u} + \tau h) - S(\tilde{u})}{\tau} \right\}_{\tau > 0} \subset H^1(0, 3) \quad (3.3)$$

*is bounded in  $L^2(0, 3)$  and every  $L^2(0, 3)$ -weak accumulation point  $\delta$  of the difference quotients in (3.3) for  $\tau \searrow 0$  satisfies*

$$\begin{aligned} & \delta = 0 \text{ a.e. in } [0, 3] \setminus (1, 2), \\ & \int_1^2 z'(z - \delta) + \delta(z - \delta) - h(z - \delta) dt \geq 0 \quad \forall z \in H_0^1(1, 2). \end{aligned} \quad (3.4)$$

*Proof.* Fix a direction  $h \in L^2(0, 3)$  and denote the difference quotients in (3.3) with  $\delta_\tau$ . Then, the global Lipschitz continuity of  $S$  as a function from  $L^2(0, 3)$  to  $L^2(0, 3)$ , see Theorem 2.3, implies that the family  $\{\delta_\tau\}$  is bounded in  $L^2(0, 3)$ . Consider now an arbitrary but fixed weak accumulation point  $\delta$  of  $\{\delta_\tau\}$  in  $L^2(0, 3)$  for  $\tau \searrow 0$ , and let  $\tau_n$  and  $\delta_n := \delta_{\tau_n}$  be sequences with  $\tau_n \searrow 0$  and  $\delta_n \rightharpoonup \delta$  in  $L^2(0, 3)$ . Then, the definition of  $\delta_n$  yields  $\tilde{y} + \tau_n \delta_n = S(\tilde{u} + \tau_n h)$  for all  $n$ , and we may test the EVI for  $S(\tilde{u} + \tau_n h)$ , i.e.,

$$\begin{aligned} & \int_0^3 (\tilde{y}' + \tau_n \delta_n' + \tilde{y} + \tau_n \delta_n - \tilde{u} - \tau_n h)(v - \tilde{y} - \tau_n \delta_n) dt \geq 0 \\ & \forall v \in L^2(0, 3), \quad v \geq 0 \text{ a.e. in } (0, 3), \end{aligned}$$

with functions of the type  $v = \tilde{y} + \tau_n z \geq 0$ ,  $z \in L^2(0, 3)$ , divide by  $\tau_n^2$ , and use (3.1) to obtain

$$\int_0^3 \delta'_n(z - \delta_n) + \delta_n(z - \delta_n) - h(z - \delta_n) + \frac{1}{\tau_n} \varphi(z - \delta_n) dt \geq 0 \quad (3.5)$$

$$\forall z \in L^2(0, 3), \quad \tilde{y} + \tau_n z \geq 0 \text{ a.e. in } [0, 3].$$

Choosing the function  $z = 0$  in (3.5) yields (in combination with the properties of  $\varphi$  and the fact that the difference quotients  $\delta_n$  are necessarily non-negative everywhere in  $[0, 3] \setminus (1, 2)$  where  $\tilde{y}$  is zero)

$$\int_0^3 \delta'_n \delta_n + \delta_n^2 + \frac{1}{\tau_n} |\varphi \delta_n| dt \leq \int_0^3 h \delta_n dt.$$

The above implies that  $\varphi \delta_n$  converges to zero in  $L^1(0, 3)$  and that  $\delta$  indeed vanishes almost everywhere in  $[0, 3] \setminus (1, 2)$ . It remains to prove the variational inequality in (3.4). To this end, we note that the function

$$z := \begin{cases} 0 & \text{a.e. in } (0, 1) \\ \phi & \text{a.e. in } [1, 2] \\ \delta_n & \text{a.e. in } (2, 3) \end{cases}$$

satisfies  $z \in L^2(0, 3)$  and  $\tilde{y} + \tau_n z \geq 0$  a.e. in  $[0, 3]$  for all sufficiently large  $n$  and all arbitrary but fixed  $\phi \in C_c^\infty(1, 2)$ . Choosing this  $z$  in (3.5) yields

$$\int_0^1 -\delta'_n \delta_n - \delta_n^2 + h \delta_n - \frac{1}{\tau_n} \varphi \delta_n dt$$

$$+ \int_1^2 (\delta'_n - \phi')(\phi - \delta_n) + (\phi' + \delta_n - h)(\phi - \delta_n) + \frac{1}{\tau_n} \varphi(\phi - \delta_n) dt \geq 0.$$

By integration and due to the properties of  $\varphi$ ,  $\phi$ , and  $\delta_n$ , we may deduce

$$\int_0^1 h \delta_n dt + \int_1^2 \phi'(\phi - \delta_n) + \delta_n(\phi - \delta_n) - h(\phi - \delta_n) dt \geq 0.$$

If we pass to the limit  $n \rightarrow \infty$  in the above (using the weak lower semicontinuity of the  $L^2$ -norm), then the claim follows immediately.  $\square$

The important point in our counterexample is the following observation:

**LEMMA 3.4.** *Consider the situation in Lemma 3.3 and the special direction  $h \equiv 1$ . Then, the solutions of (3.4) are precisely the functions*

$$\delta(t) = \begin{cases} 0 & \text{a.e. in } [0, 1) \\ 1 + ce^{-t} & \text{a.e. in } [1, 2], \\ 0 & \text{a.e. in } (2, 3] \end{cases}, \quad c \in \left[ \frac{-2e^2}{e+1}, 0 \right]. \quad (3.6)$$

*Proof.* Suppose that  $\delta \in L^2(0, 3)$  solves (3.4). Then, we may define  $\zeta := \delta - 1$  and test the variational inequality in (3.4) with functions of the form  $z(t) = \alpha \phi(t) e^t$ ,  $\phi \in C_c^\infty(1, 2)$ ,  $\alpha > 0$ , to obtain

$$0 + \int_1^2 -(\phi(t) e^t)' \zeta(t) + \zeta(t) \left( \phi(t) e^t - \frac{1}{\alpha} \delta \right) dt \geq 0 \quad \forall \phi \in C_c^\infty(1, 2).$$

Passing to the limit  $\alpha \nearrow \infty$  in the above yields

$$\int_1^2 \phi'(t)e^t \zeta(t) dt = 0 \quad \forall \phi \in C_c^\infty(1, 2).$$

This shows that there exists a constant  $c \in \mathbb{R}$  with  $\zeta(t)e^t \equiv c$  in  $(1, 2)$  and that  $\delta$  has to satisfy  $\delta = 0$  a.e. in  $[0, 3] \setminus (1, 2)$  and  $\delta(t) = 1 + ce^{-t}$  a.e. in  $(1, 2)$ . If we use this formula in (3.4), then it follows

$$\int_1^2 z'(-1 - ce^{-t}) + ce^{-t}(z - 1 - ce^{-t}) dt \geq 0 \quad \forall z \in H_0^1(1, 2)$$

and, after integration by parts,

$$\begin{aligned} \int_1^2 -ce^{-t} - c^2 e^{-2t} dt &= \left[ ce^{-2} + \frac{1}{2}c^2 e^{-4} - ce^{-1} - \frac{1}{2}c^2 e^{-2} \right] \\ &= -\frac{1}{2}e^{-4}(e^2 - 1)c \left[ \frac{2e^3 - 2e^2}{e^2 - 1} + c \right] \geq 0. \end{aligned}$$

The above entails

$$c \left[ \frac{2e^3 - 2e^2}{e^2 - 1} + c \right] = c \left[ \frac{2e^2}{e + 1} + c \right] \leq 0$$

and, consequently,

$$c \in \left[ \frac{-2e^2}{e + 1}, 0 \right]. \quad (3.7)$$

This shows that every solution  $\delta \in L^2(0, 3)$  has to have the form (3.6). If, conversely, we start with a function of the type (3.6) with a  $c$  as in (3.7), then we may use exactly the same calculations as above to infer

$$\begin{aligned} \int_1^2 (z' + \delta - h)(z - \delta) dt &= \int_1^2 z'(-1 - ce^{-t}) + ce^{-t}(z - 1 - ce^{-t}) dt \\ &\geq \int_1^2 z'(-1 - ce^{-t}) + ce^{-t}z dt = 0 \quad \forall z \in H_0^1(1, 2). \end{aligned}$$

This proves that every function of the form (3.6) solves (3.4) and yields the claim.  $\square$

Note that the functions in (3.6) all have at least one jump-discontinuity. This implies in particular that a  $\delta$  with (3.6) cannot be in  $H^s(0, 3)$ ,  $s \geq 1/2$ , cf. the following classical result:

**LEMMA 3.5.** *Let  $-\infty < a < b < \infty$ . Suppose that a  $v \in L^2(a, b)$  is given such that*

$$v(t) = \begin{cases} \phi_1(t) & \text{a.e. in } (a, c) \\ \phi_2(t) & \text{a.e. in } (c, b) \end{cases}$$

*holds for some  $c \in (a, b)$  and some  $\phi_1, \phi_2 \in C^1(\mathbb{R})$  with  $\phi_1(c) \neq \phi_2(c)$ , and let  $H^s(a, b)$ ,  $s \in [1/2, 1)$ , be the Hilbert space of all  $z \in L^2(a, b)$  with*

$$\|z\|_{H^s(a, b)}^2 := \int_a^b |z(t)|^2 dt + \int_a^b \int_a^b \frac{|z(t_1) - z(t_2)|^2}{|t_1 - t_2|^{1+2s}} dt_1 dt_2 < \infty.$$

*Then,  $v \notin H^s(a, b)$  for all  $s \in [1/2, 1)$ .*



*Proof.* We may assume w.l.o.g. that  $\phi_1 \equiv c_1 \neq c_2 \equiv \phi_2$ . If this is not the case, then we subtract suitable Lipschitz functions from  $v$  (note that  $C^{0,1}(a,b) \subset H^s(a,b)$  for all  $s \in [1/2, 1)$ ). Due to the Sobolev embeddings, see [16, Proposition 2.1], we may further restrict our attention to the case  $s = 1/2$ . A direct calculation now yields

$$\|v\|_{H^{1/2}(a,b)}^2 \geq \int_a^c \int_c^b \frac{|c_1 - c_2|^2}{(t_1 - t_2)^2} dt_1 dt_2 = |c_1 - c_2|^2 \int_a^c \frac{-1}{b - t_2} + \frac{1}{c - t_2} dt_2 = \infty.$$

This proves the claim.  $\square$

If we combine all of the above, then we arrive at:

**THEOREM 3.6.** *Consider the situation in Assumption 3.1, the EVI (3.2) and the function  $h \equiv 1$ . Then, every weak  $L^2$ -accumulation point of the difference quotients*

$$\delta_\tau := \frac{S(\tilde{u} + \tau h) - S(\tilde{u})}{\tau} \in H^1(0, 3), \quad \tau > 0, \quad (3.8)$$

for  $\tau \searrow 0$  is an element of  $L^\infty(0, 3) \setminus H^{1/2}(0, 3)$ .

Theorem 3.6 and the preceding results have several implications for the sensitivity analysis of obstacle-type EVIs:

**REMARK 3.7.**

- (i) *Since all  $L^2$ -weak accumulation points of the difference quotients (3.8) for  $\tau \searrow 0$  are in  $L^\infty(0, 3) \setminus H^{1/2}(0, 3)$ , the solution map  $S : u \mapsto y$  associated with (3.2) cannot be (locally) Lipschitz continuous or directionally differentiable as a function  $L^2(0, 3) \rightarrow H^s(0, 3)$  for any  $s \geq 1/2$ . Note that  $\tilde{u}$ ,  $\tilde{y}$  and  $h$  are smooth in our example. The regularity of the data is thus not the issue here.*
- (ii) *The variational inequality (3.4) for the  $L^2$ -weak accumulation points  $\delta$  of the difference quotients in (3.3) can also be written as*

$$\begin{aligned} \delta \in L^2(0, 3), \quad \delta \in T_{[0, \infty)}^{\text{tan}}(\tilde{y}) \cap \varphi^\perp \text{ a.e. in } [0, 3], \\ \int_0^3 z'(z - \delta) + \delta(z - \delta) - h(z - \delta) dt + \frac{1}{2}z(0)^2 \geq 0, \\ \forall z \in H^1(0, 3), \quad z \in T_{[0, \infty)}^{\text{tan}}(\tilde{y}) \cap \varphi^\perp \text{ a.e. in } [0, 3], \end{aligned} \quad (3.9)$$

where  $\varphi = \tilde{y}' + \tilde{y} - \tilde{u}$  is the function in (3.1), where  $T_{[0, \infty)}^{\text{tan}}(x)$  is the tangent cone to the set  $[0, \infty)$  at  $x \in [0, \infty)$ , i.e.,

$$T_{[0, \infty)}^{\text{tan}}(x) = \begin{cases} \mathbb{R} & \text{if } x > 0 \\ [0, \infty) & \text{if } x = 0 \end{cases},$$

and where  $a^\perp$  denotes the kernel of an  $a \in \mathbb{R}$  interpreted as a linear function on  $\mathbb{R}$ , i.e.,

$$a^\perp = \begin{cases} \{0\} & \text{if } a \neq 0 \\ \mathbb{R} & \text{if } a = 0 \end{cases}.$$

Note that (3.9) is precisely the weak form of the problem

$$\begin{aligned} \delta \in H^1(0, 3), \quad \delta(0) = 0, \quad \delta \in T_{[0, \infty)}^{\text{tan}}(\tilde{y}) \cap \varphi^\perp \text{ a.e. in } [0, 3], \\ \int_0^3 \delta'(z - \delta) + \delta(z - \delta) - h(z - \delta) dt \geq 0 \\ \forall z \in L^2(0, 3), \quad z \in T_{[0, \infty)}^{\text{tan}}(\tilde{y}) \cap \varphi^\perp \text{ a.e. in } [0, 3], \end{aligned} \quad (3.10)$$

cf. [8, Section II] and [27, Definition 2]. Further, (3.10) is the exact analogue of the EVI that is used in [29] for the characterization of the derivatives of the solution operator to (2.6) and the parabolic counterpart of the auxiliary problem that characterizes the directional derivatives of the solution map for a time-independent elliptic variational inequality with a polyhedral admissible set, see [32] and [21, Theorem 1]. Recall now that Lemma 3.4 yields that (3.9) possesses infinitely many solutions for  $h \equiv 1$  and that the existence of a strong solution to an EVI of the form (3.10) always implies the unique solvability of its weak formulation (to see this, test (3.10) with a weak solution, (3.9) with the strong solution and add the resulting inequalities). What we have constructed in this section is thus a situation in which the EVI that, in view of the classical theory for elliptic variational inequalities, should characterize the directional derivatives of the solution operator to a problem of the type (P) does not admit a strong solution and possesses infinitely many weak solutions. This and the fact that we cannot work with the space  $H^{1/2}(0, 3)$  either in the situation of Assumption 3.1 suggest that uniquely characterizing directional derivatives with an auxiliary problem is typically far from straightforward for an obstacle-type evolution variational inequality. In particular, our results indicate that it is not possible to proceed along the lines of [13, 14, 26, 29] to establish the directional differentiability of the solution map  $S$  to the general problem (P). We remark that, despite the non-uniqueness of solutions to (3.9), this variational inequality can still be used to derive strong stationarity conditions. See Section 5 for details on this topic.

- (iii) The construction that we have used in this section can be extended straightforwardly to the problems in Examples 2.5 and 2.6. In the case of the classical parabolic obstacle problem, it yields that the solution map  $S$  cannot be expected to be Lipschitz continuous or directionally differentiable w.r.t. the norm

$$\begin{aligned} & \|z\|_{H^{1/2}(0,T;L^2(\Omega))} \\ & := \left( \|z\|_{L^2(0,T;L^2(\Omega))}^2 + \int_0^T \int_0^T \frac{\|z(t_1) - z(t_2)\|_{L^2(\Omega)}^2}{|t_1 - t_2|^2} dt_1 dt_2 \right)^{1/2} \end{aligned}$$

because the weak accumulation points of the difference quotients may suddenly “jump” at some time  $t$  to the zero function, and in the case of the Signorini-type problem (2.6), it implies that the solution map typically does not satisfy a Lipschitz estimate w.r.t. the norm

$$\begin{aligned} & \|z\|_{H^{1/2}(0,T;H^1(\Omega))} \\ & := \left( \|z\|_{L^2(0,T;H^1(\Omega))}^2 + \int_0^T \int_0^T \frac{\|z(t_1) - z(t_2)\|_{H^1(\Omega)}^2}{|t_1 - t_2|^2} dt_1 dt_2 \right)^{1/2}. \end{aligned}$$

Note that we do not get a contradiction with the Lipschitz result in [29] here, since, for inequality constraints on the boundary  $\partial\Omega$ , our approach only yields a discontinuity in time of the trace of the  $L^2(0, T; H^1(\Omega))$ -weak accumulation points of the difference quotients. Such a discontinuity is not detectable by the norm  $\|\cdot\|_{H^{1/2}(0,T;L^2(\Omega))}$  used in [29]. Our results show, however, that the approach in [29] has to fail for EVIs that behave like the classical parabolic obstacle problem (2.5) due to the lacking  $H^{1/2}$ -Lipschitz continuity of  $S$ .

**4. Directional Differentiability of the Solution Map.** Having established that the solution operator  $S$  of the EVI (P) can, in general, not be expected to be directionally differentiable in any of the spaces  $H^s(0, T; H)$ ,  $s \geq 1/2$ , we now turn our attention to the differentiability properties of  $S$  in the  $L^q$ -spaces. Henceforth, we again consider the general setting that we have introduced in Section 2, i.e., we suppose that a problem of the type (P) is given and that Assumption 2.1 is satisfied. The aim of this section is to prove the following main result of this paper:

**THEOREM 4.1** (Directional Differentiability in the  $L^q$ -Spaces). *Consider the situation in Assumption 2.1. Then, the solution map  $S$  to (P) is directionally differentiable in the sense that, for every  $u \in L^2(0, T; H)$  and every  $h \in L^2(0, T; H)$ , there exists a unique  $S'(u; h) =: \delta \in L^\infty(0, T; H) \cap L^2(0, T; V)$  such that the difference quotients*

$$\delta_\tau := \frac{S(u + \tau h) - S(u)}{\tau}, \quad \tau > 0,$$

satisfy

$$\begin{aligned} \delta_\tau &\xrightarrow{*} \delta \text{ in } L^\infty(0, T; H), & \delta_\tau &\rightharpoonup \delta \text{ in } L^2(0, T; V), \\ \delta_\tau &\rightarrow \delta \text{ in } L^q(0, T; H) & \forall 1 \leq q < \infty \end{aligned} \quad (4.1)$$

for  $\tau \searrow 0$ . Further, the directional derivatives  $\delta = S'(u; h)$ ,  $u, h \in L^2(0, T; H)$ , satisfy

$$\left. \begin{aligned} \delta &\in T_{\mathcal{K}, L^2}^{\text{tan}}(y) \cap \varphi^\perp, \\ \int_0^T \langle z', z - \delta \rangle_V + \langle A\delta, z - \delta \rangle_V - \langle h, z - \delta \rangle_V dt + \frac{1}{2} \|z(0)\|_H^2 &\geq 0 \\ \forall z &\in \text{cl}_{L^2(0, T; V) \cap H^1(0, T; V^*)} (T_{\mathcal{K}}^{\text{rad}}(y) \cap \varphi^\perp). \end{aligned} \right\} \quad (4.2)$$

Here,  $y := S(u)$  is the state associated with the right-hand side  $u$ ,  $\varphi^\perp$  is the kernel of the linear map  $\varphi := y' + Ay - u \in L^2(0, T; V^*) \cong L^2(0, T; V)^*$ , i.e.,

$$\varphi^\perp := \left\{ z \in L^2(0, T; V) \mid \int_0^T \langle y' + Ay - u, z \rangle_V dt = 0 \right\},$$

$\mathcal{K}$  is the set

$$\mathcal{K} := \left\{ z \in L^2(0, T; V) \cap H^1(0, T; V^*) \mid z \in K \text{ a.e. in } (0, T) \right\},$$

and  $T_{\mathcal{K}}^{\text{rad}}(y)$  and  $T_{\mathcal{K}, L^2}^{\text{tan}}(y)$  are the radial and the  $L^2(0, T; V)$ -tangent cone to  $\mathcal{K}$  at  $y$ , respectively, i.e.,

$$T_{\mathcal{K}}^{\text{rad}}(y) := \mathbb{R}^+(\mathcal{K} - y), \quad T_{\mathcal{K}, L^2}^{\text{tan}}(y) := \text{cl}_{L^2(0, T; V)}(\mathbb{R}^+(\mathcal{K} - y)),$$

where  $\text{cl}(\cdot)$  denotes a topological closure.

Before we prove Theorem 4.1, we give some remarks:

**REMARK 4.2.**

- (i) *The convergence behavior in (4.1) fits very well to the Lipschitz estimate (2.2) in Theorem 2.3 and the observations in the previous section. Moreover, (4.2) corresponds precisely to the variational inequality (3.9).*

- (ii) Recall that the space  $L^2(0, T; V) \cap H^1(0, T; V^*)$  is continuously embedded into  $C([0, T]; H)$ , see, e.g., [40, Theorem 10.9]. The expression  $\|z(0)\|_H^2$  in (4.2) is thus well-defined.
- (iii) Since  $S$  is Lipschitz as a function  $L^2(0, T; H) \rightarrow L^\infty(0, T; H)$  and strongly directionally differentiable as a function  $L^2(0, T; H) \rightarrow L^q(0, T; H)$  for all  $1 \leq q < \infty$ ,  $S$  is Hadamard directionally differentiable in the sense of [7, Definition 2.45] from  $L^2(0, T; H)$  to  $L^q(0, T; H)$  for all  $1 \leq q < \infty$ , see [7, Proposition 2.49]. The notion of Hadamard directional differentiability is of particular importance in the fields of optimization and optimal control because it is sufficient for the derivation of chain rules, see [7, Proposition 2.47].
- (iv) As we will see in the proof of Theorem 4.1, the functions  $\delta_\tau$  converge pointwise a.e. in  $(0, T)$  monotonously from above to  $\delta$  in  $H = L^2(\Omega, \mu)$  for  $\tau \searrow 0$ . Our proof further shows that the  $C([0, T]; H)$ -representatives of  $\delta_\tau$  converge even pointwise everywhere in  $[0, T]$  in  $H$  for  $\tau \searrow 0$ , cf. (4.4) below.
- (v) If  $\mathcal{K}$  is polyhedral in  $L^2(0, T; V) \cap H^1(0, T; V^*)$ , see [21, 32, 44], then (4.2) holds for all  $z \in T_{\mathcal{K}, H^1}^{\text{tan}}(y) \cap \varphi^\perp$ , where  $T_{\mathcal{K}, H^1}^{\text{tan}}(y)$  is the tangent cone to  $\mathcal{K}$  at  $y$  w.r.t. the topology of the space  $L^2(0, T; V) \cap H^1(0, T; V^*)$ . Compare also with the discussion after Remark 5.6 in this context.

To prove the directional differentiability of the solution operator  $S$  in the  $L^q$ -spaces, we proceed in several steps. In the remainder of this section, we always tacitly assume that the conditions in Assumption 2.1 are satisfied, that an EVI of the form (P) is given, that  $u, h \in L^2(0, T; H)$  are arbitrary but fixed, and that  $S, y, \delta_\tau, \mathcal{K}$  and  $\varphi$  are defined as in Theorem 4.1. We first note:

LEMMA 4.3. For every  $z \in L^2(0, T; V)$ , it holds

$$\int_0^T \langle y' + Ay - u, z^+ \rangle_V dt = \int_0^T \langle \varphi, z^+ \rangle_V dt \geq 0.$$

*Proof.* Test the variational inequality for  $y$  with functions of the form  $v = y + z^+$ , then the claim follows immediately.  $\square$

The key observation is now the following:

LEMMA 4.4. For all  $0 < \tau_1 < \tau_2$ , it holds  $(\delta_{\tau_1} - \delta_{\tau_2})^+ = 0$  in  $L^2(0, T; V)$ .

*Proof.* Test the EVI for  $S(u + \tau h) = y + \tau \delta_\tau$ , i.e., the problem

$$\int_0^T \langle y' + \tau \delta'_\tau + Ay + \tau A \delta_\tau - u - \tau h, v - y - \tau \delta_\tau \rangle_V dt \geq 0$$

$$\forall v \in L^2(0, T; V), \quad v \in K \text{ a.e. in } (0, T),$$

with functions of the form  $y + \tau z$  and divide by  $\tau^2$  to obtain

$$y + \tau \delta_\tau \in K \text{ a.e. in } (0, T), \quad \delta_\tau(0) = 0,$$

$$\int_0^T \langle \delta'_\tau + A \delta_\tau - h, z - \delta_\tau \rangle_V + \frac{1}{\tau} \langle y' + Ay - u, z - \delta_\tau \rangle_V dt \geq 0$$

$$\forall z \in L^2(0, T; V), \quad y + \tau z \in K \text{ a.e. in } (0, T).$$

Consider now two arbitrary but fixed  $0 < \tau_1 < \tau_2$ . Then, our assumptions on  $K$  and the properties of  $\tau_1, \tau_2, \delta_{\tau_1}, \delta_{\tau_2}$  imply

$$y + \tau_2 \max(\delta_{\tau_1}, \delta_{\tau_2}) = y + \tau_2 \delta_{\tau_2} + \tau_2 (\delta_{\tau_1} - \delta_{\tau_2})^+ \in K$$

and

$$\begin{aligned}
y + \tau_1 \min(\delta_{\tau_1}, \delta_{\tau_2}) &= \min(y + \tau_1 \delta_{\tau_1}, y + \tau_1 \delta_{\tau_2}) \\
&= \min\left(y + \tau_1 \delta_{\tau_1}, y + \tau_1 \delta_{\tau_2}^+ + \tau_1 \delta_{\tau_2}^-\right) \\
&= \min\left(y + \tau_1 \delta_{\tau_1}, y + \tau_2 \delta_{\tau_2}^- - \tau_2 \delta_{\tau_2}^- + \tau_1 \delta_{\tau_2}^+ + \tau_1 \delta_{\tau_2}^-\right) \\
&= \min\left(y + \tau_1 \delta_{\tau_1}, y + \tau_2 \delta_{\tau_2}^- + \tau_1 \delta_{\tau_2}^+ - (\tau_2 - \tau_1) \delta_{\tau_2}^-\right) \\
&= \min\left(y + \tau_1 \delta_{\tau_1}, \min(y, y + \tau_2 \delta_{\tau_2}^-) + \tau_1 \delta_{\tau_2}^+ + (\tau_2 - \tau_1)(-\delta_{\tau_2}^-)^+\right) \\
&\in K
\end{aligned}$$

for a.a.  $t \in (0, T)$ , where  $z^-$  is short for  $\min(0, z)$ . Testing in the EVIs for  $\delta_{\tau_1}$  and  $\delta_{\tau_2}$  now yields

$$\begin{aligned}
\int_0^T \langle \delta'_{\tau_1} + A\delta_{\tau_1} - h, \min(\delta_{\tau_1}, \delta_{\tau_2}) - \delta_{\tau_1} \rangle_V \\
+ \frac{1}{\tau_1} \langle y' + Ay - u, \min(\delta_{\tau_1}, \delta_{\tau_2}) - \delta_{\tau_1} \rangle_V dt \geq 0
\end{aligned}$$

and

$$\begin{aligned}
\int_0^T \langle \delta'_{\tau_2} + A\delta_{\tau_2} - h, \max(\delta_{\tau_1}, \delta_{\tau_2}) - \delta_{\tau_2} \rangle_V \\
+ \frac{1}{\tau_2} \langle y' + Ay - u, \max(\delta_{\tau_1}, \delta_{\tau_2}) - \delta_{\tau_2} \rangle_V dt \geq 0.
\end{aligned}$$

Add the above to obtain

$$\begin{aligned}
\int_0^T \langle \delta'_{\tau_1} + A\delta_{\tau_1} - h, -(\delta_{\tau_1} - \delta_{\tau_2})^+ \rangle_V + \langle \delta'_{\tau_2} + A\delta_{\tau_2} - h, (\delta_{\tau_1} - \delta_{\tau_2})^+ \rangle_V \\
+ \frac{1}{\tau_1} \langle y' + Ay - u, -(\delta_{\tau_1} - \delta_{\tau_2})^+ \rangle_V + \frac{1}{\tau_2} \langle y' + Ay - u, (\delta_{\tau_1} - \delta_{\tau_2})^+ \rangle_V dt \geq 0
\end{aligned}$$

and, consequently,

$$\begin{aligned}
\int_0^T -(\delta'_{\tau_1} - \delta'_{\tau_2}, (\delta_{\tau_1} - \delta_{\tau_2})^+)_H - \langle A(\delta_{\tau_1} - \delta_{\tau_2}), (\delta_{\tau_1} - \delta_{\tau_2})^+ \rangle_V \\
- \left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right) \langle y' + Ay - u, (\delta_{\tau_1} - \delta_{\tau_2})^+ \rangle_V dt \geq 0.
\end{aligned}$$

Due to  $(1/\tau_1 - 1/\tau_2) \geq 0$ , our assumptions on  $A$ , the integration by parts formula (2.1), and Lemma 4.3, the above entails

$$-\frac{1}{2} \|(\delta_{\tau_1} - \delta_{\tau_2})^+(T)\|_H^2 - \int_0^T \langle A(\delta_{\tau_1} - \delta_{\tau_2})^+, (\delta_{\tau_1} - \delta_{\tau_2})^+ \rangle_V dt \geq 0.$$

The strong monotonicity of  $A$  now yields the claim.  $\square$

Some remarks are in order regarding the last result:

REMARK 4.5.

- (i) Since  $\{\delta_\tau\} \subset H^1(0, T; H)$  and  $L^2(0, T; V) \hookrightarrow L^2(0, T; H)$ , Lemma 4.4 implies that the  $C([0, T]; H)$ -representatives of the difference quotients  $\{\delta_\tau\}$  satisfy  $\delta_{\tau_1}(t) \leq \delta_{\tau_2}(t)$  pointwise  $\mu$ -a.e. in  $\Omega$  for all  $t \in [0, T]$  and all  $0 < \tau_1 < \tau_2$ . The sequences  $\{\delta_\tau(t)\} \subset H$ ,  $t \in [0, T]$ , are thus pointwise  $\mu$ -a.e. monotonously decreasing for  $\tau \searrow 0$ .
- (ii) A monotonicity behavior similar to that in Lemma 4.4 was observed in [2] for directional derivatives in an iteration scheme for elliptic obstacle-type quasi variational inequalities. In this paper, however, the authors only obtained monotonicity for certain directions and right-hand sides, cf. [2, Lemma 4.1], and for a different limiting process (namely that of the iteration procedure). Using the argumentation in the proof of Lemma 4.4, we are able to show that, for the EVI (P), the difference quotients  $\{\delta_\tau\}$  are always monotonously decreasing for  $\tau \searrow 0$ , regardless of the properties of  $u$ ,  $h$  and  $S(u + \tau h)$ .
- (iii) The argumentation in the proof of Lemma 4.4 can also be used to show, e.g., that the difference quotients of the solution operator to the classical time-independent obstacle problem are monotonously decreasing.

We are now in the position to prove the first part of Theorem 4.1:

LEMMA 4.6. *There exists a unique function  $\delta \in L^\infty(0, T; H) \cap L^2(0, T; V)$  such that*

$$\begin{aligned} \delta_\tau &\xrightarrow{*} \delta \text{ in } L^\infty(0, T; H), & \delta_\tau &\rightharpoonup \delta \text{ in } L^2(0, T; V), \\ \delta_\tau &\rightarrow \delta \text{ in } L^q(0, T; H) & \forall 1 \leq q < \infty & \text{ as } \tau \searrow 0. \end{aligned} \quad (4.3)$$

*Proof.* Suppose that a monotonously decreasing sequence  $\{\tau_n\} \subset \mathbb{R}^+$  with  $\tau_n \searrow 0$  is given. Then, Lemma 4.4 yields that the  $C([0, T]; H)$ -representatives of the difference quotients  $\delta_n := \delta_{\tau_n}$  satisfy  $(\delta_{n+1}(t) - \delta_n(t))^+ = 0$  in  $H$  for all  $t \in [0, T]$  and all  $n \in \mathbb{N}$ . The sequences of functions  $\{\delta_n(t)\} \subset H = L^2(\Omega, \mu)$ ,  $t \in [0, T]$ , are thus monotonously decreasing pointwise  $\mu$ -a.e. in  $\Omega$  for  $n \rightarrow \infty$ . Note that this implies in particular that the limit

$$\lim_{n \rightarrow \infty} \delta_n(t) \in [-\infty, \infty)$$

exists pointwise  $\mu$ -a.e. in  $\Omega$  for all  $t \in [0, T]$ . From the Lipschitz continuity of the map  $S$  in Theorem 2.3, it follows further that  $\|\delta_n(t)\|_H \leq C$  holds for all  $n \in \mathbb{N}$  and all  $t \in [0, T]$  with some absolute constant  $C > 0$ . Using Fatou's lemma, we may now deduce that

$$\begin{aligned} 0 &\leq \int_{\Omega} \left( \lim_{n \rightarrow \infty} \delta_n(t) \right)^2 d\mu = \int_{\Omega} \lim_{n \rightarrow \infty} \delta_n(t)^2 d\mu \\ &= \int_{\Omega} \liminf_{n \rightarrow \infty} \delta_n(t)^2 d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \delta_n(t)^2 d\mu \leq C \end{aligned}$$

for all  $t \in [0, T]$ . The above implies that the  $\mu$ -a.e.-defined function

$$\delta(t) := \lim_{n \rightarrow \infty} \delta_n(t)$$

can be identified with an element of  $L^2(\Omega, \mu)$  for all  $t \in [0, T]$ . (Note that measurability is not a problem here due to the pointwise  $\mu$ -a.e. convergence  $\delta_n(t) \rightarrow \delta(t)$  in  $\Omega$ .) From the dominated convergence theorem (with majorant  $(\delta_1(t) - \delta(t))^2$ ), we now obtain

$$\int_{\Omega} (\delta_n(t) - \delta(t))^2 d\mu \rightarrow 0 \quad (4.4)$$

for all  $t \in [0, T]$  and  $n \rightarrow \infty$ , i.e., we have  $\delta_n(t) \rightarrow \delta(t)$  in  $H$  as  $n \rightarrow \infty$  for some  $\delta(t) \in H$  and all  $t \in [0, T]$ . This pointwise convergence in time implies that the map  $[0, T] \ni t \mapsto \delta(t) \in H$  is Bochner measurable, see [24, Corollary 3.1.5]. Due to the boundedness of  $\{\delta_n\}$  in  $L^\infty(0, T; H)$ , it follows immediately that  $\delta$  can be identified with an element of  $L^\infty(0, T; H)$  and, by the dominated convergence theorem, that  $\delta_n \rightarrow \delta$  holds for  $n \rightarrow \infty$  in all  $L^q(0, T; H)$ ,  $1 \leq q < \infty$ . Note that, due to [15, Theorem IV-1], we have  $L^1(0, T; H)^* \cong L^\infty(0, T; H)$ , and that the uniform boundedness and the pointwise convergence in time of  $\delta_n$  to  $\delta$  as well as the dominated convergence theorem yield

$$\int_0^T (z, \delta_n)_H dt \rightarrow \int_0^T (z, \delta)_H dt \quad \forall z \in L^1(0, T; H).$$

It thus holds  $\delta_n \xrightarrow{*} \delta$  in  $L^\infty(0, T; H)$ . To see that we also have  $\delta_n \rightharpoonup \delta$  in  $L^2(0, T; V)$ , we recall that the sequence  $\{\delta_n\}$  is bounded in  $L^2(0, T; V)$  by Theorem 2.3. This implies that every subsequence of  $\{\delta_n\}$  contains a subsequence which converges weakly in  $L^2(0, T; V)$ . Since we already know that  $\delta_n \rightarrow \delta$  in, e.g.,  $L^2(0, T; H)$ , the weak convergence in  $L^2(0, T; V)$  now follows immediately. This proves that the difference quotients  $\delta_{\tau_n}$  converge as claimed to some  $\delta \in L^\infty(0, T; H) \cap L^2(0, T; V)$  for every monotonously decreasing  $\{\tau_n\} \subset \mathbb{R}^+$  with  $\tau_n \searrow 0$ . Consider now two arbitrary but fixed monotonously decreasing sequences  $\{\tau_n\} \subset \mathbb{R}^+$ ,  $\{\tilde{\tau}_n\} \subset \mathbb{R}^+$  with  $\tau_n \searrow 0$ ,  $\tilde{\tau}_n \searrow 0$  and associated limit points  $\delta_1, \delta_2$ . Then, by passing over to subsequences (not relabeled), we can always obtain that  $\tau_n > \tilde{\tau}_n > \tau_{n+1}$  holds for all  $n \in \mathbb{N}$ . Since this nested sequence is again monotonously decreasing, we know that the associated difference quotients converge as in (4.3). This is only possible if  $\delta_1 = \delta_2$ . The limit of the difference quotients  $\delta_{\tau_n}$  is thus the same for all monotonously decreasing  $\{\tau_n\}$  with  $\tau_n \searrow 0$ . The latter implies that there exists a uniquely determined  $\delta$  such that every  $\{\tau_n\} \subset \mathbb{R}^+$  with  $\tau_n \searrow 0$  (not necessarily monotonously) contains a subsequence such that the associated difference quotients converge to  $\delta$  as in (4.3). Using classical contradiction arguments, cf. [40, Lemmas 4.16, 4.17], we now obtain that  $\delta_{\tau_n}$  converges to  $\delta$  for all sequences  $\{\tau_n\} \subset \mathbb{R}^+$  with  $\tau_n \searrow 0$ . This proves the claim.  $\square$

Note that the proof of Lemma 4.6 does not make use of the concept of polyhedricity, cf. [21, 32, 44], but is completely elementary. It remains to show that  $\delta$  solves the variational inequality (4.2):

LEMMA 4.7. *The directional derivative  $\delta = S'(u; h)$  satisfies*

$$\begin{aligned} \delta &\in T_{\mathcal{K}, L^2}^{tan}(y) \cap \varphi^\perp, \\ \int_0^T \langle z', z - \delta \rangle_V + \langle A\delta, z - \delta \rangle_V - \langle h, z - \delta \rangle_V dt + \frac{1}{2} \|z(0)\|_H^2 &\geq 0 \\ \forall z &\in \text{cl}_{L^2(0, T; V) \cap H^1(0, T; V^*)} (T_{\mathcal{K}}^{rad}(y) \cap \varphi^\perp). \end{aligned}$$

*Proof.* The proof is along the lines of that of Lemma 3.3. By testing the EVI for  $S(u + \tau h) = y + \tau \delta_\tau$  with functions of the type  $v = y + \tau z$ , we again obtain

$$\begin{aligned} \delta_\tau &\in L^2(0, T; V) \cap H^1(0, T; H), \\ y + \tau \delta_\tau &\in K \text{ a.e. in } (0, T), \quad \delta_\tau(0) = 0, \\ \int_0^T \langle \delta'_\tau + A\delta_\tau - h, z - \delta_\tau \rangle_V + \frac{1}{\tau} \langle y' + Ay - u, z - \delta_\tau \rangle_V dt &\geq 0 \\ \forall z &\in L^2(0, T; V), \quad y + \tau z \in K \text{ a.e. in } (0, T). \end{aligned} \tag{4.5}$$

In particular, the choice  $z = 0$  yields

$$\int_0^T \langle \delta'_\tau + A\delta_\tau - h, -\delta_\tau \rangle_V dt \geq \int_0^T \frac{1}{\tau} \langle y' + Ay - u, \delta_\tau \rangle_V dt = \frac{1}{\tau} \int_0^T \langle \varphi, \delta_\tau \rangle_V dt \geq 0, \quad (4.6)$$

where the second inequality in (4.6) follows from the EVI for  $y$  with  $v = y + \tau\delta_\tau$ . From the boundedness of the sequence  $\{\delta_\tau\}$  in  $L^2(0, T; V) \cap L^\infty(0, T; H)$ , it now follows immediately that  $\delta \in \varphi^\perp$ . Since  $\mathbb{R}^+(\mathcal{K} - y) \ni \delta_\tau \rightharpoonup \delta$  in  $L^2(0, T; V)$  and since the radial cone  $T_{\mathcal{K}}^{rad}(y) = \mathbb{R}^+(\mathcal{K} - y)$  is convex, we may further apply the lemma of Mazur to deduce that  $\delta \in T_{\mathcal{K}, L^2}^{tan}(y) \cap \varphi^\perp$ . Consider now an arbitrary but fixed  $z \in T_{\mathcal{K}}^{rad}(y) \cap \varphi^\perp$ . Then, for all sufficiently small  $\tau > 0$ ,  $z$  is admissible in (4.5) and we may use (4.6) to deduce

$$\int_0^T \langle \delta'_\tau - z' + z' + A\delta_\tau - h, z - \delta_\tau \rangle_V dt \geq \int_0^T \frac{1}{\tau} \langle y' + Ay - u, \delta_\tau \rangle_V dt \geq 0.$$

The above yields in combination with [40, Theorem 10.9] that

$$\frac{1}{2} \|z(0)\|_H^2 - \frac{1}{2} \|z(T) - \delta_\tau(T)\|_H^2 + \int_0^T \langle z' + A\delta_\tau - h, z - \delta_\tau \rangle_V dt \geq 0.$$

Ignoring the  $\|z(T) - \delta_\tau(T)\|_H$ -term, passing to the limit  $\tau \searrow 0$  and using density and weak lower semicontinuity, the claim now follows immediately.  $\square$

**REMARK 4.8.** *The reader might ask at this point which of the candidates in (3.6) is the actual directional derivative  $S'(\tilde{u}; h) \in L^\infty(0, 3)$  in the scenario studied in Section 3. The answer is that it is precisely that function which does not have a discontinuity at  $t = 1$ , i.e., the function with  $c = -e$ . To see this, it suffices to note that, if we consider the end time  $\tilde{T} = 2$  in Section 3 instead of the original  $T = 3$ , then the variational inequality (4.2) has a unique strong and weak solution, namely, the function which vanishes in  $(0, 1)$  and which equals  $1 - e^{1-t}$  in  $(1, 2)$ . This function obviously has to coincide with the restriction of the directional derivative for the end time  $T = 3$  to the time interval  $(0, 2)$ . Note that, using exactly the same argumentation, it can be shown that the directional derivatives are  $H^1$ -regular in time “as long as possible”.*

**5. Strong Stationarity.** In this section, we demonstrate that Theorem 4.1 can be used to derive strong and Bouligand stationarity conditions for optimal control problems that are governed by EVIs of the type (P). Our precise assumptions are as follows:

**ASSUMPTION 5.1** (Standing Assumptions for Section 5). *We are given an optimal control problem of the form*

$$\begin{aligned} \min \mathcal{J}(y, u) \\ \text{s.t. } y = S(u), \quad u \in \mathcal{U}_{ad}, \end{aligned} \quad (\text{O})$$

such that:

- (i)  $\mathcal{U}_{ad} \subset L^2(0, T; H)$  is non-empty, convex and closed,
- (ii)  $\mathcal{J} : L^q(0, T; H) \times L^2(0, T; H) \rightarrow \mathbb{R}$  is Gâteaux differentiable in the sense of [7, Section 2.2.1] and locally Lipschitz continuous for some  $1 \leq q < \infty$ ,
- (iii)  $S : L^2(0, T; H) \rightarrow L^2(0, T; V) \cap H^1(0, T; H)$  is the solution map to an EVI of the form (P) that satisfies the conditions in Assumption 2.1.



REMARK 5.2. We could also work, e.g., with a Fréchet differentiable objective function  $\mathcal{J} : L^2(0, T; V) \times L^2(0, T; H) \rightarrow \mathbb{R}$  in this section, using a Taylor expansion and the weak  $L^2(0, T; V)$ -directional differentiability of  $S$ . We restrict our attention to functionals  $\mathcal{J} : L^q(0, T; H) \times L^2(0, T; H) \rightarrow \mathbb{R}$  for the sake of simplicity.

By invoking Theorem 4.1, it is straightforward to prove:

PROPOSITION 5.3 (Bouligand Stationarity Condition). *If  $\bar{u} \in \mathcal{U}_{ad}$  is a local minimizer of (O) with associated state  $\bar{y} := S(\bar{u})$ , then it holds*

$$\langle \partial_y \mathcal{J}(\bar{y}, \bar{u}), S'(\bar{u}; h) \rangle_{L^q(0, T; H)} + \langle \partial_u \mathcal{J}(\bar{y}, \bar{u}), h \rangle_{L^2(0, T; H)} \geq 0 \quad \forall h \in \mathbb{R}^+(\mathcal{U}_{ad} - \bar{u}). \quad (5.1)$$

Here,  $\partial_y \mathcal{J}(\bar{y}, \bar{u})$  and  $\partial_u \mathcal{J}(\bar{y}, \bar{u})$  denote the partial derivatives of  $\mathcal{J}$  at  $(\bar{y}, \bar{u})$ .

*Proof.* Due to its local Lipschitz continuity and directional differentiability,  $\mathcal{J}$  is Hadamard-Gâteaux differentiable. Using the chain rule in [7, Proposition 2.47], the claim now follows immediately from the local optimality of  $\bar{u}$  and Theorem 4.1.  $\square$

To obtain a strong stationarity system analogous to that in [32,33] for the optimal control problem (O), we proceed similarly to [11,31] and note the following:

LEMMA 5.4. *If  $\bar{u} \in \mathcal{U}_{ad}$  is Bouligand stationary for (O) in the sense of (5.1) with associated state  $\bar{y} = S(\bar{u})$  and if the set  $\mathbb{R}^+(\mathcal{U}_{ad} - \bar{u})$  is dense in  $L^2(0, T; H)$ , then  $\bar{p} := -\partial_u \mathcal{J}(\bar{y}, \bar{u})$  is an element of  $L^2(0, T; V) \cap L^\infty(0, T; H)$ .*

*Proof.* From (5.1), we know that

$$\langle -\partial_u \mathcal{J}(\bar{y}, \bar{u}), h \rangle_{L^2(0, T; H)} \leq \langle \partial_y \mathcal{J}(\bar{y}, \bar{u}), S'(\bar{u}; h) \rangle_{L^q(0, T; H)} \quad \forall h \in \mathbb{R}^+(\mathcal{U}_{ad} - \bar{u}),$$

and from (2.2), we obtain that there exists an absolute constant  $C > 0$  with

$$\|S(u_1) - S(u_2)\|_{L^q(0, T; H)} \leq C \min(\|u_1 - u_2\|_{L^2(0, T; V^*)}, \|u_1 - u_2\|_{L^1(0, T; H)})$$

for all  $u_1, u_2 \in L^2(0, T; H)$ . The latter implies

$$\|S'(\bar{u}; h)\|_{L^q(0, T; H)} \leq C \min(\|h\|_{L^2(0, T; V^*)}, \|h\|_{L^1(0, T; H)}) \quad \forall h \in \mathbb{R}^+(\mathcal{U}_{ad} - \bar{u})$$

and, consequently,

$$\langle -\partial_u \mathcal{J}(\bar{y}, \bar{u}), h \rangle_{L^2(0, T; H)} \leq \tilde{C} \min(\|h\|_{L^2(0, T; V^*)}, \|h\|_{L^1(0, T; H)}) \quad \forall h \in \mathbb{R}^+(\mathcal{U}_{ad} - \bar{u}) \quad (5.2)$$

with some  $\tilde{C} > 0$  independent of  $h$ . Since  $\mathbb{R}^+(\mathcal{U}_{ad} - \bar{u})$  is dense in  $L^2(0, T; H)$ , (5.2) entails

$$\left| \langle -\partial_u \mathcal{J}(\bar{y}, \bar{u}), h \rangle_{L^2(0, T; H)} \right| \leq \tilde{C} \min(\|h\|_{L^2(0, T; V^*)}, \|h\|_{L^1(0, T; H)}) \quad \forall h \in L^2(0, T; H).$$

This estimate implies, in combination with the density of  $L^2(0, T; H)$  in the spaces  $L^2(0, T; V^*)$  and  $L^1(0, T; H)$ , that the map

$$L^2(0, T; H) \ni h \mapsto \langle -\partial_u \mathcal{J}(\bar{y}, \bar{u}), h \rangle_{L^2(0, T; H)} \in \mathbb{R}$$

can be extended to a linear and continuous functional on  $L^2(0, T; V^*)$  and  $L^1(0, T; H)$ , respectively. Using that  $L^2(0, T; V^*)^* \cong L^2(0, T; V)$  and  $L^1(0, T; H)^* \cong L^\infty(0, T; H)$ , we now obtain that there exist  $\bar{p}_1 \in L^2(0, T; V)$  and  $\bar{p}_2 \in L^\infty(0, T; H)$  with

$$\int_0^T \langle h, \bar{p}_1 \rangle_V dt = \int_0^T (\bar{p}_2, h)_H dt = \int_0^T (-\partial_u \mathcal{J}(\bar{y}, \bar{u}), h)_H dt \quad \forall h \in L^2(0, T; H).$$

The above yields  $\bar{p}_1 = \bar{p}_2 = -\partial_u \mathcal{J}(\bar{y}, \bar{u})$  and proves the claim.  $\square$

From the last result and Theorem 4.1, we may deduce:

**THEOREM 5.5** (Strong Stationarity Conditions). *In the situation of Assumption 5.1, the following holds true:*

- (I) *For every  $\bar{u} \in \mathcal{U}_{ad}$  which satisfies the Bouligand stationarity condition (5.1) and for which the set  $\mathbb{R}^+(\mathcal{U}_{ad} - \bar{u})$  is dense in  $L^2(0, T; H)$ , there exist a unique  $\bar{p} \in L^2(0, T; V) \cap L^\infty(0, T; H)$  and a unique  $\bar{\eta} \in (L^2(0, T; V) \cap H^1(0, T; V^*))^*$  such that the following strong stationarity system holds:*

$$\begin{aligned} \bar{y} &= S(\bar{u}), \\ \bar{\varphi} &= \bar{y}' + A\bar{y} - \bar{u}, \\ -\bar{p}' + A^*\bar{p} &= \partial_y \mathcal{J}(\bar{y}, \bar{u}) - \bar{\eta}, \\ \bar{p} + \partial_u \mathcal{J}(\bar{y}, \bar{u}) &= 0, \end{aligned} \tag{5.3}$$

$$\bar{p} \in T_{\mathcal{K}, L^2}^{tan}(\bar{y}) \cap \bar{\varphi}^\perp, \quad \langle \bar{\eta}, z \rangle_{L^2(0, T; V) \cap H^1(0, T; V^*)} \geq 0 \quad \forall z \in \mathcal{C}(\bar{y}).$$

Here,  $\mathcal{C}(\bar{y})$  is defined by

$$\mathcal{C}(\bar{y}) := \left\{ z \in \text{cl}_{L^2(0, T; V) \cap H^1(0, T; V^*)} (T_{\mathcal{K}}^{rad}(\bar{y}) \cap \bar{\varphi}^\perp) \mid z(0) = 0 \right\},$$

the sets  $\mathcal{K}$ ,  $T_{\mathcal{K}, L^2}^{tan}(\bar{y})$ ,  $T_{\mathcal{K}}^{rad}(\bar{y})$  are defined as in Theorem 4.1, and the adjoint equation is understood as a formal identity in  $(L^2(0, T; V) \cap H^1(0, T; V^*))^*$ , see (5.6) below.

- (II) *If, conversely, we are given a  $\bar{u} \in \mathcal{U}_{ad}$  such that (5.3) holds with a  $\bar{p} \in \mathcal{C}(\bar{y})$  and an  $\bar{\eta} \in (L^2(0, T; V) \cap H^1(0, T; V^*))^*$ , then it holds*

$$\langle \partial_y \mathcal{J}(\bar{y}, \bar{u}), S'(\bar{u}; h) \rangle_{L^q(0, T; H)} + \langle \partial_u \mathcal{J}(\bar{y}, \bar{u}), h \rangle_{L^2(0, T; H)} \geq 0 \tag{5.4}$$

for all  $h \in L^2(0, T; H)$  with  $S'(\bar{u}; h) \in \mathcal{C}(\bar{y})$ . In particular,  $\bar{u}$  is Bouligand stationary for (O) if

$$L^2(0, T; H) = \text{cl}_{L^2(0, T; H)} \{ h \in L^2(0, T; H) \mid S'(\bar{u}; h) \in \mathcal{C}(\bar{y}) \}. \tag{5.5}$$

*Proof.* Ad (I): Suppose that a  $\bar{u} \in \mathcal{U}_{ad}$  is given such that the Bouligand stationarity condition (5.1) is satisfied and such that the set  $\mathbb{R}^+(\mathcal{U}_{ad} - \bar{u})$  is dense in  $L^2(0, T; H)$ . Then, we know from Lemma 5.4 that the derivative  $\bar{p} := -\partial_u \mathcal{J}(\bar{y}, \bar{u})$  is an element of  $L^2(0, T; V) \cap L^\infty(0, T; H)$ , and we may define

$$\begin{aligned} \langle \bar{\eta}, z \rangle_{L^2(0, T; V) \cap H^1(0, T; V^*)} &:= \langle \partial_y \mathcal{J}(\bar{y}, \bar{u}), z \rangle_{L^q(0, T; H)} - \langle z' + Az, \bar{p} \rangle_{L^2(0, T; V)} \\ &\quad \forall z \in L^2(0, T; V) \cap H^1(0, T; V^*) \end{aligned} \tag{5.6}$$

to obtain an element of  $(L^2(0, T; V) \cap H^1(0, T; V^*))^*$  which (formally) satisfies

$$-\bar{p}' + A^*\bar{p} = \partial_y \mathcal{J}(\bar{y}, \bar{u}) - \bar{\eta}$$

in  $(L^2(0, T; V) \cap H^1(0, T; V^*))^*$ . Here, we have again used the continuous embedding  $L^2(0, T; V) \cap H^1(0, T; V^*) \hookrightarrow C([0, T]; H)$ , see [40, Theorem 10.9]. To prove that the above  $\bar{p}$  and  $\bar{\eta}$  have the properties in (5.3), we note that (2.2) yields

$$\begin{aligned} \|S(u_1) - S(u_2)\|_{L^2(0, T; V)} + \|S(u_1) - S(u_2)\|_{L^q(0, T; H)} &\leq C \|u_1 - u_2\|_{L^2(0, T; V^*)} \\ &\quad \forall u_1, u_2 \in L^2(0, T; H) \end{aligned}$$

with some absolute constant  $C > 0$ . This implies in combination with the directional differentiability properties of the solution map  $S$  and the weak lower semicontinuity of the  $L^2(0, T; V)$ -norm that

$$\begin{aligned} & \|S'(\bar{u}; h_1) - S'(\bar{u}; h_2)\|_{L^2(0, T; V)} + \|S'(\bar{u}; h_1) - S'(\bar{u}; h_2)\|_{L^q(0, T; H)} \\ & \leq C \|h_1 - h_2\|_{L^2(0, T; V^*)} \quad \forall h_1, h_2 \in L^2(0, T; H) \end{aligned} \quad (5.7)$$

holds, and that the map  $h \mapsto S'(\bar{u}; h)$  admits a unique Lipschitz continuous extension  $F : L^2(0, T; V^*) \rightarrow L^2(0, T; V) \cap L^q(0, T; H)$ . As before, we write  $\bar{\varphi} := \bar{y}' + A\bar{y} - \bar{u}$ . Using Theorem 4.1, the  $L^2(0, T; V)$ -closedness of the set  $T_{\mathcal{K}, L^2}^{tan}(\bar{y}) \cap \bar{\varphi}^\perp$ , the mapping properties of the function  $F$ , and the Bouligand stationarity condition (5.1), we may deduce that

$$\begin{aligned} & F(h) \in T_{\mathcal{K}, L^2}^{tan}(\bar{y}) \cap \bar{\varphi}^\perp, \\ & \int_0^T \langle z' + AF(h) - h, z - F(h) \rangle_V dt + \frac{1}{2} \|z(0)\|_H^2 \geq 0 \\ & \quad \forall z \in \text{cl}_{L^2(0, T; V) \cap H^1(0, T; V^*)} (T_{\mathcal{K}}^{rad}(\bar{y}) \cap \bar{\varphi}^\perp) \end{aligned}$$

and

$$\langle \partial_y \mathcal{J}(\bar{y}, \bar{u}), F(h) \rangle_{L^q(0, T; H)} - \langle h, \bar{p} \rangle_{L^2(0, T; V)} \geq 0 \quad (5.8)$$

holds for all  $h \in L^2(0, T; V^*)$ . Consider now an arbitrary but fixed function  $z \in \mathcal{C}(\bar{y})$  and define  $h := z' + Az \in L^2(0, T; V^*)$ . Then, it clearly holds

$$\int_0^T \langle z' + Az - h, F(h) - z \rangle_V dt \geq 0,$$

and we obtain from the variational inequality for  $F(h)$  that

$$\int_0^T \langle z' + AF(h) - h, z - F(h) \rangle_V dt \geq 0.$$

Adding the last two inequalities yields  $F(h) = z$ , i.e., we have  $F(z' + Az) = z$  for all  $z \in \mathcal{C}(\bar{y})$ . The latter implies in combination with (5.8) that

$$\langle \partial_y \mathcal{J}(\bar{y}, \bar{u}), z \rangle_{L^q(0, T; H)} - \langle z' + Az, \bar{p} \rangle_{L^2(0, T; V)} = \langle \bar{\eta}, z \rangle_{L^2(0, T; V) \cap H^1(0, T; V^*)} \geq 0$$

for all  $z \in \mathcal{C}(\bar{y})$ . It remains to prove that  $\bar{p} \in T_{\mathcal{K}, L^2}^{tan}(\bar{y}) \cap \bar{\varphi}^\perp$ . To this end, we note that, for every  $h \in L^2(0, T; V^*)$  with

$$\langle h, z \rangle_{L^2(0, T; V)} \leq 0 \quad \forall z \in T_{\mathcal{K}, L^2}^{tan}(\bar{y}) \cap \bar{\varphi}^\perp,$$

we may test the variational inequality for  $F(h)$  with  $z = 0$  to obtain

$$\int_0^T \langle AF(h), F(h) \rangle_V dt \leq \int_0^T \langle h, F(h) \rangle_V dt \leq 0.$$

The above implies  $F(h) = 0$  and, due to (5.8),

$$-\langle h, \bar{p} \rangle_{L^2(0, T; V)} \geq 0.$$

We may thus conclude that

$$\langle h, \bar{p} \rangle_{L^2(0,T;V)} \leq 0 \quad \forall h \in L^2(0,T;V^*) \text{ with } \langle h, z \rangle_{L^2(0,T;V)} \leq 0 \quad \forall z \in T_{\mathcal{K},L^2}^{\text{tan}}(\bar{y}) \cap \bar{\varphi}^\perp$$

which may also be written as

$$\bar{p} \in (T_{\mathcal{K},L^2}^{\text{tan}}(\bar{y}) \cap \bar{\varphi}^\perp)^{\circ\circ},$$

where  $(\cdot)^\circ$  denotes a polarization in  $L^2(0,T;V)$  or  $L^2(0,T;V^*)$ , respectively. From the bipolar theorem, see [18, Section III-5], and the fact that  $T_{\mathcal{K},L^2}^{\text{tan}}(\bar{y}) \cap \bar{\varphi}^\perp$  is a closed convex pointed cone in  $L^2(0,T;V)$ , it now follows immediately that  $\bar{p} \in T_{\mathcal{K},L^2}^{\text{tan}}(\bar{y}) \cap \bar{\varphi}^\perp$ . Since  $\bar{p}$  and  $\bar{\eta}$  are trivially unique, the proof of (I) is now complete.

Ad (II): Consider an arbitrary but fixed  $\bar{u} \in \mathcal{U}_{ad}$  which satisfies the strong stationarity system (5.3) with a  $\bar{p} \in \mathcal{C}(\bar{y})$  and an  $\bar{\eta} \in (L^2(0,T;V) \cap H^1(0,T;V^*))^*$ , and suppose that an  $h \in L^2(0,T;H)$  with  $\delta := S'(\bar{u}; h) \in \mathcal{C}(\bar{y})$  is given. Then, (4.2) implies

$$\delta \in \mathcal{C}(\bar{y}), \quad \int_0^T \langle z', z - \delta \rangle_V + \langle A\delta, z - \delta \rangle_V - \langle h, z - \delta \rangle_V dt \geq 0 \quad \forall z \in \mathcal{C}(\bar{y}).$$

Since  $\mathcal{C}(\bar{y})$  is convex and since  $\delta \in \mathcal{C}(\bar{y})$ , we may test the above variational inequality with functions of the form  $\delta + \alpha(z - \delta)$ ,  $z \in \mathcal{C}(\bar{y})$ ,  $\alpha \in (0, 1)$ , divide by  $\alpha$  and pass to the limit  $\alpha \searrow 0$  to obtain

$$\delta \in \mathcal{C}(\bar{y}), \quad \int_0^T \langle \delta' + A\delta - h, z - \delta \rangle_V dt \geq 0 \quad \forall z \in \mathcal{C}(\bar{y}). \quad (5.9)$$

Using (5.9), the adjoint equation in (5.3), the properties of  $\bar{\eta}$  and  $\bar{p}$ , and the fact that  $\mathcal{C}(\bar{y})$  is a convex cone, we may now deduce that

$$\begin{aligned} \langle \partial_y \mathcal{J}(\bar{y}, \bar{u}), \delta \rangle_{L^q(0,T;H)} &= \langle \bar{\eta}, \delta \rangle_{L^2(0,T;V) \cap H^1(0,T;V^*)} + \langle \delta' + A\delta, \bar{p} \rangle_{L^2(0,T;V)} \\ &\geq \langle \delta' + A\delta - h, \delta + \bar{p} - \delta \rangle_{L^2(0,T;V)} + \langle h, \bar{p} \rangle_{L^2(0,T;V)} \\ &\geq \langle h, \bar{p} \rangle_{L^2(0,T;V)} \\ &= \langle -\partial_u \mathcal{J}(\bar{y}, \bar{u}), h \rangle_{L^2(0,T;H)}. \end{aligned}$$

This proves (5.4) for all  $h \in L^2(0,T;H)$  with  $S'(\bar{u}; h) \in \mathcal{C}(\bar{y})$ . If the density (5.5) holds, then the Bouligand stationarity condition (5.1) follows immediately from the continuity of the map  $L^2(0,T;H) \ni h \mapsto S'(\bar{u}; h) \in L^q(0,T;H)$ , see (5.7). This completes the proof of the theorem.  $\square$

REMARK 5.6. *The optimality condition (5.3) is the parabolic counterpart of the strong stationarity system derived by Mignot and Puel in [33, Theorem 2.2, Remark 2.1]. Compare also with [31, Theorem 6.8] in this context. Note that, in (5.3), the adjoint state  $\bar{p}$  only satisfies  $\bar{p} \in T_{\mathcal{K},L^2}^{\text{tan}}(\bar{y}) \cap \bar{\varphi}^\perp$  and is not necessarily contained in the set  $\mathcal{C}(\bar{y}) \subset T_{\mathcal{K},L^2}^{\text{tan}}(\bar{y}) \cap \bar{\varphi}^\perp$  on which the multiplier  $\bar{\eta}$  is non-negative. This discrepancy is a direct consequence of the asymmetry in the regularity properties of the solution and the test functions in the weakly formulated variational inequality (4.2) and responsible for the additional assumptions that are needed in part (II) of Theorem 5.5 to get from (5.3) back to the Bouligand stationarity condition (5.1). Such an effect is not present in the time-independent case where the equivalence between the Bouligand and the strong stationarity system can be established without restrictions.*

To get a feeling for the results in Theorem 5.5 and to assess the strength of the necessary optimality condition (5.3), let us consider the special case where the map  $S$  in Assumption 5.1 is the solution operator of a classical parabolic obstacle problem with homogeneous boundary conditions and zero obstacle. More precisely, let us assume that  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 1$ , is a bounded domain with a  $C^{1,1}$ -boundary, that

$$\begin{aligned} V &:= H_0^1(\Omega), & H &:= L^2(\Omega), & V^* &:= H^{-1}(\Omega), & A &:= -\Delta, \\ K &:= \{v \in H_0^1(\Omega) \mid v \geq 0 \text{ } \mathcal{L}^d\text{-a.e. in } \Omega\}, \end{aligned}$$

where  $\mathcal{L}^d$  again denotes the Lebesgue measure, that a  $T > 0$  is given, that  $y_0 = 0$ , and that  $S : L^2(0, T; L^2(\Omega)) \rightarrow L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$  is the solution map of the EVI

$$\begin{aligned} y &\in L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; L^2(\Omega)), \\ y &\in K \text{ a.e. in } (0, T), \quad y(0) = 0, \\ \int_0^T (y', v - y)_{L^2} + \langle -\Delta y, v - y \rangle_{H_0^1} - (u, v - y)_{L^2} dt &\geq 0 \\ \forall v &\in L^2(0, T; H_0^1(\Omega)), \quad v \in K \text{ a.e. in } (0, T). \end{aligned} \tag{5.10}$$

Note that the conditions in Assumption 2.1 are satisfied in the above situation by [3, Theorem 5.8.2] and [46, Lemma 3.2], cf. Examples 2.5 and 2.6. The problem (5.10) is thus covered by our analysis. From Theorem 5.5, we may now deduce that, for every  $\bar{u} \in \mathcal{U}_{ad}$  which is Bouligand stationary for (O) in the sense of (5.1) and for which the set  $\mathbb{R}^+(\mathcal{U}_{ad} - \bar{u})$  is dense in  $L^2(0, T; L^2(\Omega)) \cong L^2((0, T) \times \Omega)$ , there exist a unique adjoint state  $\bar{p} \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$  and a unique multiplier  $\bar{\eta} \in (L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)))^*$  with

$$\begin{aligned} \bar{y} &= S(\bar{u}), \\ \bar{\varphi} &= \bar{y}' - \Delta \bar{y} - \bar{u}, \\ -\bar{p}' - \Delta \bar{p} &= \partial_y \mathcal{J}(\bar{y}, \bar{u}) - \bar{\eta}, \\ \bar{p} + \partial_u \mathcal{J}(\bar{y}, \bar{u}) &= 0, \end{aligned} \tag{5.11}$$

$$\bar{p} \in T_{\mathcal{K}, L^2}^{tan}(\bar{y}) \cap \bar{\varphi}^\perp, \quad \langle \bar{\eta}, z \rangle_{L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega))} \geq 0 \quad \forall z \in \mathcal{C}(\bar{y}),$$

where the adjoint equation is again understood in the weak sense (5.6) and where

$$\begin{aligned} \mathcal{K} &:= \left\{ z \in L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)) \mid z \in K \text{ a.e. in } (0, T) \right\}, \\ T_{\mathcal{K}}^{rad}(\bar{y}) &:= \mathbb{R}^+(\mathcal{K} - \bar{y}), \\ T_{\mathcal{K}, L^2}^{tan}(\bar{y}) &:= \text{cl}_{L^2(0, T; H_0^1(\Omega))} (T_{\mathcal{K}}^{rad}(\bar{y})), \\ \mathcal{C}(\bar{y}) &:= \left\{ z \in \text{cl}_{L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega))} (T_{\mathcal{K}}^{rad}(\bar{y}) \cap \bar{\varphi}^\perp) \mid z(0) = 0 \right\}. \end{aligned} \tag{5.12}$$

To analyze the properties of the sets in (5.12) and the last line in (5.11), we note that the solutions  $y = S(u)$  to (5.10) with  $u \in L^2(0, T; L^2(\Omega))$  enjoy the additional regularity  $y \in L^2(0, T; H^2(\Omega)) \cap C([0, T]; H_0^1(\Omega))$  by [5, Corollary 4.4]. This implies that  $\bar{\varphi} = \bar{y}' - \Delta \bar{y} - \bar{u}$  is an element of  $L^2(0, T; L^2(\Omega))$ . The EVI (5.10), the density of the set of all non-negative  $L^2(0, T; H_0^1(\Omega))$ -functions in the set of all non-negative  $L^2(0, T; L^2(\Omega))$ -functions and  $L^2(0, T; L^2(\Omega)) \cong L^2((0, T) \times \Omega)$  now imply that

$$\langle \bar{\varphi}, z \rangle_{L^2((0, T) \times \Omega)} \geq 0 \quad \forall z \in L^2((0, T) \times \Omega) \text{ with } \bar{y} + z \geq 0 \text{ } \mathcal{L}^{d+1}\text{-a.e. in } (0, T) \times \Omega.$$

This yields

$$\begin{aligned}\bar{\varphi}(t) &= 0 \text{ } \mathcal{L}^d\text{-a.e. in } \{\bar{y}(t) > 0\} \text{ for a.a. } t \in (0, T), \\ \bar{\varphi}(t) &\geq 0 \text{ } \mathcal{L}^d\text{-a.e. in } \{\bar{y}(t) = 0\} \text{ for a.a. } t \in (0, T).\end{aligned}$$

From the definition of the sets in (5.12), it follows further that

$$T_{\mathcal{K}, L^2}^{tan}(\bar{y}) \subset \left\{ z \in L^2(0, T; H_0^1(\Omega)) \mid z(t) \in T_K^{tan}(\bar{y}(t)) \text{ for a.a. } t \in (0, T) \right\},$$

where  $T_K^{tan}(v) := \text{cl}_{H_0^1}(\mathbb{R}^+(K - v))$  denotes the  $H_0^1$ -tangent cone to  $K$  at  $v \in H_0^1(\Omega)$ , and by invoking standard results from capacity theory, cf. [22, Lemma 3.11] and [7, Theorem 6.57], we readily obtain that

$$T_K^{tan}(\bar{y}(t)) = \left\{ z \in H_0^1(\Omega) \mid z \geq 0 \text{ } H_0^1\text{-q.e. in } \{\bar{y}(t) = 0\} \right\} \text{ for a.a. } t \in (0, T),$$

where “ $H_0^1$ -q.e.” is short for  $H_0^1(\Omega)$ -quasi everywhere, i.e., equal up to a set of  $H_0^1(\Omega)$ -capacity zero, and where  $\bar{y}(t)$  denotes the  $H_0^1$ -quasi continuous representative of the function  $\bar{y}(t) \in H_0^1(\Omega)$ , see [7, Section 6.4.3]. Since  $H_0^1$ -q.e. implies  $\mathcal{L}^d$ -a.e., we may combine all of the above to deduce that

$$\begin{aligned}T_{\mathcal{K}, L^2}^{tan}(\bar{y}) \cap \bar{\varphi}^\perp \\ \subset \left\{ z \in L^2(0, T; H_0^1(\Omega)) \mid z(t) \geq 0 \text{ } H_0^1\text{-q.e. in } \{\bar{y}(t) = 0\} \text{ for a.a. } t \in (0, T), \right. \\ \left. z(t) = 0 \text{ } \mathcal{L}^d\text{-a.e. in } \{\bar{\varphi}(t) > 0\} \text{ for a.a. } t \in (0, T) \right\}.\end{aligned}$$

This shows that the adjoint state  $\bar{p}$  in (5.11) is for almost all  $t$  quasi everywhere non-negative on the contact set  $\{\bar{y}(t) = 0\}$  and for almost all  $t$  almost everywhere zero on the strongly active set  $\{\bar{\varphi}(t) > 0\} \subset \{\bar{y}(t) = 0\}$  (where the inclusion holds up to sets of measure zero). To study the inequality in the last line of (5.11), we observe that

$$\text{cl}_{\mathcal{W}} \left( \mathbb{R}^+ \left( \tilde{\mathcal{K}} - \bar{y} \right) \cap \bar{\varphi}^\perp \right) \subset \mathcal{C}(\bar{y}),$$

where

$$\tilde{\mathcal{K}} := \{z \in \mathcal{W} \mid z \in K \text{ a.e. in } (0, T)\} \subset \mathcal{K}$$

and where

$$\mathcal{W} := \{z \in L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; L^2(\Omega)) \mid z(0) = 0\}$$

is the Hilbert space with the product

$$(z_1, z_2)_{\mathcal{W}} := \int_0^T \int_{\Omega} \nabla z_1(t) \cdot \nabla z_2(t) + z_1(t)z_2(t) + z_1'(t)z_2'(t) d\mathcal{L}^d dt.$$

Note that [3, Theorem 5.8.2] and [46, Corollary 2.3] imply that the map  $z \mapsto z^+$  is well-defined and continuous from  $\mathcal{W}$  to  $\mathcal{W}$  and that

$$(z^+, z^-)_{\mathcal{W}} = 0 \quad \forall z \in \mathcal{W}.$$

The above yields that  $\mathcal{W}$  is a vector lattice in the sense of [44, Definition 4.6] when endowed with the cone

$$\{z \in \mathcal{W} \mid z(t) \geq 0 \text{ } \mathcal{L}^d\text{-a.e. in } \Omega \text{ for a.a. } t \in (0, T)\}$$

and a Dirichlet space in the sense of [32, Définition 3.1] (with space  $X := (0, T] \times \text{cl}(\Omega)$ )

and measure  $\xi := \mathcal{L}^{d+1}$ ). From [32, Lemme 3.2] and [44, Theorem 4.18], we may now deduce that the set  $\tilde{\mathcal{K}}$  is polyhedric in  $\mathcal{W}$  in the sense of [44, Definition 3.1] and that

$$\begin{aligned} \text{cl}_{\mathcal{W}} \left( \mathbb{R}^+ \left( \tilde{\mathcal{K}} - \bar{y} \right) \cap \bar{\varphi}^\perp \right) &= \text{cl}_{\mathcal{W}} \left( \mathbb{R}^+ \left( \tilde{\mathcal{K}} - \bar{y} \right) \right) \cap \bar{\varphi}^\perp \\ &= \{z \in \mathcal{W} \mid z \geq 0 \text{ } \mathcal{W}\text{-q.e. in } \{\bar{y} = 0\}\} \cap \bar{\varphi}^\perp, \end{aligned}$$

where “ $\mathcal{W}$ -q.e.” is short for quasi everywhere w.r.t. the capacity of the Dirichlet space  $\mathcal{W}$ , and where the set  $\{\bar{y} = 0\}$  is defined w.r.t. the  $\mathcal{W}$ -quasi continuous representative of the state  $\bar{y}$ . Proceeding as for  $\bar{p}$ , we now obtain that the multiplier

$$\bar{\eta} \in (L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)))^* \subset \mathcal{W}^*$$

satisfies

$$\langle \bar{\eta}, z \rangle_{\mathcal{W}} = \langle \bar{\eta}, z \rangle_{L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega))} \geq 0$$

for all  $z \in \mathcal{W}$  with

$$z \geq 0 \text{ } \mathcal{W}\text{-q.e. in } \{\bar{y} = 0\}, \quad z = 0 \text{ } \mathcal{L}^{d+1}\text{-a.e. in } \{\bar{\varphi} > 0\}.$$

In summary, we have now proved that, for the problem (O) and the EVI (5.10), the abstract optimality condition (5.11) gives rise to the stationarity system

$$\begin{aligned} \bar{y} &= S(\bar{u}), & \bar{\varphi} &= \bar{y}' - \Delta \bar{y} - \bar{u}, \\ \bar{p} &= -\partial_u \mathcal{J}(\bar{y}, \bar{u}), & \bar{\eta} &= \bar{p}' + \Delta \bar{p} + \partial_y \mathcal{J}(\bar{y}, \bar{u}), \\ \bar{p}(t) &\geq 0 \text{ } H_0^1\text{-q.e. in } \{\bar{y}(t) = 0\} \text{ for a.a. } t \in (0, T), \\ \bar{p}(t) &= 0 \text{ } \mathcal{L}^d\text{-a.e. in } \{\bar{\varphi}(t) > 0\} \text{ for a.a. } t \in (0, T), \\ \bar{\varphi}(t) &= 0 \text{ } \mathcal{L}^d\text{-a.e. in } \{\bar{y}(t) > 0\} \text{ for a.a. } t \in (0, T), \\ \bar{\varphi}(t) &\geq 0 \text{ } \mathcal{L}^d\text{-a.e. in } \{\bar{y}(t) = 0\} \text{ for a.a. } t \in (0, T), \\ \langle \bar{\eta}, z \rangle_{\mathcal{W}} &\geq 0 \quad \forall z \in \mathcal{W} \text{ s.t. } z \geq 0 \text{ } \mathcal{W}\text{-q.e. in } \{\bar{y} = 0\}, z = 0 \text{ } \mathcal{L}^{d+1}\text{-a.e. in } \{\bar{\varphi} > 0\}, \end{aligned} \tag{5.13}$$

where the adjoint equation is still understood in the weak sense (5.6), and where  $\bar{u}$ ,  $\bar{y}$ ,  $\bar{\varphi}$ ,  $\bar{p}$  and  $\bar{\eta}$  satisfy

$$\begin{aligned} \bar{u} &\in L^2(0, T; L^2(\Omega)), & \bar{y} &\in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)) \cap C([0, T]; H_0^1(\Omega)), \\ \bar{\varphi} &\in L^2(0, T; L^2(\Omega)), & \bar{p} &\in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)), & \bar{\eta} &\in \mathcal{W}^*. \end{aligned}$$

Several things are noteworthy regarding the optimality condition (5.13):

REMARK 5.7.

- (i) *The different notions of quasi everywhere appearing in (5.13) are again a consequence of the asymmetry in the regularity properties of  $\delta$  and  $z$  in (4.2). A similar effect cannot be observed in the time-independent case, cf., e.g., the results in [22, 33]. Note that (5.13) is still a necessary optimality condition when “ $H_0^1$ -q.e.” is replaced with “ $\mathcal{L}^d$ -a.e.”.*
- (ii) *The stationarity system (5.13) immediately yields that  $\bar{y}$ ,  $\bar{\varphi}$ ,  $\bar{p}$  and  $\bar{\eta}$  satisfy the complementarity conditions*

$$\langle \bar{\eta}, \bar{y} \rangle_{\mathcal{W}} = 0, \quad \bar{\varphi}(t)\bar{p}(t) = 0 \text{ } \mathcal{L}^d\text{-a.e. in } \Omega \text{ for a.a. } t \in (0, T).$$

(iii) *The line*

$$\bar{p}(t) \geq 0 \text{ } H_0^1\text{-q.e. in } \{\bar{y}(t) = 0\} \text{ for a.a. } t \in (0, T) \quad (5.14)$$

in (5.13) does not appear in necessary optimality conditions that are obtained by regularization, cf., e.g., the systems derived in [28, Theorem 6.2] and [5, Theorem 5.2, Equations (5.61), (5.63)]. This makes sense since (5.14) yields information about the behavior of  $\bar{p}$  on the bi-active set (i.e., the intersection of the zero level sets of  $\bar{y}$  and  $\bar{\varphi}$ ) where regularization approaches are known to be imprecise. Note that our results are in particular consistent with the observations made in [31, Theorems 6.6, 6.8] and [11, Section 4] for optimal control problems governed by non-smooth partial differential equations.

(iv) A stationarity system similar to (5.13) can be found in [34, Théorème 2]. However, a rigorous proof of the strong stationarity conditions in [34] has, at least to the author's best knowledge, never been published (and the sketch of proof in [34] seems to fail due to the inapplicability of subdifferential calculus rules). Further, the optimality system in [34] does not contain, e.g., the  $L^\infty(0, T; L^2(\Omega))$ -regularity of the adjoint state  $\bar{p}$  and the quasi everywhere (in-)equalities in (5.13). Note that the analysis in Sections 3 and 4 provides the differentiability results that are referred to as missing/unknown in [34, Remarque 3] and [7, Section 7.5, Page 582].

**6. Concluding Remarks and Open Questions.** As we have seen in this paper, the Hadamard directional differentiability of the solution operator  $S$  to an obstacle-type evolution variational inequality of the form (P) can readily be established in the spaces  $L^q(0, T; H)$ ,  $1 \leq q < \infty$ , by exploiting the monotonicity properties of the difference quotients in Lemma 4.4. Moreover, the method of proof in Section 4 allows to completely circumvent the difficulties that arise due to the lacking Lipschitz continuity properties of the solution map  $S$  in the spaces  $H^s(0, T; H)$ ,  $s \geq 1/2$ , and the non-uniqueness of solutions to the problem (4.2) when an approach analogous to [13,14,21,26,29,32] is used for the sensitivity analysis of an evolution variational inequality. What is, at least to the author's best knowledge, presently unknown is whether an argumentation similar to that in Section 4 can also be used for other classes of EVIs. The same is true for the extension of the strong stationarity conditions in Section 5 to optimal control problems with control constraints. Compare, e.g., with the results for elliptic obstacle problems in [45] in this context.

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