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Optimal control of a linear unsteady fluid-structure interaction problem

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Abstract In this paper, we consider optimal control problems governed by a linear unsteady fluid-structure interaction problem. Based on a novel symmetric monolithic formulation, we derive optimality systems and provide regularity results for optimal solutions. The proposed formulation allows for natural application of gradient based optimization algorithms and for space-time finite element discretizations.

Keywords unsteady fluid-structure interaction, optimal control, adjoint equation, optimality conditions

Mathematics Subject Classification (2000)

1 Introduction

Fluid-structure interaction (FSI) problems have been extensively studied from theoretical and numerical point of view in the last decade. More and more applications leading to optimal control, shape optimization, and parameter estimation of FSI are regarded recently. At the same time, efficient gradient based optimization algorithms for solving optimization problems governed by elliptic, parabolic, or hyperbolic equations are developed and deeply analyzed in the literature. All these algorithms are based on optimality systems containing appropriate adjoint equations and building necessary optimality conditions for considered problems. However, especially in the context of optimal control problems for *unsteady* FSI problems no optimality systems based on rigorous analysis are avaliable in the literature.

Optimization problems for unsteady FSI configurations are studied, e.g., in [6–8, 11, 23, 34], where different optimization algorithms are used, which are not based on adjoint equations. In [10, 26], adjoint equations are derived for one dimensional

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FSI configurations and in [36] for a stationary FSI problem. In [5, 11, 32], the authors discretize the FSI problem in space and time and solve in each time step a minimization problem by a gradient based algorithm, in order to estimate Young's modulus. In [31], the authors derive formally necessary optimality conditions for an optimal control problem of a nonlinear time dependent FSI configuration using shape derivatives. Further results on optimal feedback control of FSI can be found in [9, 21, 22] where corresponding Riccati equations are derived. In [25], the authors apply reduced basis methods for a shape optimization problem in context of arterial blood flow.

In this paper, we formulate a model optimal control problem governed by a linear FSI problem, establish necessary optimality conditions, and analyze the regularity of the optimal solutions. To this end, we propose a novel *symmetric* monolithic formulation for the linear FSI problem. This formulation leads to an adjoint equation with the same structure as the considered linear FSI problem, which allows for a unified analytical and numerical treatment of the state and the adjoint systems.

One of the main issues in the analysis as well as in the numerical solution of FSI problems is the incorporation of coupling conditions between the equations describing the behavior of the fluid and of the structure respectively. A correct treatment of such conditions for the adjoint system is indispensable for a precise description of the information transport across the interface between the fluid and the structure, and as a consequence, for an accurate evaluation of the derivatives required in gradient based optimization algorithms. In the framework suggested in this paper, the coupling conditions in the adjoint systems have exactly the same structure as for the state system. This is advantageous not only from the theoretical point of view but especially allows to use the same discretization schemes and the same practical solution algorithms for both the state and the adjoint systems.

The fact that the coupling conditions are directly incorporated in the variational formulation allows for a natural usage of Galerkin finite element discretizations is space and time. This is advantageous particularly for optimal control problems, since the two approaches optimize-then-discretize, i.e., the discretization of the optimality system from continuous level, and discretize-then-optimize, i.e., discretization of the state equation and subsequent construction of the optimality system on the discrete level, lead to the same discretization scheme, see, e.g., [4]. We refer, e.g., to [27–30] for a priori numerical analysis and adaptivity for Galerkin discretizations of parabolic optimal control problems. The application of these techniques to the FSI problem under consideration is a topic for future work.

Usually in FSI models, the Navier-Stokes equations are coupled with a nonlinear hyperbolic equation. As the solid motion involves large stress-induced displacements, the fluid domain is not stationary. To solve the resulting system the Navier-Stokes equations can be transformed to a reference domain, see, e.g., [33, 39]. This leads to a highly nonlinear coupled system. As our goal is to focus on the treatment of coupling conditions, we regard the linear FSI problem, where the Stokes equations are coupled with a linear wave equation on a domain with fixed interface, see a detailed description below. Although such a linear system neglects several problem relevant phenomena, we belive that our results provide an important step towards tackling optimal control problems for nonlinear FSI models. Such linear FSI configurations have been already analyzed in [12, 13], wherein the authors prove existence and regularity results. A further class of nonlinear FSI models with a fixed interface for fluid flow motion around a smooth elastic object is analyzed in [2,3,20], where in [3,20] the authors even prove the existence of strong solutions for smooth initial conditions. A similar result can be found in [19] for a problem with a solid separating two fluid domains and periodic boundary conditions. If a damping term in the wave equation is introduced, better regularity results can be achieved, see [1,17]. Long-time behavior of the linearized FSI problem is analyzed in [40].

In this article, we will analyze the following linear-quadratic optimal control problem subject to a linear fluid-structure interaction problem:

$$\min J(q,u,v) := \frac{\gamma_{\mathrm{f}}}{2}\int_I \|v - v_{\mathrm{d}}\|^2_{L^2(\Omega_{\mathrm{f}})}\,\mathrm{d}t + \frac{\gamma_{\mathrm{s}}}{2}\int_I \|u - u_{\mathrm{d}}\|^2_{L^2(\Omega_{\mathrm{s}})}\,\mathrm{d}t + \frac{\alpha}{2}\|q\|^2_Q$$

subject to

$$\begin{cases} \rho_{\mathrm{f}}\partial_t v - \nu\Delta v - \nabla p = B_{\mathrm{f}} q & \text{in } \Omega_{\mathrm{f}} \times I, \\ \operatorname{div} v = 0 & \text{in } \Omega_{\mathrm{f}} \times I, \\ \rho_{\mathrm{s}}\partial_{tt} u - \mu\Delta u = B_{\mathrm{s}} q & \text{in } \Omega_{\mathrm{s}} \times I, \\ & \\ \nu\partial_{n_{\mathrm{f}}} v - p n_{\mathrm{f}} + \mu\partial_{n_{\mathrm{s}}} u = 0 & \text{on } \Gamma_{\mathrm{i}} \times I, \\ v = \partial_t u & \text{on } \Gamma_{\mathrm{i}} \times I, \\ v = 0 & \text{on } \Gamma_{\mathrm{f}} \times I, \\ u = 0 & \text{on } \Gamma_{\mathrm{s}} \times I, \\ & \\ u(0) = u_0, \quad \partial_t u(0) = u_1, \quad v(0) = v_0 & \\ q_a \le q \le q_b. & \end{cases} \tag{1.1}$$

Thereby, $\Omega \subset \mathbb{R}^d$ with $d = 2,3$ is a domain separated in two disjoint Lipschitz sub-domains Ω_{s} and Ω_{f} with $\bar{\Omega} = \bar{\Omega}_{\mathrm{s}} \cup \bar{\Omega}_{\mathrm{f}}$ as presented in Figure 1.1. Furthermore, $I = (0,T)$ is a given time interval and $\Gamma := \partial\Omega$ denotes the outer boundary which is split into two parts $\Gamma_{\mathrm{s}} := \Gamma \cap \overline{\Omega_{\mathrm{s}}}$ and $\Gamma_{\mathrm{f}} := \Gamma \cap \overline{\Omega_{\mathrm{f}}}$, where we assume that Γ_{s} has positive measure in Γ. The interface between Ω_{f} and Ω_{s} is denoted by $\Gamma_{\mathrm{i}} := \overline{\Omega_{\mathrm{f}}} \cap \overline{\Omega_{\mathrm{s}}}$. Moreover, n_{s} is the unit outward normal vector on Γ_{i} with respect to the region Ω_{s} and $n_{\mathrm{f}} = -n_{\mathrm{s}}$ is the unit outward normal vector with respect to the region Ω_{s}.

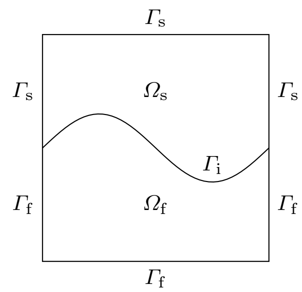

Fig. 1.1 An exemplary domain Ω with fixed interface Γ_{i}

On Ω_f, the fluid is described via the Stokes equations and the structure on Ω_s behaves according to the linear wave equation where ρ_f, ρ_s, ν and μ are given material parameters. We note, that all the results presented in this paper can be extended the model problem, where the wave equations is replaced by the linear Lamé system.

The variables v and p denote the velocity and pressure of the fluid and u denotes the structure displacement on Ω_s. At the interface Γ_i, the momentum has to be conserved. Therefore, we demand the directional derivatives to coincide (dynamic coupling condition)

$$\rho_\mathrm{f}\nu\partial_{n_\mathrm{f}} v - p n_\mathrm{f} = -\mu\partial_{n_\mathrm{s}} u \quad \text{on } \Gamma_\mathrm{i} \times I.$$

Additionally, the fluid is not allowed to enter the structure domain (kinematic coupling condition). For slow fluid flow, we can assume a no slip condition which implies that structure and fluid velocity have to be equal. Therefore, we demand at the interface

$$v = \partial_t u \quad \text{on } \Gamma_\mathrm{i} \times I.$$

At the outer boundaries Γ_f and Γ_s, we prescribe homogeneous Dirichlet boundary conditions.

The control q is going to be either time dependent or distributed in space and controlling the volume force through the linear operators B_f and B_s, see two the configurations in Section 3 for details. In addition, the control variable is subject to the control constrains with the bounds $q_a, q_b \in \mathbb{R}\cup\{\pm\infty\}$ and $q_a < q_b$. The variables v_d and u_d are the given desired states and $\alpha > 0$ is a given regularization parameter. To enable observation on both or just on one sub-domain, the parameters $\gamma_\mathrm{f}, \gamma_\mathrm{s} \geq 0$ can be chosen appropriately.

The rest of the paper is organized as follows. In Section 2, after a general discussion of a linear FSI problem, we recall known existence and regularity results from the literature for this model in Subsection 2.1. In Subsection 2.2, we introduce a novel symmetric monolithic formulation and adapt the results from Subsection 2.1 to it. Section 3 is devoted to the optimal control problem. After description of two model configurations, we discuss the existence of the optimal solution in Subsection 3.1 before we prove our main results on the optimality system in Subsection 3.2.

2 A linear FSI problem

To keep the notation as compact as possible we introduce the vector-valued spaces

$$V := \left\{ \varphi \in H^1(\Omega)^d \;\middle|\; \varphi = 0 \text{ on } \Gamma \right\},$$
$$V_\mathrm{f} := \left\{ \varphi \in H^1(\Omega_\mathrm{f})^d \;\middle|\; \varphi = 0 \text{ on } \Gamma_\mathrm{f} \right\}, \quad \text{and} \quad V_\mathrm{s} := \left\{ \varphi \in H^1(\Omega_\mathrm{s})^d \;\middle|\; \varphi = 0 \text{ on } \Gamma_\mathrm{s} \right\}$$

with trace zero on parts of the boundary. In addition, we will need the spaces of divergence free functions

$$V_\mathrm{div} := \{\, v \in V \mid \operatorname{div} v = 0 \text{ on } \Omega_\mathrm{f} \,\} \quad \text{and} \quad V_{\mathrm{f},\mathrm{div}} := \{\, v \in V_\mathrm{f} \mid \operatorname{div} v = 0 \text{ on } \Omega_\mathrm{f} \,\}.$$

Thereby, the divergence condition is only demanded on the fluid domain Ω_f. Furthermore, we introduce the vector valued L^2 spaces

$$H := L^2(\Omega)^d, \quad H_\text{f} := L^2(\Omega_\text{f})^d, \quad \text{and} \quad H_\text{s} := L^2(\Omega_\text{s})^d.$$

Finally, for the pressure variable, we introduce the following L^2 space:

$$L_\text{f} := \left\{ p \in L^2(\Omega_\text{f}) \;\middle|\; (p,1)_\text{f} = 0 \right\}.$$

Here and in what follows, let

$$\begin{aligned} (u,v) &:= (u,v)_\Omega, & \langle u,v\rangle_\text{i} &:= \langle u,v\rangle_{\Gamma_\text{i}}, \\ (u,v)_\text{f} &:= (u,v)_{\Omega_\text{f}}, & (u,v)_\text{s} &:= (u,v)_{\Omega_\text{s}} \end{aligned}$$

be the L^2 inner products on Ω, its sub-domains Ω_f and Ω_s, and on Γ_i. Furthermore, we use the following notation for inner products on the space-time cylinder:

$$\begin{aligned} (\!(u,v)\!) &= \int_I (u,v)\ \text{d}t, & \langle\!\langle u,v\rangle\!\rangle_\text{i} &= \int_I \langle u,v\rangle_\text{i}\ \text{d}t, \\ (\!(u,v)\!)_\text{f} &= \int_I (u,v)_\text{f}\ \text{d}t, & (\!(u,v)\!)_\text{s} &= \int_I (u,v)_\text{s}\ \text{d}t. \end{aligned}$$

Monolithic formulations are well known for nonlinear FSI problems and they are used for example in [14,35] to obtain robust numerical algorithms keeping errors occurring from the coupling conditions small. Such formulations are usually obtained by transforming a weak formulation of the FSI equations in a system of first order in time by introducing a structure velocity variable. Then, the kinematic coupling condition is enforced by choosing a smooth trial space for the common velocity variable defined on the whole domain. Furthermore, due to a test function defined in the same space, the dynamic coupling condition is automatically fulfilled. In the case of the here considered linear FSI problem (1.1), this leads to a velocity $v \in L^2(I;V)$, a pressure $p \in L^2(I;L_\text{f})$, and a structure displacement $u \in L^2(I;V_\text{s})$ fulfilling the weak formulation

$$\begin{aligned} &\rho_\text{f}(\!(\partial_t v,\varphi)\!)_\text{f} + \nu(\!(\nabla v,\nabla\varphi)\!)_\text{f} - (\!(p,\operatorname{div}\varphi)\!)_\text{f} \\ &\qquad + \rho_\text{s}(\!(\partial_t v,\varphi)\!)_\text{s} + \mu(\!(\nabla u,\nabla\varphi)\!)_\text{s} = (\!(B_\text{f} q,\varphi)\!)_\text{f} + (\!(B_\text{s} q,\varphi)\!)_\text{s} \quad \forall\varphi \in L^2(I;V) \end{aligned} \tag{2.1}$$

$$(\!(v,\psi)\!)_\text{s} - (\!(\partial_t u,\psi)\!)_\text{s} = 0 \qquad \forall\psi \in L^2(I;H_\text{s}) \tag{2.2}$$

$$(\!(\xi,\operatorname{div} v)\!)_\text{f} = 0 \qquad \forall\xi \in L^2(I;L_\text{f}). \tag{2.3}$$

together with the initial conditions

$$u(0) = u_0, \quad v(0)\big|_{\Omega_\text{s}} = u_1, \quad \text{and} \quad v(0)\big|_{\Omega_\text{f}} = v_0. \tag{2.4}$$

The velocity v now describes the fluid velocity on Ω_f and the velocity of the structure on Ω_s.

For optimal control, this formulation has some drawbacks due to its asymmetry, see Section 3.2.1 for details. Because of this, we favor a slightly different formulation which is motivated by an approach used by Johnson in [18] in the contex of the wave equation. He suggests to introduce a velocity variable v which fulfills

$$\mu(\!(\nabla v, \nabla\psi)\!)_{\mathrm{s}} - \mu(\!(\nabla\partial_t u, \nabla\psi)\!)_{\mathrm{s}} = 0 \quad \forall\psi \in L^2(I; V_{\mathrm{s}}) \tag{2.5}$$

instead of (2.2). As the resulting weak equations (2.1), (2.3), and (2.5) are symmetric and test and trial spaces coincide, the resulting linear FSI system is self adjoint.

2.1 Known results from the literature

The stated linear fluid-structure interaction problem (1.1) was intensively studied in [12] and [13] by Du, Gunzburger, and coworkers. The following proposition taken from there ensures existence and uniqueness of a solution.

Proposition 2.1 (Theorem 2.5 in [12] and Theorem 2.2 in [13]) *Assume that* f_{f}, f_{s}, u_0, u_1, *and* v_0 *satisfy*

$$\begin{gathered} f_{\mathrm{f}} \in L^2(I; V_{\mathrm{f}}^*), \quad f_{\mathrm{s}} \in L^2(I; H_{\mathrm{s}}) \\ u_0 \in V_{\mathrm{s}}, \quad u_1 \in V_{\mathrm{s}}, \quad v_0 \in V_{\mathrm{f,div}}, \quad v_0\big|_{\Gamma_{\mathrm{i}}} = u_1\big|_{\Gamma_{\mathrm{i}}}. \end{gathered}$$

Then, there exists a unique $\tilde{v} \in H^1(I; V_{\mathrm{div}}^*)$ *with*

$$\begin{aligned} v = \tilde{v}\big|_{\Omega_{\mathrm{f}}} &\in L^2(I; V_{\mathrm{f,div}}) \cap L^\infty(I; H_{\mathrm{f}}) \\ \text{and} \quad u = \int_0^t \tilde{v}(s)\big|_{\Omega_{\mathrm{s}}}\, ds + u_0 &\in L^\infty(I; V_{\mathrm{s}}) \cap W^{1,\infty}(I; H_{\mathrm{s}}) \end{aligned}$$

satisfying the initial conditions $v(0) = v_0$ *in* H_{f}, $u(0) = u_0$ *in* V_{s}, *and* $\partial_t u(0) = u_1$ *in* H_{s} *as well as the coupling condition*

$$\int_0^t v(s)\big|_{\Gamma_{\mathrm{i}}}\, ds = u(t)\big|_{\Gamma_{\mathrm{i}}} - u_0\big|_{\Gamma_{\mathrm{i}}} \text{ in } L^2(I; H^{\frac{1}{2}}(\Gamma_{\mathrm{i}})^d)$$

and almost everywhere in I

$$\begin{aligned} \rho_{\mathrm{f}}(\partial_t\tilde{v}, \varphi)_{\mathrm{f}} + \nu(\nabla v, \nabla\varphi)_{\mathrm{f}} + \rho_{\mathrm{s}}(\partial_t\tilde{v}, \varphi)_{\mathrm{s}} + \mu(\nabla u, \nabla\varphi)_{\mathrm{s}} \\ = (f_{\mathrm{f}}, \varphi)_{\mathrm{f}} + (f_{\mathrm{s}}, \varphi)_{\mathrm{s}} \quad \forall\varphi \in V_{\mathrm{div}}. \end{aligned}$$

Furthermore, the solution fulfills the following a priori estimate:

$$\begin{aligned} \|v\|^2_{L^2(I;H^1(\Omega_{\mathrm{f}}))} + \|v\|^2_{L^\infty(I;L^2(\Omega_{\mathrm{f}}))} + \|u\|^2_{L^\infty(I;H^1(\Omega_{\mathrm{s}}))} + \|\partial_t u\|^2_{L^\infty(I;L^2(\Omega_{\mathrm{s}}))} \\ \le C\big[\|f_{\mathrm{f}}\|^2_{L^2(I;V_{\mathrm{f}}^*)} + \|f_{\mathrm{s}}\|^2_{L^2(I;L^2(\Omega_{\mathrm{s}}))}\big] \\ + C\big[\|u_0\|^2_{H^1(\Omega_{\mathrm{s}})} + \|u_1\|^2_{H^1(\Omega_{\mathrm{s}})} + \|v_0\|^2_{H^1(\Omega_{\mathrm{f}})}\big] \end{aligned}$$

Remark 2.1 Clearly, the solution given by Proposition 2.1 fulfills also the following space-time weak formulation:

$$\begin{aligned} \rho_{\mathrm{f}}(\!(\partial_t\tilde{v}, \varphi)\!)_{\mathrm{f}} + \nu(\!(\nabla v, \nabla\varphi)\!)_{\mathrm{f}} + \rho_{\mathrm{s}}(\!(\partial_t\tilde{v}, \varphi)\!)_{\mathrm{s}} + \mu(\!(\nabla u, \nabla\varphi)\!)_{\mathrm{s}} \\ = (\!(f_{\mathrm{f}}, \varphi)\!)_{\mathrm{f}} + (\!(f_{\mathrm{s}}, \varphi)\!)_{\mathrm{s}} \quad \forall\varphi \in L^2(I; V_{\mathrm{div}}). \end{aligned} \tag{2.6}$$

Remark 2.2 The results in [2] and [20] indicate, that even for weaker initial conditions there exists a unique solution solving the linear FSI problem. To prove existence, the authors therein use maximal regularity results for the Stokes operator and hidden regularity results for the hyperbolic equation.

If the initial conditions and the right-hand side are smoother and fulfill compatibility conditions, then in [12,13] the authors provide an additional regularity result.

Proposition 2.2 (Theorems 3.2 and 3.4 in [12] and Theorem 2.3 in [13]) *Assume that f_f, f_s, u_0, u_1, and v_0 satisfy*

$$f_\mathrm{f} \in H^1(I; V_\mathrm{f}^*), \quad f_\mathrm{s} \in H^1(I; H_\mathrm{s})$$
$$u_0 \in V_\mathrm{s} \cap H^2(\Omega_\mathrm{s})^d, \quad u_1 \in V_\mathrm{s}, \quad v_0 \in V_{\mathrm{f,div}} \cap H^2(\Omega_\mathrm{f})^d \quad v_0\big|_{\Gamma_\mathrm{i}} = u_1\big|_{\Gamma_\mathrm{i}}.$$

Assume further that there exists a $p_0 \in H^1(\Omega_\mathrm{f})$ such that

$$(p_0 n_\mathrm{f} - \nu \nabla v_0^T n_\mathrm{f})\big|_{\Gamma_\mathrm{i}} = (\mu \nabla u_0^T n_\mathrm{s})\big|_{\Gamma_\mathrm{i}}.$$

Then, there exists a unique triplet (v, p, u) with

$$v \in H^1(I; V_\mathrm{f}) \cap W^{1,\infty}(I; H_\mathrm{f}), \quad p \in L^2(I; L_\mathrm{f}), \quad u \in W^{1,\infty}(I; V_\mathrm{s}) \cap W^{2,\infty}(I; H_\mathrm{s})$$

satisfying the initial conditions $v(0) = v_0$ in H_f, $u(0) = u_0$ in V_s, and $\partial_t u(0) = u_1$ in H_s as well the coupling condition

$$v\big|_{\Gamma_\mathrm{i}} = \partial_t u\big|_{\Gamma_\mathrm{i}} \text{ in } L^2(I; H^{\frac{1}{2}}(\Gamma_\mathrm{i})^d)$$

and almost everywhere in I

$$\begin{aligned} \rho_\mathrm{f}(\partial_t v, \varphi)_\mathrm{f} - (p, \operatorname{div} \varphi)_\mathrm{f} + \nu(\nabla v, \nabla \varphi)_\mathrm{f} & \\ + \rho_\mathrm{s}(\partial_{tt} u, \varphi)_\mathrm{s} + \mu(\nabla u, \nabla \varphi)_\mathrm{s} &= (f_\mathrm{f}, \varphi)_\mathrm{f} + (f_\mathrm{s}, \varphi)_\mathrm{s} \qquad \forall \varphi \in V, \\ (\xi, \operatorname{div} v)_\mathrm{f} &= 0 \qquad \forall \xi \in L_\mathrm{f}. \end{aligned}$$

Furthermore, the solution fulfills the estimates from Proposition 2.1 and the following a priori estimates:

a)
$$\begin{aligned} \|\partial_t v\|^2_{L^2(I;H^1(\Omega_\mathrm{f}))} + \|\partial_t v\|^2_{L^\infty(I;L^2(\Omega_\mathrm{f}))} + \|\partial_t u\|^2_{L^\infty(I;H^1(\Omega_\mathrm{s}))} + \|\partial_{tt} u\|^2_{L^\infty(I;L^2(\Omega_\mathrm{s}))} \\ \le C\big[\|f_\mathrm{f}\|^2_{H^1(I;V_\mathrm{f}^*)} + \|f_\mathrm{s}\|^2_{H^1(I;L^2(\Omega_\mathrm{s}))}\big] \\ + C\big[\|u_0\|^2_{H^2(\Omega_\mathrm{s})} + \|u_1\|^2_{H^1(\Omega_\mathrm{s})} + \|v_0\|^2_{H^2(\Omega_\mathrm{f})} + \|p_0\|^2_{H^1(\Omega_\mathrm{f})}\big], \end{aligned}$$

b)
$$\begin{aligned} \|p\|^2_{L^2(I;L^2(\Omega_\mathrm{f}))} \le C\big[\|f_\mathrm{f}\|^2_{H^1(I;V_\mathrm{f}^*)} + \|f_\mathrm{s}\|^2_{H^1(I;L^2(\Omega_\mathrm{s}))}\big] \\ + C\big[\|u_0\|^2_{H^2(\Omega_\mathrm{s})} + \|u_1\|^2_{H^1(\Omega_\mathrm{s})} + \|v_0\|^2_{H^2(\Omega_\mathrm{f})} + \|p_0\|^2_{H^1(\Omega_\mathrm{f})}\big]. \end{aligned}$$

Remark 2.3 As before, the solution given by Proposition 2.2 also fulfills the weak space-time formulation

$$\begin{aligned} \rho_\mathrm{f}(\!(\partial_t v, \varphi)\!)_\mathrm{f} - (\!(p, \operatorname{div} \varphi)\!)_\mathrm{f} + \nu(\!(\nabla v, \nabla \varphi)\!)_\mathrm{f} & \\ + \rho_\mathrm{s}(\!(\partial_{tt} u, \varphi)\!)_\mathrm{s} + \mu(\!(\nabla u, \nabla \varphi)\!)_\mathrm{s} &= (\!(f_\mathrm{f}, \varphi)\!)_\mathrm{f} + (\!(f_\mathrm{s}, \varphi)\!)_\mathrm{s} \qquad \forall \varphi \in L^2(I; V), \\ (\!(\xi, \operatorname{div} v)\!)_\mathrm{f} &= 0 \qquad \forall \xi \in L^2(I; L_\mathrm{f}). \end{aligned} \tag{2.7}$$

Remark 2.4 In [12,13], the authors demand $f_{\mathrm{f}} \in L^2(I; H_{\mathrm{f}})$ in Proposition 2.1 and $f_{\mathrm{f}} \in H^1(I; H_{\mathrm{f}})$ in Proposition 2.2. However, the proofs can directly be extended to $f_{\mathrm{f}} \in L^2(I; V_{\mathrm{f}}^*)$ and $f_{\mathrm{f}} \in H^1(I; V_{\mathrm{f}}^*)$ as stated above.

Remark 2.5 The results in [12,13] are more general and also apply for Stokes flow coupled with linear elasticity equations. Therefore, all the results presented in the following are also extendable to formulations with stress tensors.

2.2 Novel symmetric weak formulation

Now, we would like to regard closer the symmetric weak formulation given by (2.1), (2.3), and (2.5), which was already motivated at the beginning of this section. In doing so, we consider the following system of equations where, for the remaining part of this section, the control terms are replaced by right-hand sides f_{f} and f_{s}:

$$\begin{aligned} \rho_{\mathrm{f}}(\!(\partial_t v, \varphi)\!)_{\mathrm{f}} + \nu(\!(\nabla v, \nabla\varphi)\!)_{\mathrm{f}} - (\!(p, \operatorname{div}\varphi)\!)_{\mathrm{f}} & \\ + \rho_{\mathrm{s}}(\!(\partial_t v, \varphi)\!)_{\mathrm{s}} + \mu(\!(\nabla u, \nabla\varphi)\!)_{\mathrm{s}} &= (\!(f_{\mathrm{f}}, \varphi)\!)_{\mathrm{f}} + (\!(f_{\mathrm{s}}, \varphi)\!)_{\mathrm{s}} \quad \forall \varphi \in L^2(I; V), \\ \mu(\!(\nabla v, \nabla\psi)\!)_{\mathrm{s}} - \mu(\!(\nabla\partial_t u, \nabla\psi)\!)_{\mathrm{s}} &= (\!(g, \psi)\!)_{\mathrm{s}} \quad \forall \psi \in L^2(I; V_{\mathrm{s}}), \\ (\!(\xi, \operatorname{div} v)\!)_{\mathrm{f}} &= 0 \quad \forall \xi \in L^2(I; L_{\mathrm{f}}). \end{aligned} \tag{2.8}$$

Additionally, as before, the initial conditions given by (2.4) have to be fulfilled. Note, that the volume force g appearing on the right-hand side of the equation introducing the structure velocity has no physical interpretation but will occur later in the adjoint equation, see Section 3.

The results in Section 2.1 enable us to prove existence and uniqueness of a solution for this weak formulation.

Theorem 2.1 *Assume that u_0, u_1, and v_0 satisfy*

$$u_0 \in V_{\mathrm{s}} \cap H^2(\Omega_{\mathrm{s}})^d, \quad u_1 \in V_{\mathrm{s}}, \quad v_0 \in V_{\mathrm{f,div}} \cap H^2(\Omega_{\mathrm{f}})^d, \quad v_0\big|_{\Gamma_{\mathrm{i}}} = u_1\big|_{\Gamma_{\mathrm{i}}}.$$

and the right-hand sides f_{f}, f_{s}, and g fulfill

$$f_{\mathrm{f}} \in H^1(I; V_{\mathrm{f}}^*), \quad f_{\mathrm{s}} \in H^1(I; H_{\mathrm{s}}), \quad g \in L^2(I; H_{\mathrm{s}}).$$

Assume further that there exists a $p_0 \in H^1(\Omega_{\mathrm{f}})$ such that

$$(p_0 n_{\mathrm{f}} - \nu \nabla v_0^T n_{\mathrm{f}})\big|_{\Gamma_{\mathrm{i}}} = (\mu \nabla u_0^T n_{\mathrm{s}})\big|_{\Gamma_{\mathrm{i}}}.$$

Then, there exists a unique triplet (v, p, u) with

$$\begin{gathered} v \in L^2(I; V) \cap W^{1;\infty}(I, H), \quad v\big|_{\Omega_{\mathrm{f}}} \in H^1(I; V_{\mathrm{f}}), \quad v\big|_{\Omega_{\mathrm{s}}} \in L^\infty(I; V_{\mathrm{s}}), \\ u \in L^\infty(I; V_{\mathrm{s}}) \cap H^1(I; V_{\mathrm{s}}), \quad p \in L^2(I; L_{\mathrm{f}}), \end{gathered}$$

which satisfies the initial conditions (2.4) *and solves* (2.8). *Furthermore, the solution fulfills the following a priori estimates:*

a)
$$\begin{aligned} \|v\|^2_{L^\infty(I;L^2(\Omega))} + \|v\|^2_{L^2(I;H^1(\Omega_{\mathrm{f}}))} + \|u\|^2_{L^\infty(I;H^1(\Omega_{\mathrm{s}}))} & \\ \le C\big[\|f_{\mathrm{f}}\|^2_{L^2(I;V_{\mathrm{f}}^*)} + \|f_{\mathrm{s}}\|^2_{L^2(I;L^2(\Omega_{\mathrm{s}}))} + \|g\|^2_{L^2(I;L^2(\Omega_{\mathrm{s}}))}\big] & \\ + C\big[\|u_0\|^2_{H^1(\Omega_{\mathrm{s}})} + \|u_1\|^2_{H^1(\Omega_{\mathrm{s}})} + \|v_0\|^2_{H^1(\Omega_{\mathrm{f}})}\big], & \end{aligned}$$

b) $$\|\partial_t v\|^2_{L^\infty(I;L^2(\Omega))} + \|\partial_t v\|^2_{L^2(I;H^1(\Omega_{\mathrm{f}}))} + \|v\|^2_{L^\infty(I;H^1(\Omega_{\mathrm{s}}))} + \|\partial_t u\|^2_{L^2(I;H^1(\Omega_{\mathrm{s}}))}$$
$$\leq C\big[\|f_{\mathrm{f}}\|^2_{H^1(I;V_{\mathrm{f}}^*)} + \|f_{\mathrm{s}}\|_{H^1(I;L^2(\Omega_{\mathrm{s}}))} + \|g\|^2_{L^2(I;L^2(\Omega_{\mathrm{s}}))}\big]$$
$$+ C\big[\|u_0\|^2_{H^2(\Omega_{\mathrm{s}})} + \|u_1\|^2_{H^1(\Omega_{\mathrm{s}})} + \|v_0\|^2_{H^2(\Omega_{\mathrm{f}})} + \|p_0\|^2_{H^1(\Omega_{\mathrm{f}})}\big],$$

c) $$\|p\|^2_{L^2(I;L^2(\Omega_{\mathrm{f}}))} \leq C\big[\|f_{\mathrm{f}}\|^2_{H^1(I;V_{\mathrm{f}}^*)} + \|f_{\mathrm{s}}\|^2_{H^1(I;L^2(\Omega_{\mathrm{s}}))} + \|g\|^2_{L^2(I;L^2(\Omega_{\mathrm{s}}))}\big]$$
$$+ C\big[\|u_0\|^2_{H^2(\Omega_{\mathrm{s}})} + \|u_1\|^2_{H^1(\Omega_{\mathrm{s}})} + \|v_0\|^2_{H^2(\Omega_{\mathrm{f}})} + \|p_0\|^2_{H^1(\Omega_{\mathrm{f}})}\big].$$

Proof Let $\hat f_{\mathrm{f}} := f_{\mathrm{f}}$ and $\hat f_{\mathrm{s}} := f_{\mathrm{s}} + \int_0^t g(s)\,\mathrm{d}s$. Due to the assumptions on the data, we have $\hat f_{\mathrm{f}} \in H^1(I;V_{\mathrm{f}}^*)$ and $\hat f_{\mathrm{s}} \in H^1(I;H_{\mathrm{s}})$. Therefore, Proposition 2.2 ensures for these right-hand sides and the given initial conditions existence of a unique triplet $(\hat v_{\mathrm{f}}, \hat p, \hat u)$ solving (2.7). Next, we introduce a structure velocity $\hat v_{\mathrm{s}}$ by the setting $\hat v_{\mathrm{s}} = \partial_t \hat u \in L^\infty(I;V_{\mathrm{s}})$. Thus, $\hat v_{\mathrm{s}}$ fulfills

$$\mu(\!(\nabla \hat v_{\mathrm{s}}, \nabla\psi)\!)_{\mathrm{s}} = \mu(\!(\nabla\partial_t \hat u, \nabla\psi)\!)_{\mathrm{s}} \quad \forall \psi \in L^2(I;V_{\mathrm{s}}). \tag{2.9}$$

Now we are prepared to introduce the global velocity $\hat v$ by

$$\hat v\big|_{\Omega_{\mathrm{f}}} = \hat v_{\mathrm{f}} \quad \text{and} \quad \hat v\big|_{\Omega_{\mathrm{s}}} = \hat v_{\mathrm{s}}.$$

As $\hat v_{\mathrm{f}} \in L^2(I;V_{\mathrm{f}})$ and $\hat v_{\mathrm{s}} \in L^2(I;V_{\mathrm{s}})$, we get immediately $\hat v \in L^2(I;H)$. To obtain $\hat v \in L^2(I;V)$, we have to check that the weak partial derivatives $\hat w_i$ with $\hat w_i\big|_{\Omega_{\mathrm{f}}} := \partial_{x_i}\hat v_{\mathrm{f}}$ and $\hat w_i\big|_{\Omega_{\mathrm{s}}} := \partial_{x_i}\hat v_{\mathrm{s}}$ constitute the weak partial derivatives $\partial_{x_i}\hat v$ of $\hat v$ for $i = 1, 2, \ldots, d$. To this end, let $\varphi \in L^2(I;C_0^\infty(\Omega)^d)$. We obtain by the definition of the weak derivatives

$$\begin{aligned}(\!(\hat v, \partial_{x_i}\varphi)\!) &= (\!(\hat v_{\mathrm{f}}, \partial_{x_i}\varphi)\!)_{\mathrm{f}} + (\!(\hat v_{\mathrm{s}}, \partial_{x_i}\varphi)\!)_{\mathrm{s}} \\ &= -(\!(\partial_{x_i}\hat v_{\mathrm{f}}, \varphi)\!)_{\mathrm{f}} - (\!(\partial_{x_i}\hat v_{\mathrm{s}}, \varphi)\!)_{\mathrm{s}} + \langle\!\langle \hat v_{\mathrm{f}}, \varphi n_{\mathrm{f}}^T e_i\rangle\!\rangle_{\mathrm{i}} + \langle\!\langle \hat v_{\mathrm{s}}, \varphi n_{\mathrm{s}}^T e_i\rangle\!\rangle_{\mathrm{i}} \\ &= -(\!(\hat w_i, \varphi)\!) + \langle\!\langle \partial_t \hat u - \hat v_{\mathrm{f}}, \varphi n_{\mathrm{s}}^T e_i\rangle\!\rangle_{\mathrm{i}} = -(\!(\hat w_i, \varphi)\!)\end{aligned}$$

where the last step holds, since the kinematic coupling condition is valid due to Proposition 2.2. Therefore, it holds $\hat v \in L^2(I;V)$.

It remains to prove that $(\hat v, \hat p, \hat u)$ solves the weak formulation (2.8). Due to the construction of $\hat v_{\mathrm{s}}$ by (2.9), we directly get

$$\rho_{\mathrm{s}}(\!(\partial_{tt}\hat u, \varphi)\!)_{\mathrm{s}} = \rho_{\mathrm{s}}(\!(\partial_t \hat v, \varphi)\!)_{\mathrm{s}} \quad \forall \varphi \in L^2(I;V_{\mathrm{s}}). \tag{2.10}$$

If we enter (2.10) in the weak formulation (2.7), we immediately obtain with (2.9) that the triplet $(\hat v, \hat p, \hat u)$ solves the weak formulation (2.8) with the right-hand sides $\hat f_{\mathrm{f}}$, $\hat f_{\mathrm{s}}$ and $g = 0$.

In what follows, we construct a solution to (2.8) with the original right-hand sides f_{f}, f_{s} and g. We define $\tilde u\colon I \to V_{\mathrm{s}}$ for almost all $t \in I$ by

$$\mu(\nabla\tilde u(t), \nabla\psi)_{\mathrm{s}} = (-g(t), \psi)_{\mathrm{s}} \quad \forall \psi \in V_{\mathrm{s}}. \tag{2.11}$$

Standard elliptic theory guarantees the existence and uniqueness of $\tilde u(t)$ together with the estimate

$$\|\tilde u(t)\|_{H^1(\Omega_{\mathrm{s}})} \leq C\|g(t)\|_{L^2(\Omega_{\mathrm{s}})} \quad \text{for almost all } t \in I. \tag{2.12}$$

As $g \in L^2(I;H_\mathrm{s})$, integration in time of the above inequality immediately leads to $\tilde{u} \in L^2(I;V_\mathrm{s})$. Further, integrating (2.11) in time twice implies

$$\mu \left(\!\!\left(\nabla \int_0^t \tilde{u}(s)\ \mathrm{d}s, \nabla\psi\right)\!\!\right)_\mathrm{s} = -\left(\!\!\left(\int_0^t g(s)\ \mathrm{d}s, \psi\right)\!\!\right)_\mathrm{s} \quad \forall\psi \in L^2(I;V_\mathrm{s}).$$

Defining $u := \hat{u} + \int_0^t \tilde{u}\ \mathrm{d}s$, we directly obtain $u \in L^2(I;V_\mathrm{s})$. Since for $\varphi \in L^2(I;V)$ it holds $\psi = \varphi\big|_{\Omega_\mathrm{s}} \in L^2(I;V_\mathrm{s})$, we get for all $\varphi \in L^2(I;V)$ the identity

$$\begin{aligned}\mu(\!(\nabla\hat{u}, \nabla\varphi)\!)_\mathrm{s} &= \mu(\!(\nabla\hat{u}, \nabla\varphi)\!)_\mathrm{s} + \mu\left(\!\!\left(\nabla\int_0^t \tilde{u}(s)\ \mathrm{d}s, \nabla\varphi\right)\!\!\right)_\mathrm{s} + \left(\!\!\left(\int_0^t g(s)\ \mathrm{d}s, \varphi\right)\!\!\right)_\mathrm{s}\\ &= \mu(\!(\nabla u, \nabla\varphi)\!)_\mathrm{s} + \left(\!\!\left(\int_0^t g(s)\ \mathrm{d}s, \varphi\right)\!\!\right)_\mathrm{s}.\end{aligned}$$

Together with the definition of $\hat{f}_\mathrm{s}$, this implies that u, $v := \hat{v}$, and $p := \hat{p}$ solves the first equation of (2.8). Furthermore, since $\hat{u}$ and $\hat{v}$ solve the second equation of (2.8) with $g = 0$, we obtain for all $\psi \in L^2(I;V_\mathrm{s})$

$$\begin{aligned}\mu(\!(\nabla v, \nabla\psi)\!)_\mathrm{s} - \mu(\!(\nabla\partial_t u, \nabla\psi)\!)_\mathrm{s} &= \mu(\!(\nabla\hat{v}, \nabla\psi)\!)_\mathrm{s} - \mu(\!(\nabla\partial_t\hat{u}, \nabla\psi)\!)_\mathrm{s} - \mu(\!(\nabla\tilde{u}, \nabla\psi)\!)_\mathrm{s}\\ &= -\mu(\!(\nabla\tilde{u}, \nabla\psi)\!)_\mathrm{s} = (\!(g, \psi)\!)_\mathrm{s}.\end{aligned}$$

Therefore (v, p, u) solves the weak formulation (2.8) for the right-hand sides f_f, f_s, and g.

It remains to prove the uniqueness. Let (v_1, p_1, u_1) and (v_2, p_2, u_2) be two solutions fulfilling the weak formulation (2.8) and the regularities assumed in Theorem 2.1. Define $\bar{v} := v_1 - v_2$, $\bar{p} := p_1 - p_2$, $\bar{u} := u_1 - u_2$. It holds

$$\bar{v}\big|_{\Omega_\mathrm{f}}(0) = 0, \quad \bar{v}\big|_{\Omega_\mathrm{s}}(0) = 0, \quad \bar{u}(0) = 0$$

and for almost all $t \in I$

$$\begin{aligned}\rho_\mathrm{f}(\partial_t\bar{v}(t), \varphi)_\mathrm{f} - (\bar{p}(t), \operatorname{div}\varphi)_\mathrm{f} + \nu(\nabla\bar{v}(t), \nabla\varphi)_\mathrm{f}&\\ +\rho_\mathrm{s}(\partial_t\bar{v}(t), \varphi)_\mathrm{s} + \mu(\nabla\bar{u}(t), \nabla\varphi)_\mathrm{s} &= 0 \qquad \forall\varphi \in V,\\ \mu(\nabla\bar{v}(t), \nabla\psi)_\mathrm{s} - \mu(\nabla\partial_t\bar{u}(t), \nabla\psi)_\mathrm{s} &= 0 \qquad \forall\psi \in V_\mathrm{s},\\ (\xi, \operatorname{div}\bar{v}(t))_\mathrm{f} &= 0 \qquad \forall\xi \in L_\mathrm{f}.\end{aligned}$$

Choosing the test functions $\varphi = \bar{v}(t)$, $\psi = \bar{u}(t)$, and $\xi = \bar{p}(t)$, we get

$$\begin{aligned}\rho_\mathrm{f}(\partial_t\bar{v}(t), \bar{v}(t))_\mathrm{f} - (\bar{p}(t), \operatorname{div}\bar{v}(t))_\mathrm{f} + \nu(\nabla\bar{v}(t), \nabla\bar{v}(t))_\mathrm{f}&\\ +\rho_\mathrm{s}(\partial_t\bar{v}(t), \bar{v}(t))_\mathrm{s} + \mu(\nabla\bar{u}(t), \nabla\bar{v}(t))_\mathrm{s} &= 0,\\ \mu(\nabla\bar{v}(t), \nabla\bar{u}(t))_\mathrm{s} - \mu(\nabla\partial_t\bar{u}(t), \nabla\bar{u}(t))_\mathrm{s} &= 0,\\ (\bar{p}(t), \operatorname{div}\bar{v}(t))_\mathrm{f} &= 0.\end{aligned}$$

Because of the symmetry of the bilinear forms we obtain for almost all $t \in I$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\rho_\mathrm{f}\|\bar{v}(t)\|^2_{L^2(\Omega_\mathrm{f})} + \nu\|\nabla\bar{v}(t)\|^2_{L^2(\Omega_\mathrm{f})} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\rho_\mathrm{s}\|\bar{v}(t)\|^2_{L^2(\Omega_\mathrm{s})} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mu\|\nabla\bar{u}(t)\|^2_{L^2(\Omega_\mathrm{s})} = 0.$$

Integrating this identity in in time and noting the initial conditions, we are led to

$$\frac{1}{2}\rho_\mathrm{f}\|\bar{v}(t)\|^2_{L^2(\Omega_\mathrm{f})} + \nu\int_0^t \|\nabla\bar{v}(s)\|^2_{L^2(\Omega_\mathrm{f})}\,\mathrm{d}s + \frac{1}{2}\rho_\mathrm{s}\|\bar{v}(t)\|^2_{L^2(\Omega_\mathrm{s})} + \frac{1}{2}\mu\|\nabla\bar{u}(t)\|^2_{L^2(\Omega_\mathrm{s})} = 0$$

for almost all $t \in I$. This implies $\bar{v} = 0$ and, as $\bar{u}$ vanishes on $\Gamma_\mathrm{s} \subset \Gamma$ with $|\Gamma_\mathrm{s}| > 0$, also $\bar{u} = 0$. Thus we get in particular for almost all $t \in I$

$$(\bar{p}, \operatorname{div} \varphi)_\mathrm{f} = 0 \quad \forall \varphi \in V$$

and thus $\bar{p} = 0$ since $\bar{p} \in L_\mathrm{f}$. Therefore the solution is unique.

According to Proposition 2.2, the solution $(\hat{v}_\mathrm{f}, \hat{p}, \hat{u})$ of (2.7) fulfills the estimates given in Proposition 2.2 with right-hand side $\hat{f}_\mathrm{f} := f_\mathrm{f}$ and $\hat{f}_\mathrm{s} := f_\mathrm{s} + \int_0^t g(s)\,\mathrm{d}s$. As $\hat{v}_\mathrm{s}$ of the formulation (2.8) coincides to $\partial_t \hat{u}$ the estimates for $\partial_t \hat{u}$ from the Propositions 2.1 and 2.2 are valid for $\hat{v}\big|_{\Omega_\mathrm{s}} = \hat{v}_\mathrm{s}$, too. Hence, we have

$$\begin{aligned}
\|\hat{v}\|^2_{L^\infty(I;L^2(\Omega))} + \|\hat{v}\|^2_{L^2(I;H^1(\Omega_\mathrm{f}))} + \|\hat{u}\|^2_{L^\infty(I;H^1(\Omega_\mathrm{s}))} &\\
\le C\big[\|f_\mathrm{f}\|^2_{L^2(I;V_\mathrm{f}^*)} + \|f_\mathrm{s}\|^2_{L^2(I;L^2(\Omega_\mathrm{s}))} + \|g\|^2_{L^2(I;L^2(\Omega_\mathrm{s}))}\big]&\\
+ C\big[\|u_0\|^2_{H^1(\Omega_\mathrm{s})} + \|u_1\|^2_{H^1(\Omega_\mathrm{s})} + \|v_0\|^2_{H^1(\Omega_\mathrm{f})}\big],&\\
\|\partial_t\hat{v}\|^2_{L^\infty(I;L^2(\Omega))} + \|\partial_t\hat{v}\|^2_{L^2(I;H^1(\Omega_\mathrm{f}))} + \|\hat{v}\|^2_{L^\infty(I;H^1(\Omega_\mathrm{s}))} &\\
\le C\big[\|f_\mathrm{f}\|^2_{H^1(I;V_\mathrm{f}^*)} + \|f_\mathrm{s}\|^2_{H^1(I;L^2(\Omega_\mathrm{s}))} + \|g\|^2_{L^2(I;L^2(\Omega_\mathrm{s}))}\big]&\\
+ C\big[\|u_0\|^2_{H^2(\Omega_\mathrm{s})} + \|u_1\|^2_{H^1(\Omega_\mathrm{s})} + \|v_0\|^2_{H^2(\Omega_\mathrm{f})} + \|p_0\|^2_{H^1(\Omega_\mathrm{f})}\big],&\\
\|\hat{p}\|^2_{L^2(I;L^2(\Omega_\mathrm{f}))} \le C\big[\|f_\mathrm{f}\|^2_{H^1(I;V_\mathrm{f}^*)} + \|f_\mathrm{s}\|^2_{H^1(I;L^2(\Omega_\mathrm{s}))} + \|g\|^2_{L^2(I;L^2(\Omega_\mathrm{s}))}\big]&\\
+ C\big[\|u_0\|^2_{H^2(\Omega_\mathrm{s})} + \|u_1\|^2_{H^1(\Omega_\mathrm{s})} + \|v_0\|^2_{H^2(\Omega_\mathrm{f})} + \|p_0\|^2_{H^1(\Omega_\mathrm{f})}\big].&
\end{aligned}$$

Due to the setting $v = \hat{v}$ and $p = \hat{p}$, these estimates directly transfer to v and p. To estimate $u := \hat{u} + \int_0^t \tilde{u}(s)\,\mathrm{d}s$, $\tilde{u} \in V_\mathrm{s}$ given by (2.11) has to be bounded. By (2.12), we get for almost all $t \in I$ that

$$\begin{aligned}
\|u(t)\|^2_{H^1(\Omega_\mathrm{s})} = \left\|\hat{u}(t) + \int_0^t \tilde{u}(s)\,\mathrm{d}s\right\|^2_{H^1(\Omega_\mathrm{s})} &\le C\left[\|\hat{u}(t)\|^2_{H^1(\Omega_\mathrm{s})} + T\int_I \|\tilde{u}(s)\|^2_{H^1(\Omega_\mathrm{s})}\mathrm{d}s\right]\\
&\le C\big[\|\hat{u}(t)\|^2_{H^1(\Omega_\mathrm{s})} + \|g\|^2_{L^2(I;L^2(\Omega_\mathrm{s}))}\big],
\end{aligned}$$

which implies

$$\|u\|^2_{L^\infty(I;H^1(\Omega_\mathrm{s}))} \le C\big[\|\hat{u}\|^2_{L^\infty(I;H^1(\Omega_\mathrm{s}))} + \|g\|^2_{L^2(I;L^2(\Omega_\mathrm{s}))}\big].$$

Furthermore, we get with $\partial_t\hat{u} = \hat{v}\big|_{\Omega_\mathrm{s}}$ for almost all $t \in I$

$$\|\partial_t u(t)\|^2_{H^1(\Omega_\mathrm{s})} = \|\partial_t\hat{u}(t) + \tilde{u}(t)\|^2_{H^1(\Omega_\mathrm{s})} \le C\big[\|\hat{v}(t)\|^2_{H^1(\Omega_\mathrm{s})} + \|g(t)\|^2_{L^2(\Omega_\mathrm{s})}\big]$$

and consequently (limited through $g \in L^2(I; H_\mathrm{s})$)

$$\|\partial_t u\|^2_{L^2(I;H^1(\Omega_\mathrm{s}))} \le C\big[\|\hat{v}\|^2_{L^2(I;H^1(\Omega_\mathrm{s}))} + \|g\|^2_{L^2(I;L^2(\Omega_\mathrm{s}))}\big].$$

Together with the above estimates to for $(\hat{v}, \hat{p}, \hat{u})$, we obtain the stated estimates for (v, p, u). □

Remark 2.6 If the right-hand side g lies in $L^\infty(I; H_\mathrm{s})$, we also get an estimate for $\|\partial_t u\|_{L^\infty(I;H^1(\Omega_\mathrm{s}))}$ as in Proposition 2.2.

In what follows, we analyze in which sense the weak solution of (2.8) fulfills the original fluid-structure interaction problem and especially in which sense the coupling conditions are fulfilled. To this end, we introduce the space $\widetilde{H}^{\frac{1}{2}}(\Gamma_{\mathrm{i}})$ in the spirit of [16, Definition 1.3.2.5] by

$$\widetilde{H}^{\frac{1}{2}}(\Gamma_{\mathrm{i}}) = \left\{ v \in H^{\frac{1}{2}}(\Gamma_{\mathrm{i}}) \;\middle|\; \tilde{v} \in H^{\frac{1}{2}}(\Gamma) \right\},$$

where $\tilde{v}$ denotes the continuation of v on Γ by zero.

Theorem 2.2 *Let the assumptions of Theorem 2.1 be fulfilled and let in addition* $f_{\mathrm{f}} \in L^2(I; H_{\mathrm{f}})$ *and* (v, p, u) *be the solution of* (2.8). *Then, the kinematic coupling condition*

$$v\big|_{\Omega_{\mathrm{f}}} = v\big|_{\Omega_{\mathrm{s}}}$$

is valid in the sense of $L^2(I; H^{\frac{1}{2}}(\Gamma_{\mathrm{i}})^d) \cap H^{\frac{1}{2}}(I; L^2(\Gamma_{\mathrm{i}})^d)$. *Furthermore, the dynamic coupling condition*

$$\nu \partial_{n_{\mathrm{f}}} v - p n_{\mathrm{f}} + \mu \partial_{n_{\mathrm{s}}} u = 0$$

holds in $L^2(I; (\widetilde{H}^{\frac{1}{2}}(\Gamma_{\mathrm{i}})^d)^*)$.

Proof By Theorem 2.1, we have that $v \in L^2(I, V)$ and $v \in W^{1,\infty}(I; H) \subset H^1(I; H)$. Hence, the trace results in [24, Theorem 2.1] imply that the kinematic coupling condition $v\big|_{\Omega_{\mathrm{f}}} = v\big|_{\Omega_{\mathrm{s}}}$ holds on Γ_{i} in the space $L^2(I; H^{\frac{1}{2}}(\Gamma_{\mathrm{i}})^d) \cap H^{\frac{1}{2}}(I; L^2(\Gamma_{\mathrm{i}})^d)$.

In the remaining part of the proof, derive validity of the stated dynamic coupling condition. We choose in (2.8) test functions φ with $\varphi\big|_{\Omega_{\mathrm{s}}} = 0$ and $\varphi\big|_{\Omega_{\mathrm{f}}} \in L^2(I; C_0^\infty(\Omega_{\mathrm{f}})^d)$ and get

$$(\!(\rho_{\mathrm{f}} \partial_t v, \varphi)\!)_{\mathrm{f}} - (\!(\operatorname{div}(\nu \nabla v + p \,\mathrm{Id}), \varphi)\!)_{\mathrm{f}} = (\!(f_{\mathrm{f}}, \varphi)\!)_{\mathrm{f}} \quad \forall \varphi \in L^2(I; C_0^\infty(\Omega_{\mathrm{f}})^d),$$

where $\operatorname{div}(\nu \nabla v + p \,\mathrm{Id})$ is defined in the distributional sense. This is equivalent to

$$\operatorname{div}(\nu \nabla v + p \,\mathrm{Id}) = \rho_{\mathrm{f}} \partial_t v - f_{\mathrm{f}} \quad \text{in } L^2(I; C_0^\infty(\Omega_{\mathrm{f}})^d)^*.$$

As Theorem 2.1 yields $v\big|_{\Omega_{\mathrm{f}}} \in W^{1,\infty}(I; H_{\mathrm{f}})$, we get by the assumption on f_{f} that $\rho_{\mathrm{f}} \partial_t v + f_{\mathrm{f}} \in L^2(I; H_{\mathrm{f}})$. We immediately obtain, that

$$\operatorname{div}(\nu \nabla v + p \,\mathrm{Id}) = \rho_{\mathrm{f}} \partial_t v - f_{\mathrm{f}} \quad \text{in } L^2(I; H_{\mathrm{f}}) \tag{2.13}$$

and

$$\|\operatorname{div}(\nu \nabla v + p \,\mathrm{Id})\|_{L^2(I; L^2(\Omega_{\mathrm{f}}))} \le \|f_{\mathrm{f}}\|_{L^2(I; L^2(\Omega_{\mathrm{f}}))} + C \|\partial_t v\|_{L^2(I; L^2(\Omega_{\mathrm{f}}))}$$

where the right-hand side is bounded according to Theorem 2.1. The same approach, but choosing $\varphi\big|_{\Omega_{\mathrm{f}}} = 0$ and $\varphi\big|_{\Omega_{\mathrm{s}}} \in L^2(I; C_0^\infty(\Omega_{\mathrm{s}})^d)$ leads to

$$\operatorname{div}(\mu \nabla u) = \rho_{\mathrm{s}} \partial_t v - f_{\mathrm{s}} \quad \text{in } L^2(I; H_{\mathrm{s}}) \tag{2.14}$$

and

$$\|\operatorname{div}(\mu \nabla u)\|_{L^2(I; L^2(\Omega_{\mathrm{s}}))} \le \|f_{\mathrm{s}}\|_{L^2(I; L^2(\Omega_{\mathrm{s}}))} + C \|\partial_t v\|_{L^2(I; L^2(\Omega_{\mathrm{s}}))}$$

where the right-hand side is again bounded according to Theorem 2.1. Thus, we obtain that

$$\nu\nabla v + p\,\mathrm{Id} \in E(\Omega_\mathrm{f}) := \left\{ \varphi \in L^2(I;H_\mathrm{f}) \;\middle|\; \|\mathrm{div}\,\varphi\|_{L^2(I;L^2(\Omega_\mathrm{f}))} < \infty \right\},$$
$$\mu\nabla u \in E(\Omega_\mathrm{s}) := \left\{ \varphi \in L^2(I;H_\mathrm{s}) \;\middle|\; \|\mathrm{div}\,\varphi\|_{L^2(I;L^2(\Omega_\mathrm{s}))} < \infty \right\}.$$

According to [37, Ch. I §1 Theorem 1.1], the space $L^2(I;C_0^\infty(\Omega_\mathrm{f})^d)$ is dense in $E(\Omega_\mathrm{f})$ and $L^2(I;C_0^\infty(\Omega_\mathrm{s})^d)$ is dense in $E(\Omega_\mathrm{s})$. Therefore, following [15, p. 114] or [16, Theorems 1.5.3.10 and 1.5.3.11], we get

$$\begin{aligned}
\|(\nu\nabla v + p\,\mathrm{Id})^T n_\mathrm{f}\|_{L^2(I;(\widetilde{H}^{\frac12}(\Gamma_\mathrm{i})^d)^*)} &\le \|\nu\nabla v + p\,\mathrm{Id}\|_{L^2(I;L^2(\Omega_\mathrm{f}))} \\
&\quad + \|f_\mathrm{f}\|_{L^2(I;L^2(\Omega_\mathrm{f}))} + C\|\partial_t v\|_{L^2(I;L^2(\Omega_\mathrm{f}))} \\
\|(\mu\nabla u)^T n_\mathrm{s}\|_{L^2(I;(\widetilde{H}^{\frac12}(\Gamma_\mathrm{i})^d)^*)} &\le \|\mu\nabla u\|_{L^2(I;L^2(\Omega_\mathrm{s}))} \\
&\quad + \|f_\mathrm{s}\|_{L^2(I;L^2(\Omega_\mathrm{s}))} + C\|\partial_t v\|_{L^2(I;L^2(\Omega_\mathrm{s}))}.
\end{aligned}$$

According to [15,16], this enables us to apply Gauß' theorem in (2.8) to obtain

$$\begin{aligned}
\langle\!\langle \nu\partial_{n_\mathrm{f}} v - p n_\mathrm{f}, \varphi\rangle\!\rangle_\mathrm{i} + \langle\!\langle \mu\partial_{n_\mathrm{s}} u, \varphi\rangle\!\rangle_\mathrm{i} &= (\!(\mathrm{div}(\nu\nabla v + p\,\mathrm{Id}) - \rho_\mathrm{f}\partial_t v + f_\mathrm{f}, \varphi)\!)_\mathrm{f} \\
&\quad + (\!(\mathrm{div}(\mu\nabla u) - \rho_\mathrm{s}\partial_t v + f_\mathrm{s}, \varphi)\!)_\mathrm{s} \quad \forall \varphi \in L^2(I;V).
\end{aligned}$$

This immediately implies by (2.14) and (2.13) that

$$\langle\!\langle \nu\partial_{n_\mathrm{f}} v - p n_\mathrm{f} + \mu\partial_{n_\mathrm{s}} u, \varphi\rangle\!\rangle_\mathrm{i} = 0 \quad \forall \varphi \in L^2(I;V)$$

and thus the dynamic coupling condition is fulfilled in $L^2(I;(\widetilde{H}^{\frac12}(\Gamma_\mathrm{i})^d)^*)$. □

3 Optimal control problem

In the following, we regard the optimal control problem of a linearized FSI configuration as introduced in the introduction

$$\min J(q,u,v) := \frac{\gamma_\mathrm{f}}{2}\int_I \|v - v_\mathrm{d}\|^2_{L^2(\Omega_\mathrm{f})}\,\mathrm{d}t + \frac{\gamma_\mathrm{s}}{2}\int_I \|u - u_\mathrm{d}\|^2_{L^2(\Omega_\mathrm{s})}\,\mathrm{d}t + \frac{\alpha}{2}\|q\|_Q^2$$

subject to $q_a \le q \le q_b$ and (1.1). Thereby, the initial data are assumed to fulfill the conditions of Theorem 2.1 and for the desired states we require $v_\mathrm{d} \in H^1(I;V_\mathrm{f}^*) \cap L^2(I;H_\mathrm{f})$ and $u_\mathrm{d} \in L^2(I;H_\mathrm{s})$.

3.1 Existence and uniqueness of solutions

We analyze two concrete configurations for the considered control problem:
Configuration C1 Let the control space given by $Q := (L^2(\Omega)^d)^N$ with $N \in \mathbb{N}$ and let $B_\mathrm{f}\colon Q \to H^1(I;H_\mathrm{f})$ as well as $B_\mathrm{s}\colon Q \to H^1(I;H_\mathrm{s})$ be linear continuous operators given for $q = (q^1, q^2, \ldots, q^N) \in Q$ by

$$B_\mathrm{f} q = \sum_{i=1}^N g_\mathrm{f}^i q^i\big|_{\Omega_\mathrm{f}} \quad \text{and} \quad B_\mathrm{s} q = \sum_{i=1}^N g_\mathrm{s}^i q^i\big|_{\Omega_\mathrm{s}}.$$

Thereby, $g_\mathrm{s}^i, g_\mathrm{f}^i \in H^1(I)$, $i = 1, 2, \dots, N$, are given functions. The admissible set Q_ad is defined as

$$Q_\mathrm{ad} = \left\{\, q \in Q \;\middle|\; q_a \le q^i(x) \le q_b, \text{ for almost all } x \in \Omega \text{ and } i = 1, 2, \dots, N \,\right\}.$$

Note, that the inequality in the definition of Q_ad has to be understood componentwise for $q^i \in L^2(\Omega)^d$.

Configuration C2 Let the control space given by $Q := L^2(I)^N$ with $N \in \mathbb{N}$ and let $B_\mathrm{f} \colon Q \to L^2(I; H_\mathrm{f})$ as well as $B_\mathrm{s} \colon Q \to L^2(I; H_\mathrm{s})$ be linear continuous operators given for $q = (q^1, q^2, \dots, q^N) \in Q$ by

$$B_\mathrm{f} q = \sum_{i=1}^N q^i h^i\big|_{\Omega_\mathrm{f}} \quad \text{and} \quad B_\mathrm{s} q = \sum_{i=1}^N q^i h^i\big|_{\Omega_\mathrm{s}}.$$

Thereby, $h^i \in V_\mathrm{div}$, $i = 1, 2, \dots, N$, are given functions. The admissible set Q_ad is defined as

$$Q_\mathrm{ad} = \left\{\, q \in Q \;\middle|\; q_a \le q^i(t) \le q_b, \text{ for almost all } t \in I \text{ and } i = 1, 2, \dots, N \,\right\}.$$

The assumption that h^i has to be divergence-free in Ω_f is taken for simplicity of the presentation. All results can be extended to $h^i \in V$ using a Helmholtz decomposition.

Since for both configurations, $B_\mathrm{f} q \in L^2(I; H_\mathrm{f})$ and $B_\mathrm{s} q \in L^2(I; H_\mathrm{s})$, Proposition 2.1 ensures the well-posedness of the so-defined control to state mapping $G \colon q \mapsto (v(q), u(q))$ with $(v(q), u(q))$ the solution of (2.6) for $f_\mathrm{f} := B_\mathrm{s} q$ and $f_\mathrm{s} := B_\mathrm{s} q$. The linearity of (2.6) and the estimate given in Proposition 2.1 imply the continuity of G:

Lemma 3.1 *The control to state mapping $G \colon Q \to L^2(I; H_\mathrm{f}) \times L^2(I; H_\mathrm{s})$ is an affine linear and continuous operator for both configurations* C1 *and* C2*.*

Proof Let $(\hat{v}, \hat{u})$ be the solution of (2.6) for $f_\mathrm{f} = f_\mathrm{s} = 0$ and $G_0 \colon Q \to L^2(I; H_\mathrm{f}) \times L^2(I; H_\mathrm{s})$ be the linear part of G defined by (2.6) with zero initial data for $f_\mathrm{f} := B_\mathrm{s} q$ and $f_\mathrm{s} := B_\mathrm{s} q$. Hence, the control to state mapping $G \colon Q \to L^2(I; H_\mathrm{f}) \times L^2(I; H_\mathrm{s})$ can be expressed as

$$(v(q), u(q)) = Gq = (\hat{v}, \hat{u}) + G_0 q.$$

Proposition 2.1 yields that $(\hat{v}, \hat{u})$ is bounded in $L^\infty(I; H_\mathrm{f}) \times L^\infty(I; V_\mathrm{s})$ and G_0 is a bounded linear operator. Thus, the control to state mapping G is continuous in both considered configurations. □

By means of the control to state mapping G, the reduced cost functional $j \colon Q \to \mathbb{R}$ can be defined as

$$j(q) := J(q, u(q), v(q)) \tag{3.1}$$

and the optimal control problem under consideration can for both configurations be written in the compact form

$$\min_{q \in Q_\mathrm{ad}} j(q). \tag{3.2}$$

Theorem 3.1 *For both configurations* C1 *and* C2*, the considered optimal control problem* (3.2) *admits a unique solution.*

Proof Standard arguments, see, e.g. [38, Theorem 2.14], guarantee the existence of a unique optimal control $\bar{q} \in Q_{\mathrm{ad}}$. □

We emphasize that this existence result is also valid if the control is acting only on the domain Ω_{s} or Ω_{f} and if reference solutions are only given on sub-domains.

3.2 Necessary optimality conditions

Since the reduced functional j is convex due to the (affine) linear-quadratic structure of the considered control problem, the necessary and sufficient optimality condition for the optimal solution $\bar{q} \in Q_{\mathrm{ad}}$ of (3.2) reads as

$$j'(\bar{q})(\delta q - \bar{q}) \geq 0 \quad \forall \delta q \in Q_{\mathrm{ad}}. \tag{3.3}$$

Based on this, we derive in the sequel an optimality system separately for the configurations C1 and C2. In addition, we prove a priori estimates for the optimal state and the corresponding adjoint solution. Thereby, we make use of the self-adjoint formulation (2.8). Due to this symmetry, the derivation of an optimality system for configuration C1 is straight forward, see Section 3.2.3. For configuration C2 however, this is not directly possible since for $q \in Q = L^2(I)^N$, the right-hand sides $B_{\mathrm{f}}q$ and $B_{\mathrm{s}}q$ do not fulfill the prerequisites of Theorem 2.1. Therefore, an additional approximation step will be necessary, see Section 3.2.4.

3.2.1 Discussion of the adjoint equations for a non symmetric formulation

Before analyzing the adjoint equations of the weak formulation (2.8), we investigate for a moment the optimal control problem with the state equation formulated by (2.1), (2.2), and (2.3). For this formulation, the formal Lagrange approach leads to the following adjoint equation:

$$\begin{aligned}
-(\!(\varphi, z_t^v)\!)_{\mathrm{f}} + \nu(\!(\nabla\varphi, \nabla z^v)\!)_{\mathrm{f}} + (\!(z^p, \operatorname{div}\varphi)\!)_{\mathrm{f}} & \\
-(\!(\varphi, z_t^v)\!)_{\mathrm{s}} - (\!(\varphi, z^u)\!)_{\mathrm{s}} &= \gamma_{\mathrm{f}}(\!(v - v_{\mathrm{d}}, \varphi)\!)_{\mathrm{f}} \quad \forall \varphi \in L^2(I;V),\\
-(\!(\psi, z_t^u)\!)_{\mathrm{s}} + \mu(\!(\nabla\psi, \nabla z^v)\!)_{\mathrm{s}} &= \gamma_{\mathrm{s}}(\!(u - u_{\mathrm{d}}, \psi)\!)_{\mathrm{s}} \quad \forall \psi \in L^2(I;V_{\mathrm{s}}),\\
-(\!(\xi, \operatorname{div} z^v)\!)_{\mathrm{f}} &= 0 \qquad\qquad\qquad\quad \forall \xi \in L^2(I;L_{\mathrm{f}}).
\end{aligned}$$

Here, $z^v\big|_{\Omega_{\mathrm{f}}}$ describes the solution of an adjoint Stokes equation and $z^v\big|_{\Omega_{\mathrm{s}}}$ the solution of an adjoint linear wave equation. However, as the system (2.1), (2.2), (2.3) is not symmetric, the adjoint equation is a Stokes-wave system with new coupling conditions on Γ_{i}:

$$z^v\big|_{\Omega_{\mathrm{f}}} = z^v\big|_{\Omega_{\mathrm{s}}}, \quad \nu\partial_{n_{\mathrm{f}}} z^v - z^p n_{\mathrm{f}} = 0, \quad \text{and} \quad \mu\partial_{n_{\mathrm{s}}} z^v = 0.$$

In contrast to this, the advantage of the following optimality system lies in the fact that the adjoint equation is again a linear FSI problem and all numerical methods developed to solve the primal FSI problem can be utilized. Therefore, no additional difficulty occurs in the implementation.

3.2.2 Existence and regularity for the adjoint equation

For the symmetric weak formulation (2.8), we can derive the adjoint equation using the formal Lagrange technique. As the coupling conditions do not occur explicitly, but are embedded in the weak formulation, we obtain immediately the following adjoint equation:

$$\begin{aligned}
-\rho_\mathrm{f}(\!(\varphi, \partial_t z^v)\!)_\mathrm{f} + \nu(\!(\nabla\varphi, \nabla z^v)\!)_\mathrm{f} + (\!(z^p, \operatorname{div}\varphi)\!)_\mathrm{f}& \\
-\rho_\mathrm{s}(\!(\varphi, \partial_t z^v)\!)_\mathrm{s} + \mu(\!(\nabla\varphi, \nabla z^u)\!)_\mathrm{s} &= \gamma_\mathrm{f}(\!(v - v_\mathrm{d}, \varphi)\!)_\mathrm{f} \quad \forall\varphi \in L^2(I;V) \\
\mu(\!(\nabla\psi, \nabla z^v)\!)_\mathrm{s} + \mu(\!(\nabla\psi, \nabla\partial_t z^u)\!)_\mathrm{s} &= \gamma_\mathrm{s}(\!(u - u_\mathrm{d}, \psi)\!)_\mathrm{s} \quad \forall\psi \in L^2(I;V_\mathrm{s}) \\
-(\!(\xi, \operatorname{div} z^v)\!)_\mathrm{f} &= 0 \qquad\qquad\qquad\quad \forall\xi \in L^2(I;L_\mathrm{f})
\end{aligned} \tag{3.4}$$

Later, we will prove that together with zero terminal conditions this is indeed the correct adjoint equation appearing in the optimality system.

Due to the symmetry in (2.8), the adjoint equation is again a linear FSI problem. Therefore we can use the already proved results and we get the following result on existence of a unique adjoint solution:

Theorem 3.2 *Let $v_\mathrm{d} \in H^1(I;V_\mathrm{f}^*) \cap L^2(I;H_\mathrm{f})$, $u_\mathrm{d} \in L^2(I;H_\mathrm{s})$, and the initial data u_0, u_1, and v_0 fulfill the assumptions in Theorem 2.1. Further, let $q \in Q$ be given as in configuration* C1 *or $q \in Q \cap H^1(I)^N$ be given for configuration* C2 *and let the triple (v, p, u) be the corresponding solution of (2.8) with $f_\mathrm{f} = B_\mathrm{f} q$, $f_\mathrm{s} = B_\mathrm{s} q$, and $g = 0$. Then, there exists a unique triple (z^v, z^p, z^u) with*

$$\begin{aligned}
z^v \in L^2(I;V) \cap W^{1,\infty}(I;H), \quad z^v\big|_{\Omega_\mathrm{f}} \in H^1(I;V_\mathrm{f}), \quad z^v\big|_{\Omega_\mathrm{s}} \in L^\infty(I;V_\mathrm{s}), \\
z^u \in L^\infty(I;V_\mathrm{s}) \cap H^1(I;V_\mathrm{s}), \quad z^p \in L^2(I;L_\mathrm{f})
\end{aligned}$$

satisfying the terminal condition $z^v(T) = 0$, $z^u(T) = 0$ and the adjoint equation (3.4). Furthermore, the adjoint solution triple (z^v, z^p, z^u) fulfills the following a priori estimates:

a)
$$\begin{aligned}
\|z^v\|^2_{L^\infty(I;L^2(\Omega))} + \|z^v\|^2_{L^2(I;H^1(\Omega_\mathrm{f}))} + \|z^u\|^2_{L^\infty(I;H^1(\Omega_\mathrm{s}))} \\
\leq C\big[\|v_\mathrm{d}\|^2_{L^2(I;V_\mathrm{f}^*)} + \|u_\mathrm{d}\|^2_{L^2(I;L^2(\Omega_\mathrm{s}))}\big] \\
+ C\big[\|B_\mathrm{f} q\|^2_{L^2(I;V_\mathrm{f}^*)} + \|B_\mathrm{s} q\|^2_{L^2(I;L^2(\Omega_\mathrm{s}))}\big] \\
+ C\big[\|u_0\|^2_{H^1(\Omega_\mathrm{s})} + \|u_1\|^2_{H^1(\Omega_\mathrm{s})} + \|v_0\|^2_{H^1(\Omega_\mathrm{f})}\big],
\end{aligned}$$

b)
$$\begin{aligned}
\|\partial_t z^v\|^2_{L^\infty(I;L^2(\Omega))} + \|\partial_t z^v\|^2_{L^2(I;H^1(\Omega_\mathrm{f}))} + \|z^v\|^2_{L^\infty(I;H^1(\Omega_\mathrm{s}))} + \|\partial_t z^u\|^2_{L^2(I;H^1(\Omega_\mathrm{s}))} \\
\leq C\big[\|v_\mathrm{d}\|^2_{H^1(I;V_\mathrm{f}^*)} + \|u_\mathrm{d}\|_{L^2(I;L^2(\Omega_\mathrm{s}))}\big] \\
+ C\big[\|B_\mathrm{f} q\|^2_{H^1(I;V_\mathrm{f}^*)} + \|B_\mathrm{s} q\|_{H^1(I;L^2(\Omega_\mathrm{s}))}\big] \\
+ C\big[\|u_0\|^2_{H^2(\Omega_\mathrm{s})} + \|u_1\|^2_{H^1(\Omega_\mathrm{s})} + \|v_0\|^2_{H^2(\Omega_\mathrm{f})} + \|p_0\|^2_{H^1(\Omega_\mathrm{f})}\big],
\end{aligned}$$

c)
$$\begin{aligned}
\|z^p\|^2_{L^2(I;L^2(\Omega_\mathrm{f}))} \leq C\big[\|v_\mathrm{d}\|^2_{H^1(I;V_\mathrm{f}^*)} + \|u_\mathrm{d}\|^2_{L^2(I;L^2(\Omega_\mathrm{s}))})\big] \\
+ C\big[\|B_\mathrm{f} q\|^2_{H^1(I;V_\mathrm{f}^*)} + \|B_\mathrm{s} q\|^2_{H^1(I;L^2(\Omega_\mathrm{s}))}\big] \\
+ C\big[\|u_0\|^2_{H^2(\Omega_\mathrm{s})} + \|u_1\|^2_{H^1(\Omega_\mathrm{s})} + \|v_0\|^2_{H^2(\Omega_\mathrm{f})} + \|p_0\|^2_{H^1(\Omega_\mathrm{f})}\big].
\end{aligned}$$

Proof Based on the assumptions on the control q, Theorem 2.1 ensures a solution $v \in H^1(I;H)$, $u \in L^2(I;V_\mathrm{s})$ of (2.8) with $f_\mathrm{f} = B_\mathrm{f}q$, $f_\mathrm{s} = B_\mathrm{s}q$, and $g = 0$ Hence, the right-hand sides of the adjoint equation $f_\mathrm{f} := \gamma_\mathrm{f}(v - v_\mathrm{d})$ and $g := \gamma_\mathrm{s}(u - u_\mathrm{d})$, fulfill the required regularities

$$f_\mathrm{f} \in H^1(I;V_\mathrm{f}^*) \quad \text{and} \quad g \in L^2(I;H_\mathrm{s})$$

of Theorem 2.1. Furthermore, the initial conditions for the adjoint equation backwards in time $z_T^v = 0$ and $z_T^u = 0$ fulfill with $z_T^p = 0$ the assumptions on u_0, v_0, p_0, and u_1 of Theorem 2.1. As after the transformation $t \mapsto -t$, the considered adjoint equation (3.4) coincides with the state equation (2.8), there exists a unique adjoint solution (z^v, z^p, z^u) due to Theorem 2.1. The estimates follow immediately from Theorem 2.1, too. □

3.2.3 Control distributed in space (Configuration C1)

Here, the control $q \in Q = (L^2(\Omega)^d)^N$ acts as volume force through the linear operators B_f and B_s as described in configuration C1. Since in this case $B_\mathrm{f}q \in H^1(I;H_\mathrm{f})$ and $B_\mathrm{s}q \in H^1(I;H_\mathrm{s})$, the weak formulation (2.8) is applicable for $f_\mathrm{f} = B_\mathrm{f}q$ and $f_\mathrm{s} = B_\mathrm{s}q$ by Theorem 2.1. For the derivative of the reduced functional given by (3.1), we directly obtain the following representation:

Lemma 3.2 *Let the initial data u_0, u_1, and v_0 fulfill the assumptions of Theorem 2.1 and let $v_\mathrm{d} \in H^1(I;V_\mathrm{f}^*) \cap L^2(I;H_\mathrm{f})$ and $u_\mathrm{d} \in L^2(I;H_\mathrm{s})$. Let for given $q \in Q$ the triple (v,p,u) be the solution of (2.8) with $f_\mathrm{f} = B_\mathrm{f}q$, $f_\mathrm{s} = B_\mathrm{s}q$, and $g = 0$ guaranteed by Theorem 2.1. Further, let (z^v, z^p, z^u) be the solution of the adjoint equation (3.4) guaranteed by Theorem 3.2. Then, the directional derivative of the reduced cost functional at q in direction $\delta q \in Q$ is given by*

$$j'(q)(\delta q) = \sum_{i=1}^{N} \left[(\!(g_\mathrm{f}^i \delta q^i, z^v)\!)_\mathrm{f} + (\!(g_\mathrm{s}^i \delta q^i, z^v)\!)_\mathrm{s} + \alpha (q^i, \delta q^i) \right].$$

Proof By Theorem 2.1, the control to state map G can be understood as mapping from Q to $L^2(I;V) \times L^2(I;L_\mathrm{f}) \times L^2(I;V_\mathrm{s})$. Similar to the proof of Lemma 3.1, let $(\hat{v}, \hat{p}, \hat{u})$ be the solution of (2.8) for $f_\mathrm{f} = f_\mathrm{s} = g = 0$ and let $G_0 \colon Q \to L^2(I, H_\mathrm{f}) \times L^2(I;L_\mathrm{f}) \times L^2(I;H_\mathrm{s})$ the linear part G given by (2.8) for zero initial data and $f_\mathrm{f} = B_\mathrm{f}q$, $f_\mathrm{s} = B_\mathrm{s}q$, $g = 0$. Then G can be written for $q \in Q$ as

$$(v(q), p(q), u(q)) = Gq = (\hat{v}, \hat{p}, \hat{u}) + G_0 q.$$

Hence, we get directly

$$j'(q)(\delta q) = \gamma_\mathrm{f} (\!(v - v_\mathrm{d}, \delta v)\!)_\mathrm{f} + \gamma_\mathrm{s} (\!(u - u_\mathrm{d}, \delta u)\!)_\mathrm{s} + \alpha \sum_{i=1}^{N} (q^i, \delta q^i) \tag{3.5}$$

for all $\delta q \in Q$ where $(\delta v, \delta p, \delta u) = G_0 \delta q$.

Since $(\delta v, \delta p, \delta u)$ solves (2.8) for the right-hand sides $f_\mathrm{f} = B_\mathrm{f}\delta q$, $f_\mathrm{s} = B_\mathrm{s}\delta q$, $g = 0$ and zero initial data, we get by testing this equation with $(\varphi, \xi, \psi) = (z^v, z^p, z^u) \in L^2(I;V) \times L^2(I;L_\mathrm{f}) \times L^2(I;V_\mathrm{s})$ the following identity:

$$
\begin{aligned}
\rho_\mathrm{f}(\!(\delta v_t, z^v)\!)_\mathrm{f} - (\!(\delta p, \operatorname{div} z^v)\!)_\mathrm{f} + \nu(\!(\nabla \delta v, \nabla z^v)\!)_\mathrm{f}& \\
+\rho_\mathrm{s}(\!(\delta v_t, z^v)\!)_\mathrm{s} + \mu(\!(\nabla \delta u, \nabla z^v)\!)_\mathrm{s} &= \sum_{i=1}^N \left[(\!(g_\mathrm{f}^i \delta q^i, z^v)\!)_\mathrm{f} + (\!(g_\mathrm{s}^i \delta q^i, z^v)\!)_\mathrm{s}\right] \\
\mu(\!(\nabla \delta v, \nabla z^u)\!)_\mathrm{s} - \mu(\!(\nabla \delta u_t, \nabla z^u)\!)_\mathrm{s} &= 0 \\
(\!(z^p, \operatorname{div} \delta v)\!)_\mathrm{f} &= 0
\end{aligned}
$$

Testing with $(\varphi, \xi, \psi) = (\delta v, \delta p, \delta u) \in L^2(I;V) \times L^2(I;L_\mathrm{f}) \times L^2(I;V_\mathrm{s})$ in the adjoint equation (3.4) yields

$$
\begin{aligned}
-\rho_\mathrm{f}(\!(\delta v, \partial_t z^v)\!)_\mathrm{f} + \nu(\!(\nabla \delta v, \nabla z^v)\!)_\mathrm{f} + (\!(z^p, \operatorname{div} \delta v)\!)_\mathrm{f}& \\
-\rho_\mathrm{s}(\!(\delta v, \partial_t z^v)\!)_\mathrm{s} + \mu(\!(\nabla \delta v, \nabla z^u)\!)_\mathrm{s} &= \gamma_\mathrm{f}(\!(v - v_\mathrm{d}, \delta v)\!)_\mathrm{f} \\
-(\!(\delta p, \operatorname{div} z^v)\!)_\mathrm{f} &= 0 \\
\mu(\!(\nabla \delta u, \nabla z^v)\!)_\mathrm{s} + \mu(\!(\nabla \delta u, \nabla \partial_t z^u)\!)_\mathrm{s} &= \gamma_\mathrm{s}(\!(u - u_\mathrm{d}, \delta u)\!)_\mathrm{s}
\end{aligned}
$$

As the adjoint solution (z^v, z^p, z^u) has zero initial conditions at $t = T$ and as $(\delta v, \delta p, \delta u)$ has zero initial conditions at $t = 0$, the boundary terms vanish when using integration by parts in time. If we insert the equations into each other we obtain for any $\delta q \in Q$

$$
\gamma_\mathrm{f}(\!(v - v_\mathrm{d}, \delta v)\!)_\mathrm{f} + \gamma_\mathrm{s}(\!(u - u_\mathrm{d}, \delta u)\!)_\mathrm{s} = \sum_{i=1}^N \left[(\!(g_\mathrm{f}^i \delta q^i, z^v)\!)_\mathrm{f} + (\!(g_\mathrm{s}^i \delta q^i, z^v)\!)_\mathrm{s}\right].
$$

Together with (3.5) this implies the assertion. □

Combining the condition (3.3) and Lemma 3.2 implies the following representation and regularity for the optimal control $\bar{q}$ in terms of the pointwise projection $\mathrm{P}_{\mathrm{Q}_\mathrm{ad}}$ on the admissible set Q_ad given by

$$
\mathrm{P}_{\mathrm{Q}_\mathrm{ad}} \colon L^2(\Omega)^d \to L^2(\Omega)^d, \quad \mathrm{P}_{\mathrm{Q}_\mathrm{ad}}(r)(x) := \max(q_a, \min(q_b, r(x)))
$$

for almost all $x \in \Omega$, where the projection has to be applied componentwise for $r \in L^2(\Omega)^d$.

Lemma 3.3 *Let the assumptions of Lemma 3.2 be fulfilled. Then, the optimal solution $\bar{q} \in Q_{ad}$ of the considered optimal control problem* (3.2) *for configuration* C1 *fulfills for $i = 1, 2, \ldots, N$:*

$$
\bar{q}^i\big|_{\Omega_\mathrm{f}} = \mathrm{P}_{\mathrm{Q}_\mathrm{ad}}\left(-\frac{1}{\alpha}\int_I g_\mathrm{f}^i(t) z^v(t,\cdot)\ dt\right), \quad \bar{q}^i\big|_{\Omega_\mathrm{s}} = \mathrm{P}_{\mathrm{Q}_\mathrm{ad}}\left(-\frac{1}{\alpha}\int_I g_\mathrm{s}^i(t) z^v(t,\cdot)\ dt\right).
$$

Thus, for the optimal control holds $\bar{q}\big|_{\Omega_\mathrm{f}} \in (H^1(\Omega_\mathrm{f})^d)^N$ and $\bar{q}\big|_{\Omega_\mathrm{s}} \in (H^1(\Omega_\mathrm{s})^d)^N$.

Proof The necessary optimality condition (3.3) can be written as

$$\left(\int_I g_{\mathrm{f}}^i z^v \,\mathrm{d}t, \delta q^i - \bar{q}^i\right)_{\mathrm{f}} + \left(\int_I g_{\mathrm{s}}^i z^v \,\mathrm{d}t, \delta q^i - \bar{q}^i\right)_{\mathrm{s}} + \alpha(\bar{q}^i, \delta q^i - \bar{q}^i) \geq 0 \quad \forall \delta q^i \in Q_{\mathrm{ad}}.$$

Using the projection $\mathrm{P}_{\mathrm{Q_{ad}}}$, this can be expressed for $i = 1, 2, \ldots, N$ as

$$\bar{q}^i\big|_{\Omega_{\mathrm{f}}} = \mathrm{P}_{\mathrm{Q_{ad}}}\left(-\frac{1}{\alpha}\int_I g_{\mathrm{f}}^i(t) z^v(t,\cdot)\,\mathrm{d}t\right), \quad \bar{q}^i\big|_{\Omega_{\mathrm{s}}} = \mathrm{P}_{\mathrm{Q_{ad}}}\left(-\frac{1}{\alpha}\int_I g_{\mathrm{s}}^i(t) z^v(t,\cdot)\,\mathrm{d}t\right).$$

Theorem 3.2 ensures $z^v \in L^2(I; V)$ and together with

$$\|\mathrm{P}_{\mathrm{Q_{ad}}}(r)\|_{H^1(\Omega)} \leq \|r\|_{H^1(\Omega)}$$

we conclude that $\bar{q}^i\big|_{\Omega_{\mathrm{f}}} \in H^1(\Omega_{\mathrm{f}})^d$ and $\bar{q}^i\big|_{\Omega_{\mathrm{s}}} \in H^1(\Omega_{\mathrm{s}})^d$. □

The optimal solution $\bar{q} \in Q_{\mathrm{ad}}$ solves the optimality system presented in the following theorem:

Theorem 3.3 *Let the initial data u_0, u_1, and v_0 fulfill the assumptions of Theorem 2.1 and let $v_{\mathrm{d}} \in H^1(I; V_{\mathrm{f}}^*) \cap L^2(I; H_{\mathrm{f}})$ and $u_{\mathrm{d}} \in L^2(I; H_{\mathrm{s}})$. Then, the optimal solution $\bar{q} \in Q_{\mathrm{ad}}$ of the considered optimal control problem* (3.2) *for configuration* C1 *fulfills $\bar{q}\big|_{\Omega_{\mathrm{f}}} \in (H^1(\Omega_{\mathrm{f}})^d)^N$, $\bar{q}\big|_{\Omega_{\mathrm{s}}} \in (H^1(\Omega_{\mathrm{s}})^d)^N$ and the following necessary optimality condition:*

1. *The optimal state $(\bar{v}, \bar{p}, \bar{u}) = (v(\bar{q}), p(\bar{q}), u(\bar{q}))$ solves*

$$\begin{aligned}
&\rho_{\mathrm{f}}(\!(\partial_t \bar{v}, \varphi)\!)_{\mathrm{f}} - (\!(\bar{p}, \operatorname{div}\varphi)\!)_{\mathrm{f}} + \nu(\!(\nabla\bar{v}, \nabla\varphi)\!)_{\mathrm{f}} \\
&\qquad + \rho_{\mathrm{s}}(\!(\partial_t \bar{v}, \varphi)\!)_{\mathrm{s}} + \mu(\!(\nabla\bar{u}, \nabla\varphi)\!)_{\mathrm{f}} = (\!(B_{\mathrm{f}}\bar{q}, \varphi)\!)_{\mathrm{f}} + (\!(B_{\mathrm{s}}\bar{q}, \varphi)\!)_{\mathrm{s}} \quad \forall \varphi \in L^2(I; V), \\
&\mu(\!(\nabla\bar{v}, \nabla\psi)\!)_{\mathrm{s}} - \mu(\!(\nabla\partial_t\bar{u}, \nabla\psi)\!)_{\mathrm{s}} = 0 \quad \forall \psi \in L^2(I; V_{\mathrm{s}}), \\
&(\!(\xi, \operatorname{div}\bar{v})\!)_{\mathrm{f}} = 0 \quad \forall \xi \in L^2(I; L_{\mathrm{f}}).
\end{aligned}$$

2. *The optimal adjoint $(\bar{z}^v, \bar{z}^p, \bar{z}^u) = (z^v(\bar{q}), z^p(\bar{q}), z^u(\bar{q}))$ solves*

$$\begin{aligned}
&-\rho_{\mathrm{f}}(\!(\varphi, \partial_t \bar{z}^v)\!)_{\mathrm{f}} + \nu(\!(\nabla\varphi, \nabla\bar{z}^v)\!)_{\mathrm{f}} + (\!(\bar{z}^p, \operatorname{div}\varphi)\!)_{\mathrm{f}} \\
&\qquad - \rho_{\mathrm{s}}(\!(\varphi, \partial_t \bar{z}^v)\!)_{\mathrm{s}} + \mu(\!(\nabla\varphi, \nabla\bar{z}^u)\!)_{\mathrm{s}} = \gamma_{\mathrm{f}}(\!(\bar{v} - v_{\mathrm{d}}, \varphi)\!)_{\mathrm{f}} \quad \forall \varphi \in L^2(I; V), \\
&\mu(\!(\nabla\psi, \nabla\bar{z}^v)\!)_{\mathrm{s}} + \mu(\!(\nabla\psi, \nabla\partial_t\bar{z}^u)\!)_{\mathrm{s}} = \gamma_{\mathrm{s}}(\!(\bar{u} - u_{\mathrm{d}}, \psi)\!)_{\mathrm{s}} \quad \forall \psi \in L^2(I; V_{\mathrm{s}}), \\
&-(\!(\xi, \operatorname{div}\bar{z}^v)\!)_{\mathrm{f}} = 0 \quad \forall \xi \in L^2(I; L_{\mathrm{f}}).
\end{aligned}$$

3. *It holds for $i = 1, 2, \ldots, N$ that*

$$\bar{q}^i\big|_{\Omega_{\mathrm{f}}} = \mathrm{P}_{\mathrm{Q_{ad}}}\left(-\frac{1}{\alpha}\int_I g_{\mathrm{f}}^i(t) \bar{z}^v(t,\cdot)\,dt\right), \quad \bar{q}^i\big|_{\Omega_{\mathrm{s}}} = \mathrm{P}_{\mathrm{Q_{ad}}}\left(-\frac{1}{\alpha}\int_I g_{\mathrm{s}}^i(t) \bar{z}^v(t,\cdot)\,dt\right).$$

3.2.4 Time-dependent control (Configuration C2)

In the following, the control $q \in Q = (L^2(I))^N$ is controlling the volume force through the linear operators B_f and B_s described in configuration C2. As Theorem 2.1 does not guarantee existence of a unique solution of equation (2.8) for a right-hand side $f_\mathrm{f} = B_\mathrm{f} q \in L^2(I; H_\mathrm{f})$ and $f_\mathrm{s} = B_\mathrm{s} q \in L^2(I; H_\mathrm{s})$ we can not directly proceed as in Section 3.2.3. Therefore, we will make use of a smooth sequence in $Q \cap (H^1(I))^N$ converging against the optimal solution. For smooth controls, the symmetric formulation (2.8) can be utilized and a priori estimates for the adjoint then lead to higher regularity also for the limit. Then, we are able to derive the optimality system similar as for the configuration C1.

Lemma 3.4 *Let the initial data u_0, u_1, and v_0 fulfill the assumptions of Theorem 2.1 and let $v_\mathrm{d} \in H^1(I; V_\mathrm{f}^*) \cap L^2(I; H_\mathrm{f})$ and $u_\mathrm{d} \in L^2(I; H_\mathrm{s})$. Let for a given control $q \in Q \cap (H^1(I))^N$ the triple (v, p, u) be the solution of (2.8) with $f_\mathrm{f} = B_\mathrm{f} q$, $f_\mathrm{s} = B_\mathrm{s} q$, and $g = 0$ guaranteed by Theorem 2.1. Further, let (z^v, z^p, z^u) be the solution of the adjoint equation (3.4) guaranteed by Theorem 3.2. Then the directional derivative of the reduced cost functional j at q in direction $\delta q \in Q$ is given by*

$$j'(q)(\delta q) = \sum_{i=1}^{N} \left[(\!(h^i \delta q^i, z^v)\!) + \alpha \int_I q^i \delta q^i \, dt \right].$$

Proof Since for $q \in Q \cap (H^1(I))^N$ it holds $B_\mathrm{f} q \in H^1(I, H_\mathrm{f})$ and $B_\mathrm{s} q \in H^1(I; H_\mathrm{s})$ we proceed as in Lemma 3.2 to obtain

$$j'(q)(\delta q) = \sum_{i=1}^{N} \left[(\!(h^i \delta q^i, z^v)\!) + \alpha \int_I q^i \delta q^i \, \mathrm{d}t \right]$$

for all $\delta q \in Q \cap (H^1(I))^N$. By the density of $(H^1(I))^N$ in $(L^2(I))^N$ with respect to the $(L^2(I))^N$ topology, we obtain the assertion. □

In the next lemma, we prove that the optimal control $\bar{q}$ lies in $Q \cap (H^1(I))^N$ such that the representation derived in Lemma 3.4 is also valid for $\bar{q}$. Therefore, we will introduce also for configuration C2 the pointwise projection $\mathrm{P}_{Q_\mathrm{ad}}$ on the admissible set Q_ad given here by

$$\mathrm{P}_{Q_\mathrm{ad}} \colon L^2(I) \to L^2(I), \quad \mathrm{P}_{Q_\mathrm{ad}}(r)(t) := \max(q_a, \min(q_b, r(t))) \quad \text{for almost all } t \in I.$$

Lemma 3.5 *Let the assumptions of Lemma 3.4 be fulfilled. Then, the optimal solution $\bar{q} \in Q_\mathrm{ad}$ of the considered optimal control problem (3.2) for configuration C2 lies in $(H^1(I))^N$.*

Proof Let $\bar{q} \in Q_{ad}$ be the optimal solution. We regard a smooth sequence $q_n \in Q \cap (H^1(I))^N$ with $q_n \to \bar{q}$ in Q. As in the proof of Lemma 3.2, according to Theorem 2.1, we have that $(v_n, p_n, u_n) = G q_n$ solves (2.8) with right-hand sides $f_\mathrm{f} = B_\mathrm{f} q_n$, $f_\mathrm{s} = B_\mathrm{s} q_n$, and $g = 0$. The velocity and displacement have at least the regularities $v_n \in H^1(I; H_\mathrm{f})$ and $u_n \in L^2(I; V_\mathrm{s})$. Therefore, Theorem 3.2 guarantees the existence of a unique adjoint solution (z_n^v, z_n^p, z_n^u) of (3.4) with $v_n - v_\mathrm{d}$ and $u_n - u_\mathrm{d}$ in the right-hand side. By Lemma 3.4, we have

$$j'(q_n)(\delta q) = \sum_{i=1}^{N} \left[(\!(h_\mathrm{f}^i \delta q^i, z_n^v)\!) + \alpha \int_I q_n^i \delta q^i \, \mathrm{d}t \right]$$

for all $\delta q \in Q$. Due to estimate a) in Theorem 2.1, the linearity of (2.8), and the boundedness of h^i in H, we get for $m, n \in \mathbb{N}$ the estimate

$$\begin{aligned}
&\|v_n - v_m\|^2_{L^2(I;H^1(\Omega_\mathrm{f}))} + \|u_n - u_m\|^2_{L^2(I;H^1(\Omega_\mathrm{s}))} \\
&\le C\left[\sum_{i=1}^N \|h^i_\mathrm{f}(q^i_n - q^i_m)\|^2_{L^2(I;V^*_\mathrm{f})} + \sum_{i=1}^N \|h^i_\mathrm{s}(q^i_n - q^i_m)\|^2_{L^2(I;L^2(\Omega_\mathrm{s}))}\right] \\
&\le C\sum_{i=1}^N \|q^i_n - q^i_m\|^2_{L^2(I)}.
\end{aligned}$$

Further, due to estimate a) in Theorem 3.2, the adjoint variables fulfill the estimate

$$\begin{aligned}
&\|z^v_n - z^v_m\|^2_{L^\infty(I;L^2(\Omega))} + \|z^v_n - z^v_m\|^2_{L^2(I;H^1(\Omega_\mathrm{f}))} + \|z^u_n - z^u_m\|^2_{L^2(I;H^1(\Omega_\mathrm{s}))} \\
&\le C\left[\sum_{i=1}^N \|h^i_\mathrm{f}(q^i_n - q^i_m)\|^2_{L^2(I;V^*_\mathrm{f})} + \sum_{i=1}^N \|h^i_\mathrm{s}(q^i_n - q^i_m)\|^2_{L^2(I;L^2(\Omega_\mathrm{s}))}\right] \\
&\le C\sum_{i=1}^N \|q^i_n - q^i_m\|^2_{L^2(I)}.
\end{aligned}$$

If we regard in (3.4) test functions $\varphi \in L^2(I;V_\mathrm{div})$ that are divergence free in the fluid domain Ω_f, we get the estimate

$$\begin{aligned}
\|\partial_t z^v_n - \partial_t z^v_m\|^2_{L^2(I;V^*_\mathrm{div})} \le C\big[&\|z^v_n - z^v_m\|^2_{L^2(I;H^1(\Omega_\mathrm{f}))} + \|z^u_n - z^u_m\|^2_{L^2(I;H^1(\Omega_\mathrm{s}))} \\
&+ \|v_n - v_m\|_{L^2(I;L^2(\Omega_\mathrm{f}))}\big].
\end{aligned}$$

By combining the above estimates, we derive for the adjoint $z^v_n - z^v_m$ the bound

$$\|\partial_t z^v_n - \partial_t z^v_m\|^2_{L^2(I;V^*_\mathrm{div})} + \|z^v_n - z^v_m\|^2_{L^\infty(I;L^2(\Omega))} \le C\sum_{i=1}^N \|q^i_n - q^i_m\|^2_{L^2(I)}.$$

As $q_n \to \bar{q}$ in $(L^2(I))^N$, it holds $\|q^i_n - q^i_m\|^2_{L^2(I)} \to 0$ for $m, n \to \infty$. Thus, z^v_n is a Cauchy sequence in $H^1(I;V^*_\mathrm{div}) \cap L^\infty(I;H)$ and therefore there exists the limit $\tilde{z}^v \in H^1(I;V^*_\mathrm{div}) \cap L^\infty(I;H)$ such that

$$z^v_n \to \tilde{z}^v \quad \text{in} \quad H^1(I;V^*_\mathrm{div}) \cap L^\infty(I;H).$$

Since we assumed $h^i \in V_\mathrm{div}$ and $\delta q^i \in L^2(I)$, the product fulfills $h^i\delta q^i \in L^2(I;V_\mathrm{div})$. This implies due to $L^2(I;V_\mathrm{div}) \hookrightarrow L^2(I;H)$ and due to the convergence of z^v_n in $L^\infty(I;H)$ that

$$j'(q_n)(\delta q) = \sum_{i=1}^N \left[(\!(h^i\delta q^i, z^v_n)\!) + \alpha\int_I q^i_n \delta q^i \,\mathrm{d}t\right] \to \sum_{i=1}^N \left[(\!(h^i\delta q^i, \tilde{z}^v)\!) + \alpha\int_I \bar{q}^i \delta q^i \,\mathrm{d}t\right].$$

In addition, the directional derivative of the reduced cost functional $j'(\cdot)(\delta q)$ is continuous as the the control to state mapping $G\colon Q \to L^2(I;H_\mathrm{f}) \times L^2(I;H_\mathrm{s})$ is affine-linear and continuous accordingly to Lemma 3.1. Therefore, the convergence $q_n \to \bar{q}$ in Q implies in addition $j'(q_n)(\delta q) \to j'(\bar{q})(\delta q)$ and we obtain the identity

$$j'(\bar{q})(\delta q) = \sum_{i=1}^N \left[(\!(h^i\delta q^i, \tilde{z}^v)\!) + \alpha\int_I \bar{q}^i \delta q^i \,\mathrm{d}t\right].$$

As the optimal solution $\bar{q}$ fulfills the necessary optimality condition (3.3), we get the optimality condition

$$\sum_{i=1}^{N}\left[(\!(h^i(\delta q^i-\bar{q}^i),\tilde{z}^v)\!)+\alpha\int_I \bar{q}^i(\delta q^i-\bar{q}^i)\ \mathrm{d}t\right]\geq 0\quad \forall\delta q\in Q_{ad}.$$

Using the projection $\mathrm{P}_{\mathrm{Q_{ad}}}$ on the admissible set Q_{ad}, this can be expressed as

$$\bar{q}^i=\mathrm{P}_{\mathrm{Q_{ad}}}(r^i)\quad\text{with}\quad r^i=-\frac{1}{\alpha}\int_\Omega h^i(x)\tilde{z}^v(\cdot,x)\ \mathrm{d}x,\quad i=1,2,\dots,N.$$

The time regularity of the limit $\tilde{z}^v\in H^1(I;V_{\mathrm{div}}^*)$ and the assumed regularity of $h\in V_{\mathrm{div}}$ imply

$$\partial_t r^i(t)=-\frac{1}{\alpha}\langle h^i,\partial_t\tilde{z}^v(t,\cdot)\rangle_{V_{\mathrm{div}}\times V_{\mathrm{div}}^*},\quad i=1,2,\dots,N$$

for almost all $t\in I$ and consequently that $r^i\in H^1(I)$. Using the stability of the projection

$$\|\mathrm{P}_{\mathrm{Q_{ad}}}(r)\|_{H^1(I)}\leq\|r\|_{H^1(I)},$$

we obtain the asserted regularity $\bar{q}\in(H^1(I))^N$. □

Then, the optimal solution $\bar{q}\in Q_{ad}$ of the considered optimal control problem in configuration C2 fulfills the following theorem:

Theorem 3.4 *Let the initial data u_0, u_1, and v_0 fulfills the assumptions of Theorem 2.1 and let $v_{\mathrm{d}}\in H^1(I;V_{\mathrm{f}}^*)\cap L^2(I;H_{\mathrm{f}})$ and $u_{\mathrm{d}}\in L^2(I;H_{\mathrm{s}})$. Then, the optimal solution $\bar{q}\in Q_{ad}$ of the considered optimal control problem* (3.2) *for configuration* C2 *fulfills $\bar{q}\in(H^1(I))^N$ and the following necessary optimality condition:*

1. *The optimal state $(\bar{v},\bar{p},\bar{u})=(v(\bar{q}),p(\bar{q}),u(\bar{q}))$ solves*

$$\begin{aligned}&\rho_{\mathrm{f}}(\!(\partial_t\bar{v},\varphi)\!)_{\mathrm{f}}-(\!(\bar{p},\operatorname{div}\varphi)\!)_{\mathrm{f}}+\nu(\!(\nabla\bar{v},\nabla\varphi)\!)_{\mathrm{f}}\\&\qquad+\rho_{\mathrm{s}}(\!(\partial_t\bar{v},\varphi)\!)_{\mathrm{s}}+\mu(\!(\nabla\bar{u},\nabla\varphi)\!)_{\mathrm{f}}=(\!(B_{\mathrm{f}}\bar{q},\varphi)\!)_{\mathrm{f}}+(\!(B_{\mathrm{s}}\bar{q},\varphi)\!)_{\mathrm{s}}&&\forall\varphi\in L^2(I;V),\\&\mu(\!(\nabla\bar{v},\nabla\psi)\!)_{\mathrm{s}}-\mu(\!(\nabla\partial_t\bar{u},\nabla\psi)\!)_{\mathrm{s}}=0&&\forall\psi\in L^2(I;V_{\mathrm{s}}),\\&(\!(\xi,\operatorname{div}\bar{v})\!)_{\mathrm{f}}=0&&\forall\xi\in L^2(I;L_{\mathrm{f}}).\end{aligned}$$

2. *The optimal adjoint $(\bar{z}^v,\bar{z}^p,\bar{z}^u)=(z^v(\bar{q}),z^p(\bar{q}),z^u(\bar{q}))$ solves*

$$\begin{aligned}&-\rho_{\mathrm{f}}(\!(\varphi,\partial_t\bar{z}^v)\!)_{\mathrm{f}}+\nu(\!(\nabla\varphi,\nabla\bar{z}^v)\!)_{\mathrm{f}}+(\!(\bar{z}^p,\operatorname{div}\varphi)\!)_{\mathrm{f}}\\&\qquad-\rho_{\mathrm{s}}(\!(\varphi,\partial_t\bar{z}^v)\!)_{\mathrm{s}}+\mu(\!(\nabla\varphi,\nabla\bar{z}^u)\!)_{\mathrm{s}}=\gamma_{\mathrm{f}}(\!(\bar{v}-v_{\mathrm{d}},\varphi)\!)_{\mathrm{f}}&&\forall\varphi\in L^2(I;V),\\&\mu(\!(\nabla\psi,\nabla\bar{z}^v)\!)_{\mathrm{s}}+\mu(\!(\nabla\psi,\nabla\partial_t\bar{z}^u)\!)_{\mathrm{s}}=\gamma_{\mathrm{s}}(\!(\bar{u}-u_{\mathrm{d}},\psi)\!)_{\mathrm{s}}&&\forall\psi\in L^2(I;V_{\mathrm{s}}),\\&-(\!(\xi,\operatorname{div}\bar{z}^v)\!)_{\mathrm{f}}=0&&\forall\xi\in L^2(I;L_{\mathrm{f}}).\end{aligned}$$

3. *It holds for $i=1,2,\dots,N$ that*

$$\bar{q}^i=\mathrm{P}_{\mathrm{Q_{ad}}}\left(-\frac{1}{\alpha}\int_\Omega h^i(x)\bar{z}^v(\cdot,x)\ dx\right).$$

Proof As $\bar{q}\in Q\cap(H^1(I))^N$, one can choose in the proof of Lemma 3.5 the sequence $q_n=\bar{q}$. This immediately implies that $\tilde{z}^v=\bar{z}^v$. □

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