Journal manuscript No.
(will be inserted by the editor)

Finite Element Error Estimates in Non-Energy Norms for the Two-Dimensional Scalar Signorini Problem

Constantin Christof · Christof Haubner

Received: date / Accepted: date

Abstract This paper is concerned with error estimates for the piecewise linear finite element approximation of the two-dimensional scalar Signorini problem on a convex polygonal domain Ω. Using a Céa-type lemma, a supercloseness result and a non-standard duality argument, we prove $W^{1,p}(\Omega)$-, $L^p(\Omega)$-, $L^\infty(\Omega)$-, $W^{1,\infty}(\Omega)$- and $H^{1/2}(\partial\Omega)$-error estimates under mild assumptions on the contact sets of the continuous and the discrete solution. The obtained orders of convergence turn out to be optimal for problems with essentially bounded right-hand sides. Our results are accompanied by numerical experiments which confirm the theoretical findings.

Keywords A priori error analysis · Linear finite elements · Signorini problem · Aubin-Nitsche trick · Unilateral constraint · Dirichlet-Neumann problem

Mathematics Subject Classification (2010) 35J86 · 65K15 · 65N15 · 65N30

1 Introduction

The aim of this paper is to study the accuracy of the piecewise linear finite element method for the two-dimensional scalar Signorini problem

$$\begin{gathered} -\Delta u + u = f \text{ in } \Omega, \\ \partial_n u \geq 0, \quad u \geq 0, \quad u\partial_n u = 0 \text{ on } \partial\Omega. \end{gathered} \tag{1}$$

Using a Céa-type lemma (Lemma 3.4), a supercloseness result (Theorem 3.6) and a duality argument that is based on ideas of Mosco (cf. [24] and Section 5), we prove $W^{1,p}$- and L^p-error estimates that, in view of the $W^{2,p}$- and H^s-regularity properties of the exact solution u, are optimal for right-hand sides $f \in L^\infty(\Omega)$. In

C. Christof
Technische Universität München, Chair of Optimal Control, Center for Mathematical Sciences, Boltzmannstraße 3, 85748 Garching, Germany
Tel.: +49 89 289 19589
E-mail: christof@ma.tum.de
C. Haubner
Universität der Bundeswehr München, Institut für Mathematik und Computergestützte Simulation, Werner-Heisenberg-Weg 39, 85577 Neubiberg, Germany

particular, we obtain an L^4-error estimate of the form $\|u - u_h\|_{L^4(\Omega)} = \mathcal{O}(h^{2-\varepsilon})$ for all $\varepsilon > 0$ which explains the order of convergence in the lower L^p-norms that is typically observed in numerical experiments, cf. Section 7 and [33, Section 7]. See Corollaries 4.1, 4.2 and 4.3 and Theorem 6.1 for the main results of this paper.

Before we begin with our analysis, let us give some background: While H^1- and L^∞-error estimates for obstacle- and Signorini-type problems have been discussed by various authors, cf. [5,6,11,13,17,23,24] and [3,7,10,14,21,27,28], results on the finite element error in other norms are only rarely addressed in the literature. See, e.g., [8,24,26,32,33,35] for some of the few contributions on this topic. The reason for this is that the weak formulations of Signorini- and obstacle-type problems take the form of elliptic variational inequalities which cannot be analyzed with standard techniques. This becomes apparent, e.g., in the study of the L^2-error where classical duality arguments as the Aubin-Nitsche trick typically fail due to a lack of Galerkin orthogonality or due to a lack of regularity of the dual solution. Compare, for instance, with the approaches in [24,26,35] in this context where the regularity properties of the dual problem are the major bottleneck. Unfortunately, it turns out that these difficulties are not entirely artificial and that the behavior of the finite element error in lower L^p-norms may indeed change drastically when the problem under consideration involves additional inequality constraints. This is demonstrated in [8] where it is shown that L^∞-error estimates are optimal for the one-dimensional obstacle problem in the sense that they provide the best order of convergence obtainable with an a priori L^p-error estimate. See [8, Theorems 5, 10] for the precise results. Note that this observation implies in particular that it is not very useful to study error estimates in lower L^p-norms for one-dimensional obstacle problems (with non-linear obstacles).

In the present paper, we demonstrate that, for the Signorini problem (1), the situation is different, and that it is possible to obtain L^p-error estimates which are of significantly higher order than L^∞-error estimates if the problem at hand only involves inequality constraints on the boundary. For right-hand sides $f \in L^\infty(\Omega)$, our approach can be summarized as follows:

Using an elementary argument, we prove that the H^1-error between the finite element approximation u_h and the Ritz projection $R_h(u)$ of the exact solution u (see Definition 3.1) is smaller than the H^1-norm of every finite element function w_h with $R_h(u) - u \leq w_h \leq R_h(u)$ on $\partial\Omega$ (see Lemma 3.4). This best approximation property yields, in tandem with results on unilateral finite element approximations and the regularity of solutions to (1), that, for a solution u with a sufficiently well-behaved contact set (see Definition 2.2), it holds $\|u_h - R_h(u)\|_{H^1(\Omega)} = \mathcal{O}(h^{3/2-\varepsilon})$ for all $\varepsilon \in (0, 1/2)$ (see Theorem 3.6). By exploiting this supercloseness property, inverse estimates and standard results for the Ritz projection, we immediately obtain estimates of optimal order in $W^{1,4}(\Omega)$, $W^{1,\infty}(\Omega)$, $L^\infty(\Omega)$ and $H^{1/2}(\partial\Omega)$ (see Corollary 4.3 and Remark 4.4). To study the error in the lower L^p-norms, we follow an approach of Mosco and consider two dual problems, one for each of the components $\max(0, u - u_h)$ and $\min(0, u - u_h)$ (see Section 5). As we will see, the solutions of our dual variational inequalities suffer from the same regularity problems as those in [24,26,35] and cannot be expected to be elements of $H^2(\Omega)$. However, by invoking the results of [15,16], we can show that $W^{2,4/3-\varepsilon}$-regularity for all $\varepsilon > 0$ is obtainable instead. This observation and the fact that $q := 4/3$ is precisely the Hölder conjugate of $p := 4$ allow us to invoke our $W^{1,4}$-estimate to compensate the lack of regularity of the dual solutions and to arrive at an estimate of the type $\|u - u_h\|_{L^4(\Omega)} = \mathcal{O}(h^{2-\varepsilon})$ for all $\varepsilon > 0$ (see Theorem 6.1).

We would like to point out that the $H^{1/2}(\partial\Omega)$-error estimate in Corollary 4.3 reproduces [33, Theorem 2.2] under slightly different assumptions on the regularity of u (or the right-hand side f, respectively). Further, Corollary 4.3 shows that the order of convergence $3/2-\varepsilon$ that has been obtained in [33, Corollary 5.8] in $L^2(\Omega)$ is, in fact, even achieved in the L^∞-norm. Surprisingly, we obtain this L^∞-result without ever invoking the discrete maximum principle (which is normally used to prove pointwise error estimates for unilateral problems) and without the related assumptions on the underlying triangulation, cf. [3,7,9,10,14,21,27]. Theorem 6.1 finally improves the order of convergence in [33, Corollary 5.8] by the factor $h^{1/2}$ and yields an L^4-error estimate of optimal order. To the best of our knowledge, the L^p- and $W^{1,p}$-error estimates derived in this paper are new. Further, the duality argument in Section 5 seems to be the first of its kind that actually works without artificial assumptions on the regularity of the dual solution, cf. [24,26,35].

We conclude this introduction with a quick overview of the outline and the structure of this paper:

Section 2 is concerned with preliminaries. Here, we clarify the notation, state our precise assumptions, and collect several regularity results for the problem (1). In Section 3, we prove the Céa-type lemma and the supercloseness result that are at the heart of our error analysis. Section 4 addresses the consequences that the results of Section 3 have for the derivation of finite element error estimates. The main results of this section, Corollaries 4.1, 4.2 and 4.3, contain various $W^{1,p}(\Omega)$-, $L^p(\Omega)$- and $H^{1/2}(\partial\Omega)$-estimates that cover a large variety of different situations. Section 5 is devoted to the analysis of the L^4-error in the case $f \in L^\infty(\Omega)$. Here, we use a non-standard duality argument to prove that the continuous and the discrete solution satisfy $\|u - u_h\|_{L^4(\Omega)} = \mathcal{O}(h^{2-\varepsilon})$ for all $\varepsilon > 0$. In Section 6, we summarize our results and give some concluding remarks. Section 7 finally contains numerical experiments that confirm our theoretical findings.

2 Notations, Assumptions and Preliminaries

2.1 Basic Notation

Throughout this paper, we use the standard notations $L^p(\Omega)$, $C^{k,\gamma}(\Omega)$, $W^{s,p}(\Omega)$ and $H^s(\Omega)$ for the Lebesgue, Hölder and (fractional) Sobolev spaces on a bounded domain $\Omega \subset \mathbb{R}^2$. See, e.g., [1,2,12] for details on these spaces. The scalar products on $L^2(\Omega)$ and $H^1(\Omega)$ are denoted with $(\cdot,\cdot)_{L^2(\Omega)}$ and $(\cdot,\cdot)_{H^1(\Omega)}$, respectively, i.e.

$$(v_1, v_2)_{L^2(\Omega)} := \int_\Omega v_1 v_2 \mathrm{d}\mathcal{L}^2 \quad \text{and} \quad (v_1, v_2)_{H^1(\Omega)} := \int_\Omega \nabla v_1 \cdot \nabla v_2 + v_1 v_2 \mathrm{d}\mathcal{L}^2.$$

With $\mathcal{L}^k$ and $\mathcal{H}^k$, we denote the k-dimensional Lebesgue and Hausdorff measure, and with $\mathrm{tr}(\cdot)$ the classical trace operator, cf. [2]. For functions v with a continuous representative, we typically drop the prefix tr and simply write v instead of $\mathrm{tr}(v)$. Further, we use the symbols $\mathrm{cl}(\cdot)$ and ∂ to denote the topological closure and the boundary of a set, respectively, and the abbreviation $B_r(x)$ to denote the closed ball of radius $r > 0$ around an $x \in \mathbb{R}^2$. With $\mathcal{O}(\cdot)$, $o(\cdot)$, we denote the classical

Landau symbols, and with C a generic constant which may change within an estimate but is never dependent on crucial quantities as, e.g., the mesh width. If we want to emphasize that C depends on a quantity α, then we write $C = C(\alpha)$. Lastly, we define $a^+ := \max(0, a)$ and $a^- := \min(0, a)$ for all $a \in \mathbb{R}$.

2.2 The Continuous Setting

As already mentioned, the purpose of this paper is to study finite element error estimates for the two-dimensional scalar Signorini problem

$$\begin{gathered} -\Delta u + u = f \text{ in } \Omega, \\ \partial_n u \geq 0, \quad u \geq 0, \quad u \partial_n u = 0 \text{ on } \partial\Omega. \end{gathered} \tag{2}$$

Here and in what follows, Δ and ∂_n denote the (distributional) Laplacian and the normal derivative, respectively, and $f \in L^2(\Omega)$ is a given right-hand side. To avoid obscuring the basic ideas of our analysis with technicalities and distinctions of cases, and to reduce the notational overhead, throughout this paper, we always assume that Ω is the unit square, i.e., $\Omega := (0, 1)^2$. Our arguments can be extended straightforwardly to other convex polygonal domains with obvious modifications. The same holds true for other variants of Signorini's problem as, e.g., the version studied in [33] which involves a separate Dirichlet boundary. (In this case, however, one has to be careful since the H^2-regularity result in Theorem 2.1 is not directly applicable.) Note that a Signorini problem of the type (2) with a sufficiently regular non-zero obstacle on the boundary can always be transformed into a problem with a vanishing obstacle by an elementary translation argument. We may thus restrict our attention to the homogeneous situation in (2) without a great loss of generality.

To begin our analysis, we recall that the weak formulation of (2) is given by the elliptic variational inequality

$$u \in K, \quad (u, v - u)_{H^1(\Omega)} \geq (f, v - u)_{L^2(\Omega)} \quad \forall v \in K \tag{S}$$

with the admissible set

$$K := \left\{ v \in H^1(\Omega) \,\middle|\, \operatorname{tr}(v) \geq 0 \ \mathcal{H}^1\text{-a.e. on } \partial\Omega \right\}.$$

From [15, Theorem 3.2.3.1, Example (3,2,3,1)], we further obtain:

Theorem 2.1 (S) *admits a unique solution $u \in H^2(\Omega)$ for all $f \in L^2(\Omega)$.*

Note that Theorem 2.1 and the Sobolev embeddings imply that u possesses a representative which is continuous on the closure $\operatorname{cl}(\Omega)$ of the domain Ω. In what follows, we always use this representative when talking, e.g., about level sets. As it turns out, solutions u to (S) enjoy additional regularity properties when the contact set $\{x \in \partial\Omega \mid u(x) = 0\}$ is sufficiently well-behaved. To explore this effect, we introduce:

Definition 2.2 (Condition (A)) *A solution $u \in H^2(\Omega)$ of* (S) *is said to satisfy condition (A) if the relative boundary of the contact set $\{x \in \partial\Omega \mid u(x) = 0\}$ in $\partial\Omega$ has one-dimensional Hausdorff measure zero and if the relative interior of the contact set in $\partial\Omega$ consists of at most finitely many connected components.*

Under assumption (A), the solution u of the variational inequality (S) can be identified with the solution of a mixed Dirichlet-Neumann problem with segment-wise prescribed boundary conditions on a convex polygonal domain. This, together with the H^2-regularity result in Theorem 2.1 and the analysis in [15], yields:

Theorem 2.3 *Suppose that $u \in H^2(\Omega)$ solves* (S) *and satisfies condition (A). Then, the following holds true:*

1. *If f is in $L^p(\Omega)$ for some $2 < p < 4$, then it holds $u \in W^{2,p}(\Omega)$.*
2. *If f is in $L^p(\Omega)$ for some $4 < p < \infty$, then there exist functions $u_s, u_r \in H^2(\Omega)$ such that $u = u_s + u_r$, $u_r \in W^{2,p}(\Omega)$ and $u_s \in H^{5/2-\varepsilon}(\Omega)$ holds for all $\varepsilon \in (0, 1/2)$, and such that the restriction of the trace of u_s to each of the four sides of the square $\Omega = (0,1)^2$ has $W^{2,2-\varepsilon}$-regularity for all $\varepsilon \in (0, 1/2)$.*

Proof Since u satisfies condition (A), we may find relatively open disjoint straight line segments $\Gamma_i \subset \partial\Omega$, $i = 1, ..., N + M$, $N, M \in \mathbb{N}_0$, and a set $R \subset \partial\Omega$ of one-dimensional Hausdorff measure zero such that

$$\{x \in \partial\Omega \mid u(x) = 0\} = \bigcup_{i=1}^{N} \Gamma_i \cup R \quad \text{and} \quad \partial\Omega = \bigcup_{i=1}^{N+M} \mathrm{cl}(\Gamma_i).$$

From the variational inequality (S), the H^2-regularity of the solution u, and Green's first identity, it follows further that

$$u \in K, \quad \int_\Omega (-\Delta u + u - f)(v - u)\mathrm{d}\mathcal{L}^2 + \int_{\partial\Omega} (\partial_n u)(v - u)\mathrm{d}\mathcal{H}^1 \geq 0 \quad \forall v \in K.$$

The above yields

$$\begin{aligned} -\Delta u &= f - u \ \mathcal{L}^2\text{-a.e. in } \Omega, \\ u &= 0 \ \mathcal{H}^1\text{-a.e. on } \Gamma_i \text{ for all } i = 1, ..., N, \\ \partial_n u &= 0 \ \mathcal{H}^1\text{-a.e. on } \Gamma_i \text{ for all } i = N + 1, ..., N + M. \end{aligned} \tag{3}$$

Note that the parts of the contact set that are contained in the line segments Γ_i, $i = N + 1, ..., N + M$, are negligible here due to the properties of R. Let us suppose now that f is an element of $L^p(\Omega)$ for some $2 < p < 4$. Then, we may use [15, Theorem 4.4.3.7] to deduce that there exist real numbers c_i and trigonometric functions ϕ_i such that

$$u - \sum_{i=1,\dots,N+M} c_i \eta_i r_i^{1/2} \phi_i(\theta_i) \in W^{2,p}(\Omega) \tag{4}$$

holds, where $r_i \geq 0$ and $\theta_i \in [0, 2\pi)$ denote local polar coordinates centered at the vertices x_i, $i = 1, ..., N + M$, of the partition $\{\mathrm{cl}(\Gamma_i)\}$ of the boundary $\partial\Omega$, and where η_i is a smooth cut-off function which is identical one in a neighborhood of x_i for each i. We already know, however, that $u \in H^2(\Omega)$, and it is easy to check that the factor $r_i^{1/2}$ prevents a function of the form $\eta_i r_i^{1/2} \phi_i(\theta_i)$ to be an element of $H^2(\Omega)$. This implies that all c_i in (4) have to be zero and proves the first claim, cf. also with the discussion in [33, Remark 2.1] and [25] in this regard. To obtain the second claim, we can proceed along exactly the same lines: If f is an element of $L^p(\Omega)$ for some $4 < p < \infty$, then we may use the same arguments as above and again [15, Theorem 4.4.3.7] to deduce that there exist real numbers $\tilde{c}_i$ and trigonometric functions $\tilde{\phi}_i$ with

$$u - \sum_{i=1,\ldots,N+M} \tilde{c}_i \eta_i r_i^{3/2} \tilde{\phi}_i(\theta_i) \in W^{2,p}(\Omega),$$

where η_i, r_i and θ_i are as in (4). The functions

$$u_s := \sum_{i=1,\ldots,N+M} \tilde{c}_i \eta_i r_i^{3/2} \tilde{\phi}_i(\theta_i), \qquad u_r := u - u_s, \tag{5}$$

have all of the desired properties (see, for instance, [16, Theorem 1.2.18] for the $H^{5/2-\varepsilon}$-regularity). This completes the proof. □

We would like to point out that the solution u of (S) can, in general, not be expected to possess $W^{2,4}$- or $H^{5/2}$-regularity even for smooth right-hand sides f. Compare, e.g., with the examples in [33] and Section 7 in this context.

2.3 The Discrete Setting

As discrete counterparts of the variational inequality (S), we consider problems of the form

$$u_h \in K_h, \quad (u_h, v_h - u_h)_{H^1(\Omega)} \geq (f, v_h - u_h)_{L^2(\Omega)} \quad \forall v_h \in K_h. \tag{S$_h$}$$

Our standing assumptions on the quantities in (S$_h$) are as follows:

Assumption 2.4 (Standing Assumptions for the FE-Discretization)

1. *$\{\mathcal{T}_h\}_{0<h<1/2}$ is a quasi-uniform family of triangulations of $\Omega = (0,1)^2$,*
2. *$V_h := \{v \in C(\mathrm{cl}(\Omega)) \,|\, v$ is affine on all cells $T \in \mathcal{T}_h\}$,*
3. *$K_h := K \cap V_h = \{v_h \in V_h \,|\, v_h \geq 0$ on $\partial\Omega\}$.*

See, e.g., [7, Definition 2] or [4, Definition 4.4.13] for the definition of the term "quasi-uniform triangulation". For brevity's sake, in what follows, we often ignore the upper bound on the mesh width and simply write "for all $h > 0$" instead of "for all $0 < h < 1/2$". From standard results as, e.g., [18, Theorem 2.1], we obtain:

Theorem 2.5 (S$_h$) *admits a unique solution u_h for all $f \in L^2(\Omega)$ and all $h > 0$.*

In the remainder of this paper, our aim will be to study the approximation properties of the solution u_h of (S$_h$) for $h \searrow 0$. The main ingredients of our error analysis are:

3 A Céa-type Lemma and a Supercloseness Result

To study the error $u - u_h$, we introduce the following operator:

Definition 3.1 (Ritz-Projection) *For every $v \in H^1(\Omega)$, we define the Ritz projection $R_h(v)$ to be the unique element of V_h with*

$$(R_h(v), w_h)_{H^1(\Omega)} = (v, w_h)_{H^1(\Omega)} \quad \forall w_h \in V_h.$$

Note that $R_h : H^1(\Omega) \to V_h$ is precisely the solution operator of the unconstrained problem associated with (S_h). In particular, R_h is well-defined, linear and continuous, and we may invoke classical results to obtain:

Lemma 3.2

1. *For every $v \in W^{2,p}(\Omega)$, $2 \le p < \infty$, it holds*

$$\begin{aligned} \|v - R_h(v)\|_{L^p(\Omega)} + h\|v - R_h(v)\|_{W^{1,p}(\Omega)} + h^{1/p}\|v - R_h(v)\|_{L^p(\partial\Omega)} \\ \le Ch^2\|v\|_{W^{2,p}(\Omega)} \end{aligned} \tag{6}$$

with some constant $C > 0$ independent of h and v.

2. *If v satisfies $v \in H^{5/2-\varepsilon}(\Omega)$ for all $\varepsilon \in (0, 1/2)$, then, for every $\varepsilon \in (0, 1/2)$, there exists a constant $C > 0$ independent of h with*

$$\|v - R_h(v)\|_{H^{1/2}(\partial\Omega)} \le Ch^{3/2-\varepsilon}.$$

Proof For the domain $\Omega = (0, 1)^2$, the estimate (6) can be derived as follows: Consider an arbitrary but fixed $v \in W^{1,\infty}(\Omega)$ with associated Ritz projection $R_h(v) \in V_h$. Then, we may use reflections to extend v and $R_h(v)$ first to the rectangle $(0, 1)\times(-1, 2)$ and subsequently to the square $\tilde{\Omega} := (-1, 2)^2$ to construct functions $\tilde{v}, \tilde{v}_h \in W^{1,\infty}(\tilde{\Omega})$ with $\tilde{v}|_\Omega = v$, $\tilde{v}_h|_\Omega = R_h(v)$ and $\tilde{v}_h = \tilde{R}_h(\tilde{v})$. Here, $\tilde{R}_h$ is the Ritz projection operator on the mesh of $\tilde{\Omega}$ that is obtained from the reflection procedure. From the interior norm estimate [30, Theorem 1.2], we may now deduce that there exists a constant $C > 0$ independent of h with

$$\begin{aligned} \|v - R_h(v)\|_{W^{1,\infty}(\Omega)} &= \|\tilde{v} - \tilde{R}_h(\tilde{v})\|_{W^{1,\infty}(\Omega)} \\ &\le C\left(\|\tilde{v}\|_{W^{1,\infty}(\tilde{\Omega})} + \|\tilde{v} - \tilde{R}_h(\tilde{v})\|_{L^2(\tilde{\Omega})}\right). \end{aligned}$$

Since $\tilde{v}$ has the same $W^{1,\infty}$-norm as v, the above, the triangle inequality and the properties of $\tilde{v}$ and $\tilde{R}_h(\tilde{v})$ imply that

$$\|R_h(v)\|_{W^{1,\infty}(\Omega)} \le C\|v\|_{W^{1,\infty}(\Omega)}$$

holds with some $C > 0$ which does not depend on h. From the Theorem of Riesz-Thorin, cf. [4, Sections 14.1, 14.2], and the estimate $\|R_h(v)\|_{H^1(\Omega)} \le C\|v\|_{H^1(\Omega)}$, it now follows straightforwardly that there exists a $C > 0$ independent of h with

$$\|R_h(v)\|_{W^{1,p}(\Omega)} \le C\|v\|_{W^{1,p}(\Omega)} \qquad \forall v \in W^{1,p}(\Omega) \qquad \forall p \in [2, \infty].$$

The above estimate, the regularity results of [15], and exactly the same arguments as in the proofs of [29, Equations (1.8), (1.9)] yield that, for every $2 \le p < \infty$, there exists a constant $C > 0$ independent of h with

$$\|v - R_h(v)\|_{L^p(\Omega)} + h\|v - R_h(v)\|_{W^{1,p}(\Omega)} \le Ch^2\|v\|_{W^{2,p}(\Omega)} \quad \forall v \in W^{2,p}(\Omega).$$

It remains to prove the $L^p(\partial\Omega)$-estimate. This, however, follows immediately from the last inequality and [15, Theorem 1.5.1.10] with parameter $\varepsilon := h^p$. Note that the above argumentation only works for rectangles and squares. For more general domains Ω, (6) can be obtained by employing the techniques of [29] and [31]. To do so, however, one has to study in detail the behavior of certain Green's functions in the vicinity of the corners of the domain under consideration, cf. the comments in [31, Section 3]. Such a study is beyond the scope of this paper.

To prove the second assertion of the lemma, we suppose that a function v with $v \in H^{5/2-\varepsilon}(\Omega)$ for all $\varepsilon \in (0, 1/2)$ is given and that $\tilde{v}_h$ is the unique element of V_h with

$$\int_\Omega \tilde{v}_h - R_h(v) \mathrm{d}\mathcal{L}^2 = 0, \quad \int_\Omega \nabla \tilde{v}_h \cdot \nabla w_h \mathrm{d}\mathcal{L}^2 = \int_\Omega \nabla v \cdot \nabla w_h \mathrm{d}\mathcal{L}^2 \quad \forall w_h \in V_h.$$

By proceeding completely analogously to the proof of [22, Lemma 5.7] (with the Besov estimate in [22, Lemma 4.1] replaced with the second estimate in [19, Lemma 2.1]), we obtain that, for every $\varepsilon \in (0, 1/2)$, there exists a $C > 0$ with

$$|v - \tilde{v}_h|_{H^{1/2}(\partial\Omega)} \le C h^{3/2-\varepsilon}, \tag{7}$$

where $|\cdot|_{H^{1/2}(\partial\Omega)}$ denotes the $H^{1/2}$-seminorm on the boundary $\partial\Omega$ (as defined in [15, Sections 1.3.3, 1.5]). Testing the variational equality for $\tilde{v}_h$ with $\tilde{v}_h - R_h(v)$ and the variational equality for $R_h(v)$ with $R_h(v) - \tilde{v}_h$ and adding the resulting identities yields

$$\int_\Omega \nabla(\tilde{v}_h - R_h(v)) \cdot \nabla(\tilde{v}_h - R_h(v)) \mathrm{d}\mathcal{L}^2 = \int_\Omega (v - R_h(v))(R_h(v) - \tilde{v}_h) \mathrm{d}\mathcal{L}^2.$$

From the inequality of Poincaré-Wirtinger and the first part of the lemma, we may now deduce that

$$\|\tilde{v}_h - R_h(v)\|_{H^1(\Omega)} \le C \|v - R_h(v)\|_{L^2(\Omega)} \le C h^2$$

holds with some $C > 0$. If we combine the above with (7), the trace theorem, the triangle inequality and the $L^p(\partial\Omega)$-estimate in (6), then the claim follows immediately. □

We may now make the following observation (that has already been made in [7, Lemma 10] for the classical obstacle problem):

Lemma 3.3 *Suppose that u solves* (S) *for some $f \in L^2(\Omega)$. Then, $R_h(u)$ is the unique solution of the variational inequality*

$$\tilde{u}_h \in \tilde{K}_h, \quad (\tilde{u}_h, v_h - \tilde{u}_h)_{H^1(\Omega)} \ge (f, v_h - \tilde{u}_h)_{L^2(\Omega)} \quad \forall v_h \in \tilde{K}_h \tag{$\tilde{\mathrm{S}}_h$}$$

with

$$\tilde{K}_h := \{v_h \in V_h \mid v_h \ge R_h(u) - u \text{ on } \partial\Omega\}.$$

Proof The problem ($\tilde{\mathrm{S}}_h$) admits a unique solution $\tilde{u}_h$ by [18, Theorem 2.1]. To see that this solution is precisely $R_h(u)$, we note that $R_h(u) \in \tilde{K}_h$ and that for all $v_h \in \tilde{K}_h$, i.e., for all $v_h \in V_h$ with $v_h - R_h(u) + u \ge 0$ on $\partial\Omega$, the definition of $R_h(u)$ and the variational inequality (S) yield

$$\begin{aligned}(R_h(u), v_h - R_h(u))_{H^1(\Omega)} &= (u, v_h - R_h(u))_{H^1(\Omega)} \\ &= (u, v_h - R_h(u) + u - u)_{H^1(\Omega)} \\ &\ge (f, v_h - R_h(u) + u - u)_{L^2(\Omega)} \\ &= (f, v_h - R_h(u))_{L^2(\Omega)}.\end{aligned}$$

This proves the claim. □

The above shows that it suffices to study the error that occurs in the solution u_h of (S_h) when the original obstacle (i.e., the zero function) in (S_h) is replaced with the obstacle $R_h(u) - u$ to relate the approximate solution u_h and the Ritz projection $R_h(u)$ of the exact solution u to each other. By pursuing this approach, we obtain the following Céa-type result:

Lemma 3.4 *Let $f \in L^2(\Omega)$ be arbitrary but fixed, and let u and u_h denote the solutions of* (S) *and* (S_h)*, respectively. Then, it holds*

$$\begin{aligned} &\|R_h(u) - u_h\|_{H^1(\Omega)} \\ &\qquad \le \inf\left\{ \|w_h\|_{H^1(\Omega)} \;\middle|\; w_h \in V_h,\; R_h(u) - u \le w_h \le R_h(u) \text{ on } \partial\Omega \right\}. \end{aligned} \tag{8}$$

Proof Consider an arbitrary but fixed $w_h \in V_h$ that is contained in the set on the right-hand side of (8). (Note that this set is not empty since it contains $R_h(u)$.) Since $w_h \ge R_h(u) - u$ on $\partial\Omega$ and since $R_h(u)$ is the solution of ($\tilde{\mathrm{S}}_h$), it holds

$$(R_h(u), v_h - R_h(u))_{H^1(\Omega)} \ge (f, v_h - R_h(u))_{L^2(\Omega)}$$

for all $v_h \in V_h$ with $v_h \ge w_h$ on $\partial\Omega$. In particular, the choice $v_h := u_h + w_h$ yields

$$(R_h(u), R_h(u) - u_h)_{H^1(\Omega)} \le (R_h(u), w_h)_{H^1(\Omega)} + (f, R_h(u) - u_h - w_h)_{L^2(\Omega)}\,.$$

On the other hand, we know that $R_h(u) - w_h \ge R_h(u) - R_h(u) = 0$ on $\partial\Omega$. Thus, we may choose the test function $v_h := R_h(u) - w_h$ in (S_h) to obtain

$$(u_h, u_h - R_h(u))_{H^1(\Omega)} \le (u_h, -w_h)_{H^1(\Omega)} + (f, u_h + w_h - R_h(u))_{L^2(\Omega)}\,.$$

By addition, it now follows that

$$\|R_h(u) - u_h\|^2_{H^1(\Omega)} \le (R_h(u) - u_h, w_h)_{H^1(\Omega)}\,.$$

This proves the claim. □

Note that the above arguments work for all elliptic variational inequalities with unilateral constraints (not just for the Signorini problem). To obtain a tangible a priori H^1-error estimate, it remains to construct a function $w_h \in V_h$ which satisfies $R_h(u) - u \le w_h \le R_h(u)$ on $\partial\Omega$ and which has a small H^1-norm. This is accomplished in the following lemma by means of a unilateral approximation technique that has also been used in [7,8,23,24,34]:

Lemma 3.5

1. *Suppose that $v \in W^{2,p}(\Omega)$, $2 < p < \infty$, is a function with a non-negative trace. Then, for every $h > 0$, we can find a $w_h \in V_h$ with $R_h(v) - v \le w_h \le R_h(v)$ on $\partial\Omega$ such that $\|w_h\|_{H^1(\Omega)} \le Ch^{3/2-1/p}$ holds with a C independent of h.*
2. *Suppose that $v \in H^2(\Omega)$ is a function with a non-negative trace that can be decomposed into two parts v_s and v_r which satisfy the conditions in point 2 of Theorem 2.3 for some $4 < p < \infty$. Then, for every $\varepsilon \in (0, 1/2)$ and every $h > 0$, we can find a $w_h \in V_h$ with $R_h(v) - v \le w_h \le R_h(v)$ on $\partial\Omega$ such that $\|w_h\|_{H^1(\Omega)} \le Ch^{3/2-2/p-\varepsilon}$ holds with a C independent of h.*

Proof We first introduce some notation: Suppose that $h > 0$ is arbitrary but fixed. We denote the nodes of the triangulation $\mathcal{T}_h$ which are contained in the boundary of the square $\Omega = (0,1)^2$ with x_i, $i = 0, ..., N$, starting with $x_0 := (0,0)$ and then proceeding counterclockwise. For convenience, we additionally set $x_{-1} := x_N$ and $x_{N+1} := x_0$. Further, we define σ_i to be the closed line segment $[x_i, x_{i+1}]$, $i = -1, ..., N$, and τ_i to be the mesh cell whose boundary contains σ_i. With C we again denote a generic constant which may change within an estimate but never depends on h. We now proceed in three steps:

Step 1 (h-Independent Morrey Inequality on the Mesh Cells): Consider the reference element $T := \operatorname{conv}\{(0,0),(1,0),(0,1)\}$. Then, we know from the classical Morrey inequality that for every $2 < p < \infty$ there exists a constant $C = C(p,T)$ with

$$\max_{x,y\in T} \frac{|v(x) - v(y)|}{|x-y|^{1-2/p}} \le C\|v\|_{W^{1,p}(T)} \quad \forall v \in W^{1,p}(T).$$

From the inequality of Poincaré-Wirtinger, we obtain further that there exists another $C = C(p,T) > 0$ such that, for all $v \in W^{1,p}(T)$ with average value zero in T, we have

$$\|v\|_{L^p(T)} \le C\|\nabla v\|_{L^p(T)}.$$

By combining the last two inequalities, we may deduce that

$$\max_{x,y\in T} \frac{|v(x) - v(y)|}{|x-y|^{1-2/p}} \le C\|\nabla v\|_{L^p(T)}$$

holds for all $v \in W^{1,p}(T)$ with average value zero on T. Since the seminorms appearing here do not detect constant functions, the last estimate also holds for all $v \in W^{1,p}(T)$. Consider now an arbitrary but fixed mesh cell τ_i, $i \in \{-1, ..., N\}$, and denote with $F_i : T \to \tau_i$, $x \mapsto x_i + G_i x$, the affine linear function with $\det(G_i) > 0$ which maps the reference element T to τ_i. Then, for every $v \in W^{1,p}(\tau_i)$, we obtain

$$\begin{aligned}\max_{x,y\in T} \frac{|v(F_i(x)) - v(F_i(y))|}{|G_i^{-1}(F_i(x) - F_i(y))|^{1-2/p}} &\le C(p,T)\left(\int_T |(\nabla v)(F_i)G_i|^p \,\mathrm{d}\mathcal{L}^2\right)^{1/p} \\ &\le C(p,T)\frac{|G_i|}{\det(G_i)^{1/p}}\left(\int_{\tau_i} |\nabla v|^p \,\mathrm{d}\mathcal{L}^2\right)^{1/p}.\end{aligned}$$

The above yields

$$\max_{x,y\in \tau_i} \frac{|v(x) - v(y)|}{|x-y|^{1-2/p}} \le C(p,T)\frac{|G_i^{-1}|^{1-2/p}|G_i|}{\det(G_i)^{1/p}}\left(\int_{\tau_i} |\nabla v|^p \,\mathrm{d}\mathcal{L}^2\right)^{1/p}.$$

Due to the quasi-uniformity of the underlying family of meshes, we can find a constant C independent of h and i with

$$\frac{|G_i^{-1}|^{1-2/p}|G_i|}{\det(G_i)^{1/p}} \le C\frac{h^{-1+2/p}h}{h^{2/p}} = C.$$

We may thus conclude that there exists a constant $C > 0$ independent of i and h with

$$\max_{x,y\in \tau_i} \frac{|v(x) - v(y)|}{|x-y|^{1-2/p}} \le C\|\nabla v\|_{L^p(\tau_i)} \qquad \forall v \in W^{1,p}(\tau_i), \quad \forall i = -1, ..., N. \tag{9}$$

Step 2 (Proof in the $W^{2,p}$-Case): Suppose now that a function $v \in W^{2,p}(\Omega)$, $2 < p < \infty$, with a non-negative trace is given, let $h > 0$ be arbitrary but fixed, and consider the auxiliary problem

$$\begin{aligned} \min \sum_{i=0}^{N} v_h(x_i), \quad \text{s.t. } v_h \in V_h, \quad R_h(v) - v \le v_h \le R_h(v) \text{ on } \partial\Omega, \\ v_h = 0 \text{ at every interior node of the mesh } \mathcal{T}_h, \end{aligned} \tag{10}$$

where we use the $C(\text{cl}(\Omega))$-representatives of v and $R_h(v)$. Since (10) is a finite-dimensional minimization problem with a non-empty compact admissible set and a continuous objective functional, it admits at least one solution $\tilde{w}_h \in V_h$. Consider now an arbitrary but fixed x_i with $0 \le i \le N$. Then, the fact that $\tilde{w}_h$ solves (10) implies that we cannot reduce the function value $\tilde{w}_h(x_i)$ (while leaving the other nodal values unchanged) without violating the constraint $R_h(v) - v \le \tilde{w}_h$ on $\partial\Omega$. This implies that one of the following three cases has to hold true (as one may easily check by contradiction, cf. Figure 1 below and the analysis in [7]):

1. It holds $\tilde{w}_h(x_i) = R_h(v)(x_i) - v(x_i)$.
2. There exists an $a \in [x_{i-1}, x_i)$ with

$$\begin{aligned} \tilde{w}_h(a) &= R_h(v)(a) - v(a), \\ (\nabla \tilde{w}_h)(a) \cdot (x_i - a) &= \nabla(R_h(v) - v)(a) \cdot (x_i - a). \end{aligned} \tag{11}$$

3. There exists an $a \in (x_i, x_{i+1}]$ satisfying (11).

Here, $[x_{i-1}, x_i)$ and $(x_i, x_{i+1}]$ denote the relatively half-open straight line segments between x_{i-1} and x_i, and x_i and x_{i+1}, respectively, and with $\nabla R_h(v)(a)$ and $\nabla \tilde{w}_h(a)$ we mean the gradient of the respective finite element function on the mesh cell τ_{i-1} in case 2 and on the mesh cell τ_i in case 3. Recall in this context that $W^{2,p}(\Omega) \hookrightarrow C^{1,1-2/p}(\text{cl}(\Omega))$ for $2 < p < \infty$.

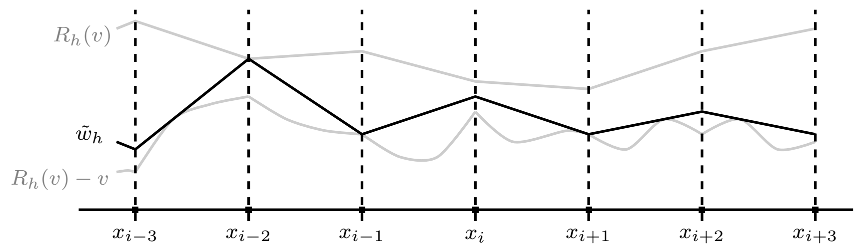

Fig. 1 Prototypical situation on the boundary mesh. The nodes x_{i-1}, x_{i+1} are covered by case 1, the nodes x_{i-2}, x_{i+2}, x_{i+3} are covered by case 2 and the nodes x_{i-3}, x_i, x_{i+1}, x_{i+2} are covered by case 3. For x_i, the point a is identical to the mesh node x_{i+1}.

Note that, in case 1, we trivially have $\tilde{w}_h(x_i) - R_h(v)(x_i) + v(x_i) = 0$. In the second case, we may apply Taylor's formula in the direction of the line segment

$[x_{i-1}, x_i)$ to compute that

$$\begin{aligned}
0 &\leq \tilde{w}_h(x_i) - R_h(v)(x_i) + v(x_i) \\
&= \int_0^1 \nabla(\tilde{w}_h - R_h(v) + v)(a + t(x_i - a)) \cdot (x_i - a)\mathrm{d}t \\
&= \int_0^1 \Big(\nabla v(a + t(x_i - a)) - \nabla v(a)\Big) \cdot (x_i - a)\mathrm{d}t \qquad (12)\\
&\leq Ch^{2-2/p} \int_0^1 \frac{|\nabla v(a + t(x_i - a)) - \nabla v(a)|}{t^{1-2/p}|x_i - a|^{1-2/p}} t^{1-2/p}\mathrm{d}t \\
&\leq Ch^{2-2/p}\|v\|_{W^{2,p}(\tau_{i-1})}
\end{aligned}$$

with some constant C independent of i and h. Here, we have used the properties of a, the regularity of v, the linearity of $\tilde{w}_h$ and $R_h(v)$ on $[x_{i-1}, x_i)$ and (9). Completely analogously, we obtain in the third case that

$$0 \leq \tilde{w}_h(x_i) - R_h(v)(x_i) + v(x_i) \leq Ch^{2-2/p}\|v\|_{W^{2,p}(\tau_i)}.$$

We have now proved that

$$0 \leq \tilde{w}_h(x_i) - R_h(v)(x_i) + I_h(v)(x_i) \leq Ch^{2-2/p}\Big(\|v\|^p_{W^{2,p}(\tau_{i-1})} + \|v\|^p_{W^{2,p}(\tau_i)}\Big)^{1/p}$$

holds for all $i = 0, ..., N$, where I_h denotes the Lagrange interpolation operator. Consider now an arbitrary but fixed $i \in \{0, ..., N\}$. Then, it follows

$$\begin{aligned}
&\int_{\sigma_i} |\tilde{w}_h - R_h(v) + I_h(v)|^p \mathrm{d}\mathcal{H}^1 \\
&\leq |x_i - x_{i+1}| \|\tilde{w}_h - R_h(v) + I_h(v)\|^p_{L^\infty(\sigma_i)} \\
&\leq Ch^{2p-1}\Big(\|v\|^p_{W^{2,p}(\tau_{i-1})} + \|v\|^p_{W^{2,p}(\tau_i)} + \|v\|^p_{W^{2,p}(\tau_{i+1})}\Big)
\end{aligned}$$

and we may deduce by summation that

$$\|\tilde{w}_h\|_{L^p(\partial\Omega)} \leq Ch^{2-1/p}\|v\|_{W^{2,p}(\Omega)} + \|v - R_h(v)\|_{L^p(\partial\Omega)} + \|v - I_h(v)\|_{L^p(\partial\Omega)}.$$

Using the inverse estimate in [20, Equation (3.1)], again [15, Theorem 1.5.1.10] (with parameter h^p), Lemma 3.2 and standard error estimates for the Lagrange interpolation operator as found in [4, Theorem 4.4.20], we now obtain

$$\begin{aligned}
&\|\tilde{w}_h\|_{H^{1/2}(\partial\Omega)} \\
&\leq Ch^{-1/2}\|\tilde{w}_h\|_{L^2(\partial\Omega)} \\
&\leq Ch^{-1/2}\|\tilde{w}_h\|_{L^p(\partial\Omega)} \\
&\leq Ch^{3/2-1/p}\|v\|_{W^{2,p}(\Omega)} + Ch^{-1/2}\left(\|v - R_h(v)\|_{L^p(\partial\Omega)} + \|v - I_h(v)\|_{L^p(\partial\Omega)}\right) \\
&\leq Ch^{3/2-1/p}\|v\|_{W^{2,p}(\Omega)} \\
&\qquad + Ch^{-1/2-1/p}\left(h\|\nabla v - \nabla I_h(v)\|_{L^p(\Omega)} + \|v - I_h(v)\|_{L^p(\Omega)}\right) \\
&\leq Ch^{3/2-1/p}\|v\|_{W^{2,p}(\Omega)}
\end{aligned}$$

with some constant $C > 0$ which may change from step to step but is always independent of h. To construct a function $w_h \in V_h$ with the desired properties, it remains to extend $\mathrm{tr}(\tilde{w}_h)$ suitably to a function in V_h. This can be accomplished, e.g., by employing the discrete harmonic extension operator $E_h : \mathrm{tr}(V_h) \to V_h$, which, according to [20, Lemma 3.2], satisfies

$$\|E_h(v_h)\|_{H^1(\Omega)} \le C\|v_h\|_{H^{1/2}(\partial\Omega)} \qquad \forall v_h \in \mathrm{tr}(V_h)$$

for some constant $C > 0$ independent of h and v_h.

Step 3 (Proof in the v_s-v_r-Case): For a function v with a non-negative trace that can be decomposed into two parts v_s and v_r which satisfy the conditions in point 2 of Theorem 2.3 for some $4 < p < \infty$, we can proceed completely analogously to Step 2 to construct a function $\tilde{w}_h \in V_h$ with $R_h(v) - v \le \tilde{w}_h \le R_h(v)$ on $\partial\Omega$ which satisfies either $\tilde{w}_h(x_i) - R_h(v)(x_i) + v(x_i) = 0$ or one of the cases 2 and 3 at each node x_i, $i = 0, ..., N$. Let us again consider case 2, fix an $\varepsilon \in (0, 1/2)$, write $q := 2/(1 + 2\varepsilon) \in (1, 2)$ and assume w.l.o.g. that the line segment σ_{i-1} is contained in $\mathbb{R} \times \{0\}$ so that $a = (\bar{a}, 0)$, $x_{i-1} = (\bar{x}_{i-1}, 0)$ and $x_i = (\bar{x}_i, 0)$ with $\bar{a}, \bar{x}_{i-1}, \bar{x}_i \in \mathbb{R}$. Then, we may use the same calculation as in (12), the regularity properties of v_s and v_r and Morrey's inequality to obtain

$$\begin{aligned}
0 &\le \tilde{w}_h(x_i) - R_h(v)(x_i) + v(x_i)\\
&= \int_0^1 \Big(\nabla v(a + t(x_i - a)) - \nabla v(a)\Big) \cdot (x_i - a)\mathrm{d}t\\
&= \int_0^1 \Big(\partial_1 v(\bar{a} + t(\bar{x}_i - \bar{a}), 0) - \partial_1 v(\bar{a}, 0)\Big)(\bar{x}_i - \bar{a})\mathrm{d}t\\
&\le \int_{\bar{a}}^{\bar{x}_i} \Big(\partial_1 v_s(t, 0) - \partial_1 v_s(\bar{a}, 0)\Big)\mathrm{d}t + Ch^{2-2/p}\|v_r\|_{C^{1,1-2/p}(\Omega)}\\
&\le \int_{\bar{a}}^{\bar{x}_i} \Big|\partial_1^2 v_s(t, 0)(t - \bar{x}_i)\Big| \mathrm{d}t + Ch^{2-2/p}\|v_r\|_{C^{1,1-2/p}(\Omega)}\\
&\le \left(\int_{\bar{a}}^{\bar{x}_i} (t - \bar{x}_i)^{\frac{q}{q-1}} \mathrm{d}t\right)^{\frac{q-1}{q}} \|\mathrm{tr}(v_s)\|_{W^{2,q}(\sigma_{i-1})} + Ch^{2-2/p}\|v_r\|_{C^{1,1-2/p}(\Omega)}\\
&\le Ch^{2-1/q}\|\mathrm{tr}(v_s)\|_{W^{2,q}(\sigma_{i-1})} + Ch^{2-2/p}\|v_r\|_{C^{1,1-2/p}(\Omega)}.
\end{aligned}$$

If we use exactly the same strategy in case 3, then it follows that $\tilde{w}_h$ satisfies

$$\begin{aligned}
0 &\le \tilde{w}_h(x_i) - R_h(v)(x_i) + I_h(v)(x_i)\\
&\le Ch^{2-1/q} \Big(\|\mathrm{tr}(v_s)\|^q_{W^{2,q}(\sigma_{i-1})} + \|\mathrm{tr}(v_s)\|^q_{W^{2,q}(\sigma_i)}\Big)^{1/q}\\
&\qquad + Ch^{2-2/p}\|v_r\|_{C^{1,1-2/p}(\Omega)}
\end{aligned}$$

for all $i = 0, ..., N$, where I_h again denotes the Lagrange interpolation operator. By integration, we may now again deduce (using the estimate $(a + b)^q \le C(a^q + b^q)$)

$$\begin{aligned}
&\int_{\sigma_i} |\tilde{w}_h - R_h(v) + I_h(v)|^q \mathrm{d}\mathcal{H}^1\\
&\le Ch^{2q} \Big(\|\mathrm{tr}(v_s)\|^q_{W^{2,q}(\sigma_{i-1})} + \|\mathrm{tr}(v_s)\|^q_{W^{2,q}(\sigma_i)} + \|\mathrm{tr}(v_s)\|^q_{W^{2,q}(\sigma_{i+1})}\Big)\\
&\qquad + Ch^{2q+1-2q/p}\|v_r\|^q_{C^{1,1-2/p}(\Omega)}
\end{aligned}$$

and, by summation,

$$\begin{aligned}&\|\tilde{w}_h - R_h(v) + I_h(v)\|_{L^q(\partial\Omega)}\\ &\le Ch^{2-2/p}\left(h^{2q/p}\|\operatorname{tr}(v_s)\|^q_{W^{2,q}(\partial\Omega)} + \|v_r\|^q_{C^{1,1-2/p}(\Omega)}\right)^{1/q}.\end{aligned}$$

Combining the above with Lemma 3.2, inverse estimates (cf. [20, Equation (3.1)]), the definition of q and standard results for the Lagrange interpolant yields

$$\begin{aligned}\|\tilde{w}_h\|_{H^{1/2}(\partial\Omega)} &\le \|\tilde{w}_h - R_h(v) + I_h(v)\|_{H^{1/2}(\partial\Omega)} + \|I_h(v_r) - R_h(v_r)\|_{H^{1/2}(\partial\Omega)}\\ &\qquad + \|I_h(v_s) - R_h(v_s)\|_{H^{1/2}(\partial\Omega)}\\ &\le h^{-1/q}\|\tilde{w}_h - R_h(v) + I_h(v)\|_{L^q(\partial\Omega)} + Ch^{3/2-1/p} + Ch^{3/2-\varepsilon}\\ &\le Ch^{2-1/q-2/p} + Ch^{3/2-1/p} + Ch^{3/2-\varepsilon}\\ &\le Ch^{3/2-2/p-\varepsilon} + Ch^{3/2-1/p} + Ch^{3/2-\varepsilon}\end{aligned}$$

with a constant C which may depend on ε but is independent of h. The claim now follows completely analogously to Step 2. □

We may now combine Lemmas 3.4 and 3.5 to arrive at the following main result of this section:

Theorem 3.6 (Supercloseness) *Suppose that u solves* (S) *for some $f \in L^2(\Omega)$. Then, the following holds true for the Ritz projection $R_h(u)$ and the finite element solution u_h of* (S_h)*:*

1. *If u satisfies $u \in W^{2,p}(\Omega)$ for some $2 < p < 4$, then there exists a constant $C > 0$ independent of h with*
$$\|R_h(u) - u_h\|_{H^1(\Omega)} \le Ch^{3/2-1/p}.$$
2. *If u admits a decomposition $u = u_s + u_r$ as in point 2 of Theorem 2.3 for some $4 < p < \infty$, then, for every $\varepsilon \in (0, 1/2)$, there exists a constant $C > 0$ independent of h with*
$$\|R_h(u) - u_h\|_{H^1(\Omega)} \le Ch^{3/2-2/p-\varepsilon}.$$

Note that Theorem 3.6 accords very well with intuition. Since the additional constraint in (S_h) is only present on the boundary, it is only natural that the finite element solution u_h is very close to the Ritz projection $R_h(u)$ which corresponds to the solution of the unconstrained problem.

4 Consequences for $W^{1,p}$-, L^p- and $H^{1/2}$-Error Estimates

Theorem 3.6 gives rise to various error estimates in a very straightforward manner:

Corollary 4.1 *Suppose that u solves* (S) *for some $f \in L^2(\Omega)$. Assume further that there exists a $2 < p < 4$ with $u \in W^{2,p}(\Omega)$. Then, for every $1 < q < \infty$, there exists a constant $C > 0$ independent of h with*

$$\|u - u_h\|_{W^{1,\frac{4p}{p+2}}(\Omega)} \le Ch, \qquad \|u - u_h\|_{W^{1,\infty}(\Omega)} \le Ch^{1/2-1/p},$$
$$\|u - u_h\|_{L^q(\Omega)} \le Ch^{3/2-1/p}, \qquad \|u - u_h\|_{L^\infty(\Omega)} \le C|\ln(h)|^{1/2}h^{3/2-1/p},$$
$$\|u - u_h\|_{H^{1/2}(\partial\Omega)} \le Ch^{3/2-1/p}.$$

Proof From Theorem 3.6, the inverse estimates in [4, Theorem 4.5.11], and Lemma 3.2, it follows

$$\begin{aligned}\|u-u_h\|_{W^{1,\frac{4p}{p+2}}(\Omega)} &\le \|R_h(u)-u_h\|_{W^{1,\frac{4p}{p+2}}(\Omega)} + \|u-R_h(u)\|_{W^{1,\frac{4p}{p+2}}(\Omega)} \\ &\le Ch^{\frac{p+2}{2p}-1}\|R_h(u)-u_h\|_{H^1(\Omega)} + C\|u-R_h(u)\|_{W^{1,p}(\Omega)} \\ &\le Ch^{\frac{p+2}{2p}-1+3/2-1/p} + Ch \\ &\le Ch.\end{aligned}$$

This proves the first estimate. Similarly, we may compute (using standard error estimates for the Lagrange interpolant $I_h(u)$, see [4, Theorem 4.4.20], Sobolev embeddings and again [4, Theorem 4.5.11] and Lemma 3.2) that

$$\begin{aligned}&\|u-u_h\|_{W^{1,\infty}(\Omega)} \\ &\le \|R_h(u)-u_h\|_{W^{1,\infty}(\Omega)} + \|I_h(u)-R_h(u)\|_{W^{1,\infty}(\Omega)} + \|u-I_h(u)\|_{W^{1,\infty}(\Omega)} \\ &\le Ch^{-1}\|R_h(u)-u_h\|_{H^1(\Omega)} + Ch^{-2/p}\|I_h(u)-R_h(u)\|_{W^{1,p}(\Omega)} + Ch^{1-2/p} \\ &\le Ch^{1/2-1/p} + Ch^{1-2/p} + Ch^{1-2/p} \\ &\le Ch^{1/2-1/p}\end{aligned}$$

and

$$\begin{aligned}&\|u-u_h\|_{L^q(\Omega)} \\ &\le \|R_h(u)-u_h\|_{L^q(\Omega)} + C\|I_h(u)-R_h(u)\|_{L^\infty(\Omega)} + C\|u-I_h(u)\|_{L^\infty(\Omega)} \\ &\le C\|R_h(u)-u_h\|_{H^1(\Omega)} + Ch^{-2/p}\|I_h(u)-R_h(u)\|_{L^p(\Omega)} + Ch^{2-2/p} \\ &\le Ch^{3/2-1/p} + Ch^{2-2/p} \\ &\le Ch^{3/2-1/p}\end{aligned}$$

holds for all $1 < q < \infty$. Further, we may use the discrete Sobolev inequality in [4, Lemma 4.9.2] and exactly the same arguments as above to obtain

$$\begin{aligned}&\|u-u_h\|_{L^\infty(\Omega)} \\ &\le \|R_h(u)-u_h\|_{L^\infty(\Omega)} + \|I_h(u)-R_h(u)\|_{L^\infty(\Omega)} + \|u-I_h(u)\|_{L^\infty(\Omega)} \\ &\le C|\ln(h)|^{1/2}\|R_h(u)-u_h\|_{H^1(\Omega)} + Ch^{2-2/p} \\ &\le C|\ln(h)|^{1/2}h^{3/2-1/p}.\end{aligned}$$

It remains to prove the $H^{1/2}(\partial\Omega)$-error estimate. To this end, we define ψ_h to be the unique element of V_h, which is one at every boundary node and zero at every interior node, and $\mathrm{supp}(\psi_h)$ to be the support of ψ_h. We may now use the classical trace theorem and Hölder's inequality to infer that

$$\begin{aligned}&\|u-I_h(u)\|^2_{H^{1/2}(\partial\Omega)} \\ &\le C\|\psi_h(u-I_h(u))\|^2_{H^1(\Omega)} \\ &\le C\left(\|\nabla\psi_h\|^2_{L^\infty(\Omega)}\|u-I_h(u)\|^2_{L^2(\mathrm{supp}(\psi_h))} + \|\nabla(u-I_h(u))\|^2_{L^2(\mathrm{supp}(\psi_h))}\right) \\ &\le Ch^2|u|^2_{H^2(\mathrm{supp}(\psi_h))} \\ &\le Ch^2|u|^2_{W^{2,p}(\mathrm{supp}(\psi_h))}\mathcal{L}^2(\mathrm{supp}(\psi_h))^{1-2/p} \\ &\le Ch^{3-2/p}.\end{aligned} \tag{13}$$

Now we may proceed as before (using Lemma 3.2, [20, Equation (3.1)], the trace theorem and again [15, Theorem 1.5.1.10] with parameter $\varepsilon := h^p$) to obtain

$$\begin{aligned}
&\|u - u_h\|_{H^{1/2}(\partial\Omega)} \\
&\leq \|R_h(u) - u_h\|_{H^{1/2}(\partial\Omega)} + \|R_h(u) - I_h(u)\|_{H^{1/2}(\partial\Omega)} + \|I_h(u) - u\|_{H^{1/2}(\partial\Omega)} \\
&\leq C\|R_h(u) - u_h\|_{H^1(\Omega)} + Ch^{-1/2}\|R_h(u) - I_h(u)\|_{L^p(\partial\Omega)} + Ch^{3/2-1/p} \\
&\leq Ch^{3/2-1/p}.
\end{aligned}$$

This proves the claim. □

Corollary 4.2 *Suppose that u solves* (S) *for some $f \in L^2(\Omega)$. Assume further that u admits a decomposition $u = u_s + u_r$ as in point 2 of Theorem 2.3 for some $4 < p < \infty$. Then, for all $\varepsilon \in (0, 1/2)$, there exists a constant $C > 0$ independent of h with*

$$\|u - u_h\|_{W^{1,\frac{4p}{p+4}}(\Omega)} \leq Ch^{1-\varepsilon}, \qquad \|u - u_h\|_{W^{1,\infty}(\Omega)} \leq Ch^{1/2-2/p-\varepsilon},$$
$$\|u - u_h\|_{L^\infty(\Omega)} \leq Ch^{3/2-2/p-\varepsilon}, \qquad \|u - u_h\|_{H^{1/2}(\partial\Omega)} \leq Ch^{3/2-2/p-\varepsilon}.$$

Proof The proof is completely along the lines of that of the last corollary and only requires some minor modifications. We include it for the sake of completeness: Note that the regularity properties of u_s and u_r imply that $u \in W^{2,q}(\Omega)$ holds for all $q \in (2, 4)$, cf. [12, Theorem 6.5]. Consider now an arbitrary but fixed $\varepsilon \in (0, 1/2)$. Then, we may invoke Theorem 3.6 and compute (using the same ideas as before)

$$\begin{aligned}
\|u - u_h\|_{W^{1,\frac{4p}{p+4}}(\Omega)} &\leq \|R_h(u) - u_h\|_{W^{1,\frac{4p}{p+4}}(\Omega)} + \|u - R_h(u)\|_{W^{1,\frac{4p}{p+4}}(\Omega)} \\
&\leq Ch^{\frac{p+4}{2p}-1}\|R_h(u) - u_h\|_{H^1(\Omega)} + Ch \\
&\leq Ch^{\frac{p+4}{2p}-1+3/2-2/p-\varepsilon} + Ch \\
&\leq Ch^{1-\varepsilon}
\end{aligned}$$

and

$$\begin{aligned}
&\|u - u_h\|_{W^{1,\infty}(\Omega)} \\
&\leq \|R_h(u) - u_h\|_{W^{1,\infty}(\Omega)} + \|I_h(u) - R_h(u)\|_{W^{1,\infty}(\Omega)} + \|u - I_h(u)\|_{W^{1,\infty}(\Omega)} \\
&\leq Ch^{-1}\|R_h(u) - u_h\|_{H^1(\Omega)} + Ch^{-\frac{p+4}{2p}}\|I_h(u) - R_h(u)\|_{W^{1,\frac{4p}{p+4}}(\Omega)} + Ch^{1-\frac{p+4}{2p}} \\
&\leq Ch^{1/2-2/p-\varepsilon} + Ch^{1-\frac{p+4}{2p}} + Ch^{1-\frac{p+4}{2p}} \\
&\leq Ch^{1/2-2/p-\varepsilon}.
\end{aligned}$$

Analogously (by [4, Lemma 4.9.2]),

$$\begin{aligned}
&\|u - u_h\|_{L^\infty(\Omega)} \\
&\leq \|R_h(u) - u_h\|_{L^\infty(\Omega)} + \|I_h(u) - R_h(u)\|_{L^\infty(\Omega)} + \|u - I_h(u)\|_{L^\infty(\Omega)} \\
&\leq C|\ln(h)|^{1/2}\|R_h(u) - u_h\|_{H^1(\Omega)} + Ch^{-\frac{p+4}{2p}}\|I_h(u) - R_h(u)\|_{L^{\frac{4p}{p+4}}(\Omega)} + Ch^{2-\frac{p+4}{2p}} \\
&\leq C|\ln(h)|^{1/2}h^{3/2-2/p-\varepsilon} + Ch^{2-\frac{p+4}{2p}} \\
&\leq Ch^{3/2-2/p-2\varepsilon}.
\end{aligned}$$

(Note that the coefficient of ε in the exponent is completely unimportant here since we may always redefine ε.) Finally, we may compute (using the trace theorem, Lemma 3.2 and again (13))

$$\begin{aligned}
&\|u - u_h\|_{H^{1/2}(\partial\Omega)} \\
&\le \|R_h(u) - u_h\|_{H^{1/2}(\partial\Omega)} + \|R_h(u) - u\|_{H^{1/2}(\partial\Omega)} \\
&\le C\|R_h(u) - u_h\|_{H^1(\Omega)} + \|R_h(u_s) - u_s\|_{H^{1/2}(\partial\Omega)} \\
&\qquad + \|R_h(u_r) - I_h(u_r)\|_{H^{1/2}(\partial\Omega)} + \|u_r - I_h(u_r)\|_{H^{1/2}(\partial\Omega)} \\
&\le Ch^{3/2-2/p-\varepsilon} + Ch^{3/2-\varepsilon} + Ch^{-1/2}\|R_h(u_r) - I_h(u_r)\|_{L^p(\partial\Omega)} + Ch^{3/2-1/p} \\
&\le Ch^{3/2-2/p-\varepsilon}.
\end{aligned}$$

This completes the proof. □

Note that the regularity assumptions in Corollaries 4.1 and 4.2 fit precisely to what we have proved in Theorem 2.3. This implies in particular that the last two results remain valid when we replace the appearing regularity assumptions on u with the respective assumptions on f and the contact set $\{x \in \partial\Omega \mid u(x) = 0\}$ in Theorem 2.3. If we do this, then we arrive, e.g., at the following result:

Corollary 4.3 *Suppose that u solves* (S) *for some $f \in L^\infty(\Omega)$ and that (A) is satisfied. Then, for all $\varepsilon \in (0, 1/2)$, there exists a $C > 0$ independent of h with*

$$\begin{aligned}
&\|u - u_h\|_{W^{1,8/3-\varepsilon}(\Omega)} \le Ch, \qquad &&\|u - u_h\|_{W^{1,4}(\Omega)} \le Ch^{1-\varepsilon}, \\
&\|u - u_h\|_{W^{1,\infty}(\Omega)} \le Ch^{1/2-\varepsilon}, \qquad &&\|u - u_h\|_{L^\infty(\Omega)} \le Ch^{3/2-\varepsilon}, \\
&\qquad \|u - u_h\|_{H^{1/2}(\partial\Omega)} \le Ch^{3/2-\varepsilon}.
\end{aligned} \tag{14}$$

Proof Theorem 2.3 yields that we may apply Corollary 4.2 with an arbitrarily large $p > 4$ and Corollary 4.1 with a p which is arbitrarily close to four. The claim now follows immediately by invoking these results and by noting that, for all $\varepsilon \in (0, 1/2)$, we have (due to the $W^{1,\infty}$-estimate and the $W^{1,q}$-estimate for all exponents $2 < q < 4$)

$$\|u - u_h\|_{W^{1,4}(\Omega)} \le C\left(\|u - u_h\|_{W^{1,4-\varepsilon}(\Omega)}\right)^{\frac{4-\varepsilon}{4}} \le Ch^{(1-\varepsilon)\frac{4-\varepsilon}{4}} = Ch^{1-5\varepsilon/4+\varepsilon^2/4}$$

with some C independent of h. This completes the proof. □

Some remarks are in order regarding the last results:

Remark 4.4

1. *Since the solution u of* (S) *cannot be expected to possess more regularity than*

$$u \in W^{2,4-\varepsilon}(\Omega) \quad \forall \varepsilon \in (0, 1/2) \quad \textit{and} \quad u \in H^{5/2-\varepsilon}(\Omega) \quad \forall \varepsilon \in (0, 1/2)$$

for $f \in L^\infty(\Omega)$, cf. the comments at the end of Section 2.2 and Theorem 2.3, and since we consider piecewise linear ansatz functions, the $W^{1,8/3-\varepsilon}$-, the $W^{1,4}$-, the $W^{1,\infty}$-, the L^∞- and the $H^{1/2}$-error estimate in (14) *are optimal. Compare, e.g., with the classical results for the Lagrange interpolation operator in [4, Theorem 4.4.20] or with the estimates in Lemma 3.2 in this context.*

2. *The H^1-estimate $\|u - u_h\|_{H^1(\Omega)} \leq Ch$, that follows from the first line in* (14), *has also been obtained, e.g., in [5, Lemma 6.1] under similar assumptions on the contact set. Only recently it was shown in [11] that this error estimate also holds in general. To the authors' best knowledge, the $W^{1,p}$-error estimates in* (14) *are new. The same seems to be the case for the L^∞- and the $W^{1,\infty}$-error estimate in* (14) *and the results in Corollaries 4.1 and 4.2. Note that we obtain the L^∞-error estimate without invoking the discrete maximum principle and without the associated assumptions on $\mathcal{T}_h$, cf. [7,9,27].*
3. *The $H^{1/2}$-estimate in* (14) *has already been obtained in [33, Theorem 2.2] in dimensions two and three under the assumption that u is in $H^{5/2-\varepsilon}(\Omega)$ for all $\varepsilon \in (0, 1/2)$. Note that this regularity can only be expected if $f \in H^{1/2-\varepsilon}(\Omega)$ holds for all $\varepsilon \in (0, 1/2)$. In our analysis, we may work with the assumptions $f \in L^p(\Omega)$ or $f \in L^\infty(\Omega)$, respectively, since we have exploited in more detail the properties of the regular and the singular part of u in Theorem 2.3. Further, we obtain the $H^{1/2}$-estimate in* (14) *with arguments that seem to be more elementary than those in [33]. However, in contrast to the analysis in [33], our approach cannot be extended straightforwardly to the three-dimensional setting since unilateral approximation results analogous to those in Lemma 3.5 are only available in limited form in higher dimensions, cf. [7].*
4. *In [33], an L^2-error estimate of order $3/2 - \varepsilon$ is obtained as a corollary of the $H^{1/2}$-error estimate on the boundary. We obtain this order of convergence even in the L^∞-norm.*

As the reader might have noticed, Corollary 4.3 only yields the suboptimal order of convergence $3/2 - \varepsilon$ for, e.g., the L^4-error. We thus miss a factor $h^{1/2}$ in comparison with the approximation properties of the Lagrange interpolation operator. In what follows, we demonstrate that a better estimate can be obtained with a non-standard duality argument, and that the order two (minus epsilon), that is observed in numerical experiments, can also be recovered analytically under reasonable assumptions on the solution u and the approximations u_h.

5 L^4-Error Estimates of Optimal Order via an Aubin-Nitsche Trick

To estimate the L^4-error, we use an approach that has been proposed by Mosco in [24, Section 7] for the one-dimensional obstacle problem and consider two dual variational inequalities - one for each of the components $(u-u_h)^+ = \max(0, u-u_h)$ and $(u - u_h)^- = \min(0, u - u_h)$.

5.1 A Duality Argument for the Component $(u - u_h)^+$

To formulate our first dual problem, we introduce the following notation:

Definition 5.1 *Given a $u \in H^2(\Omega)$ which solves* (S) *for some $f \in L^2(\Omega)$ and which satisfies condition (A), we define:*

1. *$\mathcal{A}^\circ \subset \partial\Omega$ to be the relative interior of the contact set $\{x \in \partial\Omega \mid u(x) = 0\}$,*
2. *$\mathcal{A}_h^\circ \subset \partial\Omega$ to be the union of all (closed) cells of the boundary mesh which intersect $\mathcal{A}^\circ$.*

Note that, according to condition (A), at least for all sufficiently small h, the number of connected components of $\mathcal{A}_h^\circ$ is finite and precisely the number of connected components of $\mathcal{A}^\circ$. Given a u which solves (S) and satisfies (A) and a solution u_h of (S$_h$), we now consider the following auxiliary problem:

$$z \in L, \quad (z, v - z)_{H^1(\Omega)} \geq (-\max(0, u - u_h)^3, v - z)_{L^2(\Omega)} \quad \forall v \in L. \tag{D}$$

Here,

$$L := \left\{ v \in H^1(\Omega) \,\middle|\, \operatorname{tr}(v) \geq 0 \ \mathcal{H}^1\text{-a.e. on } \mathcal{A}_h^\circ \right\}.$$

Note that the solution z of (D) depends on h (since $\mathcal{A}_h^\circ$ does). From standard results and the analysis in [15], we may deduce:

Lemma 5.2 *Suppose that u solves* (S) *for some $f \in L^2(\Omega)$, that condition (A) holds, and that u_h is the solution of* (S$_h$)*. Then, the problem* (D) *admits a unique solution $z \in H^1(\Omega)$ for all $h > 0$. This solution satisfies $z \leq 0$ $\mathcal{L}^2$-a.e. in Ω and $\operatorname{tr}(z) = 0$ $\mathcal{H}^1$-a.e. on $\mathcal{A}_h^\circ$, and, for every $\varepsilon \in (0, 1/2)$, there exists a constant $C > 0$ independent of h with*

$$\|z\|_{H^1(\Omega)} \leq C \|\max(0, u - u_h)^3\|_{L^{(4-\varepsilon)/3}(\Omega)}. \tag{15}$$

Moreover, for all $\varepsilon \in (0, 1/2)$ and all sufficiently small $h > 0$, z is in $W^{2,(4-\varepsilon)/3}(\Omega)$ and satisfies

$$\|z\|_{W^{2,(4-\varepsilon)/3}(\Omega)} \leq C \|\max(0, u - u_h)^3\|_{L^{(4-\varepsilon)/3}(\Omega)} \tag{16}$$

with some constant $C > 0$ independent of h.

Proof The unique solvability of (D) for all $h > 0$ follows from [18, Theorem 2.1]. Further, we may employ Stampacchia's lemma, see [2, Theorem 5.8.2], and use the test function $v := z^- \in L$ in (D) to deduce that

$$0 \geq -(\max(0, u - u_h)^3, z^+)_{L^2(\Omega)} \geq (z, z^+)_{H^1(\Omega)} = \|z^+\|_{H^1(\Omega)}^2.$$

This proves that we indeed have $z \leq 0$ $\mathcal{L}^2$-a.e. in Ω and, as a consequence, that $\operatorname{tr}(z) = 0$ holds $\mathcal{H}^1$-a.e. on $\mathcal{A}_h^\circ$. Moreover, by choosing the test functions $v = 0$ and $v = 2z$ in (D), and by exploiting the Sobolev embeddings, we obtain

$$\begin{aligned}
\|z\|_{H^1(\Omega)}^2 &= (-\max(0, u - u_h)^3, z)_{L^2(\Omega)} \\
&\leq \|\max(0, u - u_h)^3\|_{L^{(4-\varepsilon)/3}(\Omega)} \|z\|_{L^{(4-\varepsilon)/(1-\varepsilon)}(\Omega)} \\
&\leq C \|\max(0, u - u_h)^3\|_{L^{(4-\varepsilon)/3}(\Omega)} \|z\|_{H^1(\Omega)} \quad \forall \varepsilon \in (0, 1/2),
\end{aligned}$$

where C is the embedding constant of $H^1(\Omega) \hookrightarrow L^{(4-\varepsilon)/(1-\varepsilon)}(\Omega)$. This yields (15). It remains to prove the $W^{2,(4-\varepsilon)/3}$-regularity of z and (16). To this end, we note that the non-positivity of z in Ω, the condition $\operatorname{tr}(z) \geq 0$ on $\mathcal{A}_h^\circ$ and the properties of the set $\mathcal{A}_h^\circ$ imply that z is the unique weak solution of the problem

$$\begin{aligned}
-\Delta z &= -\max(0, u - u_h)^3 - z \ \mathcal{L}^2\text{-a.e. in } \Omega, \\
z &= 0 \ \mathcal{H}^1\text{-a.e. on } \mathcal{A}_h^\circ, \\
\partial_n z &= 0 \ \mathcal{H}^1\text{-a.e. on } \partial\Omega \setminus \mathcal{A}_h^\circ.
\end{aligned}$$

Since $\mathcal{A}_h^\circ$ and its complement can be written as the union of at most finitely many straight line segments which meet at an angle $\pi/2$ or π, we may again invoke [15, Theorem 4.4.3.7] to deduce that $z \in W^{2,(4-\varepsilon)/3}(\Omega)$ holds for all $\varepsilon \in (0, 1/2)$. To obtain the estimate (16), let us assume that $\mathcal{A}^\circ \neq \emptyset$ and $\mathcal{A}^\circ \neq \partial\Omega$ (else the proof is trivial). In this case, condition (A) implies that $\mathcal{A}^\circ$ consists of finitely many connected components, that the relative boundary of $\mathcal{A}^\circ$ in $\partial\Omega$ consists of finitely many points $b_1, ..., b_N$, $N \in \mathbb{N}$, and that we may find a $\delta > 0$ with $\mathrm{dist}(b_i, b_j) > 4\delta$ for all $i \neq j$ and $\mathrm{dist}(b_i, \{(0,0), (0,1), (1,0), (1,1)\}) > 4\delta$ for all b_i which are not themselves corner-points of the square Ω. Choose rotationally symmetric cut-off functions $\psi_i \in C_c^\infty(\mathbb{R}^2)$, $i = 1, ..., N$, such that

$$0 \leq \psi_i \leq 1, \qquad \mathrm{supp}(\psi_i) \subset B_{2\delta}(b_i), \qquad \psi_i \equiv 1 \text{ in } B_\delta(b_i)$$

holds for all $i = 1, ..., N$, where $B_r(b)$ denotes the closed ball of radius $r > 0$ around a $b \in \mathbb{R}^2$, and decompose z into the parts $z_0, z_1, ..., z_N$ defined by

$$z_i := \psi_i z \text{ for } i = 1, ..., N, \qquad z_0 := \psi_0 z, \qquad \psi_0 := 1 - \sum_{i=1}^N \psi_i.$$

Suppose further that h is so small that the set $\mathcal{A}_h^\circ \setminus \mathcal{A}^\circ$ is contained in the union of the balls $B_\delta(b_i)$, $i = 1, ..., N$. (This is the case for all sufficiently small h due to the definition of $\mathcal{A}_h^\circ$.) Then, δ and the functions ψ_i are clearly independent of h, and we may compute that

$$\begin{aligned} -\Delta z_0 = -z\Delta\psi_0 - 2\nabla z \cdot \nabla\psi_0 - \psi_0 \max(0, u - u_h)^3 - z_0 \ \mathcal{L}^2\text{-a.e. in } \Omega, \\ z_0 = 0 \ \mathcal{H}^1\text{-a.e. on } \mathcal{A}^\circ, \\ \partial_n z_0 = 0 \ \mathcal{H}^1\text{-a.e. on } \partial\Omega \setminus \mathcal{A}^\circ. \end{aligned}$$

Here, we have used that $\mathcal{A}^\circ \subset \mathcal{A}_h^\circ$, and that the rotational symmetry of the cut-off functions and the choice of δ imply $\partial_n \psi_i \equiv 0$ on $\partial\Omega$ for all i. Note that the boundary conditions in the above problem are independent of h. We may thus invoke [15, Theorem 4.3.2.4] and (15) to deduce that, for every $\varepsilon \in (0, 1/2)$, we have

$$\begin{aligned} &\|z_0\|_{W^{2,(4-\varepsilon)/3}(\Omega)} \\ &\leq C \left(\|z\Delta\psi_0 + 2\nabla z \cdot \nabla\psi_0 + \psi_0 \max(0, u - u_h)^3\|_{L^{(4-\varepsilon)/3}(\Omega)} + \|z_0\|_{W^{1,(4-\varepsilon)/3}(\Omega)} \right) \\ &\leq C \left(\|\max(0, u - u_h)^3\|_{L^{(4-\varepsilon)/3}(\Omega)} + \|z\|_{W^{1,(4-\varepsilon)/3}(\Omega)} \right) \\ &\leq C \|\max(0, u - u_h)^3\|_{L^{(4-\varepsilon)/3}(\Omega)}. \end{aligned}$$

Here, $C > 0$ is a generic constant which depends on ε and ψ_0 but not on h and which may change from step to step. Consider now a point b_i of the relative boundary of $\mathcal{A}^\circ$ which is not a corner point of the square Ω, w.l.o.g. $b_i = (a, 0)$ for some $a \in (0, 1)$. Then, it follows from our choice of δ that $(a - 4\delta, a + 4\delta) \times \{0\}$ is a subset of $\partial\Omega$ and does not contain a further point of the relative boundary of $\mathcal{A}^\circ$. This implies in particular that exactly one of the sets $(a - 4\delta, a) \times \{0\}$ and $(a, a + 4\delta) \times \{0\}$ is contained in $\mathcal{A}^\circ$. Let us assume w.l.o.g. that this is the case for $(a, a + 4\delta) \times \{0\}$. Then, it follows from the definition of $\mathcal{A}_h^\circ$ and the properties of $\{\mathcal{T}_h\}$ that there exist a constant $C > 0$ independent of h and a $\tau \in [0, C]$ (possibly dependent on h) such that the set $(a - \tau h, a + 4\delta) \times \{0\}$ is contained in $\mathcal{A}_h^\circ$, and we may calculate that the function $z_i = \psi_i z$ satisfies

$$\begin{aligned} -\Delta z_i &= -z\Delta\psi_i - 2\nabla z \cdot \nabla \psi_i - \psi_i \max(0, u - u_h)^3 - z_i \ \mathcal{L}^2\text{-a.e. in } \Omega, \\ z_i &= 0 \ \mathcal{L}^2\text{-a.e. in } \Omega \setminus B_{2\delta}(a, 0), \\ z_i &= 0 \ \mathcal{H}^1\text{-a.e. on } (a - \tau h, a + 2\delta) \times \{0\}, \\ \partial_n z_i &= 0 \ \mathcal{H}^1\text{-a.e. on } (a - 2\delta, a - \tau h) \times \{0\}. \end{aligned}$$

Here, we have again used the properties of z and the rotational symmetry of ψ_i. Since z_i and ψ_i vanish outside of the ball $B_{2\delta}(a, 0)$, we may now deduce that the (trivial extension of) the function $\bar{z}(x, y) := z_i(x + a - \tau h - 1/2, y)$ satisfies

$$\begin{aligned} -\Delta \bar{z} &= \bar{g} \ \mathcal{L}^2\text{-a.e. in } \Omega, \\ \bar{z} &= 0 \ \mathcal{H}^1\text{-a.e. on } \partial\Omega \setminus (0, 1/2) \times \{0\}, \\ \partial_n \bar{z} &= 0 \ \mathcal{H}^1\text{-a.e. on } (0, 1/2) \times \{0\} \end{aligned}$$

with

$$\bar{g}(x, y) := \Big(- z\Delta\psi_i - 2\nabla z \cdot \nabla\psi_i - \psi_i \max(0, u - u_h)^3 - z_i \Big)(x + a - \tau h - 1/2, y).$$

By invoking [15, Theorem 4.3.2.4], we may now again deduce that there exists a constant $C > 0$ independent of h with

$$\|\bar{z}\|_{W^{2,(4-\varepsilon)/3}(\Omega)} \le C \left(\|\bar{g}\|_{L^{(4-\varepsilon)/3}(\Omega)} + \|\bar{z}\|_{W^{1,(4-\varepsilon)/3}(\Omega)} \right).$$

If we express $\bar{z}$ in terms of z_i and use the same calculations as before, then we arrive at

$$\|z_i\|_{W^{2,(4-\varepsilon)/3}(\Omega)} \le C \| \max(0, u - u_h)^3 \|_{L^{(4-\varepsilon)/3}(\Omega)}$$

with a constant $C > 0$ which depends on ψ_i and ε but is independent of h. Using the same arguments as above, we can transform each of the situations occurring at the points b_i, $i = 1, ..., N$, to one of finitely many reference configurations and use [15, Theorem 4.3.2.4] as well as (15) to prove that there exist constants C_i independent of h with

$$\|z_i\|_{W^{2,(4-\varepsilon)/3}(\Omega)} \le C_i \| \max(0, u - u_h)^3 \|_{L^{(4-\varepsilon)/3}(\Omega)} \qquad \forall i = 0, ..., N.$$

Note that, if we consider a point b_i which is a corner of the square Ω, then the situation is even simpler than above since, in this case, the boundaries of $\mathcal{A}^\circ$ and $\mathcal{A}_h^\circ$ are locally the same and equal to $\{b_i\}$ so that a translation argument as above is unnecessary. To arrive at (16), it now suffices to invoke the triangle inequality. This completes the proof. □

Remark 5.3 *Note that the solution z of the auxiliary problem* (D) *cannot be expected to possess $W^{2,q}$-regularity for some $q \ge 4/3$ since it typically contains a singular part analogous to that in* (4).

By choosing a suitable test function in (D) and by exploiting the estimates in Corollary 4.3, we may now deduce:

Theorem 5.4 *Suppose that u solves* (S) *for some $f \in L^\infty(\Omega)$ and that (A) is satisfied. Then, for all $\varepsilon \in (0, 1/2)$, there exists a constant $C > 0$ independent of h such that, for all sufficiently small $h > 0$, we have*

$$\|(u - u_h)^+\|_{L^4(\Omega)} \leq C h^{2-\varepsilon}.$$

Proof Let us denote the finitely many mesh nodes in the relative boundary of the set $\mathcal{A}_h^\circ$ with x_i, $i = 1, ..., N$, $N \in \mathbb{N}_0$, and the basis functions of the nodal basis of V_h that belong to the nodes x_i with φ_i. Note that the number N is independent of h here for all sufficiently small h by our assumptions and the definition of $\mathcal{A}_h^\circ$. Consider now the function $v := (u_h - u) + \sum_{i=1}^N Ch\|u - u_h\|_{W^{1,\infty}(\Omega)}\varphi_i$, where C is supposed to be a constant independent of h with $\operatorname{diam}(T) \leq Ch$ for all $T \in \mathcal{T}_h$. We claim that this v is admissible for (D). Indeed, on $\mathcal{A}^\circ$, we have $u \equiv 0$ and thus $v \geq 0$. Further, we know that, for all small h and all $x \in \mathcal{A}_h^\circ \setminus \mathcal{A}^\circ$, we can find an $\tilde{x}$ in the relative boundary of $\mathcal{A}^\circ$ and a $j \in \{1, ..., N\}$ with $x \in [x_j, \tilde{x}]$ and $\operatorname{dist}(x_j, \tilde{x}) < Ch$, where $[x_j, \tilde{x}]$ denotes the line segment between x_j and $\tilde{x}$. This yields

$$\begin{aligned} v(x) &\geq (u_h - u)(x) - (u_h - u)(\tilde{x}) + \sum_{i=1}^N Ch\|u - u_h\|_{W^{1,\infty}(\Omega)}\varphi_i(x) \\ &\geq -\|u - u_h\|_{W^{1,\infty}(\Omega)} \operatorname{dist}(x, \tilde{x}) + Ch\|u - u_h\|_{W^{1,\infty}(\Omega)}\varphi_j(x) \\ &\geq -\|u - u_h\|_{W^{1,\infty}(\Omega)} \operatorname{dist}(\tilde{x}, x_j)\varphi_j(x) + Ch\|u - u_h\|_{W^{1,\infty}(\Omega)}\varphi_j(x) \\ &\geq 0 \end{aligned}$$

for all $x \in \mathcal{A}_h^\circ \setminus \mathcal{A}^\circ$. Thus, we indeed have $v \geq 0$ $\mathcal{H}^1$-a.e. on $\mathcal{A}_h^\circ$. Choosing the function $v + z \in L$ in (D) and using Lemma 5.2, the Sobolev embedding

$$W^{2,\frac{4-\varepsilon}{3}}(\Omega) \hookrightarrow W^{1,\frac{8-2\varepsilon}{2+\varepsilon}}(\Omega) \subset C(\operatorname{cl}(\Omega)), \qquad \varepsilon \in (0, 1/2),$$

and Hölder's inequality now gives (with a generic $C > 0$ independent of h)

$$\begin{aligned} &\int_\Omega \max(0, u - u_h)^4 \mathrm{d}\mathcal{L}^2 \\ &\leq (z, u_h - u)_{H^1(\Omega)} + \left(z, \sum_{i=1}^N Ch\|u - u_h\|_{W^{1,\infty}(\Omega)}\varphi_i\right)_{H^1(\Omega)} \\ &\quad + \int_\Omega \max(0, u - u_h)^3 \left(\sum_{i=1}^N Ch\|u - u_h\|_{W^{1,\infty}(\Omega)}\varphi_i\right) \mathrm{d}\mathcal{L}^2 \\ &\leq (z, u_h - u)_{H^1(\Omega)} + C\|z\|_{W^{1,\frac{8-2\varepsilon}{2+\varepsilon}}(\Omega)} \left(h\|u - u_h\|_{W^{1,\infty}(\Omega)} \sum_{i=1}^N \|\varphi_i\|_{W^{1,\frac{8-2\varepsilon}{6-3\varepsilon}}(\Omega)}\right) \\ &\quad + C\|\max(0, u - u_h)^3\|_{L^{\frac{4-\varepsilon}{3}}(\Omega)} \left(h\|u - u_h\|_{W^{1,\infty}(\Omega)} \sum_{i=1}^N \|\varphi_i\|_{L^{\frac{4-\varepsilon}{1-\varepsilon}}(\Omega)}\right) \\ &\leq (z, u_h - u)_{H^1(\Omega)} + C\|z\|_{W^{2,\frac{4-\varepsilon}{3}}(\Omega)} \left(h\|u - u_h\|_{W^{1,\infty}(\Omega)} h^{\frac{6-3\varepsilon}{4-\varepsilon} - 1}\right) \\ &\quad + C\|\max(0, u - u_h)^3\|_{L^{\frac{4-\varepsilon}{3}}(\Omega)} \left(h\|u - u_h\|_{W^{1,\infty}(\Omega)} h^{\frac{2-2\varepsilon}{4-\varepsilon}}\right) \\ &\leq (z, u_h - u)_{H^1(\Omega)} + Ch^{\frac{6-3\varepsilon}{4-\varepsilon}} \|\max(0, u - u_h)^3\|_{L^{\frac{4-\varepsilon}{3}}(\Omega)} \|u - u_h\|_{W^{1,\infty}(\Omega)}. \end{aligned} \tag{17}$$

Note that, since z vanishes on $\mathcal{A}_h^\circ$, since the relative boundary of $\mathcal{A}_h^\circ$ consists of mesh nodes and since $z \le 0$ in Ω, the Lagrange interpolant $I_h(z) \in V_h$ satisfies $I_h(z) = 0$ on $\mathcal{A}^\circ \subset \mathcal{A}_h^\circ$ and $I_h(z) \le 0$ in Ω. This implies in combination with (S_h) and the reformulation (3) of (S) that

$$(u_h, -I_h(z))_{H^1(\Omega)} \ge (-f, I_h(z))_{L^2(\Omega)} \quad \text{and} \quad (u, I_h(z))_{H^1(\Omega)} = (f, I_h(z))_{L^2(\Omega)},$$

i.e., we have

$$(-I_h(z), u_h - u)_{H^1(\Omega)} \ge 0.$$

Using the last inequality, standard results for the Lagrange interpolation operator and again Lemma 5.2, we can continue the estimate in (17) as follows

$$\begin{aligned}
&\|(u-u_h)^+\|^4_{L^4(\Omega)} \\
&\le (z - I_h(z), u_h - u)_{H^1(\Omega)} + Ch^{\frac{6-3\varepsilon}{4-\varepsilon}} \|\max(0, u-u_h)^3\|_{L^{\frac{4-\varepsilon}{3}}(\Omega)} \|u-u_h\|_{W^{1,\infty}(\Omega)} \\
&\le \|z - I_h(z)\|_{W^{1,\frac{4-\varepsilon}{3}}(\Omega)} \|u_h - u\|_{W^{1,\frac{4-\varepsilon}{1-\varepsilon}}(\Omega)} \\
&\qquad + Ch^{\frac{6-3\varepsilon}{4-\varepsilon}} \|\max(0, u-u_h)^3\|_{L^{\frac{4-\varepsilon}{3}}(\Omega)} \|u-u_h\|_{W^{1,\infty}(\Omega)} \\
&\le Ch\|z\|_{W^{2,\frac{4-\varepsilon}{3}}(\Omega)} \|u-u_h\|^{\frac{3\varepsilon}{4-\varepsilon}}_{W^{1,\infty}(\Omega)} \|u_h - u\|^{\frac{4-4\varepsilon}{4-\varepsilon}}_{W^{1,4}(\Omega)} \\
&\qquad + Ch^{\frac{6-3\varepsilon}{4-\varepsilon}} \|\max(0, u-u_h)^3\|_{L^{\frac{4-\varepsilon}{3}}(\Omega)} \|u-u_h\|_{W^{1,\infty}(\Omega)} \\
&\le Ch\|(u-u_h)^+\|^3_{L^{4-\varepsilon}(\Omega)} \|u-u_h\|^{\frac{3\varepsilon}{4-\varepsilon}}_{W^{1,\infty}(\Omega)} \|u_h - u\|^{\frac{4-4\varepsilon}{4-\varepsilon}}_{W^{1,4}(\Omega)} \\
&\qquad + Ch^{\frac{6-3\varepsilon}{4-\varepsilon}} \|(u-u_h)^+\|^3_{L^{4-\varepsilon}(\Omega)} \|u-u_h\|_{W^{1,\infty}(\Omega)}.
\end{aligned} \tag{18}$$

The above yields, in combination with Corollary 4.3, that there exists a constant C independent of h with

$$\begin{aligned}
\|(u-u_h)^+\|_{L^{4-\varepsilon}(\Omega)} &\le Ch\|u-u_h\|^{\frac{3\varepsilon}{4-\varepsilon}}_{W^{1,\infty}(\Omega)} \|u_h - u\|^{\frac{4-4\varepsilon}{4-\varepsilon}}_{W^{1,4}(\Omega)} \\
&\qquad\qquad + Ch^{\frac{6-3\varepsilon}{4-\varepsilon}} \|u-u_h\|_{W^{1,\infty}(\Omega)} \\
&\le Ch^{1+\frac{1}{2}(1-\varepsilon)\frac{3\varepsilon}{4-\varepsilon} + (1-\varepsilon)\frac{4-4\varepsilon}{4-\varepsilon}} + Ch^{\frac{3}{2} - \frac{3\varepsilon}{8-2\varepsilon} + \frac{1}{2}(1-\varepsilon)} \\
&\le Ch^{2-o(1)},
\end{aligned}$$

where the Landau symbol refers to the limit $\varepsilon \searrow 0$. Using the above in (18) and performing the same calculation as before yields

$$\begin{aligned}
&\|(u-u_h)^+\|^4_{L^4(\Omega)} \\
&\le Ch^{6-o(1)} \left(h\|u-u_h\|^{\frac{3\varepsilon}{4-\varepsilon}}_{W^{1,\infty}(\Omega)} \|u_h - u\|^{\frac{4-4\varepsilon}{4-\varepsilon}}_{W^{1,4}(\Omega)} + Ch^{\frac{6-3\varepsilon}{4-\varepsilon}} \|u-u_h\|_{W^{1,\infty}(\Omega)} \right) \\
&\le Ch^{8-o(1)}.
\end{aligned}$$

This proves the claim (after redefining ε). □

5.2 A Duality Argument for the Component $(u - u_h)^-$

To obtain an L^4-error estimate for the component $(u-u_h)^-$, we can proceed along roughly the same lines as in the last subsection provided the contact sets

$$\tilde{\mathcal{A}}_h := \{x \in \partial\Omega \mid u_h(x) = 0\}, \qquad h > 0,$$

of the finite element solutions u_h behave sufficiently well for $h \searrow 0$. To be more precise, we need the following assumption:

Definition 5.5 (Condition $(\mathbf{A_h})$) *We say that condition (A_h) is satisfied if there exist points $d_i \in \partial\Omega$, $i = 1, ..., N$, $N \in \mathbb{N}_0$, and numbers $\delta_i > 0$ such that the sets $B_{\delta_i}(d_i) \cap \partial\Omega$ have non-zero distance from each other and the corners of the square Ω and such that the following holds true for all sufficiently small $h > 0$:*

1. *The sets $B_{\delta_i}(d_i) \cap \partial\Omega$ cover the relative boundary of $\tilde{\mathcal{A}}_h$ and each $B_{\delta_i}(d_i) \cap \partial\Omega$ contains precisely one element of the relative boundary of $\tilde{\mathcal{A}}_h$.*
2. *Every connected component of $\tilde{\mathcal{A}}_h$ has a non-empty relative interior.*

Roughly speaking, the above condition expresses that the topological properties of the sets $\{x \in \partial\Omega \mid u_h(x) = 0\}$ and $\{x \in \partial\Omega \mid u_h(x) \neq 0\}$ do not change drastically as h passes to zero, and that the set $\{x \in \partial\Omega \mid u_h(x) = 0\}$ does not contain components which are singletons. Suppose, for example, that the contact set $\tilde{\mathcal{A}}_h$ has the form $[0, \alpha_h] \times \{0\} \cup \{0\} \times [0, \beta_h]$ with some $\alpha_h, \beta_h \in (0, 1)$ for all sufficiently small $h > 0$. Then, condition (A_h) is satisfied if and only if there exists a closed interval $E \subset (0, 1)$ with $\alpha_h, \beta_h \in E$ for all small enough h. Note that we do not need here that the sequences α_h and β_h converge or that $\tilde{\mathcal{A}}_h$ approximates the contact set of the exact solution u for $h \searrow 0$ (although this is, of course, what is typically observed in numerical experiments, cf. [33, Section 7]).

Analogously to the last section, we may now consider the following auxiliary problem:

$$\tilde{z} \in \tilde{L}, \quad (\tilde{z}, v - \tilde{z})_{H^1(\Omega)} \geq (-\max(0, u_h - u)^3, v - \tilde{z})_{L^2(\Omega)} \quad \forall v \in \tilde{L} \tag{$\tilde{\mathrm{D}}$}$$

with

$$\tilde{L} := \left\{ v \in H^1(\Omega) \,\middle|\, \mathrm{tr}(v) \geq 0 \ \mathcal{H}^1\text{-a.e. on } \tilde{\mathcal{A}}_h \right\}.$$

By invoking again the results of [15], we obtain:

Lemma 5.6 *Suppose that u solves* (S) *for some $f \in L^2(\Omega)$, that u_h is the solution of* (S_h)*, and that condition (A_h) is satisfied. Then,* ($\tilde{\mathrm{D}}$) *admits a unique solution $\tilde{z} \in H^1(\Omega)$ for all $h > 0$, and this solution satisfies $\tilde{z} \leq 0$ $\mathcal{L}^2$-a.e. in Ω and $\mathrm{tr}(\tilde{z}) = 0$ $\mathcal{H}^1$-a.e. on $\tilde{\mathcal{A}}_h$, and, for every $\varepsilon \in (0, 1/2)$, there exists a constant $C > 0$ independent of h with*

$$\|\tilde{z}\|_{H^1(\Omega)} \leq C \| \max(0, u_h - u)^3 \|_{L^{(4-\varepsilon)/3}(\Omega)}. \tag{19}$$

Moreover, for all $\varepsilon \in (0, 1/2)$ and all sufficiently small $h > 0$, $\tilde{z}$ is in $W^{2,(4-\varepsilon)/3}(\Omega)$ and satisfies

$$\|\tilde{z}\|_{W^{2,(4-\varepsilon)/3}(\Omega)} \leq C \| \max(0, u_h - u)^3 \|_{L^{(4-\varepsilon)/3}(\Omega)} \tag{20}$$

with some constant $C > 0$ independent of h.

Proof The unique solvability of $(\tilde{\mathrm{D}})$ for all $h > 0$ follows from [18, Theorem 2.1], and the non-positivity of $\tilde{z}$ and the property $\mathrm{tr}(\tilde{z}) = 0$ on $\tilde{\mathcal{A}}_h$ are obtained completely analogously to the proof of Lemma 5.2. The same is the case for the estimate (19). It remains to prove the $W^{2,(4-\varepsilon)/3}$-regularity of $\tilde{z}$ and (20) for all sufficiently small $h > 0$. The former follows immediately from (A_h), [15, Theorem 4.4.3.7] and the same arguments as in Lemma 5.2. To obtain the latter, we assume that h is so small that the conditions in (A_h) hold with some $d_i \in \partial\Omega$, $\delta_i > 0$, $i = 1, ..., N$, $N \in \mathbb{N}$ (for $N = 0$ the claim is trivial) and choose rotationally symmetric cut-off functions $\psi_i \in C_c^\infty(\mathbb{R}^2)$, $i = 1, ..., N$, such that $0 \leq \psi_i \leq 1$ holds in $\mathbb{R}^2$ for all i, such that ψ_i is identical one in $B_{\delta_i}(d_i)$ for all i, and such that the sets $\mathrm{supp}(\psi_i) \cap \partial\Omega$ have non-zero distance from each other and the corners of the square Ω. In this situation, the properties of the functions ψ_i imply that we may find another cut-off function $\phi \in C_c^\infty(\mathbb{R}^2)$ with $0 \leq \phi \leq 1$ in $\mathbb{R}^2$ and $\phi \equiv 1$ in a neighborhood of the boundary $\partial\Omega$ such that the supports of the functions $\tilde{\psi}_i := \psi_i\phi$ are disjoint. Using these $\tilde{\psi}_i$, we decompose $\tilde{z}$ into the parts $\tilde{z}_0, \tilde{z}_1, ..., \tilde{z}_N$ defined by

$$\tilde{z}_i := \tilde{\psi}_i\tilde{z} \text{ for } i = 1, ..., N, \qquad \tilde{z}_0 := \tilde{\psi}_0\tilde{z}, \qquad \tilde{\psi}_0 := 1 - \sum_{i=1}^{N} \tilde{\psi}_i.$$

Since (A_h) implies that the relative boundary of $\tilde{\mathcal{A}}_h$ is contained in the union of the balls $B_{\delta_i}(d_i)$, $i = 1, ..., N$, that each set $B_{\delta_i}(d_i) \cap \partial\Omega$ contains precisely one point of the relative boundary of $\tilde{\mathcal{A}}_h$, and that the connected components of $\tilde{\mathcal{A}}_h$ each have a non-empty relative interior, we may argue as in the proof of Lemma 5.2 to deduce that $\tilde{z}_0$ satisfies

$$\begin{aligned} -\Delta\tilde{z}_0 = -\tilde{z}\Delta\tilde{\psi}_0 - 2\nabla\tilde{z}\cdot\nabla\tilde{\psi}_0 - \tilde{\psi}_0 \max(0, u_h - u)^3 - \tilde{z}_0 \ \mathcal{L}^2\text{-a.e. in } \Omega, \\ \tilde{z}_0 = 0 \ \mathcal{H}^1\text{-a.e. on } \mathcal{B}, \\ \partial_n\tilde{z}_0 = 0 \ \mathcal{H}^1\text{-a.e. on } \partial\Omega \setminus \mathcal{B} \end{aligned}$$

with some closed set $\mathcal{B} \subset \partial\Omega$ whose connected components each have a non-empty relative interior and whose relative boundary consists precisely of the points $d_1, ..., d_N$. Note that it is (at least in theory) possible that $\mathcal{B}$ varies with h since it is not uniquely determined by the above conditions. However, it is easy to see that only two sets $\mathcal{B}$ and combinations of boundary conditions are possible here. We may thus again invoke [15, Theorem 4.3.2.4] and use (19) to deduce that, for every $\varepsilon \in (0, 1/2)$, we have

$$\begin{aligned} &\|\tilde{z}_0\|_{W^{2,(4-\varepsilon)/3}(\Omega)} \\ &\leq C\left(\|\tilde{z}\Delta\tilde{\psi}_0 + 2\nabla\tilde{z}\cdot\nabla\tilde{\psi}_0 + \tilde{\psi}_0 \max(0, u_h - u)^3\|_{L^{(4-\varepsilon)/3}(\Omega)} + \|\tilde{z}_0\|_{W^{1,(4-\varepsilon)/3}(\Omega)}\right) \\ &\leq C\|\max(0, u_h - u)^3\|_{L^{(4-\varepsilon)/3}(\Omega)} \end{aligned}$$

with a generic constant $C > 0$ which is independent of h. It remains to estimate the $W^{2,(4-\varepsilon)/3}$-norm of the functions $\tilde{z}_i$, $i = 1, ..., N$. So let us consider an arbitrary but fixed point d_i. Since d_i is not a corner of Ω by (A_h), we may assume w.l.o.g. that $d_i = (a, 0)$ holds for some $a \in (0, 1)$. Further, it follows from a straightforward calculation and the properties of $\tilde{\psi}_i$ that

$$\begin{aligned} -\Delta\tilde{z}_i = -\tilde{z}\Delta\tilde{\psi}_i - 2\nabla\tilde{z}\cdot\nabla\tilde{\psi}_i - \tilde{\psi}_i \max(0, u_h - u)^3 - \tilde{z}_i \ \mathcal{L}^2\text{-a.e. in } \Omega, \\ \tilde{z}_i = 0 \ \mathcal{L}^2\text{-a.e. in } \Omega \setminus \mathrm{supp}(\tilde{\psi}_i). \end{aligned} \tag{21}$$

Since the support $\operatorname{supp}(\tilde{\psi}_i)$ contains precisely one point $(\tilde{a}, 0) \in \partial\Omega$ of the relative boundary of $\tilde{\mathcal{A}}_h$ by our assumption (A_h), we may complement (21) with one of the boundary conditions

$$\begin{aligned} \tilde{z}_i &= 0 \ \mathcal{H}^1\text{-a.e. on } \big((-\infty, \tilde{a}) \times \{0\}\big) \cap \operatorname{supp}(\tilde{\psi}_i) \subset \partial\Omega \\ \partial_n \tilde{z}_i &= 0 \ \mathcal{H}^1\text{-a.e. on } \big((\tilde{a}, \infty) \times \{0\}\big) \cap \operatorname{supp}(\tilde{\psi}_i) \subset \partial\Omega \end{aligned}$$

and

$$\begin{aligned} \tilde{z}_i &= 0 \ \mathcal{H}^1\text{-a.e. on } \big((\tilde{a}, \infty) \times \{0\}\big) \cap \operatorname{supp}(\tilde{\psi}_i) \subset \partial\Omega \\ \partial_n \tilde{z}_i &= 0 \ \mathcal{H}^1\text{-a.e. on } \big((-\infty, \tilde{a}) \times \{0\}\big) \cap \operatorname{supp}(\tilde{\psi}_i) \subset \partial\Omega. \end{aligned}$$

Using exactly the same arguments as in the proof of Lemma 5.2, we may now transform the situation at d_i into one of finitely many reference configurations and invoke [15, Theorem 4.3.2.4] as well as (19) to deduce that there exists a constant C_i independent of h with

$$\|\tilde{z}_i\|_{W^{2,(4-\varepsilon)/3}(\Omega)} \le C_i \|\max(0, u_h - u)^3\|_{L^{(4-\varepsilon)/3}(\Omega)}.$$

Proceeding exactly along the same lines at the other points d_i and using the triangle inequality, we arrive at (20). This completes the proof. ◻

By choosing the test function $v = \tilde{z} + u - u_h \in \tilde{L}$ in ($\tilde{\text{D}}$), we now obtain:

Theorem 5.7 *Suppose that u solves* (S) *for some $f \in L^\infty(\Omega)$, that u_h is the solution of* (S_h) *and that the conditions (A) and (A_h) are satisfied. Then, for all $\varepsilon \in (0, 1/2)$, there exists a constant $C > 0$ independent of h such that, for all sufficiently small $h > 0$, we have*

$$\|(u - u_h)^-\|_{L^4(\Omega)} \le C h^{2-\varepsilon}. \tag{22}$$

Proof Note that the definition of $\tilde{\mathcal{A}}_h$ implies $u - u_h = u \ge 0$ on $\tilde{\mathcal{A}}_h$. The function $u - u_h$ is thus an element of $\tilde{L}$ and we may choose the function $v = \tilde{z} + u - u_h$ in ($\tilde{\text{D}}$) to obtain

$$\|\min(0, u - u_h)\|^4_{L^4(\Omega)} \le (\tilde{z}, u - u_h)_{H^1(\Omega)}. \tag{23}$$

Since the set $\tilde{\mathcal{A}}_h$ consists of cells of the boundary mesh, since $\tilde{z}$ vanishes in $\tilde{\mathcal{A}}_h$ and since $\tilde{z} \le 0$ holds $\mathcal{L}^2$-a.e. in Ω, we know that the Lagrange interpolant $I_h(\tilde{z})$ vanishes in $\tilde{\mathcal{A}}_h$ and that $I_h(\tilde{z})$ is non-positive everywhere. This implies that, for all small enough $s > 0$, we have $u_h \pm s I_h(\tilde{z}) \in K_h$ and, as consequence, that

$$(u_h, I_h(\tilde{z}))_{H^1(\Omega)} = (f, I_h(\tilde{z}))_{L^2(\Omega)} \quad \text{and} \quad (u, -I_h(\tilde{z}))_{H^1(\Omega)} \ge (f, -I_h(\tilde{z}))_{L^2(\Omega)}.$$

Using the above in (23) yields

$$\begin{aligned} \|\min(0, u - u_h)\|^4_{L^4(\Omega)} &\le (\tilde{z}, u - u_h)_{H^1(\Omega)} \\ &\le (\tilde{z} - I_h(\tilde{z}), u - u_h)_{H^1(\Omega)} \\ &\le \|z - I_h(z)\|_{W^{1,\frac{4-\varepsilon}{3}}(\Omega)} \|u_h - u\|_{W^{1,\frac{4-\varepsilon}{1-\varepsilon}}(\Omega)} \end{aligned}$$

for all $\varepsilon \in (0, 1/2)$. A calculation completely analogous to that at the end of the proof of Theorem 5.4 now yields (22) as claimed. ◻

6 Summary and Remarks on the Error Analysis

We may now collect our results for right-hand sides $f \in L^\infty(\Omega)$ in:

Theorem 6.1 *Suppose that u solves* (S) *for some $f \in L^\infty(\Omega)$ and that (A) is satisfied. Then, for all $\varepsilon \in (0, 1/2)$, there exists a constant $C > 0$ independent of h with*

$$\begin{aligned} \|u - u_h\|_{W^{1,8/3-\varepsilon}(\Omega)} &\le Ch, & \|u - u_h\|_{W^{1,4}(\Omega)} &\le Ch^{1-\varepsilon}, \\ \|u - u_h\|_{W^{1,\infty}(\Omega)} &\le Ch^{1/2-\varepsilon}, & \|u - u_h\|_{L^\infty(\Omega)} &\le Ch^{3/2-\varepsilon}, \\ \|u - u_h\|_{H^{1/2}(\partial\Omega)} &\le Ch^{3/2-\varepsilon}, & \|(u - u_h)^+\|_{L^4(\Omega)} &\le Ch^{2-\varepsilon}. \end{aligned} \tag{24}$$

If, additionally, the approximate solutions u_h satisfy (A_h), then we also have

$$\|(u - u_h)^-\|_{L^4(\Omega)} \le Ch^{2-\varepsilon} \qquad \forall \varepsilon \in (0, 1/2).$$

Proof Combine Corollary 4.3 and Theorems 5.4 and 5.7. □

Some remarks are in order regarding the last result:

Remark 6.2

1. *The error estimates in Theorem 6.1 are optimal in view of the $W^{2,p}$- and H^s-regularity properties of the exact solution u, cf. Remark 4.4.*
2. *Recall that one of the crucial steps in the proof of Theorem 6.1 was to use the $W^{1,4}$-error estimate in Corollary 4.3 to compensate the lack of regularity of the dual solutions in Sections 5.1 and 5.2. It is quite remarkable here that the exponent in the obtained $W^{1,p}$-error estimate (namely, $p = 4$) and the exponent in the $W^{2,p}$-regularity results for* (D) *and* ($\tilde{\text{D}}$) *(namely, $p = 4/3 - \varepsilon$) are (up to the ε) Hölder conjugates of each other and thus fit together perfectly. Even more surprisingly, $4/3 + \varepsilon$ is also the difference of the exponents in the two $W^{1,p}$-error estimates in the first line of* (24)*. This omnipresence of the exponents 4 and 4/3 indicates that the L^4- and the $W^{1,4}$-norm are a natural choice for the finite element error analysis of the problem* (S).
3. *We expect that it is possible to relax the assumption (A_h) by studying in more detail the dependence of the constant in [15, Theorem 4.3.2.4] on the boundary conditions of the underlying problem. Note that the difficult part in the proofs of Theorems 5.4 and 5.7 is to obtain the uniform bound on the $W^{2,(4-\varepsilon)/3}$-norm in* (16) *and* (20)*. Showing that the dual solutions possess $W^{2,(4-\varepsilon)/3}$-regularity for all $\varepsilon \in (0, 1/2)$ is relatively simple.*
4. *Recall that, for a classical obstacle problem with an essentially bounded right-hand side, it can be shown that the exact solution enjoys $W^{2,p}$-regularity for all $2 \le p < \infty$, cf. [7,18,21]. This implies in particular that it is possible to prove L^∞-error estimates of order $\mathcal{O}(h^{2-\varepsilon})$ for arbitrarily small $\varepsilon > 0$. For the Signorini problem, this is different since the exact solution u cannot be expected to be in $W^{2,4}(\Omega)$ even for smooth right-hand sides f. The L^4-error estimates in Theorem 6.1 thus yield an order of convergence that cannot be recovered with pointwise a priori error estimates.*

7 Numerical Experiments

We conclude this paper with numerical experiments that confirm our theoretical findings. To construct a model problem that allows us to validate our results and that possesses a known analytic solution, we proceed along the lines of [33, Section 7] and consider the function $\tilde{u} : \mathbb{R} \times (0, \infty) \to \mathbb{R}$, $x \mapsto -r^{3/2} \sin(\frac{3}{2}\theta)$. Here, r and θ denote polar coordinates centered at $(0.5, 0)$, i.e.,

$$r(x_1, x_2) := \left((x_1 - 0.5)^2 + x_2^2\right)^{1/2} \quad \text{and} \quad \theta(x_1, x_2) := \arccos\left(\frac{x_1 - 0.5}{r}\right).$$

Note that the function $\tilde{u}$ is exactly of the same type as the singular terms on the right-hand side of (5), cf. also the analysis in [15,16]. Moreover, it is easy to check that $\tilde{u}$ is an element of $H^2(U)$ for all bounded, open $U \subset \mathbb{R} \times (0, \infty)$, that $\tilde{u} = 0$ and $\partial_n \tilde{u} \geq 0$ holds on $(0.5, \infty) \times \{0\}$, that $\tilde{u} \geq 0$ and $\partial_n \tilde{u} = 0$ holds on $(-\infty, 0.5) \times \{0\}$, and that $\Delta \tilde{u}$ vanishes almost everywhere in $\mathbb{R} \times (0, \infty)$. Suppose now that $\psi : [0, \infty) \to \mathbb{R}$ is a C^4-function satisfying $\psi \equiv 0$ in $[0.45, \infty)$ and

$$\psi(0) = 1, \quad \psi'(0) = ... = \psi''''(0) = \psi(0.45) = \psi'(0.45) = ... = \psi''''(0.45) = 0.$$

(In the experiments below, this ψ was an appropriately defined ninth-order spline.) Then, the properties of $\tilde{u}$ and ψ yield that the map

$$u : \Omega \to \mathbb{R}, \qquad x \mapsto 10\, \psi(r) \tilde{u}(r, \theta), \tag{25}$$

satisfies

$$\begin{gathered} -\Delta u + u \in C(\mathrm{cl}(\Omega)), \\ u = 0 \text{ and } \partial_n u \geq 0 \text{ on } \partial\Omega \setminus (0.05, 0.5) \times \{0\}, \\ u \geq 0 \text{ and } \partial_n u = 0 \text{ on } (0.05, 0.5) \times \{0\}, \end{gathered} \tag{26}$$

where Ω again denotes the unit square $(0, 1)^2$ and where we have added the factor ten for scaling reasons. Note that the conditions in (26) imply in particular that the function u solves (S) with right-hand side $f := -\Delta u + u \in C(\mathrm{cl}(\Omega))$. What we have constructed in (25) is thus indeed an analytic solution of Signorini's problem that can be used as a reference in our numerical experiments, cf. Figure 2.

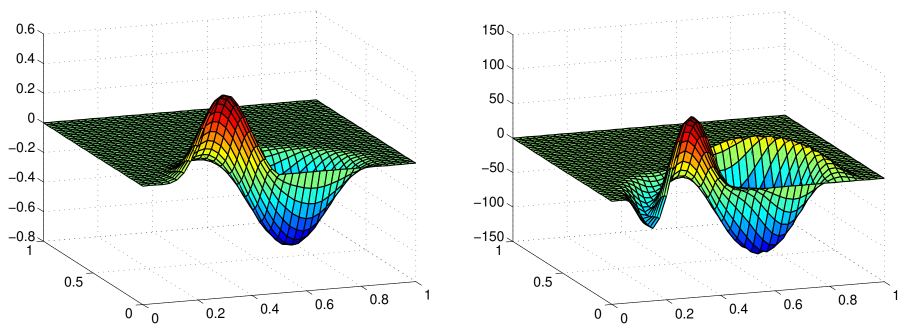

Fig. 2 Solution u (left) and right-hand side f (right) in the situation of Section 7.

The results that we have obtained for the right-hand side f associated with the solution u in (25) by means of the finite element scheme described in Section 2.3 can be seen in Tables 1 and 2 and Figure 3. Here, we have used Friedrich-Keller triangulations to discretize the continuous problem (S) and a 16-point Gauss-Legendre-type quadrature rule for triangles to evaluate the various $W^{s,p}$-errors and the integrals arising on the right-hand side of (S_h). The finite-dimensional elliptic variational inequalities obtained from the discretization scheme have been solved by means of the Matlab routine `quadprog` to a high precision. Note that the choice of mesh widths in our numerical experiments ensures that the point $(0.5, 0)$, where the analytical solution u detaches from the boundary $\partial\Omega$ and where $\nabla^2 u$ possesses a singularity, never coincides with a node of the underlying mesh. This constitutes the worst case scenario as the critical part of the contact set of u is never resolved properly. Further, it should be noted that the condition (A) is trivially satisfied in (25) by the properties of the analytic solution u. Our numerical experiments indicate that the same is true for the condition (A_h) as it can be observed that the contact sets of the finite element solutions u_h approximate their continuous counterpart, cf. the comments after Definition 5.5.

Table 1 Absolute error $u - u_h$ between the analytic solution u in (25) and the finite element approximation u_h characterized by (S_h) for different mesh widths h in various norms. The $H^{1/2}$-error was estimated by means of the interpolation property of the space $H^{1/2}(\partial\Omega)$.

h	$L^4(\Omega)$	$L^\infty(\Omega)$	$W^{1,4}(\Omega)$	$W^{1,\infty}(\Omega)$	$H^{1/2}(\partial\Omega)$
$\frac{1}{101}$	$5.8699 \cdot 10^{-4}$	$2.2877 \cdot 10^{-3}$	$1.3044 \cdot 10^{-1}$	$1.0873 \cdot 10^{-0}$	$6.3204 \cdot 10^{-3}$
$\frac{1}{201}$	$1.4167 \cdot 10^{-4}$	$7.3837 \cdot 10^{-4}$	$6.5953 \cdot 10^{-2}$	$7.7064 \cdot 10^{-1}$	$2.2274 \cdot 10^{-3}$
$\frac{1}{301}$	$6.5877 \cdot 10^{-5}$	$4.0734 \cdot 10^{-4}$	$4.4173 \cdot 10^{-2}$	$6.2834 \cdot 10^{-1}$	$1.2471 \cdot 10^{-3}$
$\frac{1}{401}$	$3.7124 \cdot 10^{-5}$	$2.6528 \cdot 10^{-4}$	$3.3226 \cdot 10^{-2}$	$5.4418 \cdot 10^{-1}$	$8.1896 \cdot 10^{-4}$
$\frac{1}{501}$	$2.3723 \cdot 10^{-5}$	$1.9010 \cdot 10^{-4}$	$2.6636 \cdot 10^{-2}$	$4.8676 \cdot 10^{-1}$	$5.9003 \cdot 10^{-4}$

Table 2 Experimental orders of convergence (EOCs) for different mesh widths h w.r.t. various norms, see (27). The row “regr.” contains the EOCs that are obtained from the experimental data by linear regression (after a double-logarithmic scaling) and the row “theo.” the orders of convergence expected from our analysis. The expected rates of convergence for the L^{16}- and the $W^{1,16}$-error have been computed with inverse estimates.

h	$L^4(\Omega)$	$L^{16}(\Omega)$	$L^\infty(\Omega)$	$W^{1,4}(\Omega)$	$W^{1,16}(\Omega)$	$W^{1,\infty}(\Omega)$	$H^{1/2}(\partial\Omega)$
$\frac{1}{101}$	-	-	-	-	-	-	-
$\frac{1}{201}$	2.0655	2.0658	1.6432	0.9910	0.6251	0.5001	1.5154
$\frac{1}{301}$	1.8962	1.8890	1.4729	0.9926	0.6306	0.5055	1.4363
$\frac{1}{401}$	1.9993	1.9374	1.4951	0.9927	0.6263	0.5012	1.4661
$\frac{1}{501}$	2.0112	1.8333	1.4966	0.9929	0.6259	0.5009	1.4726
regr.	1.9959	1.9657	1.5491	0.9920	0.6271	0.5021	1.4777
theo.	$2 - \varepsilon$	$1.625 - \varepsilon$	$1.5 - \varepsilon$	$1 - \varepsilon$	$0.625 - \varepsilon$	$0.5 - \varepsilon$	$1.5 - \varepsilon$

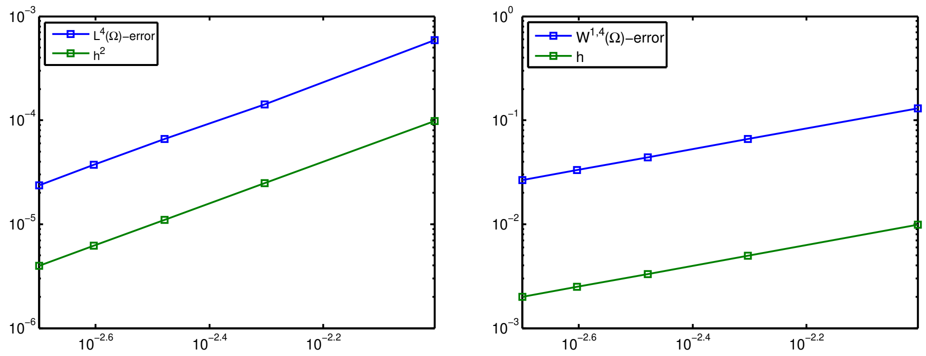

Fig. 3 $L^4(\Omega)$- and $W^{1,4}(\Omega)$-error in a double-logarithmic plot. The horizontal axis shows the mesh width h and the vertical axis the error in the respective norm.

As the results in Tables 1 and 2 and Figure 3 illustrate, the behavior observed in our numerical experiments accords very well with the analytical predictions in Theorem 6.1. (Note that this result is indeed applicable here since $f \in C(\text{cl}(\Omega))$.) In particular, the experimental orders of convergence (EOCs), i.e., the quantities

$$(\text{EOC})_{h_k, \|\cdot\|_*} := \frac{\log \|u - u_{h_k}\|_* - \log \|u - u_{h_{k-1}}\|_*}{\log h_k - \log h_{k-1}}, \tag{27}$$

fit very well to the a priori error estimates in (24). Table 2 further shows that the rates of convergence in the L^p- and the $W^{1,p}$-norms break down in the situation of (25) when p is greater than the critical exponent four. This demonstrates that, for instance, the order one (minus epsilon) is in general unobtainable when we consider the $W^{1,p}$-error for some $p > 4$.

Acknowledgements: We would like to thank Dr. Johannes Pfefferer for many interesting discussions on the topic of this paper and Prof. Dr. Boris Vexler for pointing out that a supercloseness result might be an option to prove the $W^{1,p}$-error estimates needed in the analysis of Section 5. Further, we gratefully acknowledge the support by the International Research Training Group IGDK 1754, funded by the German Science Foundation (DFG) and the Austrian Science Fund (FWF).

References

1. Adams, R.A.: *Sobolev Spaces*, Academic Press, New York, 1975
2. Attouch, H., Buttazzo, G., and Michaille, G.: *Variational Analysis in Sobolev and BV Spaces*, SIAM, Philadelphia, PA, 2006
3. Baiocchi, C.: *Estimation d'erreur dans L^∞ pour les inéquations à obstacle*, Lect. Notes Math., 606, 27-34, 1977
4. Brenner, S.C., and Scott, L.R.: *The Mathematical Theory of Finite Element Methods*, 3. Edition, Springer, New York, 2008
5. Brezzi, F., Hager, W.W., and Raviart, P.A.: *Error estimates for the finite element solution of variational inequalities, part I, primal theory*, Numer. Math., 28, 431-443, 1977
6. Brezzi, F., Hager, W.W., and Raviart, P.A.: *Error estimates for the finite element solution of variational inequalities, part II, mixed methods*, Numer. Math., 31, 1-16, 1978

7. Christof, C.: *L^∞-error estimates for the obstacle problem revisited*, Calcolo, 54, 4, 1243-1264, 2017
8. Christof, C., and Meyer, C.: *A note on a priori L^p-error estimates for the obstacle problem*, Numer. Math., 139, 1, 27-45, 2018
9. Ciarlet, P.G.: *Discrete maximum principle for finite-difference operators*, Aequationes Math., 4, 338-352, 1970
10. Cortey-Dumont, P.: *On finite element approximations in the L^∞-norm of variational inequalities*, Numer. Math., 47, 45-57, 1985
11. Drouet, G., and Hild, P.: *Optimal convergence for discrete variational inequalities modeling Signorini contact in 2d and 3d without additional assumptions on the unknown contact set*, SIAM J. Numer. Anal., 53, 3, 1488-1507, 2015
12. Di Nezza, E., Palatucci, G., and Valdinoci, E.: *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math., 136, 5, 521-573, 2012
13. Falk, R.S.: *Error estimates for the approximation of a class of variational inequalities*, Math. Comp., 28, 963-971, 1974
14. Finzi-Vita, S.: *L^∞-error estimates for variational inequalities with Hölder-continuous obstacle*, RAIRO Analyse Numérique, 16, 27-37, 1982
15. Grisvard, P.: *Elliptic Problems in Nonsmooth Domains*, Pitman, London, 1985
16. Grisvard, P.: *Singularities in Boundary Value Problems*, Springer, Berlin, 1992
17. Hild, P., and Renard, Y.: *An improved a priori error analysis for finite element approximations of Signorini's problem*, SIAM J. Numer. Anal., 50, 5, 2400-2419, 2012
18. Kinderlehrer, D., and Stampacchia, G.: *An Introduction to Variational Inequalities and Their Applications*, Classics in Applied Mathematics, 31, SIAM, Philadelphia, PA, 2000
19. Li, J., Melenk, J.M., Wohlmuth, B., and Zou, J.: *Optimal a priori estimates for higher order finite elements for elliptic interface problems*, Appl. Numer. Math., 60, 19-37, 2010
20. May, S., Rannacher, R., and Vexler, B.: *Error analysis for a finite element approximation of elliptic Dirichlet boundary control problems*, SIAM J. Control Optim., 51, 3, 2585-2611, 2013
21. Meyer, C., and Thoma, O.: *A priori finite element error analysis for optimal control of the obstacle problem*, SIAM J. Numer. Anal., 51, 605-628, 2013
22. Melenk, J.M., and Wohlmuth, B.: *Quasi-optimal approximation of surface based Lagrange multipliers in finite element methods*, SIAM J. Numer. Anal., 50, 4, 2064-2087, 2012
23. Mosco, U., and Strang, G.: *One-sided approximation and variational inequalities*, Bull. Amer. Math. Soc., 80, 308-312, 1974
24. Mosco, U.: *Error estimates for some variational inequalities*, Mathematical Aspects of Finite Element Methods, Lect. Notes Math., 606, 224-236, 1977
25. Moussaoui, M., and Khodja, K.: *Régularité des solutions d'un problème mêlé Dirichlet-Signorini dans un domaine polygonal plan*, Commun. Partial Differ. Equ., 17, 805–826, 1992
26. Natterer, F.: *Optimale L_2-Konvergenz finiter Elemente bei Variationsungleichungen*, Bonner Math. Schriften, 89, 1-13, 1976
27. Nitsche, J.: *L^∞-convergence of finite element approximations*, Lect. Notes Math., 606, 261-274, 1977
28. Nochetto, R.H.: *Pointwise accuracy of a finite element method for nonlinear variational inequalities*, Numer. Math., 54, 601-618, 1989
29. Rannacher, R., and Scott, R.: *Some optimal error estimates for piecewise linear finite element approximations*, Math. Comp., 38, 158, 437-445, 1982
30. Schatz, A.H., and Wahlbin, L.B.: *Interior maximum-norm estimates for finite element methods, part II*, Math. Comp., 64, 211, 907-928, 1995
31. Scott, R.: *Optimal L^∞-estimates for the finite element method on irregular meshes*, Math. Comp., 30, 136, 681-697, 1976
32. Steinbach, O.: *Boundary element methods for variational inequalities*, Numer. Math., 1, 126, 173-197, 2014
33. Steinbach, O., Wohlmuth, B., and Wunderlich, L.: *Trace and flux a priori error estimates in finite-element approximations of Signorini-type problems*, IMA J. Numer. Anal., 36, 1072-1095, 2016
34. Strang, G.: *One-sided approximation and plate bending*, Computing Methods in Applied Sciences and Engineering Part 1, Lect. Notes Comput. Sci., 10, 140-155, 1974
35. Suttmeier, F.-T.: *Numerical Solution of Variational Inequalities by Adaptive Finite Elements*, Vieweg-Teubner, Wiesbaden, 2008