Eigenvalue Optimization for the Transmission Problem

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1 Introduction

In this paper we consider the following shape optimization problem:

$$\min j(q) = \lambda_1(q) - \lambda_2(q) + \frac{\alpha}{2} \|q\|_{H^2(I)}^2,$$

where $\lambda_1(q)$ and $\lambda_2(q)$ are the two smallest eigenvalues of a linear partial differential operator corresponding to a transmission problem over a domain Ω_q which is parametrized via the function q. A precise formulation including a functional analytic setting is presented in Section 2.

As the eigenfunctions and eigenvalues of a partial differential operator depend on the shape of the underlying domain, it is possible to optimize functionals depending on the eigenvalues with respect to the shape of the domain. Marc Kac once asked whether it is possible to hear the shape of a drum, cf. [26]: Given all the eigenvalues of the Laplacian with homogeneous Dirichlet boundary conditions over some domain, is it possible to reconstruct the domain? Although the set of eigenvalues contains information such as the area or the diameter of the domain, the original question itself has a negative answer as has been shown in [17].

Some other questions concerning the eigenvalues are the question which domain minimizes the n-th smallest eigenvector among all domains in $\mathbb{R}^2$ with a given volume. The well-known Faber-Krahn inequality states that the unique minimizer (up to sets of capacity zero) for $n = 1$ is the ball. As proven by Krahn and Szegö, cf. [28] and [35], for $n = 2$ the solution consists of two balls of the same volume. For general $n \geq 3$, the optimal domain is not known so far, cf. [23]. Furthermore, if additional constraints like connectedness or even convexity are imposed on the admissible domains, then little is known so far. Some numerical approximations to some of these domains can be found in [4] and [34].

In most cases homogeneous Dirichlet boundary conditions are being investigated, the cases of Neumann and Robin boundary conditions are less extensively studied, for an overview we refer to [16].

Beside the Laplacian there is ongoing research concerning the eigenvalues of Schrödinger's operator, where the eigenfunctions have a physical interpretation as energy levels of quantum particles; but beside this physical meaning there are also some mathematical questions interesting on its own. An overview, including further references, can be found in [24]. In the context of nonlinear equations we would like to mention the p-Laplacian, cf. [31].

The choice of the cost functional may be motivated as follows. It is well-known that in the case of a sufficiently smooth domain, the eigenvalues with multiplicity one are Fréchet-differentiable with

respect to smooth domain perturbations, whereas eigenvalues with a higher multiplicity are only Gâteaux-differentiable, cf. [22]. This irregularity is also responsible for some physical effects. In the context of musical instruments, for example, it is possible to hear some undesired interferences if some of the lower eigenvalues are too close to each other. For a more detailed investigation onto that topic we refer to [14].

In order to solve the problem we use domains which can be parametrized as the graph of a function, cf. [20, 21, 29, 37]. We use a transformation approach, cf. [27], to formulate the partial differential equation on a fixed reference domain. This allows for the usage of the standard control theoretic approach as presented in [40] to prove existence of an optimal solution.

The main contributions of this paper are the following. First, within Subsection 2.5 we prove some general regularity results for the transmission problem. These results just rely on the regularity of the domain and the right hand side of the equation and are not restricted to eigenvalue equations. Second, we also show how to apply the transformation approach onto this transmission problem and how to prove the existence of an optimal solution. At third, within Subsection 3.2 we prove some stability estimates for the derivative of eigenfunctions with respect to domain perturbations. These results may be used for a-priori error estimation for finite-element discretizations of the transformed problem. To the best of our knowledge, such results are not yet considered in the literature.

The paper is organized as follows.

In Section 2 we give a detailed introduction on the problem under consideration, transform the problem onto a reference domain and prove the existence of an optimal solution. We prove general regularity and differentiability results and show that, given sufficient regularity, the first derivative of the reduced cost functional can be represented as a boundary integral. In addition, we show that the optimal control and the associated eigenfunctions possess some higher regularity. We finish that section with the computation of the second derivatives of the eigenvalue and eigenfunctions.

In Section 3 we first review some known results on the stability of eigenfunctions with respect to the domain and then prove some new results on the stability of the derivative of the eigenfunction with respect to domain perturbations.

In Section 4 we finally present a possible finite-element discretization of the transformed problem where we discretize the control, the state, the transformation as well as the domain.

The following notation will be used within this paper. For $p \in [1, \infty]$, $k, n \in \mathbb{N}$ and $\Omega \in \mathbb{R}^n$ let $L^p(\Omega)$ and $W^{k,p}(\Omega)$ denote the usual Lebesgue spaces with norm $\|\cdot\|_{W^{k,p}(\Omega)}$. For $s \notin \mathbb{N}$, $s = k + \sigma$ with $k = \lfloor s \rfloor \in \mathbb{N}_0$, $\sigma \in (0, 1)$ and $p \in (1, \infty)$, let $W^{s,p}(\Omega)$ be the space of all functions $u \in W^{k,p}(\Omega)$ with

$$\|u\|^p_{W^{s,p}(\Omega)} = \|u\|^p_{W^{k,p}(\Omega)} + |u|^p_{W^{s,p}(\Omega)} < \infty,$$

where

$$|u|^p_{W^{s,p}(\Omega)} = \sum_{|\alpha|=k} \left(\int_\Omega \int_\Omega \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x - y|^{n+\sigma p}} \, \mathrm{d}x \, \mathrm{d}y \right).$$

For an arbitrary domain $\Omega \subset \mathbb{R}^n$, let $(\cdot, \cdot)_\Omega$ denote the L^2-scalar product over Ω. If the domain is clear, we may skip the subindex.

2 The problem

2.1 Problem formulation

In this section we first describe the shape optimization problem under consideration, the exact definition of the cost functional to be minimized will be given in (10). Let $q \in Q = H^2_{\mathrm{per}}(I)$ with $I = (0, 2\pi)$ be the control variable. The exact definition of the control space is given as

$$H^2_{\mathrm{per}} = \overline{C^\infty_{\mathrm{per}}(I)}^{\|\cdot\|_{H^2(I)}},$$

equipped with the standard H^2-norm, where

$$C^\infty_{\mathrm{per}} = \left\{ v \in C^\infty(I) \,\middle|\, v^{(n)}(0) = v^{(n)}(2\pi) \,\forall n \in \mathbb{N}_0 \right\}.$$

The domain under consideration is now defined as

$$\Omega_q = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, -2 < x, y < 2 \right\} \subset \mathbb{R}^2,$$

i.e. Ω_q is the interior of a square with side length 4, centered at the origin and sides parallel to the axes. We divide Ω_q into an inner, star-shaped domain,

$$\Omega_{q,0} = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, r < 1 + q(\varphi), \, r = \sqrt{x^2 + y^2}, \, \varphi = \arg(x + iy) \right\},$$

and an outer domain,

$$\Omega_{q,1} = \Omega_q \backslash \overline{\Omega_{q,0}},$$

see Figure 1. In order to exclude a possible degeneracy of the domain $\Omega_{q,0}$ we fix $\overline{\varepsilon} > 0$ and define

$$\overline{Q}^{\mathrm{ad}} = \left\{ q \in Q \,\middle|\, q(\varphi) \geq -1 + \overline{\varepsilon} \,\forall \varphi \in I \text{ and } \overline{\Omega_{q,0}} \subset \Omega_q \right\}. \tag{1}$$

As $H^2(I) \hookrightarrow C^{1,1/2}(\overline{I})$, (1) is well-defined. Now let $d > 0$ be a constant which shall remain fixed
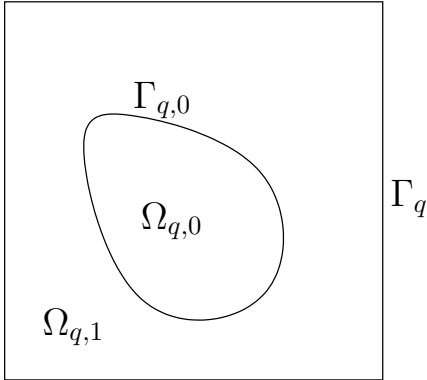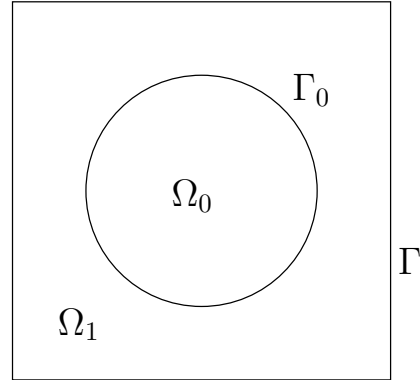

Figure 1: The original domain Ω_q

Figure 2: The transformed domain Ω

throughout this paper and let $\tilde{L} \colon H^1_0(\Omega_q) \to H^{-1}(\Omega_q)$ be the partial differential operator such that

for $u \in H_0^1(\Omega_q)$ and $f \in H^{-1}(\Omega_q)$ the equation $\tilde{L}u = f$ shall be a formulation for u being the unique weak solution to

$$\begin{cases} -d\,\Delta u = f & \text{in } \Omega_{q,0}, & -\Delta u = f & \text{in } \Omega_{q,1}, \\ [u]_q = 0 & \text{on } \Gamma_{q,0} = \partial\Omega_{q,0}, & u = 0 & \text{on } \Gamma_q = \partial\Omega_q, \\ d\,\partial_n u_{q,-} = \partial_n u_{q,+} & \text{on } \Gamma_{q,0}, \end{cases} \tag{2}$$

where $[u]_q$ is defined as follows. For $x \in \Gamma_{q,0}$ let

$$u_{q,+}(x) = \lim_{\substack{y \to x \\ y \in \Omega_{q,1}}} u(y), \qquad\qquad u_{q,-}(x) = \lim_{\substack{y \to x \\ y \in \Omega_{q,0}}} u(y), \tag{3}$$

be the function values when approaching $\Gamma_{q,0}$ from either $\Omega_{q,1}$ or $\Omega_{q,0}$ in a nontangential way, cf. [13], and let

$$[u]_q = u_{q,+} - u_{q,-},$$

be the jump of u over $\Gamma_{q,0}$. It can easily be derived that the weak formulation of (2) reads as

$$(\nabla u, \nabla v)_{\Omega_{q,1}} + d\,(\nabla u, \nabla v)_{\Omega_{q,0}} = (f, v)_{\Omega_q} \qquad\qquad \forall v \in H_0^1(\Omega_q). \tag{4}$$

As $H_0^1(\Omega_q)$ is compactly embedded into $L^2(\Omega_q)$, it follows that $\tilde{L}^{-1}$ is a compact and self-adjoint operator on $L^2(\Omega_q)$. Hence, for each fixed $q \in \overline{Q}^{\text{ad}}$, the spectral theorem yields the existence of a sequence $(\lambda_i)_{i \in \mathbb{N}} \subset \mathbb{R}^+$ with $0 < \lambda_1 \le \lambda_2 \le \ldots$ (counted with multiplicity) and

$$\lim_{i \to \infty} \lambda_i = \infty,$$

and a sequence of eigenfunctions $(u_i)_{i \in \mathbb{N}} \subset H_0^1(\Omega_q)$ with

$$\tilde{L}u_i = \lambda_i u_i \qquad\qquad \forall i \in \mathbb{N}, \tag{5}$$

and eigenfunctions to different eigenvalues are orthogonal with respect to the L^2-scalar product. In order to compute the i-th eigenvalue for general $i \in \mathbb{N}$ one may use the following lemma, a proof can be found in the survey article [7], Chapter 7.

Lemma 2.1. *Let V and H be two real Hilbert spaces with dense and continuous embedding $V \hookrightarrow H$. Let $a\colon V \times V \to \mathbb{R}$ and $b\colon H \times H \to \mathbb{R}$ be two symmetric and continuous bilinear forms. Let $a(\cdot, \cdot)$ be V-elliptic, i.e. there exists $\alpha > 0$ such that $a(v, v) \ge \alpha \|v\|_V^2$ for all $v \in V$, and let $b(\cdot, \cdot)$ define a scalar product on H. For $i \in \mathbb{N}$, let $V^{(i)}$ denote the set of all subspaces of V of dimension i. Then the i-th eigenvalue corresponding to the equation*

$$a(u_i, v) = \lambda_i\, b(u_i, v) \qquad\qquad \forall v \in V,$$

is given via

$$\lambda_i = \min_{E \in V^{(i)}} \max_{v \in E} \frac{a(v, v)}{b(v, v)}, \tag{6}$$

where the minimum with respect to the subspace is attained for E being the subspace spanned by the first i eigenfunctions, and the maximum with respect to the element of that subspace is attained for v being an eigenfunction to λ_i.

Now let
$$\mu_q = 1 + (d-1)\chi_{\Omega_{q,0}},$$
with $\chi_{\Omega_{q,0}}$ being the characteristic function of $\Omega_{q,0}$, and
$$a_q(u,v) = (\nabla u, \mu_q \nabla v)_{\Omega_q}, \tag{7}$$
$$b_q(u,v) = (u,v)_{\Omega_q}. \tag{8}$$
Then the weak formulation of (5), including a normalizing condition for u_i, reads as
$$\begin{cases} a_q(u_i, v) = \lambda_i \, b_q(u_i, v) & \forall v \in H_0^1(\Omega_q), \\ b_q(u_i, u_i) = 1. \end{cases} \tag{9}$$
From now on we consider the variational problem. The problem under consideration is now given via
$$\min_{q \in \overline{Q}^{\mathrm{ad}}} j(q) = \lambda_1(q) - \lambda_2(q) + \frac{\alpha}{2} \|q\|_{H^2(I)}^2, \tag{10}$$
subject to (6) and (9), where $\alpha > 0$ is a given constant.

Remark 2.2. With $\lambda_i(q)$ for $i \in \mathbb{N}$ and $q \in Q^{\mathrm{ad}}$ we will always denote the i-th eigenvalue for a given control q, which can be computed via (6).

In order to prove the existence of a solution to (10) we will at first show that j is uniformly bounded from below. This follows from the fact that $\lambda_2(q)$ is uniformly bounded from above for $q \in \overline{Q}^{\mathrm{ad}}$, which is a direct consequenceof the following lemma.

Lemma 2.3. *Let $i \in \mathbb{N}$, then there exists $c = c(i) > 0$ such that $\lambda_i(q) \in (0, c]$ for all $q \in \overline{Q}^{\mathrm{ad}}$.*

Proof. As all the eigenvalues are known to be positive we just have to prove the upper bound. Let $\tilde{\lambda}_i$ denote the i-th eigenvalue for the Laplacian on Ω_q which does not depend on q. Using Lemma 2.1 it follows that
$$\begin{aligned} \lambda_i(q) &= \min_{E \in V^{(i)}} \max_{v \in E} \frac{(\nabla v, \mu_q \nabla v)}{(v,v)} \\ &\leq \max\{1, d\} \min_{E \in V^{(i)}} \max_{v \in E} \frac{(\nabla v, \nabla v)}{(v,v)} \\ &= \max\{1, d\} \, \tilde{\lambda}_i. \end{aligned} \qquad \square$$

Remark 2.4. The eigenvalues for the Laplacian on a rectangle can be computed exactly. For example, on the square $D = (-2,2) \times (-2,2)$ the set of eigenvalues is given by $\left\{ \frac{\pi^2}{16}\left(m^2 + n^2\right) \,\middle|\, m,n \in \mathbb{N} \right\}$.

Lemma 2.3 ensures that
$$\lim_{\|q\|_{H^2(I)} \to \infty} j(q) = \infty.$$
It follows that there exists $\tilde{C} = \tilde{C}(\alpha)$ such that we can restrict the search for a minimum onto the set
$$Q^{\mathrm{ad}} = \left\{ q \in Q \,\middle|\, \|q\|_{H^2(I)} \leq \tilde{C} \right\}. \tag{11}$$
As within the beginning of Subsection 2.2 we have to assume that the constant $\tilde{C}$ is sufficiently small, it is reasonable to assume that $Q^{\mathrm{ad}} \subset \overline{Q}^{\mathrm{ad}}$, i.e. the elements of Q^{ad} are not degenerated in the sense of (1). Before we continue in proving the existence of a solution to (10) we will apply a transformation argument.

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2.2 Transformation of the problem

In order to solve (10) we will use a transformation T_F to transform the equation (9) onto a partitioned reference domain, see Figure 2. Let $\Omega = \Omega_q$, let Ω_0 be the open unit circle centered at the origin and let $\Omega_1 = \Omega \backslash \overline{\Omega_0}$. Let F be the weak solution to

$$
\begin{cases}
-\Delta F = 0 & \text{in } \Omega_j,\ j \in \{0,1\}, \\
F = 0 & \text{on } \Gamma = \partial\Omega, \\
F = q\,n & \text{on } \Gamma_0 = \partial\Omega_0,
\end{cases}
\tag{12}
$$

where n shall always denote the outer unit normal with respect to Ω_0. Let $T_F = \mathrm{Id} + F$ be the transformation, it now holds that

$$
\Omega_{q,j} = T_F(\Omega_j),
$$

for $j \in \{0,1\}$. It can be shown that for $\tilde{C}$ from (11) sufficiently small, T_F is a bijection from Ω_q onto Ω for all $q \in Q^{\mathrm{ad}}$.

Remark 2.5. With $F(q)$ we will always denote the solution to (12) for a given control $q \in Q^{\mathrm{ad}}$.

Lemma 2.6. *Let $q,p \in Q^{\mathrm{ad}}$ with corresponding transformations F and E, respectively. Then it holds that*

$$
F_0 = F|_{\Omega_0} \in H^{5/2}(\Omega_0) \hookrightarrow W^{2,4}(\Omega_0) \hookrightarrow C^{1,1/2}(\overline{\Omega_0}),
\tag{13}
$$

$$
F_1 = F|_{\Omega_1} \in W^{2,4}(\Omega_1) \hookrightarrow C^{1,1/2}(\overline{\Omega_1}),
\tag{14}
$$

$$
F \in W^{1,\infty}(\Omega) = C^{0,1}(\overline{\Omega}), \text{ and } \|F\|_{W^{1,\infty}(\Omega)} \le c_\varepsilon \|q\|_{H^{3/2+\varepsilon}(I)},
\tag{15}
$$

$$
\|F - E\|_{W^{1,\infty}(\Omega)} \le c_\varepsilon \|q - p\|_{H^{3/2+\varepsilon}(I)}.
\tag{16}
$$

Proof. The regularity results for F_0 and F_1, (13) and (14), follow with [19], Theorem 9.1.20, [18] and embedding theorems. As F_0 and F_1 are both continuous and coincide on the boundary Γ_0 due to (12) it follows that F is continuous on Ω, its regularity can be seen as follows. Let $x,y \in \Omega$. If either $x,y \in \Omega_0$ or $x,y \in \Omega_1$, then the regularity result within (15) follows from the regularity of F_0 and F_1. Now, without loss of generality, let $x \in \Omega_0$, $y \in \Omega_1$ and let $z \in \Gamma_0$ be the intersection of the line segment $\overline{xy}$ with Γ_0, $|x - y| = |x - z| + |z - y|$. In addition, let L_0 and L_1 be the Lipschitz constants of F_0 and F_1, respectively. Then it holds that

$$
\begin{aligned}
|F(x) - F(y)| &\le |F(x) - F(z)| + |F(z) - F(y)| \\
&\le L_0 |x - z| + L_1 |z - y| \\
&\le \max\{L_0, L_1\}\,(|x - z| + |z - y|) \\
&= \max\{L_0, L_1\}\,|x - y|.
\end{aligned}
$$

From the regularity results cited above it follows that $q \in C^{1,\varepsilon}(I)$ is sufficient for F_0 and F_1 to be Lipschitz, and L_0 and L_1 continuously depend on $\|q\|_{C^{1,\varepsilon}(I)}$. Because of $H^{3/2+\varepsilon}(I) \hookrightarrow C^{1,\varepsilon}(\overline{I})$ we end up with

$$
|F(x) - F(y)| \le c_\varepsilon \|q\|_{H^{3/2+\varepsilon}(I)}\,|x - y|,
$$

which proves (15). The last assertion, (16), follows with (15) and the fact that $q \mapsto F(q)$ is linear. $\quad\square$

For given $q \in Q^{\mathrm{ad}}$ with $F = F(q)$ and transformation T_F it is now possible to transform (9) onto the reference domain, which then reads as

$$\begin{cases} (\nabla u_i(q), \mu A_F \cdot \nabla v) = \lambda_i(q)\,(u_i(q), v\gamma_F) & \forall v \in H_0^1(\Omega), \\ \quad (u_i(q), u_i(q)\gamma_F) = 1, \end{cases} \tag{17}$$

with $\mu = 1 + (d-1)\chi_{\Omega_0}$, where χ_{Ω_0} is the characteristic function of Ω_0,

$$\begin{aligned} \gamma_F &= \det\left(\mathrm{D}T_F\right), \\ A_F &= \mathrm{D}T_F^{-1} \cdot \mathrm{D}T_F^{-T}\gamma_F. \end{aligned} \tag{18}$$

In what follows we will use the following abbreviations,

$$a(F)(u, v) = (\nabla u, \mu A_F \cdot \nabla v), \tag{19}$$

$$b(F)(u, v) = (u, v\gamma_F), \tag{20}$$

such that (17) can be rewritten as

$$\begin{cases} a(F)(u_i(q), v) = \lambda_i(q)\, b(F)(u_i(q), v) & \forall v \in H_0^1(\Omega), \\ b(F)(u_i(q), u_i(q)) = 1. \end{cases}$$

Remark 2.7. Let $u_i(q)$ denote the i-th eigenfunction for given $q \in Q^{\mathrm{ad}}$ and $i \in \mathbb{N}$, which can be computed via (17).

The transformed problem now reads as

$$\min_{q \in Q^{\mathrm{ad}}} j(q) = \lambda_1(q) - \lambda_2(q) + \frac{\alpha}{2}\|q\|_{H^2(I)}^2, \tag{21}$$

subject to (12), (6) and (17).

2.3 On the existence of eigenfunctions

Although the existence of real eigenvalues and eigenfunctions for the original equation (2) is well-known, here we give rigorous proofs for their existence in the transformed setting (17).

Definition 2.8. For given $q \in Q^{\mathrm{ad}}$ and $F = F(q)$, let $L = L_q \colon H_0^1(\Omega) \to H^{-1}(\Omega)$ be the differential operator related to the bilinear form (19),

$$Lu = -\operatorname{div}(\mu A_F \cdot \nabla u).$$

Furthermore, let $L^{-1} \colon H^{-1}(\Omega) \to H_0^1(\Omega)$ be the inverse of L with respect to the scalar product induced by the bilinear form $b(F)(\cdot, \cdot)$, i.e. $u = L^{-1}f$ is defined as the unique solution to

$$(\nabla u, \mu A_F \cdot \nabla v) = (f, v\gamma_F)_{H^{-1}, H_0^1} \qquad\qquad \forall v \in H_0^1(\Omega). \tag{22}$$

Lemma 2.9. *Let $q \in Q^{\mathrm{ad}}$, then the operator L^{-1} from Definition 2.8 is compact from $L^2(\Omega)$ onto $H_0^1(\Omega)$.*

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Proof. Let $\tilde{L}^{-1}$ be the solution operator for (2). From [36], Theorem 5 and Remark 5.1, it follows that $\tilde{L}^{-1}$ maps $L^2(\Omega)$ onto $H_0^{3/2-\varepsilon}(\Omega)$ for $\varepsilon > 0$. As $H_0^{3/2-\varepsilon}(\Omega)$ is compactly embedded into $H_0^1(\Omega)$ for $\varepsilon < 1/2$, it follows that $\tilde{L}^{-1}$ is compact from $L^2(\Omega)$ onto $H_0^1(\Omega)$. As $L^{-1}(f) = \left(\tilde{L}^{-1}(f \circ T_F^{-1})\right) \circ T_F$ is the concatenation of linear and compact operators, the result follows. $\qquad\square$

The operator L^{-1} is selfadjoint and compact over $H_0^1(\Omega)$ due to Lemma 2.9, and as the matrix μA_F is uniformly elliptic for $q \in Q^{\mathrm{ad}}$ it follows with the spectral theorem that there exists a sequence of eigenvalues $(\nu_i)_{i \in \mathbb{N}} \subset \mathbb{R}_0^+$ with 0 as only limit point, and a sequence of eigenfunctions $(u_i)_{i \in \mathbb{N}} \subset H_0^1(\Omega)$ with

$$L^{-1} u_i = \nu_i u_i.$$

Taking the $H_{0,a}^1(\Omega)$-scalar product on both sides yields

$$\left(\nabla \left(L^{-1} u_i\right), \mu A_F \cdot \nabla v\right) = \left(\nabla \left(\nu_i u_i\right), \mu A_F \cdot \nabla v\right) \qquad \forall v \in H_0^1(\Omega).$$

Setting $\lambda_i = \nu_i^{-1}$ and using the definition of L^{-1} we arrive at

$$\left(\nabla u_i, \mu A_F \cdot \nabla v\right) = \lambda_i \left(u_i, v \gamma_F\right) \qquad \forall v \in H_0^1(\Omega).$$

From

$$\lambda_i \left(u_i, u_j \gamma_F\right) = \left(\nabla u_i, \mu A_F \cdot \nabla u_j\right) = \lambda_j \left(u_i, u_j \gamma_F\right),$$

it also follows that the eigenfunctions are mutually orthogonal,

$$a(F)(u_i, u_j) = b(F)(u_i, u_j) = 0, \tag{23}$$

for $i \neq j$.

2.4 Existence of a solution

Within this subsection we are going to prove that the variational problem (10) has a solution. As the original problem is equivalent to the transformed problem (21), we will show the existence of a minimizer just for the transformed one. First we need a continuity result for the eigenvalues, the following theorem can be found in [24], Theorem 2.3.1.

Theorem 2.10. *Let T_1 and T_2 be two self-adjoint, compact and positive operators on a separable Hilbert space V. Let $i \in \mathbb{N}$, and let $\nu_i(T_1)$ and $\nu_i(T_2)$ be their i-th eigenvalues, respectively. Then it holds that*

$$|\nu_i(T_1) - \nu_i(T_2)| \leq \sup_{v \in V} \frac{(v, (T_1 - T_2)v)_V}{\|v\|_V^2}$$

$$\leq \sup_{v \in V} \frac{\|(T_1 - T_2)(v)\|_V}{\|v\|_V} = \|T_1 - T_2\|_V.$$

Lemma 2.11. *Let $q, p \in Q^{\mathrm{ad}}$ with corresponding transformations F and E, respectively. Then it holds that*

$$\|A_F - A_E\|_{L^\infty(\Omega)} \leq c_\varepsilon \|q - p\|_{H^{3/2+\varepsilon}(I)},$$

$$\|\gamma_F - \gamma_E\|_{L^\infty(\Omega)} \leq c_\varepsilon \|q - p\|_{H^{3/2+\varepsilon}(I)}.$$

Proof. This lemma follows from the definitions of A_F and γ_F and Lemma 2.6. $\qquad\square$

Lemma 2.12. *Let $i \in \mathbb{N}$ and let $q, p \in Q^{\mathrm{ad}}$ with corresponding transformations F and E, respectively. Then it holds that*

$$|\lambda_i(q) - \lambda_i(p)| \le c \sup_{u \in H_0^1(\Omega)} \frac{|(\nabla u, \mu\,(A_F - A_E) \cdot \nabla u)| + \left|\left(u^2, \gamma_F - \gamma_E\right)\right|}{\|u\|_{H_0^1(\Omega)}^2}$$

$$\le c_\varepsilon \, \|q - p\|_{H^{3/2+\varepsilon}(I)} \, .$$

Proof. This lemma follows from Theorem 2.10 and Lemma 2.11. $\qquad\square$

Theorem 2.13. *Problem (21) has a solution.*

Proof. Let $(q_n)_{n \in \mathbb{N}} \subset Q^{\mathrm{ad}}$ be a minimizing sequence with

$$\lim_{n \to \infty} j(q_n) = \inf_{q \in Q^{\mathrm{ad}}} j(q) = \overline{j}.$$

As Q^{ad} is a bounded, closed and convex subset of the Hilbert space Q it is weakly sequentially compact. It follows that there exists $\overline{q} \in Q^{\mathrm{ad}}$ and a subsequence of $(q_n)_{n \in \mathbb{N}}$, denoted in the same way, with

$$
\begin{aligned}
q_n &\rightharpoonup \overline{q} & &\text{in } H^2(I), \\
q_n &\to \overline{q} & &\text{in } H^{2-\varepsilon}(I),
\end{aligned}
$$

where the strong convergence follows from the fact that $H^2(I)$ is compactly embedded into $H^{2-\varepsilon}(I)$. With Lemma 2.12 it follows that $\lambda_i(q_n) \to \lambda_i(\overline{q})$ for $n \to \infty$ and $i \in \{1, 2\}$. As the squared norm is lower semicontinuous it follows that

$$\liminf_{n \to \infty} \|q_n\|_{H^2(I)}^2 \ge \|\overline{q}\|_{H^2(I)}^2 \, ,$$

hence

$$\liminf_{n \to \infty} j(q_n) \ge j(\overline{q}),$$

and from the definition of $\overline{j}$ it follows that

$$j(\overline{q}) = \overline{j}. \qquad\square$$

Remark 2.14. As $q \mapsto \lambda_i(q)$ is highly nonlinear, the optimal control $\overline{q}$ need not be unique.

2.5 Regularity of the eigenfunctions

The aim of this subsection is to investigate in the regularity of the eigenfunctions, i.e. the solutions (u_i, λ_i) to (17),

$$(\nabla u_i, \mu A_F \cdot \nabla v) = \lambda_i\,(u_i, v\gamma_F) \qquad\qquad \forall v \in H_0^1(\Omega),$$

where it is known that $A_F \in C^{0,1/2}(\overline{\Omega_j})$ for $j \in \{0, 1\}$. Here we will prove some general regularity results for u_i, in a later section we will show that the optimal control $\overline{q}$ possesses some even higher regularity which will also improve the regularity of the associated optimal eigenfunctions. As we just focus on the regularity of the eigenfunctions, we omit the normalizing condition in (17) within this subsection.

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Lemma 2.15. *Let $q \in Q^{\mathrm{ad}}$, $i \in \mathbb{N}$ and $u_i = u_i(q)$, then it holds that*

$$\|u_i\|_{H_0^1(\Omega)} \le c_i \|u_i\|_{L^2(\Omega)}.$$

Proof. Let $F = F(q)$ and $\lambda_i = \lambda_i(q)$, as all the matrices μA_F are uniformly elliptic for $q \in Q^{\mathrm{ad}}$ it follows that

$$c \|u_i\|_{H_0^1(\Omega)}^2 \le a(F)(u_i, u_i) = \lambda_i\, b(F)(u_i, u_i)$$
$$\le c\lambda_i \|u_i\|_{L^2(\Omega)}^2,$$

and the proof follows with Lemma 2.3. $\qquad\square$

Lemma 2.16. *There exists $p \in (2, \infty)$ such that for all $q \in Q^{\mathrm{ad}}$ and $i \in \mathbb{N}$ it holds that $u_i(q) \in W^{1,p}(\Omega)$ and*

$$\|u_i\|_{W^{1,p}(\Omega)} \le c_{i,p} \|u_i\|_{L^2(\Omega)}.$$

Proof. Let $F = F(q)$ and $\lambda_i = \lambda_i(q)$. Again we use the fact that for $q \in Q^{\mathrm{ad}}$ the ellipticity constants of the matrices μA_F can be bounded uniformly. The existence of such a $p > 2$ now follows from [32], Theorem 1. From the cited theorem it also follows that

$$\|u_i\|_{W^{1,p}(\Omega)} \le c_p \|\lambda_i u_i \gamma_F\|_{L^p(\Omega)} \le c_{i,p} \|u_i\|_{L^p(\Omega)}$$
$$\le c_{i,p} \|u_i\|_{H_0^1(\Omega)} \le c_{i,p} \|u_i\|_{L^2(\Omega)},$$

where we used the continuous embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for $p < \infty$ in dimension $n = 2$ and Lemma 2.15. $\qquad\square$

The following lemma can be proven by a direct calculation.

Lemma 2.17. *Let $f \in C^{1,\alpha}(Y, Z)$ and $g \in C^{1,\alpha}(X, Y)$ for some $\alpha \in (0, 1]$ and closed subsets X, Y and Z of some Banach spaces. Then it holds that $f \circ g \in C^{1,\alpha}(X, Z)$ and*

$$\|f \circ g\|_{C^{1,\alpha}(X,Z)} \le c \|f\|_{C^{1,\alpha}(Y,Z)} \|g\|_{C^{1,\alpha}(X,Y)}.$$

Lemma 2.18. *Let $q \in Q^{\mathrm{ad}}$, $F = F(q)$, $j \in \{0, 1\}$ and $K \subset\subset \Omega_j$. Then $F|_K$ is analytic.*

Proof. This lemma is a direct consequence of Weyl's lemma, cf. [42], Lemma 2, and the fact that F is weakly harmonic in Ω_j for $j \in \{0, 1\}$. $\qquad\square$

Lemma 2.19. *Let $i \in \mathbb{N}$, $q \in Q^{\mathrm{ad}}$, $u_i = u_i(q)$, $j \in \{0, 1\}$ and let $K \subset\subset \Omega_j$ be sufficiently smooth. Then it holds that $u_i \in C^{1,1/2}(K)$ for $i \in \mathbb{N}$ and there exists $c_i = c_i(K)$ such that*

$$\|u_i\|_{C^{1,1/2}(K)} \le c_i \|u_i\|_{L^2(\Omega)}.$$

Proof. Let $F = F(q)$ and $K' = T_F^{-1}(K)$. Due to Lemma 2.18, $F|_K$ is analytic, and as T_F is bijective it follows that K' is sufficiently smooth. On K' it holds that $u_{q,i} = u_i \circ T_F^{-1}$ solves $-\Delta u_{q,i} = \tilde{\lambda} u_{q,i}$, where $\tilde{\lambda}_i = \lambda_i$ or $\tilde{\lambda}_i = \frac{\lambda_i}{d}$, depending on whether j is either 0 or 1. Using the results presented in [24], Section 1.2.4 and the references cited therein it follows that $\|u_{i,q}\|_{W^{2,4}(K')} \le c(i, K) \|u_{i,q}\|_{L^2(\Omega_q)}$. The regularity result and the estimate for $u_i|_K = u_{i,q} \circ T_F$ follow with the continuous embedding $W^{2,4}(K') \hookrightarrow C^{1,1/2}(\overline{K'})$, Lemma 2.17 and Lemma 2.6. $\qquad\square$

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Next we are going to prove a result dealing with the regularity of the eigenfunctions up to the boundary Γ_0. The following theorem can be found in [30], Corollary 1.3.

Theorem 2.20. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{1,\alpha}$-boundary Γ with $\alpha \in (0,1)$. Let $L \in \mathbb{N}$ and for $1 \le m \le L$ let Ω_m be a subdomain of Ω with $C^{1,\alpha}$-boundary and $\overline{\Omega} = \bigcup_{m=1}^{L} \overline{\Omega_m}$. For $1 \le m \le L$ let $A^{(m)} \in C^{0,\mu}(\overline{\Omega_m})$ with $\mu \in (0,1]$ be a symmetric and positive definite matrix, and let the matrix A be defined via $A|_{\Omega_m} = A^{(m)}$. Suppose that $0 < c_1 \le A \le c_2 < \infty$ on Ω in the sense of symmetric and positive definite matrices. In a likewise manner, let $h^{(m)} \in C^{0,\mu}(\Omega_m)$ and $h|_{\Omega_m} = h^{(m)}$. At last, let $f \in L^\infty(\Omega)$ and $g \in C^{1,\mu}(\Gamma)$. Then the restriction of the weak solution u to*

$$\begin{cases} -\operatorname{div}(A \cdot \nabla u) = f + \operatorname{div}(h) & \text{in } \Omega, \\ \qquad\qquad\qquad u = g & \text{on } \Gamma, \end{cases} \tag{24}$$

onto Ω_m belongs to $C^{1,\alpha'}(\overline{\Omega_m})$ for $0 < \alpha' \le \min\left\{\mu, \frac{\alpha}{(\alpha+1)n}\right\}$ and there holds the estimate

$$\max_{1 \le m \le L} \|u\|_{C^{1,\alpha'}(\overline{\Omega_m})} \le c\left(\|f\|_{L^\infty(\Omega)} + \max_{1 \le m \le L}\left\|h^{(m)}\right\|_{C^{0,\alpha'}(\overline{\Omega_m})} + \|g\|_{C^{1,\alpha'}(\Gamma)}\right),$$

with the constant c being independent of f, $h^{(m)}$ and g.

Coming back to our situation, we obtain:

Corollary 2.21. *Let $i \in \mathbb{N}$, $q \in Q^{\mathrm{ad}}$, $u_i = u_i(q)$, $\varepsilon > 0$, and let $\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \Gamma) > \varepsilon\}$. Then it holds that*

$$u_i|_{\Omega_0} \in C^{1,1/6}(\overline{\Omega_0}), \qquad\qquad u_i|_{\Omega_1 \cap \Omega_\varepsilon} \in C^{1,1/6}(\overline{\Omega_1 \cap \Omega_\varepsilon}),$$
$$\|u_i\|_{C^{1,1/6}(\overline{\Omega_0})} \le c_i \|u_i\|_{L^2(\Omega)}, \qquad \text{and} \qquad \|u_i\|_{C^{1,1/6}(\overline{\Omega_1 \cap \Omega_\varepsilon})} \le c_{i,\varepsilon} \|u_i\|_{L^2(\Omega)}.$$

Proof. Let $K \subset\subset \Omega_1$ be sufficiently smooth such that $\partial\Omega_\varepsilon \subset K$. Lemma 2.19 now yields

$$\|u_i\|_{C^{1,1/6}(\partial\Omega_\varepsilon)} \le c \|u_i\|_{C^{1,1/2}(K)} \le c_i \|u_i\|_{L^2(\Omega)}.$$

This corollary now follows with Lemma 2.16 which ensures that $u_i \in L^\infty(\Omega)$ and Theorem 2.20. $\square$

The following lemmata are proven in order to show $H^{3/2-\varepsilon}(\Omega) \cap W^{1,p}(\Omega)$-regularity of u_i.

Lemma 2.22. *Let $q \in Q^{\mathrm{ad}}$, $i \in \mathbb{N}$, $u_i = u_i(q)$, $p \in [2, \infty)$ and $\varepsilon > 0$. Then there exists*

$$u_{i,\Gamma} \in W_0^{1,p}(\Omega) \cap W^{1+1/p-\varepsilon,p}(\Omega),$$

such that $u_{i,\Gamma}|_{\Gamma_0} = u_i|_{\Gamma_0}$ and

$$\|u_{i,\Gamma}\|_{W^{1+1/p-\varepsilon,p}(\Omega)} \le c_{i,\varepsilon,p} \|u_i\|_{L^2(\Omega)}.$$

Proof. With Corollary 2.21 it follows that $u_i|_{\Gamma_0} \in C^{1,1/6}(\Gamma_0) \hookrightarrow W^{7/6,p}(\Gamma_0)$ for all $p \le \infty$. In addition, for $\varepsilon > 0$ sufficiently small let the annulus K be defined as

$$K = \{x \in \Omega_1 \mid \operatorname{dist}(x, \Gamma_0) \le \varepsilon\} \subset \Omega_1.$$

Using the trace theorem, cf. [18], Theorem 1.5.1.2 and Theorem 1.5.2.1, it follows that for $p \in [6/5, \infty)$ there exists a function $u_{i,\Gamma} \in L^1(\Omega_0 \cup K)$ with the following properties.

$$
\begin{aligned}
u_{i,\Gamma}|_{\Omega_0} &\in W^{7/6+1/p,p}(\Omega_0) \hookrightarrow C^{1,1/6-1/p}(\overline{\Omega_0}), \\
u_{i,\Gamma}|_{\Gamma_0} &= u_i|_{\Gamma_0}, \\
\partial_n u_{i,\Gamma}|_{\Gamma_0} &= 0, \\
\|u_{i,\Gamma}\|_{W^{7/6+1/p,p}(\Omega_0)} &\leq c_p \|u_i\|_{W^{7/6,p}(\Gamma_0)},
\end{aligned}
\tag{25}
$$

and

$$
\begin{aligned}
u_{i,\Gamma}|_K &\in W^{7/6+1/p,p}(K) \hookrightarrow C^{1,1/6-1/p}(K), \\
u_{i,\Gamma}|_{\partial K \setminus \Gamma_0} &= \partial_n u_{i,\Gamma}|_{\partial K \setminus \Gamma_0} = 0, \\
\|u_{i,\Gamma}\|_{W^{7/6+1/p,p}(K)} &\leq c_p \|u_i\|_{W^{7/6,p}(\Gamma_0)}.
\end{aligned}
\tag{26}
$$

As $u_{i,\Gamma}$ is continuous along Γ_0, it follows that

$$
\begin{aligned}
\|u_{i,\Gamma}\|^p_{W^{1,p}(K \cup \Omega_0)} &= \|u_{i,\Gamma}\|^p_{W^{1,p}(K)} + \|u_{i,\Gamma}\|^p_{W^{1,p}(\Omega_0)} \\
&\leq c_p \|u_i\|^p_{W^{7/6,p}(\Gamma_0)} \\
&\leq c_p \|u_i\|^p_{C^{1,1/6}(\overline{\Omega_0})} \\
&\leq c_{i,p} \|u_i\|^p_{L^2(\Omega)},
\end{aligned}
\tag{27}
$$

where we used Corollary 2.21. From the definition of fractional norms it now follows that

$$
\begin{aligned}
|u_{i,\Gamma}|^p_{W^{1+1/p-\varepsilon,p}(K \cup \Omega_0)} &= \int_{K \cup \Omega_0} \int_{K \cup \Omega_0} \frac{|\nabla u_{i,\Gamma}(x) - \nabla u_{i,\Gamma}(y)|^p}{|x-y|^{2+p(1/p-\varepsilon)}} \, \mathrm{d}x \, \mathrm{d}y \\
&\leq c_p \left(\left(|u_{i,\Gamma}|^p_{W^{1+1/p-\varepsilon,p}(K)} + |u_{i,\Gamma}|^p_{W^{1+1/p-\varepsilon,p}(\Omega_0)} \right) + \int_K \int_{\Omega_0} \frac{|\nabla u_{i,\Gamma}(x) - \nabla u_{i,\Gamma}(y)|^p}{|x-y|^{2+p(1/p-\varepsilon)}} \, \mathrm{d}x \, \mathrm{d}y \right) \\
&\leq c_{\varepsilon,p} \|u_{i,\Gamma}\|^p_{W^{7/6,p}(\Gamma_0)} + \max \left\{ \|u_{i,\Gamma}\|^p_{C^1(K)}, \|u_{i,\Gamma}\|^p_{C^1(\overline{\Omega_0})} \right\} \int_K \int_{\Omega_0} \frac{1}{|x-y|^{2+p(1/p-\varepsilon)}} \, \mathrm{d}x \, \mathrm{d}y \\
&\leq c_{\varepsilon,p} \left(\|u_{i,\Gamma}\|^p_{C^{1,1/6}(\Gamma_0)} + \max \left\{ \|u_{i,\Gamma}\|^p_{W^{7/6+1/7,7}(K)}, \|u_{i,\Gamma}\|^p_{W^{7/6+1/7,7}(\overline{\Omega_0})} \right\} \|1_{\Omega_0}\|^p_{W^{1/p-\varepsilon,p}(K \cup \Omega_0)} \right) \\
&\leq c_{\varepsilon,p} \left(\|u_{i,\Gamma}\|^p_{C^{1,1/6}(\overline{\Omega_0})} + \|u_i\|^p_{W^{7/6,7}(\Gamma_0)} \|1_{\Omega_0}\|^p_{W^{1/p-\varepsilon,p}(K \cup \Omega_0)} \right) \\
&\leq c_{\varepsilon,p} \|u_i\|^p_{C^{1,1/6}(\overline{\Omega_0})} \left(1 + \|1_{\Omega_0}\|^p_{W^{1/p-\varepsilon,p}(K \cup \Omega_0)} \right) \\
&\leq c_{i,\varepsilon,p} \|u_i\|^p_{L^2(\Omega)},
\end{aligned}
\tag{28}
$$

where we used Corollary 2.21 and the fact that the characteristic function of every bounded C^1-domain is an element of $W^{1/p-\varepsilon,p}(\mathbb{R}^2)$, cf. [38], Proposition 2.1, and [5]. If we extend $u_{i,\Gamma}$ by zero to the whole domain Ω, one can repeat the steps undertaken in (27) and (28) to show $W^{1+1/p-\varepsilon,p}(\Omega)$-regularity as well as the stability estimate and thus finish this proof. $\qquad \square$

Lemma 2.23. *Let $q \in Q^{\mathrm{ad}}$, $i \in \mathbb{N}$, $u_i = u_i(q)$ and $p < \infty$. Then it holds that $u_i \in W^{1,p}(\Omega)$ and*

$$
\|u_i\|_{W^{1,p}(\Omega)} \leq c_{i,p} \|u_i\|_{L^2(\Omega)}.
$$

Proof. Let $\tilde{u} = (u_i - u_{i,\Gamma})$ with $u_{i,\Gamma}$ defined as in Lemma 2.22. Then $\tilde{u}$ is the weak solution to

$$\begin{cases} -\operatorname{div}(\mu A_F \cdot \nabla \tilde{u}) = \lambda_i u_i \gamma_F + \operatorname{div}(\mu A_F \cdot \nabla u_{i,\Gamma}) & \text{in } \Omega_j, \\ \tilde{u} = 0 & \text{on } \partial\Omega_j, \end{cases} \tag{29}$$

for $j \in \{0,1\}$. As μ is constant on Ω_j, one can apply [2], Theorem 1, and get $\tilde{u} \in W_0^{1,p}(\Omega_j)$, as well as

$$\begin{aligned}
\|\tilde{u}\|_{W^{1,p}(\Omega_j)} &\leq c_p \left(\|\lambda_i u_i \gamma_F\|_{L^p(\Omega_j)} + \|A_F \cdot \nabla u_{i,\Gamma}\|_{L^p(\Omega_j)} \right) \\
&\leq c_{i,p} \left(\|u_i\|_{H^1(\Omega_j)} + \|A_F\|_{L^\infty(\Omega_j)} \|u_{i,\Gamma}\|_{W^{1,p}(\Omega_j)} \right) \\
&\leq c_{i,p} \|u_i\|_{L^2(\Omega)},
\end{aligned}$$

where we used Lemma 2.15, Lemma 2.6 and Lemma 2.22. As $\tilde{u} \in W_0^{1,p}(\Omega_j)$ it also follows that $\tilde{u} \in W_0^{1,p}(\Omega)$, and the result follows. $\qquad\square$

Lemma 2.24. *Let $q \in Q^{\mathrm{ad}}$, $i \in \mathbb{N}$ and $u_i = u_i(q)$. Then it holds that $u_i \in H^{3/2-\varepsilon}(\Omega)$ and*

$$\|u_i\|_{H^{3/2-\varepsilon}(\Omega)} \leq c_{i,\varepsilon} \|u_i\|_{L^2(\Omega)}.$$

Proof. As in the proof of Lemma 2.23 let $\tilde{u} = (u_i - u_{i,\Gamma})$, then $\tilde{u}$ is the weak solution to (29) for $j \in \{0,1\}$. As both subdomains of Ω are Lipschitz it follows with [19], Theorem 9.1.25 and [33], that $\tilde{u}|_{\Omega_j} \in H^{3/2-\varepsilon}(\Omega_j)$ and

$$\begin{aligned}
\|\tilde{u}\|_{H^{3/2-\varepsilon}(\Omega_j)} &\leq c_\varepsilon \left(\|\lambda_i u_i \gamma_F\|_{L^2(\Omega_j)} + \|A_F \cdot \nabla u_{i,\Gamma}\|_{H^{1/2-\varepsilon}(\Omega_j)} \right) \\
&\leq c_{i,\varepsilon} \left(\|u_i\|_{L^2(\Omega)} + \|A_F\|_{H^{3/2}(\Omega_j)} \|u_{i,\Gamma}\|_{H^{3/2-\varepsilon}(\Omega_j)} \right) \\
&\leq c_{i,\varepsilon} \|u_i\|_{L^2(\Omega)},
\end{aligned}$$

where we used Lemma 2.22 and [18], Theorem 1.4.4.2. It remains to prove $H^{3/2-\varepsilon}(\Omega)$-regularity. This can be done in exactly the same way as shown in (28) within the proof of Lemma 2.22. $\qquad\square$

2.6 Differentiability of the eigenvalues

In the following subsection we are going to prove differentiability of the eigensystem with respect to domain perturbations. In order to do so we follow the approach presented in [11], where it is proven that eigenvalues are differentiable with respect to a specific boundary perturbation and also a representation for the derivative is given. Although our approach uses a transformation to a reference domain and our regularity assumptions differ, their proofs can be adapted to our case.

Assumption 2.25. We assume that for all $q \in Q^{\mathrm{ad}}$, the eigenvalues $\lambda_1(q)$ and $\lambda_2(q)$ have multiplicity one.

Taking into account the cost functional (21) it is reasonable to assume that $\lambda_1(q) \neq \lambda_2(q)$ for all q sufficiently close to the optimal control $\bar{q}$. Another justification is the Krein-Rutman theorem (cf. [24], Theorem 1.2.5 and Theorem 1.2.6), which states that the first eigenvalue for a uniformly elliptic partial differential operator of second order is simple. However, we do have to admit that we did not find theoretical results supporting the claim that $\lambda_2(q) \neq \lambda_3(q)$ for all $q \in Q^{\mathrm{ad}}$ with $\|\bar{q} - q\|_{H^2(I)}$ sufficiently small.

2.6.1 On the existence of the derivatives of λ_i and u_i

The proof of the existence of the derivatives of λ_i and u_i with respect to q relies on the implicit function theorem and Fredholm's alternative.

Theorem 2.26 (Fredholm's alternative). *Let X be a Banach Space over $\mathbb{K}$ with either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let T be a compact operator on X with adjoint T', and let $\lambda \in \mathbb{K}$, $\lambda \neq 0$. Then exactly one of the following two possibilities holds true.*

- *The equation*

$$\lambda x - Tx = 0, \tag{30}$$

  has $x = 0$ as its only solution and

$$\lambda x - Tx = y, \tag{31}$$

  is uniquely solvable for every $y \in X$.

- *There exist $n = \dim(\ker(\lambda \operatorname{Id} - T))$ linear independent solutions to (30), and the adjoint equation*

$$\lambda x' - T'x' = 0,$$

  also has n linear independent solutions. Furthermore, there exists a solution to (31) if and only if $y \in (\ker(\lambda \operatorname{Id} - T'))^{\perp}$.

Proof. This theorem can be found in [1], Theorem 10.8. $\qquad\square$

Lemma 2.27. *Let $q \in Q^{\mathrm{ad}}$, $F = F(q)$, $i \in \mathbb{N}$, let $(u_i = u_i(q), \lambda_i = \lambda_i(q))$ be an eigenpair to the simple eigenvalue λ_i and let $g \in H^{-1}(\Omega)$. The equation*

$$(\nabla u, \mu A_F \cdot \nabla v) = \lambda_i (u, v\gamma_F) + (g, v)_{H^{-1}, H_0^1} \qquad \forall v \in H_0^1(\Omega) \tag{32}$$

has a solution $u \in H_0^1(\Omega)$ if and only if $(g, u_i)_{H^{-1}, H_0^1} = 0$.

Proof. Again, we use the operator L from Definition 2.8. Let $h = L^{-1}(g/\gamma_F)$, then equation (32) can be written as

$$(u, v)_{H_{0,a}^1(\Omega)} = \lambda_i \left(L^{-1}u, v\right)_{H_{0,a}^1(\Omega)} + (h, v)_{H_{0,a}^1(\Omega)} \qquad \forall v \in H_{0,a}^1(\Omega),$$

which can be written as

$$\nu_i u - L^{-1}u = \nu_i h \qquad \text{in } H_{0,a}^1(\Omega),$$

with $\nu_i = \lambda_i^{-1}$. With Theorem 2.26 it now follows that (32) has a solution if and only if

$$(h, u_i)_{H_{0,a}^1(\Omega)} = 0,$$

which reads as

$$0 = (\nabla h, \mu A_F \cdot \nabla u_i) = (g, u_i)_{H^{-1}, H_0^1}. \qquad\square$$

Lemma 2.28. *The mappings* $A\colon Q^{\mathrm{ad}} \to L^\infty(\Omega)$, $q \mapsto A_{F(q)}$ *and* $\gamma\colon Q^{\mathrm{ad}} \to L^\infty(\Omega)$, $q \mapsto \gamma_{F(q)}$ *are at least two times continuously Fréchet-differentiable. For* $\delta q \in Q$ *and* $\delta F = F'(q)(\delta q)$ *it holds that*

$$A'_{F,\delta F} = \mathrm{trace}\left(\mathrm{D}T_F^{-1} \cdot \mathrm{D}\delta F\right) A_F - \mathrm{D}T_F^{-1} \cdot \mathrm{D}\delta F \cdot A_F - A_F \cdot \mathrm{D}\delta F^T \cdot \mathrm{D}T_F^{-T},$$

$$\gamma'_{F,\delta F} = \gamma_F \, \mathrm{trace}\left(\mathrm{D}T_F^{-1} \cdot \mathrm{D}\delta F\right) = \mathrm{div}\!\left(\gamma_F \, \mathrm{D}T_F^{-1} \cdot \delta F\right).$$

Proof. The mapping $q \mapsto F(q)$ is linear and hence differentiable, and the result follows with (18) and the product rule. $\qquad\square$

Theorem 2.29. *Let* $q \in Q^{\mathrm{ad}}$, $\delta q \in Q$ *and* $i \in \mathbb{N}$ *such that* $\lambda_i(q)$ *is a simple eigenvalue. Then the mappings* $q \mapsto \lambda_i(q)$ *and* $q \mapsto u_i(q)$ *are at least two times continuously Fréchet-differentiable.*

Proof. Let $F = F(q)$ and let

$$B\colon H^2(I) \times H_0^1(\Omega) \times \mathbb{R} \to H^{-1}(\Omega) \times \mathbb{R},$$

$$B(q, u, \lambda) = \begin{pmatrix} -\mathrm{div}(\mu A_F \cdot \nabla u) - \lambda u \gamma_F \\ \int_\Omega u^2 \gamma_F \, \mathrm{d}x - 1 \end{pmatrix}.$$

The functional B is at least twice differentiable, which follows with Lemma 2.28. In addition, $B(q, u_i, \lambda_i) = 0$ if and only if u_i is a normalized eigenfunction with eigenvalue λ_i corresponding to the control q. Taking the derivative of B with respect to u and λ yields

$$\mathrm{D}_{u,\lambda} B(q, u_i, \lambda_i)(v, \vartheta) = \begin{pmatrix} -\mathrm{div}(\mu A_F \cdot \nabla v) - \lambda_i v \gamma_F - \vartheta u_i \gamma_F \\ 2 \int_\Omega u_i v \gamma_F \, \mathrm{d}x \end{pmatrix}.$$

Now we show that $\mathrm{D}_{u,\lambda} B(q, u_i, \lambda_i)$ is bijective, which can be done using Theorem 2.26 and Lemma 2.27 as follows. Let $(w, \tau) \in H^{-1}(\Omega) \times \mathbb{R}$ be arbitrary, we have to show that there exists $(v, \vartheta) \in H_0^1(\Omega) \times \mathbb{R}$ such that

$$\begin{cases} (\nabla v, \mu A_F \cdot \nabla \varphi) - \lambda_i \, (v, \varphi \gamma_F) = \vartheta \, (u_i, \varphi \gamma_F) + (w, \varphi)_{H^{-1}, H_0^1} & \forall \varphi \in H_0^1(\Omega), \\ 2 \, (u_i, v \gamma_F) = \tau. \end{cases} \tag{33}$$

If we set

$$\vartheta = -\, (w, u_i)_{H^{-1}, H_0^1},$$

then Lemma 2.27 yields the existence of $v_0 \in H_0^1(\Omega)$ such that the first equation within (33) is fulfilled for $v = v_0 + c u_i$ for all $c \in \mathbb{R}$. Setting

$$c = \frac{\tau}{2} - (u_i, v_0 \gamma_F),$$

makes v also fulfill the second equation within (33), and this theorem follows with implicit function theorem, cf. [3], Theorem 2.3. $\qquad\square$

2.6.2 Representation of the derivatives λ'_i and u'_i

In this subsubsection we are going to find explicit representations for the derivatives of the eigenvalue and eigenfunction. Let $q \in Q^{\mathrm{ad}}$, $F = F(q)$, $\delta q \in Q$, $i \in \mathbb{N}$ and let $\lambda'_i = \lambda'_i(q)(\delta q)$, $\delta u_i = u'_i(q)(\delta q)$ and $\delta F = F'(q)(\delta q)$. Due to Theorem 2.29 we can differentiate (17) with respect to q, which yields

$$\begin{cases} (\nabla \delta u_i, \mu A_F \cdot \nabla v) = \lambda_i \, (\delta u_i, v \gamma_F) + \lambda'_i \, (u_i, v \gamma_F) \\ \qquad\qquad + \lambda_i \, (u_i, v \gamma'_{F,\delta F}) - (\nabla u_i, \mu A'_{F,\delta F} \cdot \nabla v) & \forall v \in H_0^1(\Omega), \\ 2 \, (\delta u_i, u_i \gamma_F) + (u_i^2, \gamma'_{F,\delta F}) = 0. \end{cases} \tag{34}$$

Remark 2.30. From the first equation within (34) it follows that δu_i can formally be seen as a solution to

$$-\operatorname{div}(\mu A_F \cdot \nabla \delta u_i) = \lambda_i \delta u_i \gamma_F + \left(\lambda_i' u_i \gamma_F + \lambda_i u_i \gamma_{F,\delta F}' + \operatorname{div}\left(\mu A_{F,\delta F}' \cdot \nabla u_i\right)\right), \tag{35}$$

which is just a "perturbed" eigenvalue equation of the form

$$L\delta u_i = \lambda_i \delta u_i \gamma_F + g,$$

with $g = g(\lambda_i, u_i, q, \delta q) \in H^{-1}(\Omega)$. Solutions to (35) are not unique: if δu_i is a solution, then so is $\delta u_i + c u_i$ for all $c \in \mathbb{R}$. Instead, uniqueness is guaranteed through the second equation within (34).

Now using u_i as a test function in (17), we get

$$(\nabla u_i, \mu A_F \cdot \nabla u_i) = \lambda_i \left(u_i^2, \gamma_F\right),$$

and differentiation yields

$$2\left(\nabla u_i, \mu A_F \cdot \nabla \delta u_i\right) + \left(\nabla u_i, \mu A_{F,\delta F}' \cdot \nabla u_i\right) = \lambda_i' \left(u_i^2, \gamma_F\right) + 2\lambda_i \left(u_i \delta u_i, \gamma_F\right) + \lambda_i \left(u_i^2, \gamma_{F,\delta F}'\right). \tag{36}$$

As $\delta u_i \in H_0^1(\Omega)$ it holds that

$$(\nabla u_i, \mu A_F \cdot \nabla \delta u_i) = \lambda_i \left(u_i, \delta u_i \gamma_F\right). \tag{37}$$

Inserting (37) and the normalizing condition $\left(u_i^2, \gamma_F\right) = 1$ into (36) yields

$$\lambda_i'(q)(\delta q) = \left(\nabla u_i, \mu A_{F,\delta F}' \cdot \nabla u_i\right) - \lambda_i \left(u_i^2, \gamma_{F,\delta F}'\right). \tag{38}$$

It can be seen that the computation of δu_i is not necessary in order to compute λ_i'. Expression (38) may be rewritten as a boundary integral, but in order to do so we need more regularity of the involved functions in order to justify partial integration. This will be shown in the next subsection.

2.7 Higher regularity of the optimal control

Our proof of the higher regularity of the optimal control exploits some first order optimality conditions, we therefore have to make the following assumption.

Assumption 2.31. We assume that the optimal control $\bar{q}$ under consideration is an element of the interior of Q^{ad}.

Due to Assumption 2.31, the first order optimality condition reads as

$$j'(\bar{q})(\delta q) = 0 \qquad\qquad \forall \delta q \in Q, \tag{39}$$

which is

$$\lambda_1'(\bar{q})(\delta q) - \lambda_2'(\bar{q})(\delta q) + \alpha \left(\bar{q}, \delta q\right)_{H^2(I)} = 0 \qquad\qquad \forall \delta q \in Q. \tag{40}$$

Lemma 2.32. *For every $q \in Q^{\mathrm{ad}}$ there exists $p_i = p_i(q) \in H^1(I)$ such that*

$$\lambda_i'(q)(\delta q) = (p_i, \delta q)_{H^1(I)} \qquad\qquad \forall \delta q \in Q.$$

Proof. Let $F = F(q)$, $\delta q \in Q$ and $\delta F = F'(q)(\delta q)$. With (38) it holds that

$$
\begin{aligned}
\lambda_i'(q)(\delta q) &= \left(\nabla u_i, \mu A'_{F,\delta F} \cdot \nabla u_i\right) - \lambda_i \left(u_i^2, \gamma'_{F,\delta F}\right) \\
&= \left(\nabla u_i, \mu A'_{F,\delta F} \cdot \nabla u_i\right)_{\Omega_0} + \left(\nabla u_i, \mu A'_{F,\delta F} \cdot \nabla u_i\right)_{\Omega_1} \\
&\quad - \lambda_i \left(u_i^2, \gamma'_{F,\delta F}\right)_{\Omega_0} - \lambda_i \left(u_i^2, \gamma'_{F,\delta F}\right)_{\Omega_1}.
\end{aligned}
\tag{41}
$$

Using Lemma 2.23 and the normalizing condition for u_i we can estimate the right hand side within (41) via

$$
\begin{aligned}
\left(u_i^2, \gamma'_{F,\delta F}\right)_{\Omega_j} &\leq \|u_i\|_{L^4(\Omega_j)}^2 \left\|\gamma'_{F,\delta F}\right\|_{L^2(\Omega_j)} \leq c \|u_i\|_{H_0^1(\Omega_j)}^2 \left\|\gamma_F \, \mathrm{D}T_F^{-1} \cdot \delta F\right\|_{H^1(\Omega_j)} \\
&\leq c \|u_i\|_{H_0^1(\Omega_j)}^2 \|\delta F\|_{H^1(\Omega_j)} \leq c_i \|\delta F\|_{H^1(\Omega)} \\
&\leq c_i \|\delta q\|_{H^1(I)},
\end{aligned}
$$

and in a similar way it holds that

$$
\begin{aligned}
\left(\nabla u_i, \mu A'_{F,\delta F} \cdot \nabla u_i\right)_{\Omega_j} &\leq c \|u_i\|_{W^{1,4}(\Omega)}^2 \left\|A'_{F,\delta F}\right\|_{L^2(\Omega_j)} \\
&\leq c \|\delta q\|_{H^1(I)},
\end{aligned}
$$

for $j \in \{0, 1\}$. As $\delta q \mapsto \lambda_i'(q)(\delta q)$ is linear, the existence of such a p_i follows with the Riesz representation theorem. $\square$

Lemma 2.33. *Let $\lambda \in H_{\mathrm{per}}^2(I)$ and $\psi \in H_{\mathrm{per}}^1(I)$ such that*

$$
(\lambda, \varphi)_{H^2(I)} = (\psi, \varphi)_{H^1(I)} \qquad\qquad \forall \varphi \in C_{\mathrm{per}}^\infty(I).
\tag{42}
$$

Then it holds that $\lambda \in H_{\mathrm{per}}^3(I)$.

Proof. The lemma follows with the definition of weak derivatives and partial integration. $\square$

Lemma 2.34. *The optimal control $\overline{q} \in Q^{\mathrm{ad}}$ has the higher regularity $\overline{q} \in H^3(I)$.*

Proof. This lemma follows from (40), Lemma 2.32 and Lemma 2.33. $\square$

Lemma 2.35. *For $\overline{F} = F(\overline{q})$ and $j \in \{0, 1\}$ it holds that $\overline{F}\big|_{\Omega_j} \in W^{2,\infty}(\Omega_j)$.*

Proof. As $\overline{q} \in H^3(I)$ due to Lemma 2.34, [19], Theorem 9.1.20 yields $\overline{F}\big|_{\Omega_0} \in H^{7/2}(\Omega_0) \hookrightarrow W^{2,\infty}(\Omega_0)$. The regularity of $\overline{F}$ on Ω_1 follows with [10], Remark 1, and [15, 41]. $\square$

In order to derive higher regularity of $\overline{u}_i = u_i(\overline{q})$ we will need some regularity results concerning spaces with bounded mean oscillation.

Definition 2.36 (Campanato-John-Nirenberg space). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with diameter d and let $\psi \colon [0, d] \to \mathbb{R}$ be a nonnegativ continuous function satisfying $r \leq c\psi(r)$ for some positive constant c. A function $f \in L^2(\Omega)$ is said to be an element of $\mathrm{BMO}_\Psi(\Omega)$, the space of bounded mean oscillation, if

$$
|f|_{\mathrm{BMO}_\psi(\Omega)} = \sup_{\substack{x_0 \in \Omega \\ 0 < \rho \leq d}} \frac{1}{\psi(\rho)} \left(\int_{\Omega(x_0, \rho)} \left| f(x) - (f)_{\Omega(x_0, \rho)} \right|^2 \, \mathrm{d}x \right)^{1/2} < \infty,
$$

where $\Omega(x_0, \rho) = \Omega \cap Q_\rho(x_0)$ with $Q_\rho(x_0)$ being a cube with center x_0, sides parallel to the axis and side length equal to 2ρ. Furthermore,

$$(f)_D = \frac{1}{|D|} \int_D f \, \mathrm{d}x,$$

shall denote the mean value of f on D.

In what follows we will focus on the case where $\psi(\rho) = \rho^\alpha$ with $\alpha > 0$ sufficiently small. As mentioned in [43], the resulting spaces are called Campanato spaces. Furthermore, in that case it even holds that

$$\mathrm{BMO}_\psi(\Omega) = C^{0,\alpha}(\overline{\Omega}),$$

cf. [39], Example 1.

Definition 2.37 (Domains of class C^{k,BMO_ψ}). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We say that $\partial\Omega \in C^{k,\mathrm{BMO}_\psi}$ for $k \in \mathbb{N}$ if for any $x_0 \in \partial\Omega$ there exists a $C^{k-1,1}$-transformation $\mathcal{T}$ and a neighborhood $\mathcal{N}_{x_0}$ of x_0 such that

$$\mathcal{T} \colon \mathcal{N}_{x_0} \cap \Omega \to B_1^+(0),$$

where $B_1^+(0)$ is the unit ball with positive last coordinate, is one to one and onto with

$$\mathcal{T}\left(\mathcal{N}_{x_0} \cap \partial\Omega\right) = \overline{B_1^+(0)} \cap \{x_n = 0\}.$$

Moreover, the norms of $\mathcal{T}$, $\mathcal{T}^{-1}$ and their derivatives $\mathrm{D}^\nu \mathcal{T}$, $\mathrm{D}^\nu\left(\mathcal{T}^{-1}\right)$ are uniformly bounded in L^∞ and BMO_ψ for $|\nu| \le k$.

From [12], Remark 3.2, it follows that domains which are locally the epigraph of a $C^{k,\alpha}$ function for $k \ge 1$ are of class $C^{k,\alpha}$. Furthermore, from the same source, Definition 3.1, it follows that if Ω is a domain of class $C^{k,\alpha}$, then it is also in C^{k,BMO_ψ} for $\psi(\rho) = \rho^\alpha$.

Theorem 2.38. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain containing $L \in \mathbb{N}$ disjoint subdomains $\Omega_m \subset\subset \Omega$ for $1 \le m \le L$, and let $\Omega_{L+1} = \Omega \backslash \bigcup_{m=1}^{L} \overline{\Omega_m}$. We consider weak solutions $u \in H^1(\Omega)$ to the equation*

$$-\operatorname{div}(A \cdot \nabla u) = -\operatorname{div}(f), \tag{43}$$

where the matrix A is uniformly elliptic. Suppose that $\partial\Omega_m \in C^{k+1,\mathrm{BMO}_\psi}$ with $k \ge 1$,

$$A|_{\Omega_m}, f|_{\Omega_m} \in C^{k-1,1}(\overline{\Omega_m}) \text{ and } \mathrm{D}^k A\Big|_{\Omega_m}, \mathrm{D}^k f\Big|_{\Omega_m} \in \mathrm{BMO}_\psi(\Omega_m).$$

Then for any $\Omega' \subset\subset \Omega$ it holds for the solution u to (43) that

$$u|_{\Omega_m} \in C^k(\overline{\Omega_m} \cap \Omega') \text{ and } \mathrm{D}^{k+1} u \in \mathrm{BMO}_\psi(\Omega' \cap \Omega_m).$$

Proof. This theorem can be found in [43], Theorem 2.3, where it is assumed that the function ψ fulfills some additional assumptions. In [25], Remark 2.2, it is shown that these assumptions hold true for $\psi(\rho) = \rho^\alpha$ with $\alpha > 0$ sufficiently small. $\qquad \square$

Lemma 2.39. *For $i \in \mathbb{N}$, $\overline{u}_i = u_i(\overline{q})$ and $j \in \{0, 1\}$ it holds that $\overline{u}_i|_{\Omega_j} \in W^{2,\infty}(\Omega_j)$.*

Proof. With Theorem 2.38 it follows that $\overline{u}_i \in C^{2,\alpha}(\overline{\Omega}_0) \hookrightarrow W^{2,\infty}(\Omega_0)$. Now let $\overline{F} = F(\overline{q})$ and $\overline{u}_{i,q} = \overline{u}_i \circ T_{\overline{F}}^{-1}$ be an eigenfunction on the untransformed domain. As

$$\overline{u}_{q,i}|_{\Omega_{q,1}} \in H^{3/2-\varepsilon}(\Omega_{q,1}) \hookrightarrow L^\infty(\Omega_{q,1})$$

as shown in the proof of Lemma 2.9, it follows with [18], Theorem 2.4.2.5 and Section 5.2 that

$$\overline{u}_{q,i}|_{\Omega_{q,1}} \in W^{2,p}(\Omega_{q,1}) \hookrightarrow C^{1,\alpha}(\Omega_{q,1})$$

for all $p < \infty$ and $\alpha = 1 - 2/p > 0$. With [10], Remark 1, and [15, 41] it now follows that

$$\overline{u}_{q,1}|_{\Omega_{q,1}} \in W^{2,\infty}(\Omega_{q,1}).$$

Due to the regularity of $T_{\overline{F}}$ on Ω_1, cf. Lemma 2.35, it follows that

$$\overline{u}_i|_{\Omega_1} = \overline{u}_{q,1} \circ T_{\overline{F}}^{-1}\Big|_{\Omega_1} \in W^{2,\infty}(\Omega_1). \qquad \square$$

2.7.1 A representation of λ' as a boundary integral

Due to the higher regularity of the optimal eigenfunctions $\overline{u}_i$, equation (17) also holds in strong form (at least on each of the subdomains Ω_0 and Ω_1), therefore it is possible to rewrite expression (38) from above, the goal is to compute $\overline{\lambda}_i'(\overline{q})(\delta q)$ as a boundary integral over Γ_0. The fact that the derivative of the reduced cost functional can be represented as a boundary integral is known as the Hadamard-Zolesio theorem and can be found in [38], Theorem 2.27. Let $\delta F = F'(\overline{q})(\delta q)$, using the representation obtained in Lemma 2.28 one can show that

$$
\begin{aligned}
\left(\nabla\overline{u}_i, \mu A'_{\overline{F},\delta F} \cdot \nabla\overline{u}_i\right) = {} & 2\left(\mathrm{div}\left(\mu A_{\overline{F}} \cdot \nabla\overline{u}_i\right), \nabla\overline{u}_i^T \cdot DT_{\overline{F}}^{-1} \cdot \delta F\right)_{\Omega_0} \\
& + 2\left(\mathrm{div}\left(\mu A_{\overline{F}} \cdot \nabla\overline{u}_i\right), \nabla\overline{u}_i^T \cdot DT_{\overline{F}}^{-1} \cdot \delta F\right)_{\Omega_1} \\
& - d\int_{\Gamma_0} \left|DT_{\overline{F}_-}^{-T} \cdot \nabla\overline{u}_{i,-}\right|^2 \gamma_{\overline{F}_-} \delta F^T \cdot DT_{\overline{F}_-}^{-T} \cdot n\,\mathrm{d}s \\
& + \int_{\Gamma_0} \left|DT_{\overline{F}_+}^{-T} \cdot \nabla\overline{u}_{i,+}\right|^2 \gamma_{\overline{F}_+} \delta F^T \cdot DT_{\overline{F}_+}^{-T} \cdot n\,\mathrm{d}s,
\end{aligned}
\tag{44}
$$

where $\overline{u}_{i,-}$ and $\overline{u}_{i,+}$ shall denote $\overline{u}_i$ approaching Γ_0 from the inside of Ω_0 and Ω_1 respectively, cf. (3), and the same for $\overline{F}_-$ and $\overline{F}_+$. It holds that

$$
\begin{aligned}
2\left(\mathrm{div}\left(\mu A_{\overline{F}} \cdot \nabla\overline{u}_i\right), \nabla\overline{u}_i^T \cdot DT_{\overline{F}}^{-1} \cdot \delta F\right)_{\Omega_j} &= -2\overline{\lambda}_i\left(\overline{u}_i \gamma_{\overline{F}}, \nabla\overline{u}_i^T \cdot DT_{\overline{F}}^{-1} \cdot \delta F\right)_{\Omega_j} \\
&= -\overline{\lambda}_i\left(\nabla\left(\overline{u}_i^2\right), \gamma_{\overline{F}} DT_{\overline{F}}^{-1} \cdot \delta F\right)_{\Omega_j},
\end{aligned}
\tag{45}
$$

for $j \in \{0,1\}$, and

$$
-\overline{\lambda}_i\left(\overline{u}_i^2, \gamma_{\overline{F},\delta F}'\right) = -\overline{\lambda}_i\left(\overline{u}_i^2, \mathrm{div}\left(\gamma_{\overline{F}} DT_{\overline{F}}^{-1} \cdot \delta F\right)\right)_{\Omega_0} - \overline{\lambda}_i\left(\overline{u}_i^2, \mathrm{div}\left(\gamma_{\overline{F}} DT_{\overline{F}}^{-1} \cdot \delta F\right)\right)_{\Omega_1}.
\tag{46}
$$

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Summing up (45) and (46) yields

$$
\sum_{j=0}^{1} \left(2 \left(\operatorname{div} \left(\mu A_{\overline{F}} \cdot \nabla \overline{u}_i \right), \nabla \overline{u}_i^T \cdot \mathrm{D}T_{\overline{F}}^{-1} \cdot \delta F \right)_{\Omega_j} - \overline{\lambda}_i \left(\overline{u}_i^2, \gamma_{\overline{F},\delta F}' \right)_{\Omega_j} \right)
$$

$$
= -\overline{\lambda}_i \int_{\Omega_0} \operatorname{div} \left(\overline{u}_i^2 \gamma_{\overline{F}} \, \mathrm{D}T_{\overline{F}}^{-1} \cdot \delta F \right) \mathrm{d}x - \overline{\lambda}_i \int_{\Omega_1} \operatorname{div} \left(\overline{u}_i^2 \gamma_{\overline{F}} \, \mathrm{D}T_{\overline{F}}^{-1} \cdot \delta F \right) \mathrm{d}x
$$

$$
= -\overline{\lambda}_i \int_{\Gamma_0} \overline{u}_i^2 \gamma_{\overline{F}_-} \delta F^T \cdot \mathrm{D}T_{\overline{F}_-}^{-T} \cdot n \, \mathrm{d}s + \overline{\lambda}_i \int_{\Gamma_0} \overline{u}_i^2 \gamma_{\overline{F}_+} \delta F^T \cdot \mathrm{D}T_{\overline{F}_+}^{-T} \cdot n \, \mathrm{d}s
$$

$$
- \overline{\lambda}_i \int_{\Gamma} \overline{u}_i^2 \gamma_{\overline{F}} \delta F^T \cdot \mathrm{D}T_{\overline{F}}^{-T} \cdot n \, \mathrm{d}s,
$$

(47)

where the last term vanishes due to $\overline{u}_i \in H_0^1(\Omega)$. Inserting (47) back into (44) finally yields

$$
\overline{\lambda}_i'(\overline{q})(\delta q) = - \int_{\Gamma_0} \left(d \left| \mathrm{D}T_{\overline{F}_-}^{-T} \cdot \nabla \overline{u}_{i,-} \right|^2 + \overline{\lambda}_i \overline{u}_i^2 \right) \gamma_{\overline{F}_-} \delta F^T \cdot \mathrm{D}T_{\overline{F}_-}^{-T} \cdot n \, \mathrm{d}s
$$

$$
+ \int_{\Gamma_0} \left(\left| \mathrm{D}T_{\overline{F}_+^{-T}} \cdot \nabla \overline{u}_{i,+} \right|^2 + \overline{\lambda}_i \overline{u}_i^2 \right) \gamma_{\overline{F}_+} \delta F^T \cdot \mathrm{D}T_{\overline{F}_+}^{-T} \cdot n \, \mathrm{d}s.
$$

(48)

Remark 2.40. As $\delta F|_{\Gamma_0} = \delta q \, n$, it is not necessary to compute δF in order to compute $\overline{\lambda}_i'(\overline{q})(\delta q)$ via (48).

Lemma 2.41. *The optimal control $\overline{q} \in Q^{\mathrm{ad}}$ has the higher regularity $\overline{q} \in H^4(I)$.*

Proof. As $\delta F|_{\Gamma_0} = \delta q \, n$ and using the higher regularity of $\overline{u}_i$ and $\overline{F}$ as shown in Lemma 2.39 and Lemma 2.35, it follows similar to the proof of Lemma 2.32 that there exists $p_i = p_i(\overline{q}) \in L^2(I)$ with

$$
\overline{\lambda}_i'(\overline{q})(\delta q) = (p_i, \delta q)_{L^2(I)} \qquad\qquad \forall \delta q \in Q,
$$

and this lemma follows similar to Lemma 2.34. $\qquad\square$

2.8 The second derivative

Within this subsection we are going to compute the second derivative of $u_i = u_i(q)$ and $\lambda_i = \lambda_i(q)$ with respect to perturbations in q, which exist due to Theorem 2.29.

Taking the second derivative of the first equation of (17) with respect to q yields the equation for $\delta\tau u_i = u_i''(q)(\delta q, \tau q)$,

$$
\begin{aligned}
(\nabla \delta\tau u_i, \mu A_F \cdot \nabla v) &= \lambda_i \left(\delta\tau u_i, v\gamma_F \right) + \lambda_{i,\delta q,\tau q}'' \left(u_i, v\gamma_F \right) + \lambda_{i,\delta q}' \left(\tau u_i, v\gamma_F \right) + \lambda_{i,\delta q}' \left(u_i, v\gamma_{F,\tau F}' \right) \\
&\quad + \lambda_{i,\tau q}' \left(\delta u_i, v\gamma_F \right) + \lambda_{i,\tau q}' \left(u_i, v\gamma_{F,\delta F}' \right) \\
&\quad + \lambda_i \left(\delta u_i, v\gamma_{F,\tau F}' \right) + \lambda_i \left(\tau u_i, v\gamma_{F,\delta F}' \right) + \lambda_i \left(u_i, v\gamma_{F,\delta F,\tau F}'' \right) \\
&\quad - \left(\nabla \delta u_i, \mu A_{F,\tau F}' \cdot \nabla v \right) - \left(\nabla \tau u_i, \mu A_{F,\delta F}' \cdot \nabla v \right) \\
&\quad - \left(\nabla u_i, \mu A_{F,\delta F,\tau F}'' \cdot \nabla v \right) \qquad \forall v \in H_0^1(\Omega),
\end{aligned}
$$

(49)

where we used the abbreviations $\lambda_i = \lambda_i(q)$, $\lambda_{i,\delta q}' = \lambda_i'(q)(\delta q)$, $\lambda_{i,\tau q}' = \lambda_i'(q)(\tau q)$, $\lambda_{i,\delta q,\tau q}'' = \lambda_i''(q)(\delta q, \tau q)$, $\delta u_i = u_i'(q)(\delta q)$ and $\tau u_i = u_i'(q)(\tau q)$. Note that (49) can again be regarded as a "perturbed" eigenfunction equation.

Using u_i itself as a test function within (17) and then taking the second derivative with respect to q yields

$$
\begin{aligned}
2 &\left(\nabla \tau u_i, \mu A_F \cdot \nabla \delta u_i\right) + 2\left(\nabla u_i, \mu A'_{F,\tau F} \cdot \nabla \delta u_i\right) + 2\left(\nabla u_i, \mu A_F \cdot \nabla \delta \tau u_i\right) \\
&+ 2\left(\nabla u_i, \mu A'_{F,\delta F} \cdot \nabla \tau u_i\right) + \left(\nabla u_i, \mu A''_{F,\delta F,\tau F} \cdot \nabla u_i\right) \\
&= \lambda''_{i,\delta q,\tau q}\left(u_i^2, \gamma_F\right) + 2\lambda'_{i,\delta q}\left(u_i, \tau u_i \gamma_F\right) + \lambda'_{i,\delta q}\left(u_i^2, \gamma'_{F,\tau F}\right) + 2\lambda'_{i,\tau q}\left(u_i, \delta u_i \gamma_F\right) \\
&+ 2\lambda_i\left(\delta u_i, \tau u_i \gamma_F\right) + 2\lambda_i\left(u_i, \delta \tau u_i \gamma_F\right) + 2\lambda_i\left(u_i, \delta u_i \gamma'_{F,\tau F}\right) + \lambda'_{i,\tau q}\left(u_i^2, \gamma'_{F,\delta F}\right) \\
&+ 2\lambda_i\left(u_i, \tau u_i \gamma'_{F,\delta F}\right) + \lambda_i\left(u_i^2, \gamma''_{F,\delta F,\tau F}\right).
\end{aligned}
\tag{50}
$$

Using τu_i as a test function in (34), and vice versa for δu_i in the equation for τu_i, yields

$$
\begin{aligned}
\left(\nabla \tau u_i, \mu A_F \cdot \nabla \delta u_i\right) + \left(\nabla u_i, \mu A'_{F,\delta F} \cdot \nabla \tau u_i\right) &= \lambda'_{i,\delta q}\left(u_i, \tau u_i \gamma_F\right) \\
&+ \lambda_i\left(\tau u_i, \delta u_i \gamma_F\right) + \lambda_i\left(u_i, \tau u_i \gamma'_{F,\delta F}\right),
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
\left(\nabla \delta u_i, \mu A_F \cdot \nabla \tau u_i\right) + \left(\nabla u_i, \mu A'_{F,\tau F} \cdot \nabla \delta u_i\right) &= \lambda'_{i,\tau q}\left(u_i, \delta u_i \gamma_F\right) \\
&+ \lambda_i\left(\delta u_i, \tau u_i \gamma_F\right) + \lambda_i\left(u_i, \delta u_i \gamma'_{F,\tau F}\right).
\end{aligned}
\tag{52}
$$

The second derivative of the normalizing condition within (17) with respect to q reads as

$$
2\left(\delta u_i, \tau u_i \gamma_F\right) + 2\left(u_i, \delta \tau u_i \gamma_F\right) + 2\left(u_i, \delta u_i \gamma'_{F,\tau F}\right) + 2\left(u_i, \tau u_i \gamma'_{F,\delta F}\right) + \left(u_i^2, \gamma''_{F,\delta F,\tau F}\right) = 0.
\tag{53}
$$

Now subtracting (51) and (52) twice from (50) finally yields

$$
\begin{aligned}
\lambda''_{i,\delta q,\tau q} &= \left(\nabla u_i, \mu A''_{F,\delta F,\tau F} \cdot \nabla u_i\right) - 2\left(\nabla \tau u_i, \mu A_F \cdot \nabla \delta u_i\right) \\
&- \lambda'_{i,\delta q}\left(u_i^2, \gamma'_{F,\delta F}\right) - \lambda'_{i,\tau q}\left(u_i^2, \gamma'_{F,\tau F}\right) + 2\lambda_i\left(\delta u_i, \tau u_i \gamma_F\right) - \lambda_i\left(u_i^2, \gamma''_{F,\delta F,\tau F}\right).
\end{aligned}
\tag{54}
$$

3 Stability estimates for eigenvalues and eigenfunctions

In order to estimate the error between eigenfunctions and their discretized counterparts, the application of the "standard" techniques is not possible, this is due to the fact that eigenfunctions appear on the left-, as well as on the right hand side of the corresponding equation, cf. (17). Hence we have to deal with different concepts which will be presented in this section.

The results of Subsection 3.1 will be needed to estimate terms like $\|u_i(q) - u_i(p)\|$ for $q, p \in Q^{\mathrm{ad}}$, whereas the results of Subsection 3.2 will be needed to estimate terms like $\|u'_i(q)(\delta q) - u'_i(p)(\delta q)\|$ for $q, p \in Q^{\mathrm{ad}}$ and $\delta q \in Q$.

3.1 Gap between operators

If u is an eigenfunction of a linear partial differential operator L, then for arbitrary $c \in \mathbb{R} \setminus \{0\}$, cu is also an eigenfunction. Due to this fact it is not clear how to estimate the difference $\|u_{1,i} - u_{2,i}\|$ between the i-th eigenfunctions $u_{1,i}$ and $u_{2,i}$ corresponding to different differential operators L_1 and L_2, for even normalized eigenfunctions are only unique up to their sign. In order to deal with this difficulty we will have a closer look at the concept of the so-called gap between operators. What follows is mainly based on [6] and [9].

Definition 3.1. Let M and N be linear subspaces of a normed space Z. The gap *from M to N* is defined via

$$
\delta(M, N) = \sup_{\substack{u \in M \\ \|u\|_Z = 1}} \mathrm{dist}(u, N),
$$

where for $u \in Z$ we have

$$\mathrm{dist}(u, N) = \inf_{v \in N} \|u - v\|_Z \,.$$

Furthermore, the gap *between* M and N is defined via

$$\hat{\delta}(M, N) = \max \{\delta(M, N), \delta(N, M)\} \,.$$

Lemma 3.2. *Let M and N be linear subspaces of a Hilbert space Z, and let P and Q be the orthogonal projections onto the closures of M and N, respectively. Then it holds that*

$$\delta(M, N) = \|(1 - Q)P\|_Z \,,$$
$$\hat{\delta}(M, N) = \|P - Q\|_Z \,.$$

Proof. This lemma can be found in [9], Theorem 2.2. □

Definition 3.3. Let $T \colon D(T) \subset X \to Y$ be a linear operator whose domain $D(T)$ is a subset of the Hilbert space X and maps onto the Hilbert space Y. The graph G of the operator T is defined as

$$G(T) = \{(u, Tu) \mid u \in D(T)\} \,.$$

Definition 3.4. Let X and Y be Hilbert spaces and let

$$S \colon D(S) \subset X \to Y,$$
$$T \colon D(T) \subset X \to Y,$$

be linear operators mapping subsets of X onto Y. The gap *from* S to T is defined by

$$\delta(S, T) = \delta(G(S), G(T)),$$

whereas the gap *between* S and T is defined by

$$\hat{\delta}(S, T) = \hat{\delta}(G(S), G(T)).$$

More explicitly,

$$\delta(S, T) = \sup_{\substack{u \in D(S) \\ \|u\|_X^2 + \|Su\|_Y^2 = 1}} \inf_{v \in D(T)} \left(\|u - v\|_X^2 + \|Su - Tv\|_Y^2 \right)^{1/2} . \tag{55}$$

Lemma 3.5. *Let X be a Hilbert space and let S and T be selfadjoint on X. Then it holds that*

$$\delta(S, T) = \delta(T, S) = \hat{\delta}(S, T).$$

Proof. This lemma can be found in [9], Corollary 2.6. □

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let T be a selfadjoint operator over $L^2(\Omega)$ with compact resolvent bounded from below and let $i \in \mathbb{N}$ such that the i-th eigenvalue λ_i of T is simple. Then there exist $c_0, \delta_0 > 0$ such that for each selfadjoint operator S over $L^2(\Omega)$ whose compact resolvent is bounded from below, for which $\delta(S, T) < \delta_0$ and normalized eigenfunction $\tilde{u}_i$ corresponding to the i-th eigenvalue $\tilde{\lambda}_i$ of S, there exists a normalized eigenfunction u_i of T, corresponding to λ_i, such that*

$$\|u_i - \tilde{u}_i\|_{L^2(\Omega)} \leq c_0 \delta(S, T).$$

Proof. This theorem can be found in [9], Theorem 2.14. □

3.2 Stability estimates for affine eigenvectors

In what follows let X be a Hilbert space over $\mathbb{R}$ with scalar product $(\cdot, \cdot)_X$, norm $\|u\|_X = \sqrt{(u, u)_X}$, and let L be a compact linear operator over X. The ordered eigenvalues of L shall be denoted with $(\nu_i)_{i \in \mathbb{N}}$, where $\lim_{i \to \infty} \nu_i = 0$. The eigenspace corresponding to ν_i will be denoted with $N_i(L)$, its orthogonal complement $N_i(L)^\perp$ has to be understood with respect to the X-scalar product. From Theorem 2.26 it follows that for $g \in X$ and $i \in \mathbb{N}$ there exists a solution $u \in X$ to

$$Lu = \nu_i u + g, \tag{56}$$

if and only if $g \in N_i(L)^\perp$. This solution, if it exists, is not unique. If u solves (56), so does $u + c u_i$ for all $u_i \in N_i(L)$ and $c \in \mathbb{R}$. In what follows we are going to prove that there exists $c_i > 0$ such that for all $g \in N_i(L)^\perp$ there exists a solution to (56) with $\|u\|_X \le c_i \|g\|_X$.

Lemma 3.7. *The subspace $N_i(L)^\perp$ is closed in X.*

Proof. Using the spectral theorem it follows that $N_i(L) \subset X$ is of finite dimension and closed. From [1], Lemma 7.17, it follows that $X = N_i(L) \oplus N_i(L)^\perp$. As $N_i(L)$ is closed, there exists a continuous orthogonal projection P onto $N_i(L)$ with $N_i(L)^\perp = \mathcal{N}(P)$, and this lemma follows with the closed complement theorem, cf. [1], Theorem 7.15. $\qquad\square$

Lemma 3.8. *Let $g \in N_i(L)^\perp$ and let u_g be a solution to (56). Then u_g minimizes the X-norm among all solutions of (56) if and only if $u_g \in N_i(L)^\perp$.*

Proof. Let u_g be a solution to (56). Then u_g has minimal X-norm if and only if for all $u_i \in N_i(L)$, the solution to

$$\arg\min_{t \in \mathbb{R}} \|u_g + t u_i\|_X^2, \tag{57}$$

is $t = 0$. The proof now follows by taking the first and second derivative of the squared norm within (57) with respect to t. $\qquad\square$

Lemma 3.9. *Let $g \in N_i(L)^\perp$. Then there exists exactly one solution u_g to (56) that minimizes the X-norm among all solutions to (56).*

Proof. Let $u_{g,1}$ and $u_{g,2}$ be two solutions to (56) with minimal X-norm, and let $u_g = u_{g,1} - u_{g,2}$. With Lemma 3.8 it follows that $u_g \in N_i(L)^\perp$. As L is linear we get $L u_g = \nu_i u_g$, hence $u_g \in N_i(L)$. It follows that $u_g \in N_i(L) \cap N_i(L)^\perp = \{0\}$, and the result follows. $\qquad\square$

Corollary 3.10. *Let $g \in N_i(L)^\perp$, let u_g be an arbitrary solution to (56) and let $\{u_\nu^1, \ldots, u_\nu^N\}$ be an orthogonal basis for $N_i(L)$. Then the solution $\overline{u}_g$ of (56) with minimal X-norm is given via*

$$\overline{u}_g = u_g - \sum_{i=1}^N \frac{(u_g, u_\nu^i)_X}{\|u_\nu^i\|_X^2} u_\nu^i.$$

Proof. By definition of $\overline{u}_g$ it follows that $(\overline{u}_g, u_\nu^i)_X = 0$ for all $i \in \{1, \ldots, N\}$ and the result follows with Lemma 3.8. $\qquad\square$

Definition 3.11. For $i \in \mathbb{N}$ let $T = T_i \colon N_i(L)^\perp \subset X \to X$, $Tg = u_g$ such that u_g is a solution to (56) corresponding to g with minimal X-norm, i.e. for all solutions $\tilde{u}_g$ to (56) with $u_g \neq \tilde{u}_g$ it holds that $\|u_g\|_X < \|\tilde{u}_g\|_X$.

Remark 3.12. The fact that the operator T from Definition 3.11 is well-defined follows with Thorem 2.26 and Lemma 3.9.

Lemma 3.13. *The operator T from Definition 3.11 is linear.*

Proof. Let $g, h \in N_i(L)^\perp$, let u_g and u_h be arbitrary solutions to the corresponding perturbed eigenvalue equations (56), let $\{u_\nu^1, \ldots, u_\nu^N\}$ be an orthogonal basis for $N_i(L)$ and let $\alpha \in \mathbb{R}$. As

$$L(\alpha u_g) = \alpha L u_g = \alpha(\nu_i u_g + g) = \nu_i (\alpha u_g) + \alpha g,$$

it follows with Corollary 3.10 that

$$T(\alpha g) = \alpha u_g - \sum_{i=1}^N \frac{(\alpha u_g, u_\nu^i)_X}{\|u_\nu^i\|_X^2} u_\nu^i = \alpha \left(u_g - \sum_{i=1}^N \frac{(u_g, u_\nu^i)_X}{\|u_\nu^i\|_X^2} u_\nu^i \right) = \alpha T(g).$$

Furthermore,

$$L(u_g + u_h) = L u_g + L u_h = (\nu_i u_g + g) + (\nu_i u_h + h) = \nu_i (u_g + u_h) + (g + h),$$

and again we use Corollary 3.10 to get

$$\begin{aligned}
T(g + h) &= (u_g + u_h) - \sum_{i=1}^N \frac{\left((u_g + u_h), u_\nu^i\right)_X}{\|u_\nu^i\|_X^2} u_\nu^i \\
&= \left(u_g - \sum_{i=1}^N \frac{(u_g, u_\nu^i)_X}{\|u_\nu^i\|_X^2} u_\nu^i \right) + \left(u_h - \sum_{i=1}^N \frac{(u_h, u_\nu^i)_X}{\|u_\nu^i\|_X^2} u_\nu^i \right) \\
&= T(g) + T(h). \qquad \square
\end{aligned}$$

Lemma 3.14. *Let T be as in Definition 3.11 and let $G(T) = \left\{ (g, Tg) \mid g \in N_i(L)^\perp \right\} \subset (X \times X)$ be the graph of T. Then $G(T)$ is closed.*

Proof. Let $(g_n)_{n \in \mathbb{N}} \subset N_i(L)^\perp$, $u_n = T(g_n)$ with $g_n \to g$ and $u_n \to u$ in X for some elements $g, u \in X$. We have to show that $g \in N_i(L)^\perp$ and $u = Tg$. From Lemma 3.8 it follows that $(u_n)_{n \in \mathbb{N}} \subset N_i(L)^\perp$. As $N_i(L)^\perp$ is closed due to Lemma 3.7 it follows that $u, g \in N_i(L)^\perp$. As L is compact it follows that

$$Lu \leftarrow L u_n = \nu_i u_n + g_n \to \nu_i u + g,$$

hence $u = Tg$. $\qquad \square$

Lemma 3.15. *The operator T from Definition 3.11 is bounded.*

Proof. As T is a linear operator with closed graph due to Lemma 3.13 and Lemma 3.14, this lemma follows with the closed graph theorem, cf. [1], Theorem 5.9. $\qquad \square$

Corollary 3.16. *There exists $c_i > 0$, independent of $g \in N_i(L)^\perp$, such that*

$$\|Tg\|_X \le c_i \|g\|_X,$$

for all $g \in N_i(L)^\perp$.

4 Discretization

Within this section we are going to discretize problem (21) with respect to the control, the state and the transformation.

4.1 Discretization of the control

We split the interval $I = (0, 2\pi)$ into $N \in \mathbb{N}$ subintervals I_j for $j \in \{0, \dots, N-1\}$ with maximal length σ, and introduce the space of (admissible) discretized controls as

$$Q_\sigma = \left\{ q_\sigma \in Q \mid q_\sigma|_{I_j} \in \mathcal{P}^3(I_j) \, \forall j \in \{0, \dots, N-1\} \right\},$$
$$Q_\sigma^{\mathrm{ad}} = Q_\sigma \cap Q^{\mathrm{ad}},$$

where $\mathcal{P}^3(I)$ shall denote the set of all polynomials of degree at most 3 over the interval I. The first partially discretized problem now reads as

$$\min_{q_\sigma \in Q_\sigma^{\mathrm{ad}}} j(q_\sigma) = \lambda_1(q_\sigma) - \lambda_2(q_\sigma) + \frac{\alpha}{2} \|q_\sigma\|_{H^2(I)}^2. \tag{58}$$

subject to (12) and

$$\begin{cases} a(F)(u_i, v) = \lambda_i \, b(F)(u_i, v) & \forall v \in H_0^1(\Omega), \\ b(F)(u_i, u_i) = 1, \end{cases}$$

where $i \in \{1, 2\}$ and λ_i is the i-th eigenvalue given via (6).

4.2 Discretization of the state

For $h > 0$ let $\Omega_{0,h} \subset \Omega_0$ be a polygonal approximation of Ω_0 where we assume that all the vertices of $\Gamma_{0,h} = \partial \Omega_{0,h}$ lie on Γ_0. In addition, let $\Omega_{1,h} = \Omega \setminus \overline{\Omega_{0,h}} \supset \Omega_1$ be a polygonal approximation of Ω_1. Let $\{\pi_h\}_{h>0}$ be a family of admissible triangulations of Ω using triangles with maximal diameter h, fulfilling the usual regularity assumptions like shape regularity and quasiuniformity. In addition we assume that each member of this family can be represented as the union of a triangulation of $\Omega_{0,h}$ with a triangulation of $\Omega_{1,h}$. We define the usual linear finite elements,

$$\begin{aligned} V_h &= \left\{ v_h \in H^1(\Omega) \mid v_h|_{K_h} \in \mathcal{P}^1(K_h) \, \forall K_h \in \pi_h \right\}, \\ V_{h,0} &= V_h \cap H_0^1(\Omega), \end{aligned} \tag{59}$$

where $\mathcal{P}^1(K_h)$ is the set of all polynomials of degree at most 1 over K. Now let

$$\begin{aligned} \mu_h &= 1 + (d-1)\chi_{\Omega_{0,h}}, \\ a_h(F)(u, v) &= (\nabla u, \mu_h A_F \cdot \nabla v), \end{aligned} \tag{60}$$

with $\chi_{\Omega_{0,h}}$ being the characteristic function of $\Omega_{0,h}$. In addition, let $\lambda_{i,h}(q)$ be the i-th eigenvalue with respect to the bilinear forms $a_h(F)(\cdot, \cdot)$ and $b(F)(\cdot, \cdot)$ which can be computed via (6). The second partially discretized problem, where we additionally discretize the state, now reads as

$$\min_{q_\sigma \in Q_\sigma^{\mathrm{ad}}} j_h(q_\sigma) = \lambda_{1,h}(q_\sigma) - \lambda_{2,h}(q_\sigma) + \frac{\alpha}{2} \|q_\sigma\|_{H^2(I)}^2, \tag{61}$$

subject to (12) and

$$\begin{cases} a_h(F)(u_{i,h}, v_h) = \lambda_{i,h} \, b(F)(u_{i,h}, v_h) & \forall v_h \in V_{h,0}, \\ b(F)(u_{i,h}, u_{i,h}) = 1, \end{cases}$$

with $i \in \{1, 2\}$.

4.3 Discretization of the transformation

As in Subsection 4.2, let $\Omega_{0,k} \subset \Omega_0$ be a polygonal approximation to Ω_0, let $\Gamma_{0,k} = \partial\Omega_{0,k}$, let $\Omega_{1,k} = \Omega \backslash \overline{\Omega_{0,k}} \supset \Omega_1$ be a polygonal approximation to Ω_1 and let $\{\pi_k\}_{k>0}$ be a family of admissible triangulations of Ω using triangles with maximal diameter k, and fulfilling the usual regularity assumptions like shape regularity and quasiuniformity. Again we assume that every triangulation π_k can be considered as the union of a triangulation of $\Omega_{0,k}$ with a triangulation of $\Omega_{1,k}$. Similar to (59) let

$$V_k = \left\{ v_k \in H^1(\Omega) \big| \ v_k|_{K_k} \in \mathcal{P}^1(K_k) \ \forall K_k \in \pi_k \right\}. \tag{62}$$

The transformation is now being discretized as follows.

$$\begin{cases} -\Delta_k F_k = 0 & \text{in } \Omega_{j,k}, \ j \in \{0,1\}, \\ \quad F_k = 0 & \text{on } \Gamma \\ \quad F_k = \Pi_k \left(q\,n \right) & \text{on } \Gamma_{0,k} = \partial\Omega_{0,k}, \end{cases} \tag{63}$$

where Δ_k is to be interpreted as the discretized weak Laplacian and Π_k first maps Γ_0 onto $\Gamma_{0,k}$, and then projects these values into the set of boundary values of V_k. For a more detailled investigation into that topic we refer to [8].

If $F_k = F_k(q)$ denotes the discrete transformation corresponding to the control q, then the fully discretized i-th eigenvalue $\lambda_{i,h,k}$ is given via (6) using the forms $a_h(F_k)(\cdot,\cdot)$ and $b(F_k)(\cdot,\cdot)$. Finally, the fully discretized problem, where also the transformation is being discretized, reads as

$$\min_{q_\sigma \in Q_\sigma^{\mathrm{ad}}} j_{h,k}(q_\sigma) = \lambda_{1,h,k}(q_\sigma) - \lambda_{2,h,k}(q_\sigma) + \frac{\alpha}{2} \|q_\sigma\|_{H^2(I)}^2, \tag{64}$$

subject to (63) and

$$\begin{cases} a_h(F_k)(u_{i,h}, v_h) = \lambda_{i,h,k}\, b(F_k)(u_{i,h}, v_h) & \forall v_h \in V_{h,0}, \\ b(F_k)(u_{i,h}, u_{i,h}) = 1, \end{cases}$$

where again $i \in \{1,2\}$.

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