WELL-POSEDNESS AND EXPONENTIAL EQUILIBRATION OF A VOLUME-SURFACE REACTION-DIFFUSION SYSTEM WITH NONLINEAR BOUNDARY COUPLING

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Abstract. We consider a model system consisting of two reaction-diffusion equations, where one species diffuses in a volume while the other species diffuses on the surface which surrounds the volume. The two equations are coupled via nonlinear reversible Robin-type boundary conditions for the volume species and a matching reversible source term for the boundary species. As a consequence of the coupling, the total mass of the two species is conserved. The considered system is motivated for instance by models for asymmetric stem cell division.

Firstly we prove the existence of a unique weak solution via an iterative method of converging upper and lower solutions to overcome the difficulties of the nonlinear boundary terms. Secondly, we show explicit exponential convergence to equilibrium via an entropy method after deriving a suitable entropy entropy-dissipation estimate.

1. Introduction

In this paper, we consider a nonlinear volume-surface reaction-diffusion system, which couples a non-negative volume-concentration $u(x,t)$ diffusing on a bounded domain $\Omega \subset \mathbb{R}^N (N \geq 1)$ with a non-negative surface-concentration $v(x,t)$ diffusing on the sufficiently smooth boundary $\Gamma := \partial\Omega$ of Ω (e.g. $\partial\Omega \in C^{2+\epsilon}$ for $\epsilon > 0$).

The interface conditions connecting these two concentrations are a nonlinear Robin-type boundary condition for the volume-concentration $u(x,t)$ and a matching reversible reaction source term in the equation for the surface-concentration $v(x,t)$:

$$\begin{cases} u_t - \delta_u \Delta u = 0, & x \in \Omega, t > 0, \\ \delta_u \frac{\partial u}{\partial \nu} = -\alpha(k_u u^\alpha - k_v v^\beta), & x \in \Gamma, t > 0, \\ v_t - \delta_v \Delta_\Gamma v = \beta(k_u u^\alpha - k_v v^\beta), & x \in \Gamma, t > 0, \\ u(0,x) = u_0(x) \geq 0, & x \in \Omega, \\ v(0,x) = v_0(x) \geq 0, & x \in \Gamma. \end{cases} \tag{1.1}$$

Here, we denote by Δ the Laplace operator on Ω with a positive diffusion coefficient $\delta_u > 0$ and by Δ_Γ the Laplace-Beltrami operator on Γ (see e.g. [26]) with a non-negative diffusion coefficients $\delta_v \geq 0$ and $\nu(x)$ denotes the unit outward normal vector of Γ at the point x. Moreover, we shall consider nonnegative initial concentrations $u_0(x) \geq 0$ on Ω and $v_0(x) \geq 0$ on Γ.

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The stoichiometric coefficients $\alpha, \beta \in [1, \infty)$ together with the positive, bounded reaction rates $k_u(t,x), k_v(t,x) \in L^\infty_+([0,\infty) \times \Gamma)$ characterise the key feature of the model system (1.1), which is the nonlinear reversible reaction between the volume density $u(t,x)$ and the surface density $v(t,x)$ located at the boundary Γ.

We emphasise that the reversible reaction between volume and boundary in system (1.1) *preserves the total initial mass* M, which shall be assumed positive in the following:

$$\begin{aligned} M &= \beta \int_\Omega u(t,x)\,dx + \alpha \int_\Gamma v(t,x)\,dS, \qquad \forall t \geq 0 \\ &= \beta \int_\Omega u_0(x)\,dx + \alpha \int_\Gamma v_0(x)\,dS > 0. \end{aligned} \tag{1.2}$$

The study of system (1.1) is motivated by models of *asymmetric stem cell division.* In stem cells undergoing asymmetric cell division, particular proteins (so-called cell-fate determinants) are localised in only one of the two daughter cells during mitosis. These cell-fate determinants trigger in the following the differentiation of one daughter cell into specific tissue while the other daughter cell remains a stem cell.

In Drosophila, SOP stem cells provide a well-studied biological example model of asymmetric stem cell division, see e.g. [4, 32, 43] and the references therein. The mechanism of asymmetric cell division in SOP stem cells operates around a key protein called Lgl (Lethal giant larvae), which exists in two conformational states: a non-phosphorylated form which regulates the localisation of the cell-fate-determinants in the membrane of one daughter cell, and a phosphorylated form which is inactive.

First mathematical models describing the evolution and localisation of phosphorylated and non-phosphorylated Lgl in SOP stem cells were presented in [2, 38] under the assumption of linear phosphorylation and de-phosphorylation kinetics. However, it is known that Lgl offers three phosphorylation sites [4]. Thus, if more than one site needs to be phosphorylated in order to effectively deactivate Lgl, a realistic model should rather consider nonlinear kinetics.

The system (1.1) formulates a nonlinear mathematical core model, which strongly simplifies the biological model for SOP stem cells by focussing only on the concentration $u(x,t)$ of the phosphorylated Lgl in the cytoplasm (i.e. in the cell volume) and the concentration $v(x,t)$ of non-phosphorylated Lgl at the cortex/membrane of the cell. The exchange of phosphorylated Lgl $u(x,t)$ and non-phosphorylated Lgl $v(x,t)$ is described by the above nonlinear reaction located at the boundary. The considered evolution process conserves the total mass of Lgl as quantified in the conservation law (1.2).

Models related to (1.1) have recently gained rapidly increasing attention as they occur naturally in many areas of cell-biology and also fluid-dynamics, see e.g. [1, 22, 25, 31, 33, 36] and references therein.

The first aim of this paper is to prove the global existence of a unique weak solution to the model system (1.1) (see Theorem 2.10 below). The main difficulties arise from the arbitrary power-law nonlinearities located at the boundary Γ and shall be overcomed by applying an iteration method of converging upper and lower solutions, in the spirit of e.g. [37]. This method is based on proving a comparison principle for upper and lower solutions (see e.g. [3]), which so far - up to our knowledge - has not been established for volume-surface reaction-diffusion systems. Once the comparison principle is shown, the existence of weak solutions to (1.1) follows from an iteration argument, which uses that the involved nonlinearities are quasi-monotone non-decreasing. The existence of solutions to related linear models was proven in [22, 25] by fix-point methods. Our approach has the advantage of providing intrinsic a-priori bounds, which allows to obtain global solutions of the superlinear problem (1.1).

The second part of the manuscript proves *explicit exponential convergence to equilibrium* for the system (1.1) via the so-called *entropy method*. The basic idea of the entropy method consists of studying the large-time asymptotics of a dissipative PDE model by looking for a nonnegative Lyapunov functional $E(f)$ and its nonnegative dissipation

$$D(f) = -\frac{d}{dt}E(f(t))$$

along the flow of the PDE model, which is well-behaved in the following sense: firstly, all states with $D(f) = 0$, which also satisfy all the involved conservation laws, identify a unique entropy-minimising equilibrium f_∞, i.e.

$$D(f) = 0 \quad \text{and} \quad \text{conservation laws} \iff f = f_\infty,$$

and secondly, there exists an *entropy entropy-dissipation estimate* of the form

$$D(f) \geq \Phi(E(f) - E(f_\infty)), \qquad \Phi(x) \geq 0, \qquad \Phi(x) = 0 \iff x = 0,$$

for some nonnegative function Φ. Generally, such an inequality can only hold when all the conserved quantities are taken into account. If $\Phi'(0) \neq 0$, one usually gets exponential convergence toward f_∞ in relative entropy $E(f) - E(f_\infty)$ with a rate, which can be explicitly estimated.

The entropy method is a fully nonlinear alternative to arguments based on linearisation around the equilibrium and has the advantage of being quite robust with respect to variations and generalisations of the model system. This is due to the fact that the entropy method relies mainly on functional inequalities which have no direct link with the original PDE model. Generalised models typically feature related entropy and entropy-dissipation functionals and previously established entropy entropy-dissipation estimates may very usefully be re-applied.

The entropy method has previously been used for scalar equations: nonlinear diffusion equations (such as fast diffusions [6, 14], Landau equation [18]), integral equations (such as the spatially homogeneous Boltzmann equation [40, 41, 42]), kinetic equations (see e.g. [19, 20, 24]), or coagulation-fragmentation equations (see e.g. [7, 8]). For certain systems of drift-diffusion-reaction equations in semiconductor physics, an entropy entropy-dissipation estimate has been shown indirectly via a compactness-based contradiction argument in [27, 28, 30].

A first proof of entropy entropy-dissipation estimates for systems with explicit rates and constants was established in [15, 16, 17] in the case of reversible reaction-diffusion equations. Recently, a new idea of proving entropy entropy-dissipation estimates based on a convexification argument was presented in [34].

In this paper, we shall prove a new entropy entropy-dissipation estimate for the model system (1.1), which entails exponential convergence to equilibrium with explicitly computable constants and rates (see Theorem 3.2 below).

We remark two novelties: i) this is (up to our knowledge) the first entropy entropy-dissipation estimate for a mixed volume-surface reaction-diffusion system, and ii) secondly, we introduce a new idea in the proof of entropy entropy-dissipation estimates for a system with general, superlinear, power-like nonlinearities, which we hope to turn out very useful when proving entropy entropy-dissipation estimates in more general settings.

Moreover, we remark that, although the existence of weak solutions is obtained for general rates, which can depend on time and space, we restrict for the sake of clarity the proof of explicit exponential convergence to equilibrium to the case of constant rates k_u and k_v. The case of non-constant (in space and/or e.g. periodic in time) reactions rates leads to non-constant equilibria and requires a more involved formalism which shall be treated in future works.

We emphasise that we distinguish two cases in the equilibration analysis of (1.1): The non-degenerate diffusion case $\delta_v > 0$ and the degenerate diffusion case $\delta_v = 0$. If $\delta_v > 0$, then the surface diffusion term $-\delta_v \Delta_\Gamma v$ enables us to obtain an entropy-entropy dissipation estimate by only using the natural a-priori estimates derived from mass conservation, entropy and entropy dissipation. In the case of degenerate boundary diffusion $\delta_v = 0$, we derive an entropy entropy-dissipation estimate by using L^∞ a-priori bounds of the solution. While such L^∞-bounds can be shown to hold for the model (1.1), they are often out of reach for more general systems with more concentrations in higher space dimensions, see e.g. [5]. However, we conjecture that in some (yet not all) cases of stoichiometric coefficients α, β, the use of L^∞-bounds should not be essential for the proof and could be avoided by more careful estimates. An example of such an estimate is presented in Proposition 3.7 when $\alpha = \beta = 1$.

For future work, we hope that the robustness of the entropy method will enable us to study the large time behaviour of more complicated and realistic models of asymmetric cell division by reusing the entropy entropy-dissipation estimate derived in Lemma 3.3 for the non-degenerate case $\delta_v > 0$ and Lemma 3.6 for the degenerate case $\delta_v = 0$. Thus, the considered mathematical core problem (1.1) is also motivated by the goal of deriving core entropy entropy-dissipation estimates, which encompasses the nonlinear boundary dynamics featured by the system (1.1).

The rest of the paper is organised as follows. In Section 2, we prove the global existence of a unique weak solution for system (1.1). Section 3 is devoted to the entropy method, establishing entropy entropy-dissipation estimates and proving explicit exponential convergence to equilibrium.

2. Existence of a global solution

In this section, we will prove global existence of a unique weak solution to system (1.1) by the method of converging upper and lower solutions. We define first our notion of weak solutions:

Definition 2.1. *A pair of functions* (u, v) *is called a weak solution to system* (1.1) *on* $(0,T)$ *if*

$$u \in C([0,T]; L^2(\Omega)), \quad \textit{and} \quad u \in L^\infty(0,T; L^\infty(\Omega)) \cap L^2(0,T; H^1(\Omega)), \tag{2.1}$$

$$v \in C([0,T]; L^2(\Gamma)), \quad \textit{and} \quad v \in L^\infty(0,T; L^\infty(\Gamma)) \cap L^2(0,T; H^1(\Gamma)), \tag{2.2}$$

and the following weak formulation holds for all testfunctions $\varphi \in C^1([0,T];L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$ *and* $\psi \in C^1([0,T];L^2(\Gamma))\cap L^2(0,T;H^1(\Gamma))$ *with* $\varphi \geq 0$, $\psi \geq 0$ *and* $\varphi(T)=\psi(T)=0$*:*

$$\begin{cases}\int_0^T\int_\Omega[-u\varphi_t+\delta_u\nabla u\nabla\varphi]dxdt = \int_\Omega u_0\varphi(0)dx - \alpha\int_0^T\int_\Gamma(k_uu^\alpha - k_vv^\beta)\varphi dSdt,\\ \int_0^T\int_\Gamma[-v\psi_t+\delta_v\nabla_\Gamma v\nabla_\Gamma\psi]dSdt = \int_\Gamma v_0\psi(0)dS + \beta\int_0^T\int_\Gamma(k_uu^\alpha - k_vv^\beta)\psi dSdt,\end{cases} \tag{2.3}$$

in which ∇_Γ *is the tangential gradient on* Γ*.*

Remark 2.1. *With the regularity of* u *and* v *as stated in* (2.1) *and* (2.2)*, all left hand terms in* (2.3) *are clearly well defined. For the nonlinear reaction terms* $\int_\Gamma k_uu^\alpha\varphi dS$ *on the right hand side of* (2.3)*, we proceed as follows: First, if* $u \in H^1(\Omega)\cap L^\infty(\Omega)$*, we have*

$$\begin{aligned}\int_\Gamma |u|^{2\alpha}dx &= \|u^\alpha\|^2_{L^2(\Gamma)}\\ &\leq C(\|\nabla(u^\alpha)\|^2_{L^2(\Omega)} + \|u^\alpha\|^2_{L^2(\Omega)}) \qquad \textit{(by using the Trace Theorem)}\\ &\leq C(\alpha^2\|u\|^{2\alpha-2}_{L^\infty(\Omega)}\|\nabla u\|^2_{L^2(\Omega)} + |\Omega|\|u\|^{2\alpha}_{L^\infty(\Omega)}).\end{aligned}$$

Hence, $u^\alpha|_\Gamma \in L^2(\Gamma)$*. Therefore, with the help of the estimate*

$$\int_\Gamma k_u(t,x)u^\alpha\varphi dS \leq \|k_u\|_\infty\|u^\alpha\|^2_{L^2(\Gamma)}\|\varphi\|^2_{L^2(\Gamma)},$$

the weak formulation in Definition 2.1 makes sense.

Definition 2.2. *We shall use the following short notation*

$$(u_1,v_1)\geq(u_2,v_2)$$

for two pairs of functions (u_1,v_1) *and* (u_2,v_2) *where* $u_i(t,x): I\times\Omega\to\mathbb{R}$ *and* $v_i(t,x): I\times\Gamma\to\mathbb{R}$, $i=1,2, I\subset\mathbb{R}$*, which means that*

$$\begin{aligned}u_1(t,x)\geq u_2(t,x) \qquad &\textit{for a.e.} \quad (t,x)\in I\times\Omega,\\ v_1(t,x)\geq v_2(t,x) \qquad &\textit{for a.e.} \quad (t,x)\in I\times\Gamma.\end{aligned}$$

Next, we define the notation

$$F(t,x,u,v) := -\alpha\big(k_u(t,x)u^\alpha - k_v(t,x)v^\beta\big), \qquad (t,x)\in[0,\infty)\times\Gamma,$$

and

$$G(t,x,u,v) := \beta\big(k_u(t,x)u^\alpha - k_v(t,x)v^\beta\big), \qquad (t,x)\in[0,\infty)\times\Gamma.$$

Lemma 2.1. *We have the following properties for the nonlinearities* F *and* G*:*

- i) $F(t,x,u,\cdot)$, $G(t,x,\cdot,v)$ *are non-decreasing for all* $(t,x)\in[0,\infty)\times\Gamma$,
- ii) $F(t,x,\cdot,v)$, $G(t,x,u,\cdot)$ *are non-increasing for all* $(t,x)\in[0,\infty)\times\Gamma$,
- iii) $F(t,x,\cdot,\cdot)$ *and* $G(t,x,\cdot,\cdot)$ *are locally Lipschitz. In particular, given a pair of non-negative functions* $(\overline{u},\overline{v})\geq(0,0)$*, there exist two non-negative bounded functions* $L_u(t,x)$, $L_v(t,x)\in L^\infty([0,\infty)\times\Gamma)$ *such that, for all* $(\overline{u},\overline{v})\geq(u_1,v_1),(u_2,v_2)\geq(0,0)$*, the followings hold pointwise in* $(t,x)\in[0,\infty)\times\Gamma$*:*

$$F(t,x,u_1,v_1) - F(t,x,u_2,v_2) \leq \alpha L_u(t,x)(u_2-u_1)_+ + \alpha L_v(t,x)(v_1-v_2)_+, \tag{2.4}$$

$$F(t,x,u_1,v_1) - F(t,x,u_2,v_2) \geq -\alpha L_u(t,x)(u_1-u_2)_+ - \alpha L_v(t,x)(v_2-v_1)_+, \tag{2.5}$$

and

$$G(t,x,u_1,v_1) - G(t,x,u_2,v_2) \le \beta L_u(t,x)(u_1-u_2)_+ + \beta L_v(t,x)(v_2-v_1)_+, \quad (2.6)$$

$$G(t,x,u_1,v_1) - G(t,x,u_2,v_2) \ge -\beta L_u(t,x)(u_2-u_1)_+ - \beta L_v(t,x)(v_1-v_2)_+, \quad (2.7)$$

and $(\cdot)_+$ *denotes the positive part, that is* $(w)_+ = w$ *if* $w \ge 0$ *and* $(w)_+ = 0$ *otherwise.*

Proof. The proof of (i) and (ii) follows trivially from the positivity of the reaction rates $k_u(t,x)$, $k_v(t,x) \in L^\infty_+([0,\infty)\times\Gamma)$. To prove (iii), we apply (after suppressing the pointwise dependency on t and x) the mean-value theorem

$$\begin{aligned} F(u_1,v_1) - F(u_2,v_2) &= -\alpha k_u(u_1^\alpha - u_2^\alpha) + \alpha k_v(v_1^\beta - v_2^\beta) \\ &= -\alpha^2 k_u(\theta_u)^{\alpha-1}(u_1-u_2) + \alpha^2 k_v(\theta_v)^{\beta-1}(v_1-v_2), \end{aligned}$$

with $\theta_u(t,x) = \theta u_1 + (1-\theta)u_2$ and $\theta_v = \theta v_1 + (1-\theta)v_2$ for various $\theta(t,x)\in(0,1)$ pointwise for all $(t,x)\in[0,\infty)\times\Gamma$. Thus (2.4) and (2.5) follow with $L_u(t,x) = \alpha k_u(t,x)\overline{u}^{\alpha-1}$ and $L_v(t,x) = \beta k_v(t,x)\overline{v}^{\beta-1}$. The proof of (2.6) and (2.7) follows analogously. □

In the following, we will prove the existence of a unique weak solution to the system (1.1) by the method of converging upper and lower solutions, see e.g. [37]. In order to apply this method, we need to prove the comparison principle for system (1.1) for pairs of upper and lower solutions. Before we derive such a comparison principle, we recall the following Trace inequality:

Lemma 2.2. [29, Theorem 1.5.1.10] *For any* $\varepsilon > 0$*, there exists* $C_\varepsilon > 0$ *such that*

$$\int_\Gamma |u|^2 dS \le \varepsilon \int_\Omega |\nabla u|^2 dx + C_\varepsilon \int_\Omega |u|^2 dx$$

for all $u \in H^1(\Omega)$.

Definition 2.3. *A pair* $(\overline{u},\overline{v})$ *is called an upper solution to the problem* (1.1) *if* $(\overline{u},\overline{v})$ *satisfy the regularity* (2.1) *and* (2.2) *and that for all testfunctions* $\varphi \in C^1(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$, $\psi \in C^1(0,T;L^2(\Gamma))\cap L^2(0,T;H^1(\Gamma))$ *with* $\varphi,\psi \ge 0$ *and* $\varphi(T)=\psi(T)=0$*, we have*

$$\begin{cases} \int_0^T\int_\Omega[-\overline{u}\varphi_t + \delta_u\nabla\overline{u}\nabla\varphi]dxdt - \int_0^T\int_\Gamma F(t,x,\overline{u},\overline{v})\varphi dSdt \ge \int_\Omega \overline{u}(0)\varphi dx, \\ \int_0^T\int_\Gamma[-\overline{v}\psi_t + \delta_v\nabla_\Gamma\overline{v}\nabla_\Gamma\psi]dSdt - \int_0^T\int_\Gamma G(t,x,\overline{u},\overline{v})\psi dSdt \ge \int_\Gamma \overline{v}(0)\psi dS, \\ \overline{u}(0,x) \ge u_0(x) \ \ a.e. \ \ x\in\Omega, \\ \overline{v}(0,x) \ge v_0(x) \ \ a.e. \ \ x\in\Gamma. \end{cases} \quad (2.8)$$

Similarly for a lower solution $(\underline{u},\underline{v})$*:*

$$\begin{cases} \int_0^T\int_\Omega[-\overline{u}\varphi_t + \delta_u\nabla\overline{u}\nabla\varphi]dxdt - \int_0^T\int_\Gamma F(t,x,\overline{u},\overline{v})\varphi dSdt \le \int_\Omega \overline{u}(0)\varphi dx, \\ \int_0^T\int_\Gamma[-\overline{v}\psi_t + \delta_v\nabla_\Gamma\overline{v}\nabla_\Gamma\psi]dSdt - \int_0^T\int_\Gamma G(t,x,\overline{u},\overline{v})\psi dSdt \le \int_\Gamma \overline{v}(0)\psi dS, \\ \overline{u}(0,x) \le u_0(x) \ \ a.e. \ \ x\in\Omega, \\ \overline{v}(0,x) \le v_0(x) \ \ a.e. \ \ x\in\Gamma. \end{cases} \quad (2.9)$$

We are now going to prove a comparison principle for upper and lower solutions. The idea of the proof is motivated by [3].

Lemma 2.3 (Comparison Principle for Pairs of Upper and Lower Solutions)**.** *Let $0 < T < \infty$, $\underline{u}$, $\overline{u}$ satisfy (2.1) and $\underline{v}$, $\overline{v}$ satisfy (2.2). Assume that for all testfunctions $\varphi \in C^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \psi \in C^1(0,T;L^2(\Gamma)) \cap L^2(0,T;H^1(\Gamma))$ with $\varphi, \psi \geq 0$ and $\varphi(T) = \psi(T) = 0$, we have*

$$\begin{cases} \int_0^T\int_\Omega[-(\underline{u}-\overline{u})\varphi_t + \delta_u\nabla(\underline{u}-\overline{u})\nabla\varphi]dxdt \\ \qquad -\int_0^T\int_\Gamma(F(t,x,\underline{u},\underline{v}) - F(t,x,\overline{u},\overline{v}))\,\varphi\,dSdt \leq \int_\Omega(\underline{u}(0)-\overline{u}(0))\varphi dx, \\ \int_0^T\int_\Gamma[-(\underline{v}-\overline{v})\psi_t + \delta_v\nabla_\Gamma(\underline{v}-\overline{v})\nabla_\Gamma\psi]dSdt \\ \qquad -\int_0^T\int_\Gamma(G(t,x,\underline{u},\underline{v}) - G(t,x,\overline{u},\overline{v}))\,\psi\,dSdt \leq \int_\Gamma(\underline{v}(0)-\overline{v}(0))\psi dS, \\ \underline{u}(0,x) \leq \overline{u}(0,x),\ x\in\Omega, \\ \underline{v}(0,x) \leq \overline{v}(0,x),\ x\in\Gamma. \end{cases} \tag{2.10}$$

Then, $(\underline{u},\underline{v}) \leq (\overline{u},\overline{v})$ in the sense of Definition 2.2.

Proof. We denote $w = \underline{u} - \overline{u}$ and $z = \underline{v} - \overline{v}$ and rewrite system (2.10) as

$$\begin{cases} \int_0^T\int_\Omega[-w\varphi_t + \delta_u\nabla w\nabla\varphi]dxdt \\ \qquad -\int_0^T\int_\Gamma(F(t,x,\underline{u},\underline{v}) - F(t,x,\overline{u},\overline{v}))\varphi dSdt \leq \int_\Omega w(0)\varphi dx \leq 0, \\ \int_0^T\int_\Gamma(-z\psi_t + \delta_v\nabla_\Gamma z\nabla_\Gamma\psi)dS \\ \qquad -\int_0^T\int_\Gamma(G(t,x,\underline{u},\underline{v}) - G(t,x,\overline{u},\overline{v}))\psi dSdt \leq \int_\Gamma z(0)\psi dS \leq 0, \\ w(0,x)\leq 0,\ z(0,x)\leq 0. \end{cases} \tag{2.11}$$

Taking φ as the positive part $w_+ \in L^2((0,T);H^1(\Omega))$ in (2.11), we get for a.e. $t \in (0,T)$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega|w_+|^2dx + \delta_u\int_\Omega|\nabla w_+|^2dx \leq \int_\Gamma(F(t,x,\underline{u},\underline{v}) - F(t,x,\overline{u},\overline{v}))w_+dS \\ &\leq \alpha\int_\Gamma L_u(-w)_+w_+dS + \alpha\|L_v\|_\infty\int_\Gamma z_+w_+dS \qquad \text{(by (2.4))} \\ &\leq \frac{\alpha\|L_v\|_\infty}{2}\int_\Gamma|z_+|^2dS + \frac{\alpha\|L_v\|_\infty}{2}\int_\Gamma|w_+|^2dS \qquad \text{(by Young's inequality)} \\ &\leq \frac{\alpha\|L_v\|_\infty}{2}\int_\Gamma|z_+|^2dS + \delta_u\int_\Omega|\nabla w_+|^2dx + C\int_\Omega|w_+|^2dx, \end{aligned} \tag{2.12}$$

where we have applied the Trace inequality as in Lemma 2.2 in the last step and the constant $C = C(\alpha, \|L_v\|_\infty, \delta_u, \Omega)$. Hence,

$$\frac{d}{dt}\int_\Omega|w_+|^2dx \leq C\left(\int_\Gamma|z_+|^2dS + \int_\Omega|w_+|^2dx\right), \tag{2.13}$$

for a constant C and similarly, using (2.6)

$$\frac{d}{dt}\int_\Gamma|z_+|^2dS \leq C\left(\int_\Gamma|z_+|^2dS + \int_\Omega|w_+|^2dx\right). \tag{2.14}$$

Thus, combining (2.13) and (2.14) and using Gronwall's inequality with $w_+(0) = 0$ and $z_+(0) = 0$ yields $w_+(t) = 0$ and $z_+(t) = 0$ for a.e. $t \in (0,T)$, which completes the proof. $\square$

By subtracting (2.9) from (2.8), the comparison Lemma 2.3 yields the following.

Lemma 2.4. *If* $(\overline{u},\overline{v})$ *is an upper solution and* $(\underline{u},\underline{v})$ *is a lower solution to* (1.1), *then we have* $(\overline{u},\overline{v}) \geq (\underline{u},\underline{v})$ *in the sense of Definition 2.2.*

Proposition 2.5. *There exists an upper solution* $(\overline{u},\overline{v})$ *and a lower solution* $(\underline{u},\underline{v})$ *to the system* (1.1).

Proof. Clearly $(\underline{u},\underline{v}) = (0,0)$ is a lower solution. To find an upper solution, we choose a function $B_0 \in L^\infty(\Gamma)$ which satisfies $B_0 \geq v_0$ a.e. in Γ. Then we let $B(t,x)$ be the unique solution to the equation

$$\begin{cases} B_t(t,x) - \delta_v \Delta_\Gamma B(t,x) = 0, & 0 < t < T, x \in \Gamma, \\ B(0,x) = B_0(x), & x \in \Gamma. \end{cases}$$

By the classical smoothing effect for this homogeneous linear heat equation, we get that B is smooth in $(0,T) \times \Gamma$. Since $k_u(t,x), k_v(t,x)$ are uniformly bounded above and away from zero, we can define

$$A_{bd}(t,x) = \left(\frac{k_v(t,x)}{k_u(t,x)} B^\beta(t,x)\right)^{1/\alpha} \quad \text{for } (t,x) \in (0,T) \times \Gamma.$$

We now choose $A_0 \in L^\infty(\Omega)$ satisfying $A_0 \geq u_0$ a.e. in Ω and then let $A(t,x)$ be the unique solution to the following heat equation in Ω with non-homogeneous Dirichlet boundary condition:

$$\begin{cases} A_t(t,x) - \delta_u \Delta A(t,x) = 0, & 0 < t < T, x \in \Omega, \\ A(t,x) = A_{bd}(t,x), & 0 < t < T, x \in \Gamma, \\ A(0,x) = A_0(x), & x \in \Omega. \end{cases}$$

It is now easy to verify that $(\overline{u},\overline{v}) = (A,B)$ is an upper solution to system (1.1). □

Remark 2.2. *In the case where* k_u *and* k_v *are constants, we can define the upper solution by setting* $\overline{u} = A$ *and* $\overline{v} = B$ *where* A *and* B *are two positive constants satisfying*

$$A \geq \|u_0\|_{L^\infty(\Omega)}, \; B \geq \|v_0\|_{L^\infty(\Gamma)} \text{ and } k_u A^\alpha = k_v B^\beta.$$

Remark 2.3. *The proof of the comparison principle of pairs of upper and lower solutions in Lemma 2.3 can readily be generalised to locally Lipschitz functions* F, G, *which satisfy* $F(\underline{u},\underline{v}) - F(\overline{u},\overline{v}) \leq L(|\underline{u}-\overline{u}| + (\underline{v}-\overline{v})_+)$ *and* $G(\underline{u},\underline{v}) - G(\overline{u},\overline{v}) \leq L((\underline{u}-\overline{u})_+ + |\underline{v}-\overline{v}|)$ *pointwise in* (t,x). *However, for general* F *and* G, *the existence of an upper solution is unclear and can not be expected in general. Otherwise, the below construction of global solutions from converging upper and lower solutions would lead to global existence of solutions, where no such existence can be expected. Nevertheless, we refer to* [37, Section 1.8] *for finding upper and lower solutions for several specific problems.*

In order to prove our existence result, we introduce the following auxiliary functions, which will be useful for proving the monotonicity of the sequences of upper and lower solutions.

$$\begin{aligned} f(t,x,u,v) &= F(t,x,u,v) + \alpha L_u(t,x)\, u \\ g(t,x,u,v) &= G(t,x,u,v) + \beta L_v(t,x)\, v. \end{aligned} \tag{2.15}$$

Lemma 2.6. *Functions f and g inherit the following properties from functions F and G:*

(i) *The functions $f(t,x,u,\cdot)$ and $g(t,x,\cdot,v)$ are non-decreasing for any $t,x \in \mathbb{R}\times\Gamma$ and any $u,v\in\mathbb{R}$.*

(ii) *For all $(\overline{u},\overline{v}) \geq (u_1,v_1) \geq (u_2,v_2) \geq (0,0)$ there holds:*

$$f(t,x,u_1,v) - f(t,x,u_2,v) \geq \\ -\alpha L_u(t,x)(u_1-u_2)_+ + \alpha L_u(t,x)(u_1-u_2) \geq 0,$$

and

$$g(t,x,u,v_1) - g(t,x,u,v_2) \geq \\ -\beta L_v(t,x)(v_1-v_2)_+ + \beta L_v(t,x)(v_1-v_2) \geq 0.$$

Thus, the functions $f(t,x,\cdot,v)$ and $g(t,x,u,\cdot)$ are monotone non-decreasing for all $(\overline{u},\overline{v}) \geq (u_1,v_1) \geq (u_2,v_2) \geq (0,0)$ contrary to F and G.

Proof. The statements of the above Lemma follow directly from Lemma 2.1, in particular from (2.5) and (2.7). □

With the notation (2.15), the system (1.1) rewrites as

$$\begin{cases} u_t - \delta_u \Delta u = 0, & x\in\Omega, t>0,\\ \delta_u \frac{\partial u}{\partial \nu} + \alpha L_u(t,x)\, u = f(t,x,u,v), & x\in\Gamma, t>0 \\ v_t - \delta_v \Delta_\Gamma v + \beta L_v(t,x)\, v = g(t,x,u,v), & x\in\Gamma, t>0. \end{cases} \tag{2.16}$$

Hereafter, we write $f(u,v)$ and $g(u,v)$ for $f(t,x,u,v)$ and $g(t,x,u,v)$ respectively except where it is stated otherwise.

Starting from the pair of lower and upper solutions $(\underline{u},\underline{v}) \leq (\overline{u},\overline{v})$ as constructed in Proposition 2.5, we will construct a sequence of lower solutions $\{(\underline{u}^{(k)},\underline{v}^{(k)})\}_{k\geq 0}$ as follows:

$$(\underline{u}^{(0)},\underline{v}^{(0)}) = (\underline{u},\underline{v}) \tag{I.0}$$

and for all $k\geq 1$, $\underline{u}^{(k)}$ and $\underline{v}^{(k)}$ are the solutions of the following heat equation with inhomogeneous Robin boundary condition:

$$\begin{cases} \partial_t \underline{u}^{(k)} - \delta_u \Delta \underline{u}^{(k)} = 0, & x\in\Omega, t>0,\\ \delta_u \frac{\partial \underline{u}^{(k)}}{\partial \nu} + \alpha L_u \underline{u}^{(k)} = f(\underline{u}^{(k-1)},\underline{v}^{(k-1)}), & x\in\Gamma, t>0,\\ \underline{u}^{(k)}(0,x) = u_0(x) \in L^\infty(\Omega), & x\in\Omega, \end{cases} \tag{I.1}$$

and the following linear inhomogeneous equation:

$$\begin{cases} \partial_t \underline{v}^{(k)} - \delta_v \Delta_\Gamma \underline{v}^{(k)} + \beta L_v \underline{v}^{(k)} = g(\underline{u}^{(k-1)},\underline{v}^{(k-1)}), & x\in\Gamma, t>0,\\ \underline{v}^{(k)}(0,x) = v_0(x) \in L^\infty(\Gamma), & x\in\Gamma. \end{cases} \tag{I.2}$$

Similarly, we construct a sequence of upper solutions $\{(\overline{u}^{(k)},\overline{v}^{(k)})\}_{k\geq 0}$:

$$(\overline{u}^{(0)},\overline{v}^{(0)}) = (\overline{u},\overline{v}) \tag{II.0}$$

and for all $k \geq 1$, $\overline{u}^{(k)}$ and $\overline{v}^{(k)}$ are the solutions of the following heat equation with inhomogeneous Robin boundary condition:

$$\begin{cases} \partial_t \overline{u}^{(k)} - \delta_u \Delta \overline{u}^{(k)} = 0, & x \in \Omega, t > 0, \\ \delta_u \frac{\partial \overline{u}^{(k)}}{\partial \nu} + \alpha L_u \overline{u}^{(k)} = f(\overline{u}^{(k-1)}, \overline{v}^{(k-1)}), & x \in \Gamma, t > 0, \\ \overline{u}^{(k)}(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{II.1}$$

and the following linear inhomogeneous equation:

$$\begin{cases} \partial_t \overline{v}^{(k)} - \delta_v \Delta_\Gamma \overline{v}^{(k)} + \beta L_v \overline{v}^{(k)} = g(\overline{u}^{(k-1)}, \overline{v}^{(k-1)}), & x \in \Gamma, t > 0, \\ \overline{v}^{(k)}(0,x) = v_0(x), & x \in \Gamma. \end{cases} \tag{II.2}$$

The existence of unique sequences of lower and upper solutions $\underline{u}^{(k)}$ and $\underline{v}^{(k)}$ follows from classical arguments in an iterative way starting from (I.0) and (II.0). Given for instance $(\underline{u}^{(k-1)}, \underline{v}^{(k-1)})$, the system (I.1) is a heat equation with inhomogeneous Robin boundary condition and bounded coefficients $L_u(t,x) \in L^\infty(\mathbb{R}_+ \times \Gamma)$. Thus, the existence of a unique weak solution in the sense of Definition 2.1 follows e.g. from [12, 13, 35]. Moreover, the equation (I.2) is a linear heat equation on a manifold without boundary and existence of a unique weak solutions follows e.g. from [39, Chapter 6].

Moreover, if $\underline{u}^{(k-1)}$ and $\underline{v}^{(k-1)}$ satisfy the regularity (2.1) and (2.2), then by the locally Lipschitz properties of f and g, we get $f(\underline{u}^{(k-1)}, \underline{v}^{(k-1)})$, $g(\underline{u}^{(k-1)}, \underline{v}^{(k-1)}) \in L^\infty(\Gamma)$, which implies from (I.1) that $\underline{u}^{(k)}$ satisfies (2.1) and from (I.2) that $\underline{v}^{(k)}$ satisfies (2.2).

An analogous argument can be applied to the equations (II.1) and (II.2) in order to get the unique existence and the regular properties of $(\overline{u}^{(k)}, \overline{v}^{(k)})$.

Lemma 2.7. *The sequence $\{(\underline{u}^{(k)}, \underline{v}^{(k)})\}_{k\geq 0}$ is an increasing sequence of lower solutions while $\{(\overline{u}^{(k)}, \overline{v}^{(k)})\}_{k\geq 0}$ is a decreasing sequence of upper solutions. More precisely, for all $k \geq 0$,*

$$(\overline{u}^{(k)}, \overline{v}^{(k)}) \geq (\overline{u}^{(k+1)}, \overline{v}^{(k+1)}) \geq (\underline{u}^{(k+1)}, \underline{v}^{(k+1)}) \geq (\underline{u}^{(k)}, \underline{v}^{(k)})$$

in the sense of Definition 2.2.

Proof.
Claim 1: $(\underline{u}^{(k+1)}, \underline{v}^{(k+1)}) \geq (\underline{u}^{(k)}, \underline{v}^{(k)})$ for all $k \geq 0$. We proceed by induction. Denote $\underline{w}^{(k)} = \underline{u}^{(k+1)} - \underline{u}^{(k)}$ and $\underline{z}^{(k)} = \underline{v}^{(k+1)} - \underline{v}^{(k)}$. From (I.1) and (I.2), by noticing that $(\underline{u}^{(0)}, \underline{v}^{(0)}) = (0,0)$ and thus $f(\underline{u}^{(0)}, \underline{v}^{(0)}) = g(\underline{u}^{(0)}, \underline{v}^{(0)}) = 0$, we have

$$\begin{cases} \partial_t \underline{w}^{(0)} - \delta_u \Delta \underline{w}^{(0)} \geq 0, \\ \delta_u \frac{\partial \underline{w}^{(0)}}{\partial \nu} + \alpha L_u \underline{w}^{(0)} \geq 0, \\ \underline{w}^{(0)}(0) \geq 0 \\ \partial_t \underline{z}^{(0)} - \delta_v \Delta_\Gamma \underline{z}^{(0)} + \beta L_v \underline{z}^{(0)} \geq 0, \\ \underline{z}^{(0)}(0) \geq 0. \end{cases} \tag{2.17}$$

and the maximum principle for weak solutions (see e.g. [9, Theorem 11.9]) implies $(\underline{w}^{(0)}, \underline{z}^{(0)}) \geq 0$.

Now, assume that $(\underline{u}^{(i)}, \underline{v}^{(i)}) \geq (\underline{u}^{(i-1)}, \underline{v}^{(i-1)})$ for all $i = 1, 2, \ldots, k$. Then, a pair $(\underline{w}^{(k)}, \underline{z}^{(k)})$ satisfies

$$\begin{cases} \partial_t \underline{w}^{(k)} - \delta_u \Delta \underline{w}^{(k)} = 0, \\ \delta_u \frac{\partial \underline{w}^{(k)}}{\partial \nu} + \alpha L_u \underline{w}^{(k)} = f(\underline{u}^{(k)}, \underline{v}^{(k)}) - f(\underline{u}^{(k-1)}, \underline{v}^{(k-1)}), \\ \partial_t \underline{z}^{(k)} - \delta_v \Delta_\Gamma \underline{z}^{(k)} + \beta L_v \underline{z}^{(k)} = g(\underline{u}^{(k)}, \underline{v}^{(k)}) - g(\underline{u}^{(k-1)}, \underline{v}^{(k-1)}), \\ \underline{w}^{(k)}(0) = 0, \underline{z}^{(k)}(0) = 0. \end{cases} \tag{2.18}$$

Using Lemma 2.6 we have

$$\begin{aligned} f(\underline{u}^{(k)}, \underline{v}^{(k)}) - f(\underline{u}^{(k-1)}, \underline{v}^{(k-1)}) &= f(\underline{u}^{(k)}, \underline{v}^{(k)}) - f(\underline{u}^{(k)}, \underline{v}^{(k-1)}) \\ &\quad + f(\underline{u}^{(k)}, \underline{v}^{(k-1)}) - f(\underline{u}^{(k-1)}, \underline{v}^{(k-1)}) \geq 0 \end{aligned} \tag{2.19}$$

since $(\underline{u}^{(k)}, \underline{v}^{(k)}) \geq (\underline{u}^{(k-1)}, \underline{v}^{(k-1)})$. Analogously, we have also the estimate

$$g(\underline{u}^{(k)}, \underline{v}^{(k)}) - g(\underline{u}^{(k-1)}, \underline{v}^{(k-1)}) \geq 0.$$

Thus, by using the maximum principle again, we have

$$(\underline{w}^{(k)}, \underline{z}^{(k)}) \geq 0$$

or equivalently

$$(\underline{u}^{(k+1)}, \underline{v}^{k+1}) \geq (\underline{u}^{(k)}, \underline{v}^{(k)}).$$

Claim 2: $(\overline{u}^{(k+1)}, \overline{v}^{(k+1)}) \leq (\overline{u}^{(k)}, \overline{v}^{(k)})$ for all $k \geq 0$. The proof is similar to Claim 1 and is omitted.

Claim 3: For each $k \geq 0$, $(\overline{u}^{(k)}, \overline{v}^{(k)})$ is an upper solution and $(\underline{u}^{(k)}, \underline{v}^{(k)})$ is a lower solution. We will show the result for $(\underline{u}^{(k)}, \underline{v}^{(k)})$ and that of $(\overline{u}^{(k)}, \overline{v}^{(k)})$ follows similarly. We again prove this claim by induction: The case $k = 0$ follows directly from (I.0). Assume that $(\underline{u}^{(i)}, \underline{v}^{(i)})$ are lower solutions for all $i = 0, 1, \ldots, k-1$. We will check that $(\underline{u}^{(k)}, \underline{v}^{(k)})$ is also a lower solution by using Definition 2.3 and the monotonicity of $\{(\underline{u}^{(j)}, \underline{v}^{(j)})\}_{j \geq 0}$. Taking the weak formulation of (I.1), we have

$$\begin{aligned} &\int_0^T \int_\Omega [-\underline{u}^{(k)} \varphi_t + \delta_u \nabla \underline{u}^{(k)} \nabla \varphi] dx dt \\ &\qquad - \int_0^T \int_\Gamma [-\alpha L_u \underline{u}^{(k)} + f(\underline{u}^{(k-1)}, \underline{v}^{(k-1)})] \varphi dS dt = \int_\Omega \underline{u}^{(k)}(0) \varphi dx. \end{aligned}$$

Hence, we have by (2.15)

$$\begin{aligned} &\int_0^T \int_\Omega [-\underline{u}^{(k)} \varphi_t + \delta_u \nabla \underline{u}^{(k)} \nabla \varphi] dx dt - \int_0^T \int_\Gamma F(\underline{u}^{(k)}, \underline{v}^{(k)}) \varphi dS dt \\ &\quad = \int_\Omega \underline{u}^{(k)}(0) \varphi dx + \int_0^T \int_\Gamma [f(\underline{u}^{(k-1)}, \underline{v}^{(k-1)}) - f(\underline{u}^{(k)}, \underline{v}^{(k)})] \varphi dS dt \\ &\quad \leq \int_\Omega \underline{u}^{(k)}(0) \varphi dx, \end{aligned}$$

where we have used Lemma 2.6 as above in estimate (2.19) and the non-decreasing monotonicity of the sequence $\{\underline{u}^{(j)},\underline{v}^{(j)}\}_{j\geq 0}$. An analogous argument provides

$$\int_0^T\int_\Omega[-\underline{v}^{(k)}\psi_t+\delta_v\nabla_\Gamma\underline{v}^{(k)}\nabla_\Gamma\psi]dSdt-\int_0^T\int_\Gamma G(\underline{u}^{(k)},\underline{v}^{(k)})\psi dSdt$$
$$=\int_\Gamma\underline{v}^{(k)}(0)\psi dS+\int_0^T\int_\Gamma[g(\underline{u}^{(k-1)},\underline{v}^{(k-1)})-g(\underline{u}^{(k)},\underline{v}^{(k)})]dSdt$$
$$\leq\int_\Gamma\underline{v}^{(k)}(0)\psi dS.$$

Taking into account that $\underline{u}^{(k)}(0)=u_0$ and $\underline{v}^{(k)}(0)=v_0$, we have that $(\underline{u}^{(k)},\underline{v}^{(k)})$ is a lower solution. □

From Lemma 2.7 with the help of the monotone convergence theorem, we have the following almost everywhere pointwise limits in $(0,T)\times\Omega$ and $(0,T)\times\Gamma$ respectively:

$$\lim_{k\to\infty}(\underline{u}^{(k)},\underline{v}^{(k)})=(\underline{u}^*,\underline{v}^*)\text{ and }\lim_{k\to\infty}(\overline{u}^{(k)},\overline{v}^{(k)})=(\overline{u}^*,\overline{v}^*).\tag{2.20}$$

The following *a priori estimates* are uniform in k pointwise for all times $t\in(0,T)$, and will allow us to pass to the limit $k\to\infty$:

Lemma 2.8. *The sequences $\{\overline{u}^{(k)}\}_{k\geq 0}$ and $\{\overline{v}^{(k)}\}_{k\geq 0}$ are bounded uniformly in k in $L^\infty(0,T;L^\infty(\Omega))\cap L^2(0,T;H^1(\Omega))$ and $L^\infty(0,T;L^\infty(\Gamma))\cap L^2(0,T;H^1(\Gamma))$, respectively, for any given $T>0$. Moreover, the sequence $\{(\overline{u}^{(k)})^\alpha|_\Gamma\}_{k\geq 0}$ is bounded in $L^2(0,T;L^2(\Gamma))$. We also have analogous estimates for $\{\underline{u}^{(k)}\}_{k\geq 0}$ and $\{\underline{v}^{(k)}\}_{k\geq 0}$.*

Proof. We will prove only for $\{\overline{u}^{(k)}\}_{k\geq 0}$ and $\{\overline{v}^{(k)}\}_{k\geq 0}$. The estimate

$$(\overline{u}^{(k)},\overline{v}^{(k)})\leq(\overline{u},\overline{v})$$

yields that $\{\overline{u}^{(k)}\}_{k\geq 0}$ is bounded in $L^\infty(0,T;L^\infty(\Omega))$ and $\{\overline{v}^{(k)}\}_{k\geq 0}$ is bounded in $L^\infty(0,T;L^\infty(\Gamma))$. More precisely, there exists $C_0>0$ independent of k such that

$$\|\overline{u}^{(k)}\|_{L^\infty(0,T;L^\infty(\Omega))}\leq C_0\text{ and }\|\overline{v}^{(k)}\|_{L^\infty(0,T;L^\infty(\Gamma))}\leq C_0\text{ for all }k\geq 0,T>0.$$

We now rewrite the equation for $\overline{u}^{(k)}$ from (II.1)

$$\begin{cases}\partial_t\overline{u}^{(k)}-\delta_u\Delta\overline{u}^{(k)}=0,\\ \delta_u\frac{\partial\overline{u}^{(k)}}{\partial\nu}+\alpha L_u\overline{u}^{(k)}=-\alpha[k_u(\overline{u}^{(k-1)})^\alpha-k_v(\overline{v}^{(k-1)})^\beta]+\alpha L_u\overline{u}^{(k-1)}.\end{cases}$$

By taking inner product with $\overline{u}^{(k)}$ in $L^2(\Omega)$, we get

$$\frac{1}{2}\frac{d}{dt}\|\overline{u}^{(k)}\|^2_{L^2(\Omega)}+\delta_u\|\nabla\overline{u}^{(k)}\|^2_{L^2(\Omega)}$$
$$=\int_\Gamma\Big(-\alpha L_u\overline{u}^{(k)}-\alpha[k_u(\overline{u}^{(k-1)})^\alpha-k_v(\overline{v}^{(k-1)})^\beta]+\alpha L_u\overline{u}^{(k-1)}\Big)\overline{u}^{(k)}dS$$
$$\leq\alpha\int_\Gamma k_v(\overline{v}^{(k-1)})^\beta\overline{u}^{(k)}dS+\alpha\int_\Gamma L_u\overline{u}^{(k-1)}\overline{u}^{(k)}dS\tag{2.21}$$

thanks to the nonnegativity of $\overline{u}^{(k)}$, $\overline{u}^{(k-1)}$ and $k_u(t,x) \geq 0$ and $L_u = \alpha k_u \overline{u}^{\alpha-1} \geq 0$. In order to estimate the right hand side of (2.21), we first have

$$\begin{aligned}
&\alpha \int_\Gamma k_v (\overline{v}^{(k-1)})^\beta \overline{u}^{(k)} dS \\
&\quad \leq 2\alpha \|k_v\|_\infty \left(\int_\Gamma |\overline{v}^{(k-1)}|^{2\beta} dS + \int_\Gamma |\overline{u}^{(k)}|^2 dS \right) \\
&\qquad\quad \text{(by using the modified Trace inequality in Lemma 2.2)} \\
&\quad \leq 2\alpha \|k_v\|_\infty |\Gamma| \|\overline{v}^{(k-1)}\|_{L^\infty(\Gamma)}^{2\beta} + \frac{\delta_u}{4} \|\nabla \overline{u}^{(k)}\|_{L^2(\Omega)}^2 + C \|\overline{u}^{(k)}\|_{L^\infty(\Omega)}^2.
\end{aligned} \tag{2.22}$$

Moreover, by using $L_u(t,x) = \alpha k_u \overline{u}^{\alpha-1}(t,x) \leq \alpha \|k_u\|_\infty \|\overline{u}\|_{L^\infty(0,T;L^\infty(\Omega))}^{\alpha-1} =: C_1$, we get

$$\begin{aligned}
&\alpha \int_\Gamma L_u \overline{u}^{(k-1)} \overline{u}^{(k)} dS \\
&\quad \leq 2\alpha C_1 \left(\int_\Gamma |\overline{u}^{(k-1)}|^2 dS + \int_\Gamma |\overline{u}^{(k)}|^2 dS \right) \\
&\quad \leq 2\alpha C_1 \left(\frac{\delta_u}{8\alpha C_1} \|\nabla \overline{u}^{(k-1)}\|_{L^2(\Omega)}^2 + C \|\overline{u}^{(k-1)}\|_{L^2(\Omega)}^2 \right. \\
&\qquad\qquad\qquad \left. + \frac{\delta_u}{8\alpha C_1} \|\nabla \overline{u}^{(k)}\|_{L^2(\Omega)}^2 + C \|\overline{u}^{(k)}\|_{L^2(\Omega)}^2 \right) \\
&\quad \leq \frac{\delta_u}{4} \|\nabla \overline{u}^{(k-1)}\|_{L^2(\Omega)}^2 + \frac{\delta_u}{4} \|\nabla \overline{u}^{(k)}\|_{L^2(\Omega)}^2 + C(\|\overline{u}^{(k-1)}\|_{L^\infty(\Omega)}^2 + \|\overline{u}^{(k)}\|_{L^\infty(\Omega)}^2),
\end{aligned} \tag{2.23}$$

with a constant $C = C(C_1, \delta_u, \delta_v, \alpha)$.

By applying (2.22) and (2.23) to (2.21), we obtain

$$\begin{aligned}
&\frac{d}{dt} \|\overline{u}^{(k)}\|_{L^2(\Omega)}^2 + \frac{\delta_u}{2} \|\nabla \overline{u}^{(k)}\|_{L^2(\Omega)}^2 \\
&\quad \leq \frac{\delta_u}{4} \|\nabla \overline{u}^{(k-1)}\|_{L^2(\Omega)}^2 + C \left(\|\overline{u}^{(k)}\|_{L^\infty(\Omega)}^2 + \|\overline{u}^{(k-1)}\|_{L^\infty(\Omega)}^2 + \|\overline{v}^{(k-1)}\|_{L^\infty(\Gamma)}^{2\beta} \right).
\end{aligned} \tag{2.24}$$

Integrating (2.24) on $(0,T)$ yields

$$\begin{aligned}
\|\nabla \overline{u}^{(k)}\|_{L^2(0,T;L^2(\Omega))}^2 &\leq \frac{2}{\delta_u} \|\overline{u}^{(k)}(0)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla \overline{u}^{(k-1)}\|_{L^2(0,T;L^2(\Omega))}^2 \\
&\quad + C \left(\|\overline{u}^{(k)}\|_{L^\infty(0,T;L^\infty(\Omega))}^2 + \|\overline{u}^{(k-1)}\|_{L^\infty(0,T;L^\infty(\Omega))}^2 + \|\overline{v}^{(k-1)}\|_{L^\infty(0,T;L^\infty(\Gamma))}^{2\beta} \right) \\
&\leq \frac{1}{2} \|\nabla \overline{u}^{(k-1)}\|_{L^2(0,T;L^2(\Omega))}^2 + \frac{2}{\delta_u} \|u_0\|_{L^2(\Omega)}^2 + C(2C_0^2 + C_0^{2\beta}) \\
&\leq \frac{1}{2} \|\nabla \overline{u}^{(k-1)}\|_{L^2(0,T;L^2(\Omega))}^2 + C.
\end{aligned} \tag{2.25}$$

Thus, we can have

$$\begin{aligned}
\|\nabla\overline{u}^{(k)}\|^2_{L^2(0,T;L^2(\Omega))} &\le \frac{1}{2}\|\nabla\overline{u}^{(k-1)}\|^2_{L^2(0,T;L^2(\Omega))} + C \\
&\le \frac{1}{4}\|\nabla\overline{u}^{(k-2)}\|^2_{L^2(0,T;L^2(\Omega))} + C\left(1+\frac{1}{2}\right) \\
&\le \ldots \\
&\le \frac{1}{2^k}\|\nabla\overline{u}^{(0)}\|^2_{L^2(0,T;L^2(\Omega))} + C\left(1+\frac{1}{2}+\ldots+\frac{1}{2^{k-1}}\right) \\
&\le \frac{1}{2^k}\|\nabla\overline{u}\|^2_{L^2(0,T;L^2(\Omega))} + 2C
\end{aligned}$$

Thefore, we have $\{|\nabla\overline{u}^{(k)}|\}_{k\ge0}$ is bounded in $L^2(0,T;L^2(\Omega))$ uniformly in k. Taking into account that $\{\overline{u}^{(k)}\}_{k\ge0}$ is bounded in $L^\infty(0,T;L^\infty(\Omega)) \hookrightarrow L^2(0,T;L^2(\Omega))$, we conclude that $\{\overline{u}^{(k)}\}_{k\ge0}$ is uniformly bounded in $L^2(0,T;H^1(\Omega))$.

We next prove that $\{(\overline{u}^{(k)})^\alpha|_\Gamma\}_{k\ge0}$ is bounded in $L^2(0,T;L^2(\Gamma))$. Indeed, adapting the estimate in Remark 2.1, we get

$$\begin{aligned}
\|(\overline{u}^{(k)})^\alpha\|^2_{L^2(0,T;L^2(\Gamma))} &= \int_0^T\int_\Gamma (\overline{u}^{(k)})^{2\alpha}dSdt \\
&\le C\int_0^T\left(\|\overline{u}^{(k)}\|^{2\alpha-2}_{L^\infty(\Omega)}\|\nabla\overline{u}^{(k)}\|^2_{L^2(\Omega)} + \|\overline{u}^{(k)}\|^{2\alpha}_{L^\infty(\Omega)}\right)dt \\
&\le C\|\overline{u}^{(k)}\|^{2\alpha-2}_{L^\infty(0,T;L^\infty(\Omega))}\|\nabla\overline{u}^{(k)}\|^2_{L^2(0,T;L^2(\Omega))} + C\|\overline{u}^{(k)}\|^{2\alpha}_{L^\infty(0,T;L^\infty(\Omega))}.
\end{aligned}$$

Thus, the boundedness of $\{(\overline{u}^{(k)})^\alpha|_\Gamma\}_{k\ge0}$ in $L^2(0,T;L^2(\Gamma))$ follows from the boundedness of $\{\overline{u}^{(k)}\}_{k\ge0}$ in $L^\infty(0,T;L^\infty(\Omega))$ and $L^2(0,T;H^1(\Omega))$.

It remains to prove that $\{\overline{v}^{(k)}\}_{k\ge0}$ is bounded in $L^2(0,T;H^1(\Omega))$. Multiplying the equation of $\overline{v}^{(k)}$

$$\partial_t\overline{v}^{(k)} - \delta_v\Delta_\Gamma\overline{v}^{(k)} + \beta L_v\overline{v}^{(k)} = \beta[k_u(\overline{u}^{(k-1)})^\alpha - k_v(\overline{v}^{(k-1)})^\beta] + \beta L_v\overline{v}^{(k-1)}$$

by $\overline{v}^{(k)}$ in $L^2(\Gamma)$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\overline{v}^{(k)}\|^2_{L^2(\Gamma)} + \delta_v\|\nabla_\Gamma\overline{v}^{(k)}\|^2_{L^2(\Gamma)} + \beta\int_\Gamma L_v|\overline{v}^{(k)}|^2dS \\
&\quad = \beta\int_\Gamma [k_u(\overline{u}^{(k-1)})^\alpha - k_v(\overline{v}^{(k-1)})^\beta]\overline{v}^{(k)}dS + \beta\int_\Gamma L_v\overline{v}^{(k-1)}\overline{v}^{(k)}dS \\
&\quad \le \beta\|k_u\|_\infty\int_\Gamma(\overline{u}^{(k-1)})^\alpha\overline{v}^{(k)}dS + \beta C_2\int_\Gamma\overline{v}^{(k-1)}\overline{v}^{(k)}dS \qquad (2.26)
\end{aligned}$$

since $k_v(\overline{v}^{(k-1)})^\beta\overline{v}^{(k)} \ge 0$ and $L_v = \beta k_v\overline{v}^{\beta-1} \le \beta\|k_v\|_\infty\|\overline{v}\|^{\beta-1}_{L^\infty(0,T;L^\infty(\Gamma))} =: C_2$. By Young's inequality, we obtain

$$\int_\Gamma(\overline{u}^{(k-1)})^\alpha\overline{v}^{(k)}dS \le \frac{1}{2}\|(\overline{u}^{(k-1)})^\alpha\|^2_{L^2(\Gamma)} + \frac{1}{2}\|\overline{v}^{(k)}\|^2_{L^2(\Gamma)},$$

and

$$\int_\Gamma\overline{v}^{(k-1)}\overline{v}^{(k)}dS \le \frac{1}{2}\|\overline{v}^{(k-1)}\|^2_{L^2(\Gamma)} + \frac{1}{2}\|\overline{v}^{(k)}\|^2_{L^2(\Gamma)}.$$

Therefore, it follows from (2.26) that

$$\frac{d}{dt}\|\overline{v}^{(k)}\|^2_{L^2(\Gamma)} + 2\delta_v\|\nabla_\Gamma\overline{v}^{(k)}\|^2_{L^2(\Gamma)} \\ \leq \beta\|k_u\|_\infty\|(\overline{u}^{(k-1)})^\alpha\|^2_{L^2(\Gamma)} + \beta(\|k_u\|_\infty + C_2)\|\overline{v}^{(k)}\|^2_{L^2(\Gamma)} + \beta C_2\|\overline{v}^{(k-1)}\|^2_{L^2(\Gamma)}. \tag{2.27}$$

By integrating (2.27) over $(0,T)$ and by using that $\{(\overline{u}^{(k)})^\alpha|_\Gamma\}_{k\geq 0}$ is uniformly bounded in $L^2(0,T;L^2(\Gamma))$ and $\{\overline{v}^{(k)}\}_{k\geq 0}$ is uniformly bounded in $L^\infty(0,T;L^\infty(\Gamma))$, we conclude that $\{\overline{v}^{(k)}\}_{k\geq 0}$ is uniformly bounded in $L^2(0,T;H^1(\Gamma))$. This completes the proof of the Lemma. □

Proposition 2.9. *Both a.e. pointwise limits* $(\underline{u}^*,\underline{v}^*)$ *and* $(\overline{u}^*,\overline{v}^*)$ *of* (2.20) *are solutions of* (1.1).

Proof. We will only prove that $(\underline{u}^*,\underline{v}^*)$ is a solution of (1.1), since the proof for $(\overline{u}^*,\overline{v}^*)$ is analog. Taking the weak formulation of (I.1), we have

$$\int_0^T\int_\Omega[-\underline{u}^{(k)}\varphi_t + \delta_u\nabla\underline{u}^{(k)}\nabla\varphi]dxdt \\ = \int_\Omega u_0\varphi dx + \int_0^T\int_\Gamma[-\alpha L_u\underline{u}^{(k)} + f(\underline{u}^{(k-1)},\underline{v}^{(k-1)})]\varphi dSdt \\ = \int_\Omega u_0\varphi dx + \int_0^T\int_\Gamma[-\alpha L_u\underline{u}^{(k)} - \alpha(k_u(\underline{u}^{(k-1)})^\alpha - k_v(\underline{v}^{(k-1)})^\beta) + \alpha L_u\underline{u}^{(k-1)}]\varphi dSdt. \tag{2.28}$$

Now using Lemma 2.8, we can apply the Dominated convergence Theorem in order to pass to the limit as $k\to+\infty$ for all terms in (2.28) to get

$$\int_0^T\int_\Omega[-\underline{u}^*\varphi_t + \delta_u\nabla\underline{u}^*\nabla\varphi]dxdt \\ = \int_\Omega u_0\varphi dx + \int_0^T\int_\Gamma[-\alpha L_u\underline{u}^* - \alpha(k_u(\underline{u}^*)^\alpha - k_v(\underline{v}^*)^\beta) + \alpha L_u\underline{u}^*]\varphi dSdt \\ = \int_\Omega u_0\varphi dx - \int_0^T\int_\Gamma \alpha[k_u(\underline{u}^*)^\alpha - k_v(\underline{v}^*)^\beta]\varphi dSdt. \tag{2.29}$$

Similarly, we can pass to the limit as $k\to+\infty$ in

$$\int_0^T\int_\Gamma[-\underline{v}^{(k)}\psi_t + \delta_v\nabla_\Gamma\underline{v}^{(k)}\nabla_\Gamma\psi + \beta L_v\underline{v}^{(k)}]dSdt \\ = \int_\Gamma v_0\psi dS + \int_0^T\int_\Gamma[\beta(k_u(\underline{u}^{(k-1)})^\alpha - k_v(\underline{v}^{(k-1)})^\beta) + \beta L_v\underline{v}^{(k-1)}]\psi dSdt$$

to derive that

$$\int_0^T\int_\Gamma[-\underline{v}^*\psi_t + \delta_v\nabla_\Gamma\underline{v}^*\nabla_\Gamma\psi]dSdt = \int_\Gamma v_0\psi dS + \int_0^T\int_\Gamma[\beta(k_u(\underline{u}^*)^\alpha - k_v(\underline{v}^*)^\beta)]\psi dSdt. \tag{2.30}$$

Equation (2.29) together with (2.30) means that $(\underline{u}^*,\underline{v}^*)$ is a weak solution to the system (1.1). □

Theorem 2.10. *For all non-negative initial data* $(u_0, v_0) \in L^\infty_+(\Omega) \times L^\infty_+(\Gamma)$*, there exists a unique non-negative global weak solution* (u, v) *for the system* (1.1).

Proof. The existence of a solution is implied from Proposition 2.9. The non-negativity of solutions follows from the Comparison Theorem, see Lemma 2.3, since $(\underline{u}, \underline{v}) = (0, 0)$ is a lower solution. To prove the uniqueness, it's enough to show that $(\underline{u}^*, \underline{v}^*) = (\overline{u}^*, \overline{v}^*)$. The technique we use is similar with the one used in the proof of Lemma 2.3. Setting $w = \overline{u}^* - \underline{u}^*$ and $z = \overline{v}^* - \underline{v}^*$, we have $w \geq 0, z \geq 0$, $w(0) = z(0) = 0$ and

$$\begin{cases} \int_0^T \int_\Omega [-w\varphi_t + \delta_u \int_\Omega \nabla w \nabla \varphi] dxdt = \int_0^T \int_\Gamma (F(\overline{u}^*, \overline{v}^*) - F(\underline{u}^*, \underline{v}^*))\varphi dSdt, \\ \int_0^T \int_\Gamma [-z\psi_t + \delta_v \int_\Gamma \nabla_\Gamma z \nabla_\Gamma \psi] dSdt = \int_0^T \int_\Gamma (G(\overline{u}^*, \overline{v}^*) - G(\underline{u}^*, \underline{v}^*))\psi dSdt. \end{cases} \tag{2.31}$$

Choosing $\varphi = w(t)$ in (2.31), we obtain, for almost every $t \in (0, T)$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |w|^2 dx + \delta_u \int_\Omega |\nabla w|^2 dx \\ &\quad = \int_\Gamma (F(\overline{u}^*, \overline{v}^*) - F(\underline{u}^*, \underline{v}^*)) w dS \\ &\quad \leq C \int_\Gamma zw dS \quad \text{(by using the locally Lipschitz property of } F \text{ (2.4))} \\ &\quad \leq \frac{\delta_u}{2}\int_\Omega |\nabla w|^2 dx + C\int_\Omega |w|^2 dx + C\int_\Gamma |z|^2 dS \quad \text{(by Young and Lemma 2.2).} \end{aligned} \tag{2.32}$$

Similarly, by choosing $\psi = z$, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Gamma |z|^2 dS + \delta_v \int_\Gamma |\nabla_\Gamma v|^2 dS \leq C\int_\Gamma |z|^2 dS + C\int_\Omega |w|^2 dx + \frac{\delta_u}{2}\int_\Omega |\nabla w|^2 dx. \tag{2.33}$$

Combining (2.32) and (2.33) implies

$$\frac{d}{dt}\left(\int_\Omega |w|^2 dx + \int_\Gamma |z|^2 dS\right) \leq C\left(\int_\Omega |w|^2 dx + \int_\Gamma |z|^2 dS\right). \tag{2.34}$$

Therefore, $w(t) = 0$ and $z(t) = 0$ for a.e. $t \in (0, T)$ since $w(0) = 0, z(0) = 0$. This completes the proof. □

3. Convergence to equilibrium

In this section, we assume that the reaction rates k_u and k_v, and thus the equilibrium state (u_∞, v_∞) (see (3.3) below) are constant. Moreover, for the sake of readability of the arguments, we shall assume normalised rates $k_u = k_v = 1$ (w.l.o.g. thanks to a rescaling in the cases $\alpha \neq \beta$). In any case, the following proofs can be readily generalised to arbitrary constants $k_u > 0$, $k_v > 0$.

We shall apply the entropy method to prove that the unique solution to (1.1) converges exponentially fast to the equilibrium (u_∞, v_∞) for any initial data $(u_0, v_0) \in L^\infty(\Omega) \times L^\infty(\Gamma)$. While the entropy method is certainly expected to apply to general reaction rates, the case of non-constant equilibria requires a more complicated formalism (see e.g. [21]), which we omit here for the sake of clarity of the argument and leave it for further work.

In the following, we will first consider the non-degenerate case $\delta_v > 0$ and later the degenerate case $\delta_v = 0$. We remark that in the first case with non-degenerate surface diffusion, our method relies only on natural a-priori bounds which are entailed by well-defined entropy and entropy-dissipation functionals along the flow of the solution. However, in the case of degenerate diffusion, we require additional L^∞-bounds of the solution. Since L^∞-bounds of solutions for general systems are often unknown, the degenerate surface diffusion case poses more difficulties to be generalised than the non-degenerate case, which seems readily generalisable.

The system (1.1) satisfies the *mass conservation law* (1.2), that is,

$$M = \beta \int_\Omega u(t,x)dx + \alpha \int_\Gamma v(t,x)dS = \beta \int_\Omega u_0(x)dx + \alpha \int_\Gamma v_0(x)dS > 0,$$

where we assume that the initial mass is positive ($M > 0$).

The equilibrium of non-negative solutions of the system (1.1) are the unique positive constants (u_∞, v_∞), which balance the reaction rates, i.e.

$$u_\infty^\alpha = v_\infty^\beta, \tag{3.1}$$

and satisfy the mass conservation law

$$\beta|\Omega|u_\infty + \alpha|\Gamma|v_\infty = M. \tag{3.2}$$

We remark that the uniqueness of the equilibrium follows from the monotonicity of the right hand sides of the equilibrium conditions

$$u_\infty^\alpha = \Big(\frac{1}{\alpha|\Gamma|}(M - \beta|\Omega|u_\infty)\Big)^\beta, \qquad v_\infty^\beta = \Big(\frac{1}{\beta|\Omega|}(M - \alpha|\Gamma|v_\infty)\Big)^\alpha \tag{3.3}$$

on the intervals of equilibrium values, which are admissible for non-negative solutions of systems (1.1), i.e. $0 < u_\infty < \frac{M}{\beta|\Omega|}$ and $0 < v_\infty < \frac{M}{\alpha|\Gamma|}$.

As mentioned in the introduction, we prove the convergence to equilibrium by means of the entropy method. The method is based on the logarithmic entropy (free energy) functional

$$E(u,v) = \int_\Omega u(\log u - 1)dx + \int_\Gamma v(\log v - 1)dS \tag{3.4}$$

and its non-negative entropy dissipation

$$\begin{aligned} D(u,v) &= -\frac{d}{dt}E(u,v) \\ &= \delta_u \int_\Omega \frac{|\nabla u|^2}{u}dx + \delta_v \int_\Gamma \frac{|\nabla_\Gamma v|^2}{v}dS + \int_\Gamma (v^\beta - u^\alpha)\log\frac{v^\beta}{u^\alpha}dS. \end{aligned} \tag{3.5}$$

Our goal is to show that there exists a constant $C_0 > 0$ such that (see Lemma 3.3 below)

$$D(u,v) \geq C_0\,(E(u,v) - E(u_\infty, v_\infty))$$

for all non-negative (u,v), which satisfy the mass conservation law (1.2). Compared to previous related results on the entropy method for reaction-diffusion systems with quadratic nonlinearities (see [15, 16, 17]), there are two main difficulties to overcome: the first is the treatment of the surface concentration v and the associated boundary integrals and the second is the general nonlinear term $(v^\beta - u^\alpha)\log\frac{v^\beta}{u^\alpha}$ for any $\alpha, \beta \geq 1$. It is in particular the general nonlinearities, which necessitates a new proof compared to the quadratic nonlinearities considered in [15, 16, 17]. We expect this new proof to constitute a more general approach. For a recent alternative approach for establishing entropy entropy-dissipation estimates based on a convexification argument we refer to [34].

In the sequel, we will frequently use the following notations and inequalities:

Spatial averages and square-root abbreviation:

$$\overline{u} = \frac{1}{|\Omega|}\int_\Omega u\,dx, \qquad U = \sqrt{u}, \qquad U_\infty = \sqrt{u_\infty}, \qquad \overline{U} = \frac{1}{|\Omega|}\int_\Omega U\,dx,$$
$$\overline{v} = \frac{1}{|\Gamma|}\int_\Gamma v\,dS, \qquad V = \sqrt{v}, \qquad V_\infty = \sqrt{v_\infty}, \qquad \overline{V} = \frac{1}{|\Gamma|}\int_\Gamma V\,dS.$$

Norms: $\|\cdot\|_\Omega$ and $\|\cdot\|_\Gamma$ are the norms in $L^2(\Omega)$ and $L^2(\Gamma)$ respectively. For a Banach space X, we denote by $\|\cdot\|_X$ its norm.

Constants: A generic constant will be denoted by $C(M,\Omega,\dots)$ and may depend besides the arguments $M,\Omega,\dots$ also on α and β without explicitly stating the dependence on α and β. Moreover, the constants $C_i(\dots)$ and $K_i(\dots)$ for $i = 0,1,2,\dots$ are specific constants, for which the same rules of dependency hold.

Inequalities:

- Poincare's inequality in Ω

$$P(\Omega)\int_\Omega |\nabla u|^2 dx \ge \int_\Omega |u-\overline{u}|^2 dx,$$

- Poincare's inequality on Γ

$$P(\Gamma)\int_\Gamma |\nabla_\Gamma v|^2 dS \ge \int_\Gamma |v-\overline{v}|^2 dS,$$

- Trace Theorem

$$T(\Omega)\int_\Omega |\nabla u|^2 dx \ge \int_\Gamma |u-\overline{u}|^2 dS. \tag{3.6}$$

The mass conservation (1.2) allows to rewrite the relative entropy towards the equilibrium as

$$\begin{aligned} E(u,v) - E(u_\infty,v_\infty) &= \int_\Omega u\log\frac{u}{\overline{u}}dx + \int_\Gamma v\log\frac{v}{\overline{v}}dS \\ &+ \int_\Omega\Big(\overline{u}\log\frac{\overline{u}}{u_\infty} - (\overline{u}-u_\infty)\Big)dx + \int_\Gamma\Big(\overline{v}\log\frac{\overline{v}}{v_\infty} - (\overline{v}-v_\infty)\Big)dS \\ &= I_1 + I_2, \end{aligned} \tag{3.7}$$

where we define

$$I_1 := \int_\Omega u\log\frac{u}{\overline{u}}\,dx + \int_\Gamma v\log\frac{v}{\overline{v}}\,dS,$$

and

$$I_2 := \int_\Omega\Big(\overline{u}\log\frac{\overline{u}}{u_\infty} - (\overline{u}-u_\infty)\Big)dx + \int_\Gamma\Big(\overline{v}\log\frac{\overline{v}}{v_\infty} - (\overline{v}-v_\infty)\Big)dS.$$

The following lemma proves, similarly to [15], a Czisz\'ar-Kullback-Pinsker type inequality, which quantifies that the relative entropy to equilibrium controls an L^1-distance:

Lemma 3.1. *For all measurable functions $u : \Omega \to \mathbb{R}_+$ and $v : \Gamma \to \mathbb{R}_+$ satisfying*

$$M = \beta \int_\Omega u\,dx + \alpha \int_\Gamma v\,dS > 0,$$

we have

$$E(u,v) - E(u_\infty, v_\infty) \geq C_{CKP}\left(\|u - u_\infty\|^2_{L^1(\Omega)} + \|v - v_\infty\|^2_{L^1(\Gamma)}\right), \tag{3.8}$$

where $C_{CKP} > 0$ is the following (non-optimal) constant depending only on the mass $M > 0$ and $\alpha, \beta \geq 1$:

$$C_{CKP} = \frac{\min\{\alpha, \beta\}}{8M}.$$

Proof. By (3.7), we have that

$$E(u,v) - E(u_\infty, v_\infty) = I_1 + I_2.$$

Considering the term I_1 at first, we use the classic Cziszár-Kullback-Pinsker inequality (see e.g. [10]) and the mass constraints $\overline{u} \leq \frac{M}{\beta|\Omega|}$ and $\overline{v} \leq \frac{M}{\alpha|\Gamma|}$ to estimate

$$\int_\Omega u \log \frac{u}{\overline{u}} dx \geq \frac{1}{2|\Omega|\overline{u}} \|u - \overline{u}\|^2_{L^1(\Omega)} \geq \frac{\beta}{2M}\|u - \overline{u}\|^2_{L^1(\Omega)},$$

and

$$\int_\Gamma v \log \frac{v}{\overline{v}} dS \geq \frac{1}{2|\Gamma|\overline{v}} \|v - \overline{v}\|^2_{L^1(\Gamma)} \geq \frac{\alpha}{2M}\|v - \overline{v}\|^2_{L^1(\Gamma)},$$

and, thus

$$I_1 \geq \frac{\beta}{2M}\|u - \overline{u}\|^2_{L^1(\Omega)} + \frac{\alpha}{2M}\|v - \overline{v}\|^2_{L^1(\Gamma)}. \tag{3.9}$$

Next, we rewrite I_2 in (3.7) by introducing $q(x) = x\log x - x$, i.e.

$$I_2 = |\Omega|(q(\overline{u}) - q(u_\infty)) + |\Gamma|(q(\overline{v}) - q(v_\infty)),$$

where we have used that the mass conservation law (1.2) implies

$$\int_\Omega (\overline{u} - u_\infty)\log u_\infty\,dx + \int_\Gamma (\overline{v} - v_\infty)\log v_\infty\,dS = 0$$

since $\frac{\log u_\infty}{\beta} = \frac{\log u_\infty^\alpha}{\alpha\beta} = \frac{\log v_\infty^\beta}{\alpha\beta} = \frac{\log v_\infty}{\alpha}$. Then, using again the conservation law (1.2), we denote

$$Q(\overline{u}) = |\Omega|q(\overline{u}) + |\Gamma|\underbrace{q\left(\frac{M - \beta|\Omega|\overline{u}}{\alpha|\Gamma|}\right)}_{=q(\overline{v})} \quad \text{and} \quad R(\overline{v}) = |\Gamma|q(\overline{v}) + |\Omega|\underbrace{q\left(\frac{M - \alpha|\Gamma|\overline{v}}{\beta|\Omega|}\right)}_{=q(\overline{u})}.$$

Thus, we have the following two equivalent ways of writing I_2:

$$I_2 = Q(\overline{u}) - Q(u_\infty) = R(\overline{v}) - R(v_\infty). \tag{3.10}$$

Moreover, direct computations give

$$Q'(u_\infty) = |\Omega|q'(u_\infty) - \frac{\beta}{\alpha}|\Omega|q'\left(\frac{M - \beta|\Omega|u_\infty}{\alpha|\Gamma|}\right) = |\Omega|\log u_\infty - \frac{\beta}{\alpha}|\Omega|\log v_\infty = 0$$

since $u_\infty^\alpha = v_\infty^\beta$. Moreover, for any $\overline{u}_\theta$ satisfying the mass constraints $0 \le \overline{u}_\theta \le \frac{M}{\beta|\Omega|}$

$$Q''(\overline{u}_\theta) = |\Omega| q''(\overline{u}_\theta) + \frac{\beta^2}{\alpha^2}\frac{|\Omega|^2}{|\Gamma|} q''\left(\frac{M-\beta|\Omega|\overline{u}_\theta}{\alpha|\Gamma|}\right) = |\Omega|\frac{1}{\overline{u}_\theta} + \frac{\beta^2}{\alpha^2}\frac{|\Omega|^2}{|\Gamma|}\frac{\alpha|\Gamma|}{M-\beta|\Omega|\overline{u}_\theta}$$
$$\ge \frac{\beta|\Omega|^2}{M} + \frac{\beta^2}{\alpha}\frac{|\Omega|^2}{M} = \frac{\beta}{\alpha}\frac{|\Omega|^2}{M}(\alpha+\beta).$$

In a similar way, for any $0 \le \overline{v}_\theta \le \frac{M}{\alpha|\Gamma|}$, we estimate

$$R'(v_\infty) = 0 \qquad \text{and} \qquad R''(\overline{v}_\theta) \ge \frac{\alpha}{\beta}\frac{|\Gamma|^2}{M}(\alpha+\beta).$$

Thus, altogether, Taylor expansion in (3.10) with $\overline{u}_\theta = \theta\overline{u} + (1-\theta)u_\infty$ and $\overline{v}_\theta = \theta\overline{v} + (1-\theta)v_\infty$ for some $\theta \in (0,1)$ yields

$$\begin{aligned} I_2 &= \frac{1}{2}(Q(\overline{u}) - Q(u_\infty)) + \frac{1}{2}(R(\overline{v}) - R(v_\infty)) \\ &\ge \frac{1}{4}\frac{\beta}{\alpha}\frac{|\Omega|^2}{M}(\alpha+\beta)(\overline{u}-u_\infty)^2 + \frac{1}{4}\frac{\alpha}{\beta}\frac{|\Gamma|^2}{M}(\alpha+\beta)(\overline{v}-v_\infty)^2 \\ &= \frac{1}{4}\frac{\alpha+\beta}{M}\left(\frac{\beta}{\alpha}\|\overline{u}-u_\infty\|^2_{L^1(\Omega)} + \frac{\alpha}{\beta}\|\overline{v}-v_\infty\|^2_{L^1(\Gamma)}\right). \end{aligned} \tag{3.11}$$

Combining (3.9) and (3.11) with $\|u-\overline{u}\|^2_{L^1(\Omega)} + \|\overline{u}-u_\infty\|^2_{L^1(\Omega)} \ge \frac{1}{2}\|u-u_\infty\|^2_{L^1(\Omega)}$ by Jensen's inequality, we get

$$I_1 + I_2 \ge \frac{\beta}{8M}\|u-u_\infty\|^2_{L^1(\Omega)} + \frac{\alpha}{8M}\|v-v_\infty\|^2_{L^1(\Omega)}$$

we obtain (3.8) with $C_{\text{CKP}} = \frac{\min\{\alpha,\beta\}}{8M}$. □

We now state our main result of this section, which is the exponential convergence to equilibrium with explicit rates and constants via the entropy method. The proof uses an entropy entropy-dissipation estimate, which is proven in Lemma 3.3 below.

Theorem 3.2 (Explicit Exponential Convergence to Equilibrium). *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\Gamma = \partial\Omega$ (e.g. $\partial\Omega \in C^{2+\epsilon}$ for any $\epsilon > 0$). Then, the unique weak solution (u,v) of system (1.1) subject to any initial data $(u_0, v_0) \in L^\infty(\Omega) \times L^\infty(\Gamma)$ satisfies the following exponential convergence to equilibrium*

$$\|u(t)-u_\infty\|^2_{L^1(\Omega)} + \|v(t)-v_\infty\|^2_{L^1(\Gamma)} \le C_{CKP}^{-1} e^{-C_0 t}\left(E(u_0,v_0) - E(u_\infty,v_\infty)\right), \tag{3.12}$$

where C_0 and C_{CKP}^{-1} are positive constants as defined in Lemma 3.3 below and Lemma 3.1 above and depend only on reaction rates $\alpha, \beta \ge 1$, the positive diffusion rates $\delta_u, \delta_v > 0$, the domain Ω, the boundary Γ and the positive initial mass $M > 0$.

Proof. On the one hand, we have

$$\frac{d}{dt}\left(E(u,v) - E(u_\infty,v_\infty)\right) = \frac{d}{dt}E(u,v) = -D(u,v). \tag{3.13}$$

On the other hand, by the Lemma 3.3, there exists $C_0 > 0$ such that

$$D(u,v) \ge C_0\left(E(u,v) - E(u_\infty,v_\infty)\right). \tag{3.14}$$

Then, from (3.13), (3.14) and the classical Gronwall inequality, we obtain

$$E(u(t),v(t)) - E(u_\infty,v_\infty) \le e^{-C_0 t}\left(E(u_0,v_0) - E(u_\infty,v_\infty)\right). \tag{3.15}$$

Finally, the estimate (3.12) follows directly from (3.15) and Lemma 3.1. □

Remark 3.1. *The same technique of this paper can be used to get the explicit exponential convergence to equilibrium for systems of the form:*

$$\begin{cases} u_t - d_u \Delta u = -\alpha(u^\alpha - v^\beta), & t > 0, x \in \Omega, \\ v_t - d_v \Delta v = \beta(u^\alpha - v^\beta), & t > 0, x \in \Omega, \\ \partial u/\partial \nu = \partial v/\partial \nu = 0, & t > 0, x \in \partial\Omega, \\ u(0,x) = u_0(x), v(0,x) = v_0(x), & x \in \Omega, \end{cases}$$

subject to non-negative initial data $u_0, v_0 \in L^\infty_+(\Omega)$ and for all stoichiometric coefficients $\alpha, \beta \geq 1$ and positive diffusion coefficients d_u, d_v. By using Poincare's inequality $P(\Omega)\|\nabla v\|^2_\Omega \geq \|v - \overline{v}\|^2_\Omega$ instead of the Trace inequality $T(\Omega)\|\nabla v\|^2_\Omega \geq \|v - \overline{v}\|^2_\Gamma$, all the following arguments can be directly reproduced in the same way. Thus, the result of this paper, in a certain sense, completely solves the problem of trend to equilibrium for concentrations of the reversible chemical reaction of two species $\mathcal{U}$ and $\mathcal{V}$:

$$\alpha\,\mathcal{U} \rightleftharpoons \beta\,\mathcal{V}.$$

We shall now prove the key entropy entropy-dissipation estimate.

Lemma 3.3 (Entropy Entropy-Dissipation Estimate)**.**
For all measurable, non-negative functions $u : \Omega \to \mathbb{R}_+$ with trace $u|_\Gamma \in L^2(\Gamma)$ and $v : \Gamma \to \mathbb{R}_+$, which satisfy the mass conservation law

$$\beta \int_\Omega u\, dx + \alpha \int_\Gamma v\, dS = M, \tag{3.16}$$

there exists a constant $C_0 > 0$ such that

$$D(u,v) \geq C_0 \left(E(u,v) - E(u_\infty, v_\infty) \right),$$

where C_0 depends only on M, $|\Omega|$, $P(\Omega)$, $T(\Omega)$, $|\Gamma|$, $P(\Gamma)$ as well as α and β.

Proof of Lemma 3.3.
We divide the proof into two cases: $\delta_v > 0$ in Section 3.1 and $\delta_v = 0$ in Section 3.2.

In the first case, we don't require any additional a-priori estimates on the solution besides well defined entropy and entropy-dissipation functionals in order to obtain the entropy-entropy dissipation estimate.

In the second case, since the diffusion term in v is missing, we shall require a-priori L^∞-bounds on the solution. However, we strongly believe that one might be able to avoid the use of L^∞-bounds in some cases of the exponents α and β.

3.1. **The non-degenerate case: $\delta_v > 0$.**
We will show in the sequel that both I_1 and I_2 as defined in (3.7) are bounded by the entropy dissipation. First, by using the Logarithmic-Sobolev inequality

$$C_L(\Omega) \int_\Omega \frac{|\nabla u|^2}{u} dx \geq \int_\Omega u \log \frac{u}{\overline{u}}\, dx, \quad \text{and} \quad C_L(\Gamma) \int_\Gamma \frac{|\nabla_\Gamma v|^2}{v} dS \geq \int_\Gamma v \log \frac{v}{\overline{v}}\, dS,$$

we immediately get the following

Lemma 3.4. *For all $t \geq 0$, we have*

$$I_1 \leq C_2 \frac{D(u,v)}{2}, \tag{3.17}$$

where

$$C_2 = 2\max\left\{\frac{C_L(\Omega)}{\delta_u}, \frac{C_L(\Gamma)}{\delta_v}\right\}.$$

Remark 3.2. *The factor 2 in constant C_2 is chosen to still have $\frac{1}{2}D(u,v)$ left to estimate term I_2, which is done in the following Lemma 3.5.*

Lemma 3.5. *There exists $C_3 > 0$ such that, for all $t \geq 0$,*

$$I_2 \leq C_3 \frac{D(u,v)}{2}. \tag{3.18}$$

Proof. In a preliminary step, we observe that the function $\Phi : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\Phi(x,y) = \frac{x\log\frac{x}{y} - (x-y)}{(\sqrt{x}-\sqrt{y})^2} = \Phi\Big(\frac{x}{y},1\Big) \tag{3.19}$$

is continuous on $(0,\infty)^2$. Moreover, for all $y \in (0,\infty)$, the function $\Phi(\cdot,y)$ is strictly increasing on $(0,\infty)$ and satisfies $\lim_{x\to 0}\Phi(x,y) = 1$, $\Phi(y,y) = 2$, see [15].

In a first step, we use now the mass conservation $\beta|\Omega|\overline{u} + \alpha|\Gamma|\overline{v} = M$ to obtain the following bounds for I_2:

$$\int_\Omega \Big(\overline{u}\log\frac{\overline{u}}{u_\infty} - (\overline{u}-u_\infty)\Big)\,dx \leq |\Omega|\,\Phi\Big(\frac{M}{\beta|\Omega|}, u_\infty\Big)\left(\sqrt{\overline{u}} - \sqrt{u_\infty}\right)^2 \tag{3.20}$$

and

$$\int_\Gamma \Big(\overline{v}\log\frac{\overline{v}}{v_\infty} - (\overline{v}-v_\infty)\Big)\,dS \leq |\Gamma|\,\Phi\Big(\frac{M}{\alpha|\Gamma|}, v_\infty\Big)\left(\sqrt{\overline{v}} - \sqrt{v_\infty}\right)^2. \tag{3.21}$$

Therefore, we have from (3.20) and (3.21) that

$$I_2 \leq K_0\left[\left(\sqrt{\overline{v}} - \sqrt{v_\infty}\right)^2 + \left(\sqrt{\overline{u}} - \sqrt{u_\infty}\right)^2\right], \tag{3.22}$$

where

$$K_0 := \max\left\{|\Omega|\,\Phi\Big(\frac{M}{\beta|\Omega|}, u_\infty\Big), |\Gamma|\,\Phi\Big(\frac{M}{\alpha|\Gamma|}, v_\infty\Big)\right\}.$$

Next, considering the entropy dissipation $D(u,v)$, we observe first that

$$\delta_u \int_\Omega \frac{|\nabla u|^2}{u}dx = 4\delta_u \int_\Omega |\nabla\sqrt{u}|^2 dx = 4\delta_u\|\nabla U\|_\Omega^2, \tag{3.23}$$

and

$$\delta_v \int_\Gamma \frac{|\nabla_\Gamma v|^2}{v}dS = 4\delta_v\|\nabla_\Gamma v\|_\Gamma^2 \geq 4\delta_v\, P^{-1}(\Gamma)\|V - \overline{V}\|_\Gamma^2. \tag{3.24}$$

Moreover, the elementary inequality $(a-b)\log\frac{a}{b} \geq 4(\sqrt{a}-\sqrt{b})^2$ yields

$$\int_\Gamma (v^\beta - u^\alpha)\log\frac{v^\beta}{u^\alpha}dS \geq 4\|V^\beta - U^\alpha\|_\Gamma^2. \tag{3.25}$$

Hence,

$$\frac{D(u,v)}{2} \geq 2\delta_u\|\nabla U\|_\Omega^2 + 2\delta_v P^{-1}(\Gamma)\|V - \overline{V}\|_\Gamma^2 + 2\|V^\beta - U^\alpha\|_\Gamma^2. \tag{3.26}$$

Combining (3.22) and (3.26), we see that in order to prove (3.18) it is sufficient to find positive constants $K_1 \le 2$ and K_2 such that

$$2\delta_u\|\nabla U\|_\Omega^2 + 2\delta_v P^{-1}(\Gamma)\|V-\overline{V}\|_\Gamma^2 + K_1\|V^\beta - U^\alpha\|_\Gamma^2 \\ \ge K_2K_0\left[\left(\sqrt{\overline{U^2}}-U_\infty\right)^2 + \left(\sqrt{\overline{V^2}}-V_\infty\right)^2\right], \quad (3.27)$$

where we denote $\overline{U^2} = \frac{1}{|\Omega|}\int_\Omega U^2\,dx$ and $\overline{V^2} = \frac{1}{|\Gamma|}\int_\Gamma V^2\,dS$.

In the following, we divide the proof of the key estimate (3.27) into several steps. As a preliminary remark, we recall that the estimate (3.27) can only hold because of the constraint imposed by the conservation law (3.16) on U and V, i.e.

$$\beta|\Omega|\overline{U^2} + \alpha|\Gamma|\overline{V^2} = M, \quad (3.28)$$

since without (3.28), the left hand side of (3.27) vanishes for all constant states U, V satisfying $V^\beta = U^\alpha$, while the right hand side of (3.27) vanishes only at the equilibrium U_∞, V_∞. Thus, the following steps are designed as a chain of estimates, which allows for the conservation law (3.16) rewritten as (3.28) to enter into the proof of estimate (3.27).

<u>*Step 1:*</u> The goal of this step is to show that there exists a constant $K_3 > 0$ such that

$$\|V^\beta - U^\alpha\|_\Gamma^2 \ge \frac{1}{2}\|\overline{V}^\beta - \overline{U}^\alpha\|_\Gamma^2 - K_3(\|U-\overline{U}\|_\Gamma^2 + \|V-\overline{V}\|_\Gamma^2), \quad (3.29)$$

which establishes a lower bound of the *reaction entropy dissipation term* in terms of a *reaction entropy dissipation term for the space averaged concentrations* $\overline{U}$ and $\overline{V}$ at the cost of two terms, which can ultimately be controlled by the *diffusion entropy dissipation*.

At first, we remark that the averaged concentrations $\overline{U}$ and $\overline{V}$ are bounded by Jensen's inequality and the conservation law (3.28)

$$\overline{U}^2 \le \overline{U^2} \le \frac{M}{\beta|\Omega|} \le \max\left\{1, \frac{M}{\beta|\Omega|}\right\} =: M_\Omega, \quad (3.30)$$

$$\overline{V}^2 \le \overline{V^2} \le \frac{M}{\alpha|\Gamma|} \le \max\left\{1, \frac{M}{\alpha|\Gamma|}\right\} =: M_\Gamma. \quad (3.31)$$

Next, we consider the following deviations around the spatially averaged concentrations:

$$\delta_1(x) := U - \overline{U}, \quad \forall x \in \Omega,$$

and

$$\delta_2(x) := V - \overline{V}, \quad \forall x \in \Gamma$$

and divide the boundary Γ into two disjoint sets:

$$\Gamma = S \cup S^\perp,$$

where

$$S := \{x \in \Gamma : \ -\overline{U} \le \delta_1(x) \le \sqrt{M_\Omega}, \ -\overline{V} \le \delta_2(x) \le \sqrt{M_\Gamma}\}.$$

Note that $\delta_1 \in L^2(\Gamma)$ is well-defined by (3.23) and the Trace Theorem (3.6).

Due to the boundedness of δ_1 and δ_2 in S, we estimate readily using Taylor expansion and Young's inequality

$$\begin{aligned}\|V^\beta - U^\alpha\|^2_{L^2(S)} &= \|(\overline{V}+\delta_2)^\beta - (\overline{U}+\delta_1)^\alpha\|^2_{L^2(S)}\\ &\geq \frac{1}{2}\|\overline{V}^\beta - \overline{U}^\alpha\|^2_{L^2(S)} - \|\beta(\overline{V}+\theta_2)^{\beta-1}\delta_2 - \alpha(\overline{U}+\theta_1)^{\alpha-1}\delta_1\|^2_{L^2(S)}\\ &\geq \frac{1}{2}\|\overline{V}^\beta - \overline{U}^\alpha\|^2_{L^2(S)} - C_3\big(M_\Omega^{\alpha-1}, M_\Gamma^{\beta-1}\big)\left(\|\delta_1\|^2_\Gamma + \|\delta_2\|^2_\Gamma\right), \quad (3.32)\end{aligned}$$

where we have used that $\theta_1(x) \leq \delta_1(x) \leq \sqrt{M_\Omega}$, $\theta_2(x) \leq \delta_2(x) \leq \sqrt{M_\Gamma}$ and which proves (3.29) on the set S.

It remains to consider the set

$$S^\perp = \{x \in \Gamma : \ \delta_1(x) > \sqrt{M_\Omega} \quad \text{or} \quad \delta_2(x) > \sqrt{M_\Gamma}\}.$$

By using Chebyshev's inequality and by observing that for $\delta_1 > \sqrt{M_\Omega} \geq \overline{U}$, the set $\{x \in \Gamma : \delta_1^2 > M_\Omega\}$ coincides with the set $\{x \in \Gamma : \delta_1 > \sqrt{M_\Omega}\}$ and analog for $\delta_2 > \sqrt{M_\Gamma} \geq \overline{V}$, we get

$$|\{x \in \Gamma : \delta_1 > \sqrt{M_\Omega}\}| = |\{x \in \Gamma : \delta_1^2 \geq M_\Omega\}| \leq \frac{\|\delta_1\|^2_\Gamma}{M_\Omega},$$

and

$$|\{x \in \Gamma : \delta_2 > \sqrt{M_\Gamma}\}| = |\{x \in \Gamma : \delta_2^2 \geq M_\Gamma\}| \leq \frac{\|\delta_2\|^2_\Gamma}{M_\Gamma}.$$

Thus, it follows that

$$|S^\perp| \leq \frac{\|\delta_1\|^2_\Gamma}{M_\Omega} + \frac{\|\delta_2\|^2_\Gamma}{M_\Gamma}.$$

By the bounds (3.30) and (3.31), we have moreover that $|\overline{V}^\beta - \overline{U}^\alpha| \leq C(M_\Omega^{\frac{\alpha}{2}}, M_\Gamma^{\frac{\beta}{2}})$. Hence, since $M_\Omega \geq 1$ and $M_\Gamma \geq 1$

$$\|\overline{V}^\beta - \overline{U}^\alpha\|^2_{L^2(S^\perp)} \leq C(M_\Omega^\alpha, M_\Gamma^\beta)|S^\perp| \leq C(M_\Omega^\alpha, M_\Gamma^\beta)\left(\|\delta_1\|^2_\Gamma + \|\delta_2\|^2_\Gamma\right)$$

thus

$$\|V^\beta - U^\alpha\|^2_{L^2(S^\perp)} \geq 0 \geq \frac{1}{2}\|\overline{V}^\beta - \overline{U}^\alpha\|^2_{L^2(S^\perp)} - C_4(M_\Omega^\alpha, M_\Gamma^\beta)(\|\delta_1\|^2_\Gamma + \|\delta_2\|^2_\Gamma). \quad (3.33)$$

Finally, the estimate (3.29) is obtained from (3.32) and (3.33) for a constant $K_3(M_\Omega^\alpha, M_\Gamma^\beta) = \max\{C_3, C_4\}$.

With estimate (3.29), we proceed in estimating the left hand side of (3.27) in the following way: We shall look for a positive constant $K_1 \leq 2$ small enough, such that the following two conditions hold:

$$\begin{cases}\delta_u T^{-1}(\Omega) - K_1K_3 \geq 0,\\ \delta_v P^{-1}(\Gamma) - K_1K_3 \geq 0,\end{cases} \qquad \Rightarrow \qquad K_1 \leq \min\left\{\frac{\delta_u}{K_3T(\Omega)}, \frac{\delta_v}{K_3P(\Gamma)}, 2\right\}.$$

Here, $T(\Omega)$ denotes the constant of the Trace inequality $T(\Omega)\|\nabla U\|_\Omega^2 \geq \|U-\overline{U}\|_\Gamma^2$. We can then estimate the left hand side of (3.27) by using (3.29)

$$\begin{aligned}
&2\delta_u\|\nabla U\|_\Omega^2 + 2\delta_v P^{-1}(\Gamma)\|V-\overline{V}\|_\Gamma^2 + K_1\|V^\beta - U^\alpha\|_\Gamma^2 \\
&\qquad \geq \delta_u\|\nabla U\|_\Omega^2 + \delta_v P^{-1}(\Gamma)\|V-\overline{V}\|_\Gamma^2 + \frac{K_1}{2}\|\overline{V}^\beta - \overline{U}^\alpha\|_\Gamma^2 \\
&\qquad\quad + (\delta_u T^{-1}(\Omega) - K_1K_3)\|U-\overline{U}\|_\Gamma^2 + (\delta_v P^{-1}(\Gamma) - K_1K_3)\|V-\overline{V}\|_\Gamma^2 \\
&\qquad \geq \delta_u\|\nabla U\|_\Omega^2 + \delta_v P^{-1}(\Gamma)\|V-\overline{V}\|_\Gamma^2 + \frac{K_1}{2}\|\overline{V}^\beta - \overline{U}^\alpha\|_\Gamma^2.
\end{aligned}$$

Therefore, in order to show (3.27) it is sufficient in the following Step 2 to find suitable constants $K_4 = \min\{\frac{2\delta_u}{K_1}, \frac{2\delta_v}{K_1P(\Gamma)}\}$ and $K_5 = \frac{2K_2K_0}{K_1}$ such that:

$$\|\overline{V}^\beta - \overline{U}^\alpha\|_\Gamma^2 + K_4(\|\nabla U\|_\Omega^2 + \|V-\overline{V}\|_\Gamma^2) \geq K_5\left[(\sqrt{\overline{U^2}} - U_\infty)^2 + (\sqrt{\overline{V^2}} - V_\infty)^2\right]. \tag{3.34}$$

<u>*Step 2:*</u> To prove (3.34), we use the following change of variables with respect to the equilibrium

$$\overline{U^2} = U_\infty^2(1+\mu_1)^2 \qquad \text{and} \qquad \overline{V^2} = V_\infty^2(1+\mu_2)^2, \tag{3.35}$$

which is well-adapted to the mass conservation law (3.28) in the sense that

$$\beta|\Omega|U_\infty^2(1+\mu_1)^2 + \alpha|\Gamma|V_\infty^2(1+\mu_2)^2 = \beta|\Omega|U_\infty^2 + \alpha|\Gamma|V_\infty^2. \tag{3.36}$$

From (3.36), it follows that the new variables μ_1 and μ_2 vary only in a bounded range of admissible values, i.e. $\mu_1 \in [-1, +\mu_{1,m})$ and $\mu_2 \in [-1, +\mu_{2,m})$, where a straightforward estimate shows $0 < \mu_{1,m} < \frac{\alpha|\Gamma|V_\infty^2}{\beta|\Omega|U_\infty^2}$ and $0 < \mu_{2,m} < \frac{\beta|\Omega|U_\infty^2}{\alpha|\Gamma|V_\infty^2}$.

Moreover, equation (3.36) implies that μ_1 can be expressed as a continuous, bounded function of μ_2 (or the other way round), i.e.

$$\mu_1(\mu_2) = -1 + \sqrt{1 - \frac{\alpha|\Gamma|V_\infty^2}{\beta|\Omega|U_\infty^2}(2\mu_2 + \mu_2^2)} = -R(\mu_2)\mu_2, \tag{3.37}$$

where

$$R(\mu_2) := \frac{\frac{\alpha|\Gamma|V_\infty^2}{\beta|\Omega|U_\infty^2}(\mu_2+2)}{1 + \sqrt{1 - \frac{\alpha V_\infty^2|\Gamma|}{\beta U_\infty^2|\Omega|}(2\mu_2+\mu_2^2)}}.$$

We obviously have that $\mu_1(\mu_2 = 0) = 0$, which represents the case $\overline{U^2} = U_\infty^2$ and $\overline{V^2} = V_\infty^2$. Moreover, $R(\mu_2)$ is a positive, monotone increasing function with

$$0 < R(-1) = \frac{\frac{\alpha|\Gamma|V_\infty^2}{\beta|\Omega|U_\infty^2}}{1 + \sqrt{1 + \frac{\alpha|\Gamma|V_\infty^2}{\beta|\Omega|U_\infty^2}}} \leq R(\mu_2) \leq R(\mu_{2,m}) < 2\frac{\alpha|\Gamma|V_\infty^2}{\beta|\Omega|U_\infty^2} + 1.$$

Hence $R(\mu_2)$ for $\mu_2 \in [-1, +\mu_{2,m})$ is uniformly bounded below and above by positive constants.

Next, we notice that

$$\|\delta_1\|_\Omega^2 = \|U - \overline{U}\|_\Omega^2 = \overline{U^2} - \overline{U}^2,$$

and thus

$$\overline{U} = \sqrt{\overline{U^2}} - \frac{1}{\sqrt{\overline{U^2}} + \overline{U}}\|\delta_1\|_\Omega^2 = U_\infty(1+\mu_1) - \frac{1}{\sqrt{\overline{U^2}} + \overline{U}}\|\delta_1\|_\Omega^2. \tag{3.38}$$

Similarly,

$$\overline{V} = \sqrt{\overline{V^2}} - \frac{1}{\sqrt{\overline{V^2}} + \overline{V}}\|\delta_2\|_\Gamma^2 = V_\infty(1+\mu_2) - \frac{1}{\sqrt{\overline{V^2}} + \overline{V}}\|\delta_2\|_\Gamma^2. \tag{3.39}$$

We denote

$$R_1(U) := \frac{1}{\sqrt{\overline{U^2}} + \overline{U}} \quad \text{and} \quad R_1(V) := \frac{1}{\sqrt{\overline{V^2}} + \overline{V}}$$

and remark that due to the lack of lower bounds for $\overline{U^2} \geq \overline{U}^2 \geq 0$ or $\overline{V^2} \geq \overline{V}^2 \geq 0$, we have no a-priori bounds to prevent $R_1(U)$ or $R_1(V)$ from being arbitrary large. Thus, we have to distinguish two cases, where the first assumes a lower bound $\varepsilon > 0$:

Case 1) $\overline{U^2} \geq \varepsilon^2, \overline{V^2} \geq \varepsilon^2$:

By (3.38) and (3.39), left hand side of (3.34) is estimated as follows

$$\begin{aligned}
&\|\overline{V}^\beta - \overline{U}^\alpha\|_\Gamma^2 + K_4(\|\nabla U\|_\Omega^2 + \|V - \overline{V}\|_\Gamma^2) \\
&\quad = \left\|(V_\infty(1+\mu_2) - R_1(V)\|\delta_2\|_\Gamma^2)^\beta - (U_\infty(1+\mu_1) - R_1(U)\|\delta_1\|_\Omega^2)^\alpha\right\|_\Gamma^2 \\
&\qquad + K_4(\|\nabla U\|_\Omega^2 + \|\delta_2\|_\Gamma^2) \\
&\quad \geq |\Gamma|\left(V_\infty^\beta(1+\mu_2)^\beta - U_\infty^\alpha(1+\mu_1)^\alpha\right)^2 - C(\varepsilon^2, M)(\|\delta_2\|_\Gamma^2 + \frac{1}{P(\Omega)}\|\delta_1\|_\Omega^2) \\
&\qquad + K_4(\|\nabla U\|_\Omega^2 + \|\delta_2\|_\Gamma^2) \\
&\quad \geq |\Gamma|\left(V_\infty^\beta(1+\mu_2)^\beta - U_\infty^\alpha(1+\mu_1)^\alpha\right)^2 - C(\varepsilon^2, M, \Omega)(\|\delta_2\|_\Gamma^2 + \|\nabla U\|_\Omega^2) \\
&\qquad + K_4(\|\nabla U\|_\Omega^2 + \|\delta_2\|_\Gamma^2)
\end{aligned} \tag{3.40}$$

by using the boundedness of U_∞, V_∞, μ_1, μ_2, $R_1(U)$ and $R_1(V)$ and by using Poincare's inequality. Choosing $K_4 \geq C(\varepsilon^2, M)$ in (3.40) (by recalling that $K_4 = \min\{\frac{2\delta_u}{K_1}, \frac{2\delta_v}{K_1 P(\Gamma)}\}$, this implies an additional constraint to choose K_1 small enough), we have

$$\|\overline{V}^\beta - \overline{U}^\alpha\|_\Gamma^2 + K_4(\|\nabla U\|_\Omega^2 + \|V - \overline{V}\|_\Gamma^2) \geq |\Gamma|(V_\infty^\beta(1+\mu_2)^\beta - U_\infty^\alpha(1+\mu_1)^\alpha)^2. \tag{3.41}$$

Therefore, in order to prove (3.34), it's enough to find K_5 such that

$$|\Gamma|(V_\infty^\beta(1+\mu_2)^\beta - U_\infty^\alpha(1+\mu_1)^\alpha)^2 \geq K_5\left(U_\infty^2\mu_1^2 + V_\infty\mu_2^2\right)$$

or equivalently,

$$\frac{U_\infty^2\mu_1^2 + V_\infty^2\mu_2^2}{V_\infty^{2\beta}\left((1+\mu_2)^\beta - (1+\mu_1)^\alpha\right)^2} \leq \frac{|\Gamma|}{K_5}. \tag{3.42}$$

In order to estimate the denominator of (3.42), we consider the following two cases:

In the first case, we assume that $-1 \leq \mu_2 < 0$, from (3.37) we have $\mu_1 > 0$. Then

$$(1+\mu_2)^\beta \leq 1 + \mu_2 < 1 \quad \text{and} \quad (1+\mu_1)^\alpha \geq 1 + \mu_1 > 1.$$

Hence,

$$|(1+\mu_2)^\beta - (1+\mu_1)^\alpha| \geq (1+\mu_1) - (1+\mu_2) = \mu_1 - \mu_2 = (1 + R(\mu_2))|\mu_2|. \tag{3.43}$$

In the second case, we consider $\mu_2 \geq 0$ and thus $\mu_1 \leq 0$ by (3.37). We estimate

$$(1+\mu_2)^\beta \geq (1+\mu_2) \text{ and } (1+\mu_1)^\alpha \leq 1+\mu_1,$$

and obtain therefore,

$$|(1+\mu_2)^\beta - (1+\mu_1)^\alpha| \geq (1+\mu_2) - (1+\mu_1) = \mu_2 - \mu_1 = (1+R(\mu_2))|\mu_2|. \quad (3.44)$$

Altogether, (3.43) and (3.44) yield

$$V_\infty^{2\beta}\left((1+\mu_2)^\beta - (1+\mu_1)^\alpha\right)^2 \geq V_\infty^{2\beta}(1+R(\mu_2))^2\mu_2^2. \quad (3.45)$$

For the numerator of (3.42), we use the expression (3.37) to get

$$U_\infty^2\mu_1^2 + V_\infty^2\mu_2^2 = \left(V_\infty^2 + U_\infty^2\, R(\mu_2)^2\right)\mu_2^2, \quad (3.46)$$

and combining (3.46) and (3.45) completes the proof of (3.42) with a constant

$$\frac{|\Gamma|}{K_5} \geq \frac{V_\infty^2 + U_\infty^2\, R(\mu_2)^2}{(1+R(\mu_2))^2)V_\infty^{2\beta}}$$

Finally, by recalling that $K_5 = \frac{2K_2K_0}{K_1}$ and that K_1 was chosen small enough in the previous step, we conclude the first part of the proof of the Lemma by choosing $K_2 \leq \frac{K_1K_5}{2K_0}$.

Case 2) $\overline{U^2} \leq \varepsilon^2$ or $\overline{V^2} \leq \varepsilon^2$:

For the second case, which considers states away from the equilibrium $U \approx U_\infty$, $V \approx V_\infty$ for sufficiently small ε, we expect to be able to derive a positive lower bound for the entropy dissipation in terms of ε. At first, we observe that the right hand side of (3.34) is bounded by

$$K_5\left[(\sqrt{\overline{U^2}} - U_\infty)^2 + (\sqrt{\overline{V^2}} - V_\infty)\right]^2 \leq 2K_5(\overline{u}+\overline{v}+u_\infty+v_\infty) \leq K_5C(M). \quad (3.47)$$

In the following, we consider two subcases of lower bounds of the entropy dissipation. The first subcase considers the situation where there is a lower bound of the diffusion entropy dissipation since U and V are not close to their spacial averages $\overline{U}$ and $\overline{V}$:

Subcase 2.1) $\|\delta_1\|_\Omega^2 \geq \eta$ or $\|\delta_2\|_\Gamma^2 \geq \eta$:

By using Poincare's inequality $P(\Omega)\|\nabla U\|_\Omega^2 \geq \|\delta_1\|_\Omega^2$, we see that the left hand side of (3.34) is bounded below by

$$\begin{cases} K_4P^{-1}(\Omega)\eta & \text{in the case } \|\delta_1\|_\Omega^2 \geq \eta, \\ K_4\eta & \text{in the case } \|\delta_2\|_\Gamma^2 \geq \eta. \end{cases} \quad (3.48)$$

Thus, from (3.47) and (3.48), we can obtain (3.34) by choosing

$$K_4 \geq \max\left\{\frac{C(M)}{P(\Omega)\eta}, \frac{C(M)}{\eta}\right\}.$$

Subcase 2.2) $\|\delta_1\|_\Omega^2 \leq \eta$ and $\|\delta_2\|_\Gamma^2 \leq \eta$:

This subcase concerns the situation where U and V are close to their spacial averages $\overline{U}$ and $\overline{V}$. Thus, since U and V are not close to the equilibrium U_∞ and V_∞ for sufficiently small ε in **Case 2)**, there has to be a lower bound for the reaction entropy dissipation.

Let us assume first $\overline{V^2} \leq \varepsilon^2$, thus $\overline{V}^2 \leq \overline{V^2} \leq \varepsilon^2$. From

$$\beta|\Omega|\overline{U^2} + \alpha|\Gamma|\overline{V^2} = M, \quad \text{and} \quad \overline{U^2} = \|\delta_1\|_\Omega^2 + \overline{U}^2,$$

we estimate

$$\overline{U}^2 = \frac{1}{\beta|\Omega|}(M - \alpha|\Gamma|\overline{V^2}) - \|\delta_1\|_\Omega^2 \geq \frac{M}{\beta|\Omega|} - \frac{\alpha|\Gamma|}{\beta|\Omega|}\varepsilon^2 - \eta.$$

Hence, we can expand the reaction term as follows

$$\begin{aligned}\|\overline{U}^\alpha - \overline{V}^\beta\|_\Gamma^2 &\geq |\Gamma|\left(\frac{1}{2}\overline{U}^{2\alpha} - \overline{V}^{2\beta}\right) \geq |\Gamma|\left(\frac{1}{2}\left(\frac{M}{\beta|\Omega|} - \frac{\alpha|\Gamma|}{\beta|\Omega|}\varepsilon^2 - \eta\right)^\alpha - \varepsilon^{2\beta}\right)\\ &\geq \frac{|\Gamma|}{2^{\alpha+2}}\left(\frac{M}{\beta|\Omega|}\right)^\alpha \qquad (3.49)\end{aligned}$$

for small enough ε and η.

The case $\overline{U^2} \leq \varepsilon^2$ can be treated similarly and yields

$$\|\overline{U}^\alpha - \overline{V}^\beta\|_\Gamma^2 \geq \frac{|\Gamma|}{2^{\beta+2}}\left(\frac{M}{\alpha|\Gamma|}\right)^\beta. \qquad (3.50)$$

From (3.48), (3.49) and (3.50), we have for both cases $\overline{U^2} \leq \varepsilon^2$ or $\overline{V^2} \leq \varepsilon^2$ that the left hand side of (3.34) is estimated below as

$$\begin{aligned}&\|\overline{V}^\beta - \overline{U}^\alpha\|_\Gamma^2 + K_4(\|\nabla U\|_\Omega^2 + \|V - \overline{V}\|_\Gamma^2)\\ &\quad\geq K_6 = \min\left\{K_4 P^{-1}(\Omega)\eta, K_4\eta, \frac{|\Gamma|}{2^{\alpha+2}}\left(\frac{M}{\beta|\Omega|}\right)^\alpha, \frac{|\Gamma|}{2^{\beta+2}}\left(\frac{M}{\alpha|\Gamma|}\right)^\beta\right\}. \qquad (3.51)\end{aligned}$$

Then, (3.34) follows from (3.47), (3.51) by choosing $K_5 \leq \frac{K_6}{C(M)}$, which means to choose $K_2 \leq \frac{K_1 K_5}{2K_0}$ small enough. □

Remark 3.3. *The Step 2 in the proof of Lemma 3.5 can be significantly shortened if we consider the stoichiometric coefficients $\alpha \geq 2$ and $\beta \geq 2$, since we can prove* (3.41) *without case distinction as follows.*

By noting that $\|\delta_1\|_\Omega^2 = \|U - \overline{U}\|_\Omega^2 = |\Omega|(\overline{U^2} - \overline{U}^2)$, we derive the expressions

$$\overline{U} = \sqrt{\overline{U^2} - \|\delta_1\|_\Omega^2/|\Omega|}, \qquad \overline{V} = \sqrt{\overline{V^2} - \|\delta_2\|_\Gamma^2/|\Gamma|}.$$

Thus, by (3.3)*, we apply again Taylor expansion to estimate the first term on the left hand side of* (3.34) *below by*

$$\begin{aligned}\|\overline{V}^\beta - \overline{U}^\alpha\|_\Gamma^2 &= \left\|\left(\overline{V^2} - \|\delta_2\|_\Gamma^2/|\Gamma|\right)^{\frac{\beta}{2}} - \left(\overline{U^2} - \|\delta_1\|_\Omega^2/|\Omega|\right)^{\frac{\alpha}{2}}\right\|_\Gamma^2\\ &\geq \left\|\overline{V^2}^{\frac{\beta}{2}} - \overline{U^2}^{\frac{\alpha}{2}}\right\|_\Gamma^2 - 2\int_\Gamma \left(\overline{V^2}^{\frac{\beta}{2}} - \overline{U^2}^{\frac{\alpha}{2}}\right)\left(\frac{\beta}{2}\Big(\overline{V^2} - \frac{\theta_2}{|\Gamma|}\Big)^{\frac{\beta}{2}-1}\frac{\|\delta_2\|_\Gamma^2}{|\Gamma|}\right.\\ &\qquad\left. - \frac{\alpha}{2}\Big(\overline{U^2} - \frac{\theta_1}{|\Omega|}\Big)^{\frac{\alpha}{2}-1}\frac{\|\delta_1\|_\Omega^2}{|\Omega|}\right) \qquad (3.52)\end{aligned}$$

for some $\theta_1/|\Omega| \leq \|\delta_1\|_\Omega^2/|\Omega| \leq \overline{U^2} \leq M_\Omega$ and $\theta_2/|\Gamma| \leq \|\delta_2\|_\Gamma^2/|\Gamma| \leq \overline{V^2} \leq M_\Gamma$. Note that $\frac{\beta}{2} - 1 \geq 0$ and $\frac{\alpha}{2} - 1 \geq 0$, then the last term on the right hand side of (3.52) *can be estimated below by*

$$C(M_\Omega^\alpha, M_\Gamma^\beta, \Omega)\left(\frac{\|\delta_1\|_\Omega^2}{P(\Omega)} + \|\delta_2\|_\Gamma^2\right).$$

Thus, from (3.35) *and* (3.52)*, we have*

$$
\begin{aligned}
&\|\overline{V}^{\beta} - \overline{U}^{\alpha}\|_{\Gamma}^2 \\
&\quad \geq |\Gamma| \left(V_{\infty}^{\beta}(1+\mu_2)^{\beta} - U_{\infty}^{\alpha}(1+\mu_1)^{\alpha}\right)^2 - C(M_{\Omega}^{\alpha}, M_{\Gamma}^{\beta}, \Omega)\left(\frac{\|\delta_1\|_{\Omega}^2}{P(\Omega)} + \|\delta_2\|_{\Gamma}^2\right) \\
&\quad \geq |\Gamma| \left(V_{\infty}^{\beta}(1+\mu_2)^{\beta} - U_{\infty}^{\alpha}(1+\mu_1)^{\alpha}\right)^2 - C(M_{\Omega}^{\alpha}, M_{\Gamma}^{\beta}, \Omega)\left(\|\nabla U\|_{\Omega}^2 + \|\delta_2\|_{\Gamma}^2\right).
\end{aligned}
$$

Therefore, by choosing $K_4 \geq C(M_{\Omega}^{\alpha}, M_{\Gamma}^{\beta})$*, we have proved* (3.41)*:*

$$\|\overline{V}^{\beta} - \overline{U}^{\alpha}\|_{\Gamma}^2 + K_4(\|\nabla U\|_{\Omega}^2 + \|V - \overline{V}\|_{\Gamma}^2) \geq |\Gamma|\left(V_{\infty}^{\beta}(1+\mu_2)^{\beta} - U_{\infty}^{\alpha}(1+\mu_1)^{\alpha}\right)^2.$$

The rest of the proof follows exactly as the end of **Case 1)** *in Lemma 3.5.*

3.2. **The degenerate case:** $\delta_v = 0$**.** By Remark 2.2 we know that (A, B) is an upper solution to (1.1), then, by the comparison principle we have that, for all $t \geq 0$,

$$\|u(t)\|_{L^{\infty}(\Omega)} \leq A, \quad \text{and} \quad \|v(t)\|_{L^{\infty}(\Gamma)} \leq B.$$

Then, by using the same function Φ as (3.19), we have

$$
\begin{aligned}
&E(u,v) - E(u_{\infty}, v_{\infty}) \\
&\qquad = \int_{\Omega}\left(u\log\frac{u}{u_{\infty}} - (u - u_{\infty})\right)dx + \int_{\Gamma}\left(v\log\frac{v}{v_{\infty}} - (v - v_{\infty})\right)dS \\
&\qquad \leq \Phi(A, u_{\infty})\int_{\Omega}(\sqrt{u} - \sqrt{u_{\infty}})^2 dx + \Phi(B, v_{\infty})\int_{\Gamma}(\sqrt{v} - \sqrt{v_{\infty}})^2 dS \\
&\qquad \leq \max\{\Phi(A, u_{\infty}), \Phi(B, v_{\infty})\}\left(\|U - U_{\infty}\|_{\Omega}^2 + \|V - V_{\infty}\|_{\Gamma}^2\right). \qquad (3.53)
\end{aligned}
$$

The following lemma, roughly speaking, shows that the diffusion of u in Ω and the reversible reaction of u and v on Γ lead to a diffusion-effect of v on Γ:

Lemma 3.6. *There exists* $C_1, C_2 > 0$ *such that*

$$C_1\|U^{\alpha} - V^{\beta}\|_{\Gamma}^2 + C_2\left(\|\nabla U\|_{\Omega}^2 + \|U - \overline{U}\|_{\Gamma}^2\right) \geq C_3\|V - \overline{V}\|_{\Gamma}^2. \qquad (3.54)$$

Proof. Note that, by the Trace Theorem $T(\Omega)\|\nabla U\|_{\Omega}^2 \geq \|U - \overline{U}\|_{\Gamma}^2$, we could neglect the term $\|U - \overline{U}\|_{\Gamma}^2$ in (3.54). We write it here for the sake of readability.

We will prove the inequality (3.54) by distinguishing cases:

<u>*Case 1:* $\overline{U} \geq \varepsilon$*.*</u> Applying the ansatz

$$V(x) = \overline{U}^{\frac{\alpha}{\beta}}(1 + \delta(x)), \qquad \delta(x) \in [-1, +\infty) \qquad \forall x \in \Gamma,$$

we get

$$
\begin{aligned}
C_1\|U^{\alpha} - V^{\beta}\|_{\Gamma}^2 = C_1\|U^{\alpha} - \overline{U}^{\alpha}\|_{\Gamma}^2 - 2C_1\int_{\Gamma}(U^{\alpha} - \overline{U}^{\alpha})\overline{U}^{\alpha}[(1+\delta)^{\beta} - 1]dS \\
+ C_1\overline{U}^{2\alpha}\|(1+\delta)^{\beta} - 1\|_{\Gamma}^2. \quad (3.55)
\end{aligned}
$$

Since $\|U\|_{L^{\infty}(\Gamma)} \leq A$, we have

$$\|U^{\alpha} - \overline{U}^{\alpha}\|_{\Gamma}^2 \leq C(A)\|U - \overline{U}\|_{\Gamma}^2. \qquad (3.56)$$

From (3.55) and (3.56), we can estimate the left hand side of (3.54) as follows

$$\begin{aligned} C_1\|U^\alpha - V^\beta\|_\Gamma^2 + C_2\|U-\overline{U}\|_\Gamma^2 &\geq \left(C_1 + \frac{C_2}{C(A)}\right)\|U^\alpha - \overline{U}^\alpha\|_\Gamma^2 \\ &- 2C_1\int_\Gamma (U^\alpha - \overline{U}^\alpha)\overline{U}^\alpha[(1+\delta)^\beta - 1]dS + C_1\overline{U}^{2\alpha}\|(1+\delta)^\beta - 1\|_\Gamma^2 \\ &\geq \frac{C_1C_2}{C_1C(A)+C_2}\overline{U}^{2\alpha}\|(1+\delta)^\beta - 1\|_\Gamma^2, \end{aligned} \tag{3.57}$$

where we have used Young's inequality

$$\begin{aligned} &2C_1\int_\Gamma (U^\alpha - \overline{U}^\alpha)\overline{U}^\alpha[(1+\delta)^\beta - 1]dS \\ &\leq \left(C_1 + \frac{C_2}{C(A)}\right)\|U^\alpha - \overline{U}^\alpha\|_\Gamma^2 + \frac{C_1^2C(A)}{C_1C(A)+C_2}\overline{U}^{2\alpha}\|(1+\delta)^\beta - 1\|_\Gamma^2. \end{aligned} \tag{3.58}$$

Next, we observe that the function $R(\delta) := \frac{(1+\delta)^\beta - 1}{\delta}$ is continuous on $\delta \in [-1,\infty)$ with $R(0) = \beta \geq 1$ and bounded below by $R(\delta) \geq R(-1) = 1$ for $\delta \in [-1,\infty)$. Thus,

$$\|(1+\delta)^\beta - 1\|_\Gamma^2 = \int_\Gamma R(\delta)^2\delta^2\,dS \geq \int_\Gamma \delta^2\,dS. \tag{3.59}$$

On the other hand, we have

$$\begin{aligned} \|V-\overline{V}\|_\Gamma^2 &= |\Gamma|\left(\overline{V^2} - \overline{V}^2\right) = |\Gamma|\overline{U}^{\frac{2\alpha}{\beta}}\left(\overline{(1+\delta)^2} - \overline{1+\delta}^2\right) \\ &= |\Gamma|\overline{U}^{\frac{2\alpha}{\beta}}\left(1 + 2\overline{\delta} + \overline{\delta^2} - (1+\overline{\delta})^2\right) \leq |\Gamma|\overline{U}^{\frac{2\alpha}{\beta}}\overline{\delta^2} \\ &\leq \overline{U}^{\frac{2\alpha}{\beta}}\int_\Gamma \delta^2\,dS. \end{aligned} \tag{3.60}$$

Now, keeping in mind that $\overline{U} \geq \varepsilon$, we obtain (3.54) from (3.57), (3.59) and (3.60), by choosing

$$C_3 \leq \frac{C_1C_2}{C_1C(M)+C_2}\min\{1;\varepsilon^{2\alpha(1-1/\beta)}\}.$$

Case 2: $\overline{U} \leq \varepsilon$. We begin by considering $\overline{U} \leq \varepsilon$, for which the contribution of $\|U-\overline{U}\|_\Gamma^2$ in (3.54) can be arbitrary small when U is close to $\overline{U}$. However, for ε sufficiently small, we shall show that the estimate (3.54) still holds because the reaction term $\|U^\alpha - V^\beta\|_\Gamma^2$ can only be "small" if the $\|V-\overline{V}\|_\Gamma^2$ is of the "same order of smallness".

We will treat two subcases: a) $\overline{U^2}$ is "small" and b) $\overline{U^2}$ is "big".

Case 2a): $\overline{U^2} \leq \frac{M}{2\beta|\Omega|}$. A direct consequence of the conservation law (3.28) yields

$$\overline{V^2} = \frac{1}{\alpha|\Gamma|}\left(M - \beta|\Omega|\overline{U^2}\right) \geq \frac{M}{2\alpha|\Gamma|}. \tag{3.61}$$

Next, we estimate similarly to (3.55)–(3.58) the left hand side of (3.54) as

$$C_1\|U^\alpha - V^\beta\|_\Gamma^2 + C_2\|U-\overline{U}\|_\Gamma^2 \geq C_4\|V^\beta - \overline{U}^\alpha\|_\Gamma^2.$$

where $C_4 = \frac{C_1C_2}{C_1C(M)+C_2}$. Then, since $\overline{U} \leq \varepsilon$

$$C_4\|V^\beta - \overline{U}^\alpha\|_\Gamma^2 \geq C_4\int_\Gamma V^{2\beta}dS - 2C_4\varepsilon^\alpha\int_\Gamma V^\beta dS.$$

On the other hand, the right hand side of (3.54) is bounded by

$$C_3 \int_\Gamma |V - \overline{V}|_\Gamma^2 = C_3|\Gamma| \left(\overline{V^2} - \overline{V}^2\right) \le C_3|\Gamma|\overline{V^2}. \tag{3.62}$$

Therefore, in order to obtain (3.54), it is sufficient to prove that

$$C_4 \int_\Gamma V^{2\beta} dS - 2C_4\varepsilon^\alpha \int_\Gamma V^\beta dS \ge C_3|\Gamma|\overline{V^2}. \tag{3.63}$$

If $\beta \in [1,2]$, by using Jensen's inequality (and noting that the function $f(x) = x^{\frac{\beta}{2}}$ is concave), we can estimate the left hand side of (3.63) as

$$\begin{aligned} &C_4 \int_\Gamma V^{2\beta} dS - 2C_4\varepsilon^\alpha \int_\Gamma V^\beta dS \\ &\qquad \ge C_4 \left(\int_\Gamma V^2 dS\right)^\beta - 2C_4\varepsilon^\alpha \left(\int_\Gamma V^2 dS\right)^{\beta/2} \\ &\qquad = C_4|\Gamma|^\beta \overline{V^2}^\beta - 2C_4\varepsilon^\alpha|\Gamma|^{\beta/2}\overline{V^2}^{\beta/2}. \end{aligned} \tag{3.64}$$

Since $\overline{V^2} \ge \frac{M}{2\alpha|\Gamma|}$, we can choose ε and C_3 small enough such that

$$C_4|\Gamma|^\beta \overline{V^2}^{\beta-1} \ge 2C_4\varepsilon^\alpha|\Gamma|^{\beta/2}\overline{V^2}^{\beta/2-1} + C_3|\Gamma|.$$

After choosing $\varepsilon \le \frac{1}{4(2\alpha)^{\beta/2\alpha}} M^{\frac{\beta}{2\alpha}}$ and $C_3 \le \frac{1}{2}\frac{C_4 M^{\beta-1}}{(2\alpha)^{\beta-1}}$, this gives together with (3.64) the inequality (3.63).

If $\beta \ge 2$, by using Jensen's inequality, we have

$$C_3|\Gamma|\overline{V^2} \le C_3|\Gamma|\overline{V^\beta}^{2/\beta} \quad \text{and} \quad C_4 \int_\Gamma V^{2\beta} dS \ge C_4|\Gamma|^2\overline{V^\beta}^2. \tag{3.65}$$

Making use of (3.65), relation (3.63) can be proven since

$$C_4|\Gamma|^2\overline{V^\beta}^2 - 2C_4\varepsilon^\alpha|\Gamma|\overline{V^\beta} \ge C_3|\Gamma|\overline{V^\beta}^{2/\beta}$$

or equivalently

$$C_4|\Gamma|\overline{V^\beta} \ge 2C_4\varepsilon^\alpha + C_3\overline{V^\beta}^{(2-\beta)/\beta}.$$

This can be satisfied if we choose, for instance, $\varepsilon \le \frac{|\Gamma|^{1/\alpha}}{4^{1/\alpha}}\left(\frac{M}{2\alpha|\Gamma|}\right)^{\beta/2\alpha}$ and $C_3 \le \frac{1}{2}C_2|\Gamma|\left(\frac{M}{2\alpha|\Gamma|}\right)^{\beta-1}$ and keeping in mind that $\overline{V^2} \ge \frac{M}{2\alpha|\Gamma|}$ and $\beta \ge 2$.

Case 2b): $\overline{U^2} \ge \frac{M}{2\beta|\Omega|}$. Similarly to (3.61), we deduce from the conservation of mass that $\overline{V^2} \le M/(2\alpha|\Gamma|)$. We estimate

$$C_2\|\nabla U\|_\Omega^2 \ge \frac{C_2}{P(\Omega)}\|U - \overline{U}\|_\Omega^2 \ge \frac{C_2|\Omega|}{P(\Omega)}\left(\overline{U^2} - \overline{U}^2\right) \ge \frac{C_2}{P(\Omega)}\left(\frac{M}{2\beta} - \varepsilon^2\right) \ge \frac{C_2}{P(\Omega)}\frac{M}{4\beta},$$

if we chose $\varepsilon^2 < \frac{M}{4\beta}$. Next, recalling (3.62), we estimate

$$\frac{C_2}{P(\Omega)}\frac{M}{4\beta} \ge \frac{C_2}{P(\Omega)}\frac{M}{4\beta}\frac{\overline{V^2}}{M_\Gamma} \ge C_3\|V - \overline{V}\|_\Gamma^2,$$

if we choose $C_3 \le \frac{C_2\alpha|\Gamma|}{2\beta P(\Omega)}$.

Altogether, the proof of (3.54) is complete by choosing ε and C_3 small enough in order to satisfy the various constraints from the above cases. □

We are now ready to prove Lemma 3.3 for degenerate case, that is when $\delta_v = 0$, there exists $C_0 > 0$ such that

$$D(u,v) \geq C_0(E(u,v) - E(u_\infty, v_\infty)).$$

Proof. We begin by estimating $D(u,v)$ below, that is

$$\begin{aligned} D(u,v) &= \delta_u \int_\Omega \frac{|\nabla u|^2}{u} dx + \int_\Gamma (u^\alpha - v^\beta) \log \frac{u^\alpha}{v^\beta} dS \\ &\geq 4\delta_u \|\nabla U\|_\Omega^2 + 4\|U^\alpha - V^\beta\|_\Gamma^2, \end{aligned}$$

where we have used the elementary inequality $(a-b)\log(a/b) \geq 4(\sqrt{a}-\sqrt{b})^2$. Then, by applying the Trace inequality $\|U - \overline{U}\|_\Gamma^2 \leq \|\nabla U\|_\Omega^2 T(\Omega)$ and Lemma 3.6, we get

$$\begin{aligned} D(u,v) &\geq 4\delta_u \|\nabla U\|_\Omega^2 + 4\|U^\alpha - V^\beta\|_\Gamma^2 \\ &\geq \theta \left[C_1 \|U^\alpha - V^\beta\|_\Gamma^2 + C_2(\|\nabla U\|_\Omega^2 + \|U - \overline{U}\|_\Gamma^2) \right] \\ &\quad + \left[4\delta_u - \theta C_2(1 + T(\Omega))\right] \|\nabla U\|_\Omega^2 + (4 - \theta C_1)\|U^\alpha - V^\beta\|_\Gamma^2 \\ &\geq \theta C_3 \|V - \overline{V}\|_\Gamma^2 + [4\delta_u - \theta C_2(1 + T(\Omega))] \|\nabla U\|_\Omega^2 + (4 - \theta C_1)\|U^\alpha - V^\beta\|_\Gamma^2 \\ &\geq C_4 \|\nabla U\|_\Omega^2 + C_5 \|V - \overline{V}\|_\Gamma^2 + C_6 \|U^\alpha - V^\beta\|_\Gamma^2. \end{aligned} \tag{3.66}$$

where we denote $C_4 = 4\delta_u - \theta C_2(1 + T(\Omega))$, $C_5 = \theta C_3$ and $C_6 = (4 - \theta C_1)$, where $\theta > 0$ is chosen such that the constants C_4 and C_6 are positive.

In the following, we estimate the relative entropy $E(u,v) - E(u_\infty, v_\infty)$ above by using (3.53)

$$\begin{aligned} E(u,v) - E(u_\infty, v_\infty) &\leq \max\{\Phi(A, u_\infty), \Phi(B, v_\infty)\}(\|U - U_\infty\|_\Omega^2 + \|V - V_\infty\|_\Gamma^2) \\ &\leq C_8(\|U - \overline{U}\|_\Omega^2 + \|V - \overline{V}\|_\Gamma^2 + \|\overline{U} - U_\infty\|_\Omega^2 + \|\overline{V} - V_\infty\|_\Gamma^2) \end{aligned} \tag{3.67}$$

with $C_8 = 2\max\{\Phi(A, u_\infty), \Phi(B, v_\infty)\}$. By using (3.67), we continue to estimate (3.66) below and obtain by using Poincaré's inequality, the Trace Theorem and for $0 < \varepsilon < 1$ to be chosen

$$\begin{aligned} D(u,v) &\geq \frac{C_4}{2P(\Omega)}\|U - \overline{U}\|_\Omega^2 + \frac{C_4}{2T(\Omega)}\|U - \overline{U}\|_\Gamma^2 + C_5\|V - \overline{V}\|_\Gamma^2 + C_6\|U^\alpha - V^\beta\|_\Gamma^2 \\ &\geq \varepsilon \min\left\{\frac{C_4}{2P(\Omega)}, C_5\right\} (\|U - \overline{U}\|_\Omega + \|V - \overline{V}\|_\Gamma^2) \\ &\quad + \frac{C_4}{2T(\Omega)}\|U - \overline{U}\|_\Gamma^2 + C_6\|U^\alpha - V^\beta\|_\Gamma^2 + C_5(1-\varepsilon)\|V - \overline{V}\|_\Gamma^2 \\ &\geq \varepsilon \min\left\{\frac{C_4}{2P(\Omega)}, C_5\right\} \left(\frac{E(u,v) - E(u_\infty, v_\infty)}{C_8} - \|\overline{U} - U_\infty\|_\Omega^2 - \|\overline{V} - V_\infty\|_\Gamma^2 \right) \\ &\quad + \frac{C_4}{2T(\Omega)}\|U - \overline{U}\|_\Gamma^2 + C_6\|U^\alpha - V^\beta\|_\Gamma^2 + C_5(1-\varepsilon)\|V - \overline{V}\|_\Gamma^2 \\ &\geq \frac{\varepsilon}{C_8} \min\left\{\frac{C_4}{2P(\Omega)}, C_5\right\} (E(u,v) - E(u_\infty, v_\infty)) \\ &\quad + \frac{C_4}{2T(\Omega)}\|U - \overline{U}\|_\Gamma^2 + C_6\|U^\alpha - V^\beta\|_\Gamma^2 + C_5(1-\varepsilon)\|V - \overline{V}\|_\Gamma^2 \\ &\quad - \varepsilon \min\left\{\frac{C_4}{2P(\Omega)}, C_5\right\} \left(\|\overline{U} - U_\infty\|_\Omega^2 + \|\overline{V} - V_\infty\|_\Gamma^2\right). \end{aligned} \tag{3.68}$$

Now, by applying (3.27) with $4\delta_v P^{-1}(\Gamma) = C_5(1-\varepsilon)$, we can find a positive constant $\varepsilon > 0$ small enough such that

$$\frac{C_4}{2T(\Omega)}\|U - \overline{U}\|_{\Gamma}^2 + C_6\|U^\alpha - V^\beta\|_{\Gamma}^2 + C_5(1-\varepsilon)\|V - \overline{V}\|_{\Gamma}^2 \geq \varepsilon \min\left\{\frac{C_4}{2P(\Omega)}, C_5\right\}\left(\|\overline{U} - U_\infty\|_{\Omega}^2 + \|\overline{V} - V_\infty\|_{\Gamma}^2\right) \quad (3.69)$$

holds and we conclude from (3.68) and (3.69) that

$$D(u,v) \geq \frac{\varepsilon}{C_8}\min\left\{\frac{C_4}{2P(\Omega)}, C_5\right\}(E(u,v) - E(u_\infty, v_\infty)),$$

which finishes the proof of the Lemma in the case of degenerate diffusion $\delta_v = 0$. □

As we can see in the proof the degenerate case, we used L^∞-bounds of the solution, which are usually unavailable for more general systems. However, we believe that in some cases of stoichiometric coefficients α and β, there will be a way to show the exponential convergence to equilibrium without using the L^∞ bounds. As example, we show that it is possible for the linear case, that is $\alpha = \beta = 1$.

Proposition 3.7. *Assume that $\alpha = \beta = 1$ and $\delta_v = 0$. The solution to the system* (1.1)*, which rewrites as*

$$\begin{cases} u_t - \delta_u \Delta u = 0, & x \in \Omega,\\ \delta_u \frac{\partial u}{\partial \nu} = -u + v, & x \in \Gamma,\\ v_t = u - v, & x \in \Gamma,\\ u(0,x) = u_0(x), & x \in \Omega,\\ v(0,x) = v_0(x), & x \in \Gamma, \end{cases} \quad (3.70)$$

converges exponentially to the equilibrium in $L^2(\Omega) \times L^2(\Gamma)$.

Remark 3.4. *Due to the lack of the surface diffusion $\delta_v \Delta_\Gamma v$, when establishing an entropy-entropy dissipation estimate, we need to prove an inequality analogous to* (3.54)*, that is*

$$C_1\|U^\alpha - V^\beta\|_{\Gamma}^2 + C_2\left(\|\nabla U\|_{\Omega}^2 + \|U - \overline{U}\|_{\Gamma}^2\right) \geq C_3\|V - \overline{V}\|_{\Gamma}^2. \quad (3.71)$$

The main point of this proposition is that, thanks to the linearity of the system, we can use the quadratic structure of the entropy to prove the existence of such an estimate without using the L^∞-bounds of the solution (see (3.77) *below). For general α and β an estimate like* (3.71) *seems highly unclear: consider for instance a state $V = U^{\frac{\alpha}{\beta}}$. Then, $\|U^\alpha - V^\beta\|_\Gamma = 0$ and the two remaining terms $\|\nabla U\|_\Omega^2$ and $\|U - \overline{U}\|_\Gamma^2$ on the left hand side of* (3.71) *seem not strong enough to ensure the integrability of V for $\alpha \gg \beta$. Such cases remain open problems to be treated in a future work.*

Proof. The unique equilibrium (u_∞, v_∞) satisfies

$$\begin{cases} u_\infty = v_\infty,\\ |\Omega| u_\infty + |\Gamma| v_\infty = M \end{cases} \quad (3.72)$$

where

$$M = \int_\Omega u_0(x)dx + \int_\Gamma v_0(x)dS$$

is the initial mass.

For the sake of simplicity, we consider the quadratic entropy (which is only an admissible entropy functional since (3.70) is linear)

$$E(u,v) = \|u\|_\Omega^2 + \|v\|_\Gamma^2, \tag{3.73}$$

its entropy dissipation

$$D(u,v) = -\frac{d}{dt}E(u,v) = 2d_u\|\nabla u\|_\Omega^2 + 2\|u-v\|_\Gamma^2, \tag{3.74}$$

and the relative entropy

$$E(u,v) - E(u_\infty, v_\infty) = \|u\|_\Omega^2 + \|v\|_\Gamma^2 - \|u_\infty\|_\Omega^2 - \|v_\infty\|_\Gamma^2. \tag{3.75}$$

Similarly to (3.7), we decompose the relative entropy as follow:

$$\begin{aligned} E(u,v) - E(u_\infty, v_\infty) &= [E(u,v) - E(\overline{u},\overline{v})] + [E(\overline{u},\overline{v}) - E(u_\infty, v_\infty)] \\ &= [\|u-\overline{u}\|_\Omega^2 + \|v-\overline{v}\|_\Gamma^2] + [\|\overline{u}-u_\infty\|_\Omega^2 + \|\overline{v}-v_\infty\|_\Gamma^2]. \end{aligned}$$

In the spirit of Lemma 3.6, we want to have an estimate similar to (3.54):

$$C_1\|\nabla u\|_\Omega^2 + C_2\|u-v\|_\Gamma^2 \geq C_3\|v-\overline{v}\|_\Gamma^2. \tag{3.76}$$

This can be done by estimating

$$\begin{aligned} C_1\|\nabla u\|_\Omega^2 + C_2\|u-v\|_\Gamma^2 &\geq \frac{C_1}{T(\Omega)}\|u-\overline{u}\|_\Gamma^2 + C_2\|u-v\|_\Gamma^2 \geq \frac{C_1C_2/T(\Omega)}{C_2 + C_1/T(\Omega)}\|\overline{u}-v\|_\Gamma^2 \\ &= \frac{C_1C_2/T(\Omega)}{C_2 + C_1/T(\Omega)}(\|v-\overline{v}\|_\Gamma^2 + \|\overline{u}-\overline{v}\|_\Gamma^2), \end{aligned} \tag{3.77}$$

where we have used that $\int_\Gamma(\overline{u}-\overline{v})(v-\overline{v})\,dS = 0$. Now, we can proceed similarly to the degenerate case to get the exponential convergence to equilibrium. For completeness, we sketch the entropy method as follows:

$$\begin{aligned} D(u,v) &= 2d_u\|\nabla u\|_\Omega^2 + 2\|u-v\|_\Gamma^2 \\ &\geq \frac{d_u}{P(\Omega)}\|u-\overline{u}\|_\Omega^2 + \theta\left(C_1\|\nabla u\|_\Omega^2 + C_2\|u-v\|_\Gamma^2\right) \\ &\geq \frac{d_u}{P(\Omega)}\|u-\overline{u}\|_\Omega^2 + \theta C_3\|v-\overline{v}\|_\Gamma^2 + \theta C_3\|\overline{u}-\overline{v}\|_\Gamma^2. \end{aligned} \tag{3.78}$$

By the mass conservation and the definition of the equilibrium (3.72), we get

$$\|\overline{u}-\overline{v}\|_\Gamma^2 = \frac{(|\Omega|+|\Gamma|)^2}{2|\Omega|^2}(\|\overline{u}-u_\infty\|_\Omega^2 + \|\overline{v}-v_\infty\|_\Gamma^2). \tag{3.79}$$

Combining (3.78) and (3.79) yields

$$\begin{aligned} D(u,v) &\geq \frac{d_u}{P(\Omega)}\|u-\overline{u}\|_\Omega^2 + \theta C_3\|v-\overline{v}\|_\Gamma^2 \\ &\quad + \theta C_3\frac{(|\Omega|+|\Gamma|)^2}{2|\Omega|^2}(\|\overline{u}-u_\infty\|_\Omega^2 + \|\overline{v}-v_\infty\|_\Gamma^2) \\ &\geq C_4(E(u,v) - E(u_\infty, v_\infty)). \end{aligned}$$

Hence, the solution satisfies the exponential convergence to equilibrium:

$$\|u-u_\infty\|_\Omega^2 + \|v-v_\infty\|_\Gamma^2 \leq e^{-C_4t}(\|u_0-u_\infty\|_\Omega^2 + \|v_0-v_\infty\|_\Gamma^2).$$

□

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