Fast reaction limit of a volume-surface reaction-diffusion systems towards a heat equation with dynamical boundary conditions

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Abstract

The fast reaction limit of a volume-surface reaction-diffusion system is rigorously investigated. We show that as the reaction rate constant goes to infinity, the original system converges to a heat equation with dynamical boundary condition. As a consequence, a dynamical boundary condition can be interpreted as a fast reaction limit of a volume-surface reaction-diffusion system.

Keywords: volume-surface reaction-diffusion systems, fast reaction limit, dynamical boundary condition.
2010 MSC: 35K57, 35B4, 35F05, 35Q92

1. Introduction and main results

In this paper, we will investigate the behaviour of the following reversible chemical reaction

$$\mathcal{U} \underset{k}{\overset{k}{\rightleftharpoons}} \mathcal{V} \tag{1.1}$$

when the reaction rate constant $k > 0$ tends to infinity, where $\mathcal{U}$ is a volume-substance in Ω and $\mathcal{V}$ is a surface-substance on $\partial\Omega$. Here we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\Gamma := \partial\Omega$ (e.g. $\Gamma \in C^{2+\epsilon}$ for some $\epsilon > 0$).

To set up a mathematical model for the reaction (1.1), we denote by $u(x,t)$ the volume-concentration of $\mathcal{U}$ and by $v(x,t)$ the surface-concentration of $\mathcal{V}$. The linear mass action volume-surface reaction-diffusion system modelling (1.1) reads as

$$\begin{cases} u_t - d_u\Delta u = 0, & x \in \Omega, \quad t > 0, \\ d_u\partial_\nu u = -k(u-v), & x \in \Gamma, \quad t > 0, \\ v_t - d_v\Delta_\Gamma v = k(u-v), & x \in \Gamma, \quad t > 0, \end{cases} \tag{1.2}$$

with initial data $u(x,0) = u_0(x), x \in \Omega$ and $v(x,0) = v_0(x), x \in \Gamma$, where ∂_ν is directional derivative corresponding to the unit outward normal vector ν of Γ, and Δ_Γ denotes the Laplace–Beltrami operator on Γ, which will be specified later. The system (1.2) has the following conservation of the total mass

$$\int_\Omega u(x,t)\,\mathrm{d}x + \int_\Gamma v(x,t)\,\mathrm{d}\sigma = \int_\Omega u_0(x)\,\mathrm{d}x + \int_\Gamma v_0(x)\,\mathrm{d}\sigma \qquad \text{for all} \quad t > 0.$$

Volume-surface reaction-diffusion (VSRD) systems are used to describe many realistic models. For example, they are used in biology to model e.g. the exchange of proteins during stem cell division [1] or signing networks [2, 3], in chemical and physical industries to model surface active agents (surfactants) [4], or also in crystal growth [5].

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On the other hand, fast reaction limits for reaction-diffusion systems have recently attracted a lot of interest. In a reactive system, it frequently happens that some reaction processes happen much faster the other processes (e.g. diffusion processes or convection processes) and thus reach the (reaction) steady state quasi immediately. In such a case, the fast reactions can be eliminated to obtain a reduced system. The fast reaction limits of reactive systems occur commonly in chemical engineering and although applying such approximation has been routinely done by chemical engineers since a long time, the mathematical theory of fast reaction limits is usually missing. Recent however, a lot of mathematical attention has been paid to rigorously prove fast reaction limit approximations (see e.g. [1, 6, 7, 8] and references therein).

In the present paper, we investigate the fast reaction limits $k \to +\infty$ for the VSRD system (1.2). We prove that, as the reaction rate constant $k \to +\infty$, the solutions to (1.2) converge to solutions of a limit equation, which in fact is a heat equation with a particular dynamical boundary condition (see e.g. [9]). As a consequence, the dynamical boundary condition for the heat equation can be interpreted as the fast reaction limit of a volume-surface reaction-diffusion system; see [10] for an alternative derivation. Up to the best of our knowledge, [1] is the only existed result concerning fast reaction limits for VSRD systems.

A problem similar to this work was studied in [7] where the authors proved the fast reaction limit for $\alpha\mathcal{U} \leftrightharpoons \beta\mathcal{V}$ with $\mathcal{U}$ and $\mathcal{V}$ are both volume-concentrations. We remark that because of the volume-surface coupling of (1.2), the technique used in [7] is not applicable here. This difficulty will be resolved in this paper by first applying an energy equation technique to prove the fast reaction convergence in $L^2(0,T;L^2(\Omega)\times L^2(\Gamma))$, then exploiting the result that solutions of (1.2) and (1.3) converge to a common equilibrium to show the fast reaction convergence in $L^2(0,T;H^1(\Omega)\times H^1(\Gamma))$. The main results of this paper are stated in the following theorem.

Theorem 1.1. *Denote by (u^k, v^k) the unique solution to system (1.2) subject to initial data $(u_0, v_0) \in L^2(\Omega)\times L^2(\Gamma)$ and reaction rate constant $k > 0$. Then for $k \to +\infty$ there holds*

$$(u^k, v^k) \to (w, w|_\Gamma) \quad \text{in} \quad L^2(0,T;H^1(\Omega)\times H^1(\Gamma)),$$

where w is the unique weak solution to the following heat equation with dynamical boundary condition

$$\begin{cases} w_t - d_u\Delta w = 0, & x\in\Omega, \quad t>0,\\ d_u\partial_\nu w = -w_t + d_v\Delta_\Gamma w, & x\in\Gamma, \quad t>0,\\ w(x,0) = u_0(x), & x\in\Omega,\\ w|_\Gamma(x,0) = v_0(x), & x\in\Gamma. \end{cases} \tag{1.3}$$

Proof. Here we sketch the proof of the main results based on some essential lemmas which will be proved in the next section.

By Lemma 2.3 we have $(u^k, v^k) \rightharpoonup (w, z)$ weakly in $L^2(0,T;H^1(\Omega)\times H^1(\Gamma))$ with $w|_\Gamma = z$ and w is a weak solution to the limit equation (1.3). The Lemma 2.4 shows the strong convergence in $L^2(0,T;L^2(\Omega)\times L^2(\Gamma))$ of $(u^k, v^k) \to (w, z)$. Finally, by using Lemma 2.9, we obtain that the $(u^k, v^k) \to (w, z)$ strongly in $L^2(0,T;H^1(\Omega)\times H^1(\Gamma))$. That completes the proof of the Theorem.

The remainder of this paper is structured as follows: In Section 2, we prove the main Theorem 1.1 by proving the Lemmas 2.3, 2.4 and 2.9 consecutively. We also briefly discuss a related nonlinear problem in the last Section 3.

2. Proof of Theorem 1.1

For the sake of brevity, throughout this paper, we denote by $\mathcal{L}^2 = L^2(\Omega)\times L^2(\Gamma)$ and $\mathcal{H}^1 = H^1(\Omega)\times H^1(\Gamma)$. The inner product in $\mathcal{L}^2$ is defined by

$$\langle (u,v);(\varphi,\psi)\rangle := \int_\Omega u\varphi\,\mathrm{d}x + \int_\Gamma v\psi\,\mathrm{d}\sigma$$

which deduces the norm $\|(u,v)\|_{\mathcal{L}^2} = \sqrt{\langle (u,v);(u,v)\rangle}$. For $t>0$, we denote by $\Omega_t = \Omega\times[0,t]$, $\Gamma_t = \Gamma\times[0,t]$ and $L^2(\Omega_t) = L^2(0,t;L^2(\Omega))$, $L^2(\Gamma_t) = L^2(0,t;L^2(\Gamma))$.

Since the domain Ω is assumed to be smooth enough, Γ is Riemann manifold without boundary with the natural metric inherited $\mathbb{R}^n$, given in local coordinates by $(g_{ij})_{i,j=1,\dots,n-1}$. Hence we can define the Laplace–Beltrami operator on Γ by

$$\Delta_\Gamma u = g^{-1/2}\sum_{i,j=1}^{n-1}\frac{\partial}{\partial y_i}\left(g^{ij}g^{1/2}\frac{\partial u}{\partial y_j}\right)$$

where $g=\det(g_{ij})$ and $(g^{ij})=(g_{ij})^{-1}$ as usual. Throughout this paper, we will use the following identity

$$\int_\Gamma(-\Delta_\Gamma u)v\,\mathrm{d}\sigma = \int_\Gamma \nabla_\Gamma u\nabla_\Gamma v\,\mathrm{d}\sigma$$

where $\mathrm{d}\sigma$ is the natural volume element on Γ, given in local coordinates by $\sqrt{g}dy_1\dots dy_{n-1}$, and ∇_Γ is the Riemannian gradient. For more details of the Laplace–Beltrami operator, we refer the reader to [11].

2.1. Well Posedness and Limiting System

In this section we provide definitions and existence results for the system (1.2) and the limiting equation (1.3). We also establish the weak convergence of solutions of (1.2) to solutions of (1.3) when $k\to\infty$.

Definition 2.1. *For fixed $T>0$, a pair of functions $(u,v)\in C([0,T];\mathcal{L}^2)\cap L^2(0,T;\mathcal{H}^1)$ is called a weak solution to (1.2) if for all $(\varphi,\psi)\in C^1([0,T];\mathcal{H}^1)$ satisfying $\varphi(T)=\psi(T)=0$ we have*

$$-\int_0^T\langle(u,v);(\varphi_t,\psi_t)\rangle\,\mathrm{d}t+\int_0^T a(u,v;\varphi,\psi)\,\mathrm{d}t=\langle(u_0,v_0);(\varphi(0),\psi(0))\rangle \tag{2.1}$$

where

$$a(u,v;\varphi,\psi)=d_u\int_\Omega\nabla u\nabla\varphi\,\mathrm{d}x+d_v\int_\Gamma\nabla_\Gamma v\nabla_\Gamma\psi\,\mathrm{d}\sigma+k\int_\Gamma(u-v)(\varphi-\psi)\,\mathrm{d}\sigma. \tag{2.2}$$

Proposition 2.1. *For any $(u_0,v_0)\in\mathcal{L}^2$, the system (1.2) possesses a unique weak solution (u,v) in the sense of Definition 2.1.*

Proof. It's easy to show that the bilinear $a\colon\mathcal{H}^1\times\mathcal{H}^1\to\mathbb{R}$ is continuous and satisfies

$$a(u,v;u,v)\geq\alpha\|(u,v)\|^2_{\mathcal{H}^1}-c\|(u,v)\|^2_{\mathcal{L}^2}$$

for some $\alpha,c>0$. The existence of a weak solution (u,v) to (1.2) then follows from standard theory of linear parabolic problems (see e.g. [12, XVIII §3)]).

Similarly, we can show the existence of a unique weak solution to the limit equation (1.3). The proof of the following Lemma is hence omitted.

Proposition 2.2. *For any $(u_0,v_0)\in\mathcal{L}^2$, the equation (1.3) possesses a unique global weak solution w which satisfies*

$$(w,w|_\Gamma)\in C([0,T];\mathcal{L}^2)\cap L^2(0,T;\mathcal{H}^1)$$

and, for all test functions φ with $(\varphi,\varphi|_\Gamma)\in C^1([0,T];\mathcal{H}^1)$ and $\varphi(T)=0$ we have

$$\begin{aligned}-\int_0^T\langle(w,w|_\Gamma);(\varphi_t,\varphi_t|_\Gamma)\rangle\,\mathrm{d}t+d_u\int_0^T\int_\Omega\nabla w\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t&\\+d_v\int_0^T\int_\Gamma\nabla_\Gamma w|_\Gamma\nabla_\Gamma\varphi|_\Gamma\,\mathrm{d}\sigma\,\mathrm{d}t=\langle(u_0,v_0);(\varphi(0),\varphi|_\Gamma(0))\rangle.&\end{aligned} \tag{2.3}$$

Later in this work, we will denote by (u^k, v^k) the unique weak solution to (1.2) corresponding to the reaction rate constant $k > 0$. Thanks to (2.1), we have

$$\|(u^k(t), v^k(t))\|_{\mathcal{L}^2}^2 + 2d_u\|\nabla u^k\|_{L^2(\Omega_t)}^2 + 2d_v\|\nabla_\Gamma v^k\|_{L^2(\Gamma_t)}^2 + 2k\|u^k - v^k\|_{L^2(\Gamma_t)}^2 = \|(u_0, v_0)\|_{\mathcal{L}^2}^2. \tag{2.4}$$

for all $t \in (0, T]$. The relation (2.4) gives us the following important *a priori* estimates

$$\{u^k\}_{k>0} \text{ is bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \tag{2.5}$$

$$\{v^k\}_{k>0} \text{ is bounded in } L^\infty(0, T; L^2(\Gamma)) \cap L^2(0, T; H^1(\Gamma)) \tag{2.6}$$

and

$$\|u^k - v^k\|_{L^2(\Gamma_T)} = O(k^{-1/2}) \text{ as } k \to +\infty. \tag{2.7}$$

Combining (2.5), (2.6) and (2.7) allows us to have the following weak convergence result.

Lemma 2.3. *There exist a function $w \in L^2(0, T; H^1(\Omega))$ and a function $z \in L^2(0, T; H^1(\Gamma))$ such that*

$$u^k \rightharpoonup w \text{ weakly in } L^2(0, T; H^1(\Omega)) \tag{2.8}$$

and

$$v^k \rightharpoonup z \text{ weakly in } L^2(0, T; H^1(\Gamma)) \tag{2.9}$$

as $k \to +\infty$. Moreover, we have $w|_\Gamma = z$ and w is the unique weak solution to the heat equation with dynamical boundary condition (1.3).

Proof. The existence of (w, z) and (2.8) and (2.9) follow from (2.5) and (2.6). We now verify that $w|_\Gamma = z$ and w solves (1.3) in weak sense.

From (2.7) we have, as $k \to +\infty$, $u^k - v^k \to 0$ strongly in $L^2(\Gamma_T)$ thus $u^k - v^k \rightharpoonup 0$ weakly in $L^2(\Gamma_T)$. It follows that, for any $\xi \in L^2(\Gamma_T)$, we have

$$\lim_{k \to +\infty} (u^k - v^k, \xi)_{L^2(\Gamma_T)} = 0. \tag{2.10}$$

Since $v^k \to z$ weakly in $L^2(0, T; H^1(\Gamma))$, we have $(v^k, \xi)_{L^2(\Gamma_T)} \to (z, \xi)_{L^2(\Gamma_T)}$. On the other hand, $u^k \to w$ weakly in $L^2(0, T; H^1(\Omega))$ then thus, $u^k|_\Gamma \to w|_\Gamma$ weakly in $L^2(\Gamma_T)$ thanks to the Trace Theorem. Therefore, $(u^k, \xi)_{L^2(\Gamma_T)} \to (w, \xi)_{L^2(\Gamma_T)}$. Hence, it follows from (2.10) that $(w - z, \xi) = 0$ for all $\xi \in L^2(\Gamma_T)$, which means $w|_\Gamma = z$.

We will show that w is the solution to equation (1.3) subject to initial data (u_0, v_0). By choosing a test function $\varphi \in C^1([0, T]; H^1(\Omega))$ satisfying $\varphi|_\Gamma \in C^1([0, T]; H^1(\Gamma))$ and $\varphi(T) = 0$, it follows from (2.1) that

$$-\int_0^T \langle (u^k, v^k); (\varphi_t, \varphi_t|_\Gamma)\rangle \,\mathrm{d}t + \int_0^T a(u^k, v^k; \varphi, \varphi|_\Gamma) \,\mathrm{d}t = \langle (u_0, v_0); (\varphi(0), \varphi|_\Gamma(0))\rangle$$

or equivalently

$$-\int_0^T \langle (u^k, v^k); (\varphi_t, \varphi_t|_\Gamma)\rangle \,\mathrm{d}t + d_u \int_0^T \int_\Omega \nabla u^k \nabla \varphi \,\mathrm{d}x \,\mathrm{d}t \\ + d_v \int_0^T \int_\Gamma \nabla_\Gamma v^k \nabla_\Gamma \varphi|_\Gamma \,\mathrm{d}\sigma \,\mathrm{d}t = \langle (u_0, v_0); (\varphi(0), \varphi|_\Gamma(0))\rangle. \tag{2.11}$$

Passing to limit in (2.11) as $k \to +\infty$ and recalling that $u^k \rightharpoonup w$ weakly in $L^2(0, T; H^1(\Omega))$ and $v^k \rightharpoonup w|_\Gamma$ weakly in $L^2(0, T; H^1(\Gamma))$, we obtain

$$-\int_0^T \langle (w, w|_\Gamma); (\varphi_t, \varphi_t|_\Gamma)\rangle \,\mathrm{d}t + d_u \int_0^T \int_\Omega \nabla w \nabla \varphi \,\mathrm{d}x \,\mathrm{d}t \\ + d_v \int_0^T \int_\Gamma \nabla_\Gamma w|_\Gamma \nabla_\Gamma \varphi|_\Gamma \,\mathrm{d}\sigma \,\mathrm{d}t = \langle (u_0, v_0); (\varphi(0), \varphi|_\Gamma(0))\rangle.$$

This verifies that w is the weak solution to the equation (1.3) and hence the proof of the lemma is completed.

2.2. Strong convergence in $L^2(0,T;\mathcal{L}^2)$

Lemma 2.4. *For any $T>0$ we have*

$$(u^k, v^k) \to (w,z) \quad \text{strongly in} \quad L^2(0,T;\mathcal{L}^2)$$

as $k \to +\infty$, where $z = w|_\Gamma$.

Proof. First, from Lemma 2.3 we have $u^k \rightharpoonup w$ weakly in $L^2(\Omega_T)$ and $v^k \rightharpoonup z$ weakly in $L^2(\Gamma_T)$, then

$$\liminf_{k\to+\infty} \left(\|u^k\|^2_{L^2(\Omega_T)} + \|v^k\|^2_{L^2(\Gamma_T)} \right) \geq \|w\|^2_{L^2(\Omega_T)} + \|z\|^2_{L^2(\Gamma_T)}$$

thanks to $\liminf(x_n + y_n) \geq \liminf x_n + \liminf y_n$, or equivalently

$$\liminf_{k\to+\infty} \|(u^k, v^k)\|^2_{L^2(0,T;\mathcal{L}^2)} \geq \|(w,z)\|^2_{L^2(0,T;\mathcal{L}^2)}. \tag{2.12}$$

With the help of (2.12), to show $(u^k, v^k) \to (w,z)$ strongly in $L^2(0,T;\mathcal{L}^2)$ we only need to prove

$$\limsup_{k\to+\infty} \|(u^k, v^k)\|^2_{L^2(0,T;\mathcal{L}^2)} \leq \|(w,z)\|^2_{L^2(0,T;\mathcal{L}^2)}. \tag{2.13}$$

We first note that, for any $0 < t \leq T$,

$$\liminf_{k\to+\infty} \|\nabla u^k\|^2_{L^2(\Omega_t)} \geq \|\nabla w\|^2_{L^2(\Omega_t)} \text{ and } \liminf_{k\to+\infty} \|\nabla_\Gamma v^k\|^2_{L^2(\Gamma_t)} \geq \|\nabla_\Gamma z\|^2_{L^2(\Gamma_t)}$$

thanks to the weak convergence (2.8) and (2.9). From (2.4), we have, for all $t \in (0,T]$,

$$\|(u^k(t), v^k(t))\|^2_{\mathcal{L}^2} + 2d_u \|\nabla u^k\|^2_{L^2(\Omega_t)} + 2d_v \|\nabla_\Gamma v^k\|^2_{\Gamma} \leq \|(u_0, v_0)\|^2_{\mathcal{L}^2}.$$

Hence,

$$\begin{aligned}
\limsup_{k\to+\infty} \|(u^k(t), v^k(t))\|^2_{\mathcal{L}^2} &\leq \limsup_{k\to+\infty} \left(-2d_u \|\nabla u^k\|^2_{L^2(\Omega_t)} - 2d_v \|\nabla_\Gamma v^k\|^2_{L^2(\Gamma_t)} \right) + \|(u_0,v_0)\|^2_{\mathcal{L}^2} \\
&\leq -2d_u \liminf_{k\to+\infty} \|\nabla u^k\|^2_{L^2(\Omega_t)} - 2d_v \liminf_{k\to+\infty} \|\nabla_\Gamma v^k\|^2_{L^2(\Gamma_t)} + \|(u_0,v_0)\|^2_{\mathcal{L}^2} \\
&\leq -2d_u \|\nabla w\|^2_{L^2(\Omega_t)} - 2d_v \|\nabla_\Gamma z\|^2_{L^2(\Gamma_t)} + \|(u_0,v_0)\|^2_{\mathcal{L}^2} \\
&= \|(w(t), z(t))\|^2_{\mathcal{L}^2}
\end{aligned} \tag{2.14}$$

for all $t \in (0,T]$. The last equality is due to the fact that w is the solution to (1.3) and $z = w|_\Gamma$. By using Fatou's lemma and (2.14) we have

$$\begin{aligned}
\limsup_{k\to+\infty} \|(u^k, v^k)\|^2_{L^2(0,T;\mathcal{L}^2)} &\leq \int_0^T \limsup_{k\to+\infty} \|(u^k(t), v^k(t))\|^2_{\mathcal{L}^2} \, \mathrm{d}t \\
&\leq \|(w,z)\|^2_{L^2(0,T;\mathcal{L}^2)}
\end{aligned}$$

and therefore obtain the desired inequality (2.13).

2.3. Convergence in $L^2(0,T;\mathcal{H}^1)$

In order to prove that $(u^k, v^k) \to (w,z)$ strongly in $L^2(0,T;\mathcal{H}^1)$, we first show that (1.2) and (1.3) share a common unique equilibrium and both trajectories of (1.2) and (1.3) converge exponentially to this equilibrium as $t \to +\infty$. Then combining these convergences to equilibrium with another energy equation method, we will be able to show that $(|\nabla u^k|, |\nabla_\Gamma v^k|) \to (|\nabla w|, |\nabla_\Gamma z|)$ strongly in $L^2(0,T;\mathcal{L}^2)$, which combined with $(u^k, v^k) \to (w,z)$ strongly in $L^2(0,T;\mathcal{L}^2)$ leads to the strong convergence $(u^k, v^k) \to (w,z)$ in $L^2(0,T;\mathcal{H}^1)$.

2.3.1. Convergence to equilibrium

Lemma 2.5. *Denote by* $M := \int_\Omega u_0(x)\,\mathrm{d}x + \int_\Gamma v_0(x)\,\mathrm{d}\sigma$ *the initial mass. Then* (1.2) *and* (1.3) *obey the conservations of mass*

$$\int_\Omega u^k(x,t)\,\mathrm{d}x + \int_\Gamma v^k(x,t)\,\mathrm{d}\sigma = \int_\Omega w(x,t)\,\mathrm{d}x + \int_\Gamma w(x,t)\,\mathrm{d}\sigma = M$$

for all $t > 0$*. Moreover,* (1.2) *and* (1.3) *possess a common equilibrium* (u_∞, v_∞)*, which solves*

$$\begin{cases} u_\infty = v_\infty \\ |\Omega|u_\infty + |\Gamma|v_\infty = M. \end{cases}$$

Proof. The proof relies on direct computations, so we omit it.

The convergence to equilibrium for (1.2) is the consequence of an entropy-entropy dissipation (EED) estimate between the entropy functional defined by

$$\mathcal{E}(f,g) = \|(f,g)\|_{\mathcal{L}^2}^2$$

and its entropy dissipation defined by

$$\mathcal{D}^k(f,g) = 2d_u\|\nabla f\|_{L^2(\Omega)}^2 + 2d_v\|\nabla_\Gamma g\|_{L^2(\Gamma)}^2 + 2k\|f-g\|_{L^2(\Gamma)}^2$$

where $f:\Omega\to\mathbb{R}$ and $g:\Gamma\to\mathbb{R}$. The following EED estimate is the main tool in showing the convergence to equilibrium.

Lemma 2.6. *For* $k_0 > 0$ *there exists* $\lambda_0 > 0$ *such that for all* $k \geq k_0$ *and all measurable functions* $f\colon \Omega\to\mathbb{R}$ *and* $g\colon\Gamma\to\mathbb{R}$ *satisfying* $\int_\Omega f\,\mathrm{d}x + \int_\Gamma g\,\mathrm{d}\sigma = 0$ *we have*

$$\mathcal{D}^k(f,g) \geq \lambda_0\,\mathcal{E}(f,g). \tag{2.15}$$

Proof. By using the notation $\overline{f} = \frac{1}{|\Omega|}\int_\Omega f(x)\,\mathrm{d}x$ and $\overline{g} = \frac{1}{|\Gamma|}\int_\Gamma g(x)\,\mathrm{d}\sigma$ for the spatial averages of f and g, the assumption $\int_\Omega f\,\mathrm{d}x + \int_\Gamma g\,\mathrm{d}\sigma = 0$ can be rewritten as

$$|\Omega|\overline{f} + |\Gamma|\overline{g} = 0.$$

First, we compute

$$\begin{aligned}\mathcal{E}(f,g) = \|(f,g)\|_{\mathcal{L}^2}^2 &= \|(f-\overline{f}, g-\overline{g})\|_{\mathcal{L}^2}^2 + \|(\overline{f},\overline{g})\|_{\mathcal{L}^2}^2 \\ &= \mathcal{E}(f-\overline{f}, g-\overline{g}) + \mathcal{E}(\overline{f},\overline{g}).\end{aligned} \tag{2.16}$$

By using the Poincaré inequalities

$$\|\nabla f\|_{L^2(\Omega)}^2 \geq C_P(\Omega)\|f-\overline{f}\|_{L^2(\Omega)}^2 \quad \text{and} \quad \|\nabla_\Gamma g\|_{L^2(\Gamma)}^2 \geq C_P(\Gamma)\|g-\overline{g}\|_{L^2(\Gamma)}^2$$

for functions in $H^1(\Omega)$ and $H^1(\Gamma)$, respectively, we have

$$\begin{aligned}\frac{1}{2}\mathcal{D}^k(f,g) &\geq d_u\|\nabla f\|_{L^2(\Omega)}^2 + d_v\|\nabla_\Gamma g\|_{L^2(\Gamma)}^2 \\ &\geq d_u C_P(\Omega)\|f-\overline{f}\|_{L^2(\Omega)}^2 + d_v C_P(\Gamma)\|g-\overline{g}\|_{L^2(\Gamma)}^2 \\ &\geq \min\{d_u C_P(\Omega), d_v C_P(\Gamma)\}\mathcal{E}(f-\overline{f}, g-\overline{g}).\end{aligned} \tag{2.17}$$

On the other hand, by using the trace inequality $\|\nabla f\|_{L^2(\Omega)}^2 \geq C_T(\Gamma)\|f-\overline{f}\|_{L^2(\Gamma)}^2$ for functions in $H^1(\Omega)$, we get

$$\begin{aligned}\frac{1}{2}\mathcal{D}^k(f,g) &\geq d_u C_T(\Gamma)\|f-\overline{f}\|_{L^2(\Gamma)}^2 + 2k\|f-g\|_{L^2(\Gamma)}^2 \\ &\geq \frac{1}{2}\min\{d_u C_T(\Gamma), 2k_0\}\|\overline{f}-g\|_{L^2(\Gamma)}^2 \\ &\geq \frac{1}{2}\min\{d_u C_T(\Gamma), 2k_0\}\left(\|\overline{f}-\overline{g}\|_{L^2(\Gamma)}^2 + \|\overline{g}-g\|_{L^2(\Gamma)}^2\right) \\ &\geq \frac{1}{2}\min\{d_u C_T(\Gamma), 2k_0\}\|\overline{f}-\overline{g}\|_{L^2(\Gamma)}^2 \\ &= \frac{1}{2}\min\{d_u C_T(\Gamma), 2k_0\}\left(\frac{|\Gamma|}{|\Omega|}+1\right)\mathcal{E}(\overline{f},\overline{g})\end{aligned} \tag{2.18}$$

where we used $|\Omega|\overline{f}+|\Gamma|\overline{g}=0$ for the last equality. Now, by combining (2.16), (2.17) and (2.18) it yields (2.15) with

$$\lambda_0=\min\{d_uC_P(\Omega),d_vC_P(\Gamma),1/2d_uC_T(\Gamma),k_0\}.$$

Proposition 2.7. *The solution (u^k,v^k) to (1.2) obeys the following convergence to equilibrium*

$$\|(u^k(t),v^k(t))-(u_\infty,v_\infty)\|^2_{\mathcal{L}^2}\le \mathrm{e}^{-\lambda_0 t}\|(u_0,v_0)-(u_\infty,v_\infty)\|^2_{\mathcal{L}^2} \tag{2.19}$$

for all $t>0$, where $k\ge k_0>0$ and λ_0 is independent of t and k.

Proof. By defining $f=u^k-u_\infty$ and $g=v^k-v_\infty$, we have

$$\begin{cases}\partial_t f-d_u\Delta f=0, & x\in\Omega,\quad t>0,\\ d_u\partial_\nu f=-k(f-g), & x\in\Gamma,\quad t>0,\\ \partial_t g-d_v\Delta_\Gamma g=k(f-g), & x\in\Gamma,\quad t>0,\end{cases}$$

with initial data $f(x,0)=u_0(x)-u_\infty$ and $g(x,0)=v_0(x)-v_\infty$. We calculate that f and g satisfies the mass conservation, for all $t>0$,

$$|\Omega|\overline{f}(t)+|\Gamma|\overline{g}(t)=|\Omega|\overline{f}(0)+|\Gamma|\overline{g}(0)=0.$$

Therefore, we can apply Lemma 2.6 to have

$$\mathcal{D}^k(f,g)\ge\lambda_0\,\mathcal{E}(f,g).$$

Note that, after direct computations, we have $\mathcal{D}^k(f,g)=-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(f,g)$. It then follows from Gronwall's lemma that

$$\mathcal{E}(f,g)(t)\le \mathrm{e}^{-\lambda_0 t}\mathcal{E}(f(0),g(0))\qquad\text{for all } t>0. \tag{2.20}$$

The proof of the lemma is complete since (2.20) is equivalent to (2.19).

Proposition 2.8. *The solution u to (1.3) satisfies the following convergence to equilibrium*

$$\|w(t)-u_\infty\|^2_\Omega+\|w|_\Gamma(t)-v_\infty\|^2_\Gamma\le \mathrm{e}^{-\lambda_1 t}(\|u_0-u_\infty\|^2_\Omega+\|v_0-v_\infty\|^2_\Gamma) \tag{2.21}$$

for all $t>0$ where λ_1 is independent of t.

Proof. The proof is similar to Lemma 2.7 with slight modifications so we omit it here.

2.3.2. Strong convergence in $L^2(0,T;\mathcal{H}^1)$

Lemma 2.9. *As $k\to+\infty$ we have*

$$(u^k,v^k)\to(w,z)\quad\textit{strongly in}\quad L^2(0,T;\mathcal{H}^1)$$

for all $T>0$, where $z=w|_\Gamma$.

Proof. We will first prove that

$$(|\nabla u^k|,|\nabla_\Gamma v^k|)\to(|\nabla w|,|\nabla_\Gamma z|)\quad\text{in}\quad L^2([0,+\infty);\mathcal{L}^2).$$

From the energy equation (2.4), we have

$$\|(u^k(t),v^k(t))\|^2_{\mathcal{L}^2}+2d_u\|\nabla u^k\|^2_{L^2(\Omega_t)}+2d_v\|\nabla_\Gamma v^k\|^2_{L^2(\Gamma_t)}\le\|(u_0,v_0)\|^2_{\mathcal{L}^2}. \tag{2.22}$$

From the limit equation (1.3), we have

$$\|(w(t),z(t))\|^2_{\mathcal{L}^2}+2d_u\|\nabla w\|^2_{L^2(\Omega_t)}+2d_v\|\nabla_\Gamma z\|^2_{L^2(\Gamma_t)}=\|(u_0,v_0)\|^2_{\mathcal{L}^2}. \tag{2.23}$$

Combining (2.22) and (2.23) yields

$$\begin{aligned}&\|(u^k(t),v^k(t))\|_{\mathcal{L}^2}^2+2d_u\|\nabla u^k\|_{L^2(\Omega_t)}^2+2d_v\|\nabla_\Gamma v^k\|_{L^2(\Gamma_t)}^2\\&\le\|(w(t),z(t))\|_{\mathcal{L}^2}^2+2d_u\|\nabla w\|_{L^2(\Omega_t)}^2+2d_v\|\nabla_\Gamma z\|_{L^2(\Gamma_t)}^2.\end{aligned}\tag{2.24}$$

Letting $k\to+\infty$ and $t\to+\infty$ in (2.24), and using $(u^k(t),v^k(t))\to(u_\infty,v_\infty)$ independently of k and $(w(t),z(t))\to(u_\infty,v_\infty)$ as $t\to+\infty$ we get

$$\begin{aligned}&\limsup_{k\to+\infty}\left(2d_u\|\nabla u^k\|_{L^2([0,+\infty)\times\Omega)}^2+2d_v\|\nabla_\Gamma v^k\|_{L^2([0,+\infty)\times\Gamma)}^2\right)\\&\qquad\le 2d_u\|\nabla w\|_{L^2([0,+\infty)\times\Omega)}^2+2d_v\|\nabla_\Gamma z\|_{L^2([0,+\infty)\times\Gamma)}^2\end{aligned}\tag{2.25}$$

On the other hand, (2.22) implies that

$$\{(|\nabla u^k|,|\nabla_\Gamma v^k|)\}\text{ is bounded in }L^2([0,+\infty);\mathcal{L}^2),$$

thus

$$(|\nabla u^k|,|\nabla_\Gamma v^k|)\rightharpoonup(|\nabla w|,|\nabla_\Gamma z|)\quad\text{weakly in }L^2([0,+\infty);\mathcal{L}^2).\tag{2.26}$$

Therefore, we have

$$\liminf_{k\to+\infty}\|(|\nabla u^k|,|\nabla_\Gamma v^k|)\|_{L^2([0,+\infty);\mathcal{L}^2)}^2\ge\|(|\nabla w|,|\nabla_\Gamma z|)\|_{L^2([0,+\infty);\mathcal{L}^2)}^2,$$

which, combined with (2.25), infers that

$$\lim_{k\to+\infty}\|(|\nabla u^k|,|\nabla_\Gamma v^k|)\|_{L^2([0,+\infty);\mathcal{L}^2)}^2=\|(|\nabla w|,|\nabla_\Gamma z|)\|_{L^2([0,+\infty);\mathcal{L}^2)}^2.\tag{2.27}$$

From (2.26) and (2.27), we get

$$(|\nabla u^k|,|\nabla_\Gamma v^k|)\to(|\nabla w|,|\nabla_\Gamma z|)\quad\text{strongly in }L^2([0,+\infty);\mathcal{L}^2).$$

This strong convergence, together with $(u^k,v^k)\to(w,z)$ strongly in $L^2(0,T;\mathcal{L}^2)$, gives us the main result

$$(u^k,v^k)\to(w,z)\quad\text{strongly in }L^2(0,T;\mathcal{H}^1)$$

for all $T>0$.

3. Discussion for a Nonlinear Problem

As a final remark to this paper, we consider the reversible reaction

$$\alpha\mathcal{U}\underset{k}{\overset{k}{\rightleftharpoons}}\beta\mathcal{V}$$

where $\mathcal{U}$ and $\mathcal{V}$ are volume- and surface-concentrations respectively, and the stoichiometric coefficients α,β are positive. The system considered in the present paper is thus a special case of this reaction when $\alpha=\beta=1$. By applying the mass action law, this reaction results the following nonlinear VSRD system

$$\begin{cases}u_t-d_u\Delta u=0, & x\in\Omega,\quad t>0,\\ d_u\partial_\nu u=-\alpha k(u^\alpha-v^\beta), & x\in\Gamma,\quad t>0,\\ v_t-d_v\Delta_\Gamma v=\beta k(u^\alpha-v^\beta), & x\in\Gamma,\quad t>0,\end{cases}\tag{3.1}$$

with suitable initial data. This system was proved to have a global weak solution which converges exponentially to equilibrium in [13].

Let the reaction rate constant k tend to infinity, it is expected at least formally, that $(u^k, v^k) \to (w, w|_\Gamma)$ where w is a solution to the following *heat equation with nonlinear dynamical boundary condition*

$$\begin{cases} w_t - d_u \Delta w = 0, & x \in \Omega, \quad t > 0, \\ d_u \partial_\nu w = \frac{\alpha}{\beta}\left(-(w^{\alpha/\beta})_t + d_v \Delta_\Gamma (w^{\alpha/\beta})\right), & x \in \Gamma, \quad t > 0, \\ w(x,0) = u_0(x), & x \in \Omega, \\ w|_\Gamma(x,0) = v_0(x)^{\alpha/\beta}, & x \in \Gamma. \end{cases}$$

The analysis of this problem is much more involved compared to the linear case. The existence of solution to the limit equation with $\beta \neq \alpha$, up to the best of our knowledge, has not been shown in literature. The fast reaction limit problem for (3.1) remains as an interesting open problem.

Acknowledgements. The authors are supported by International Research Training Group IGDK 1754. The authors thank Prof. Fellner for his helpful comments to improve the paper.

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