A Hopf-Lax Formula for the Time Evolution of the Level-Set Equation and a New Approach to Shape Sensitivity Analysis

Daniel Kraft
University of Graz
Institute of Mathematics, NAWI Graz
Universitätsplatz 3, 8010 Graz, Austria
Email: daniel.kraft@uni-graz.at

May 12th, 2015

Abstract

The level-set method is used in many different applications to describe the propagation of shapes and domains. When scalar speed fields are used to encode the desired shape evolution, this leads to the classical level-set equation. We present a concise Hopf-Lax representation formula that can be used to characterise the evolved domains at arbitrary times. This result is also applicable for the case of speed fields without a fixed sign, even though the level-set equation has a non-convex Hamiltonian in these situations. The representation formula is based on the same idea that underpins the Fast-Marching Method, and it provides a strong theoretical justification for a generalised *Composite Fast-Marching* method.

Based on our Hopf-Lax formula, we are also able to present new theoretical results. In particular, we show non-fattening of the zero level-set in a measure-theoretic sense, derive a very general shape sensitivity calculus that does not require the usual regularity assumptions on the domains, prove optimal Lipschitz constants for the evolved level-set function and discuss the effect of perturbations in both the speed field and the initial geometry.

2010 Mathematics Subject Classification: Primary 49L25; Secondary 49Q10.
Keywords: Level-Set Method; Shape Optimisation; Hopf-Lax Formula; Viscosity Solutions; Shape Sensitivity Analysis.

Preprint submitted for publication in "Interfaces and Free Boundaries".

1 Introduction

Many problem classes in applied mathematics require the manipulation of shapes and geometries (shape optimisation being an obvious one, but also free-boundary problems fall into this category). For this, it is necessary to "encode" the geometric information in a suitable way. Very thorough general discussions of this topic can be found in the classical books [11], [19] and [31]. For this paper, we focus on the *level-set method*. It was introduced in [24] by Osher and Sethian. Today, there exists a vast literature about it, covering various aspects. A general introduction can be found in [28]. For selected theoretical results, we would like to highlight [16] and [4]. Applications of the level-set method to concrete problems can be found, for instance, in [8], [29], [12] and [5].

The basic idea in the level-set framework is to introduce an auxiliary *level-set function* $\phi : \mathbb{R}^n \to \mathbb{R}$ to describe an open set $\Omega \subset \mathbb{R}^n$. The domain

$$\Omega = \phi^{-1}\left((-\infty, 0)\right)$$

is given as the sub-zero level-set of ϕ. This set is obviously open if ϕ is continuous. Of course, many different level-set functions can describe a single open set. A possible choice for the level-set function of some given Ω is its *signed distance function* (see chapter 5 of [11]). Since this works for arbitrary open

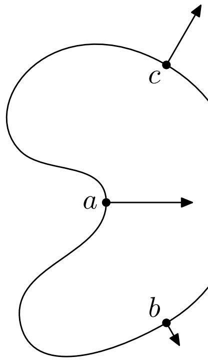


Figure 1: Illustration of the speed method. In the shown situation, $F(a) < 0 < F(b) < F(c)$.

sets, we immediately see that even Lipschitz continuity of the level-set function implies no regularity of Ω. (Throughout this work, we will concentrate on Lipschitz continuous level-set functions. Note that higher-order regularity of a level-set function *does*, in fact, imply boundary regularity of the described domain. See, for instance, Theorem 4.2 on page 77 of [11].) The other way round, however, this also means that the level-set method is very flexible and can describe a wide range of shapes.

In order to describe not only geometries themselves but also *changes* to them, let us consider for the moment shape deformations by the classical *speed method*: Given a scalar speed field $F : \mathbb{R}^n \to \mathbb{R}$, we move the boundary of Ω in normal direction according to this speed field. Positive speed corresponds to outward movement of the boundary (growth of Ω), while negative speed leads to local shrinking. This is illustrated in Figure 1. In terms of the level-set function, the corresponding time evolution is described by the *level-set equation*

$$\phi_t(x,t) + F(x) \left|\nabla \phi(x,t)\right| = 0 \text{ in } \mathbb{R}^n \times (0,\infty), \quad \phi(x,0) = \phi_0(x) \text{ for } x \in \mathbb{R}^n \text{ and } t = 0 \tag{1}$$

as introduced in [24]. Throughout this work we will assume that F is Lipschitz continuous with Lipschitz constant L_F and that it has compact support. Some of our results could be proven without these assumptions, but we make them nevertheless for simplicity. They are easy to justify in many concrete situations and necessary for the more interesting conclusions anyway. The function ϕ_0 shall also be Lipschitz continuous, and we denote its Lipschitz constant by L_{ϕ_0}. As mentioned already, a canonical choice of ϕ_0 is the signed distance function of an initial (bounded) open set Ω_0. With this choice, $L_{\phi_0} = 1$.

The main result of this paper will be a *Hopf-Lax representation formula* for the time evolution of both $\phi(\cdot, t)$ itself and the corresponding evolving set Ω_t. Section 2 recalls and introduces some necessary properties of the level-set equation (1) and its solution ϕ. Section 3 discusses the Eikonal equation and shortest paths induced by a speed field F. Based on these foundations, we can then introduce the Hopf-Lax formula itself in Section 4. Theorem 5 and Corollary 1 are the main theorems. They allow us to analyse the geometric evolution on a more abstract level, without the need to work with the level-set equation itself. This is a very useful simplification, and we will draw interesting further conclusions from it in Section 5.

Somewhat related to our results is the Generalised Fast Marching Method described in [7]. However, note that [7] is focused solely on the algorithmic analysis of the introduced numerical method. It does not state a general-purpose Hopf-Lax formula for the time evolution. We, on the other hand, would like to focus on the representation formula and the conclusions it enables from a theoretical point of view. Nevertheless, it is also important to remark that our results give a theoretical justification for the Fast Marching Method (see [27] and chapter 8 of [28]), and also generalise it to our *Composite Fast-Marching* method that is able to handle sign changes in the speed field. For an application of the latter to PDE-constrained shape optimisation, see [21]. Our numerical implementation is available together with additional tools for the level-set method as free software in the `level-set` package [22] for GNU Octave [13].

2 Preliminaries about the Level-Set Equation

Since the proper solution concept for the level-set equation (1) is that of *viscosity solutions*, we quickly recall their definition. More details can be found, for instance, in [9] and [10].

Definition 1. Let $D = \mathbb{R}^n \times (0, \infty)$ be the open space-time cylinder, $\phi : D \to \mathbb{R}$ and $(x, t) \in D$. Then $J^{1+}\phi(x, t)$ is the set of all $(p, a) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$\phi(y, s) \le \phi(x, t) + a(s - t) + p \cdot (y - x) + o\left(|s - t| + |y - x|\right)$$

as $(y, s) \to (x, t)$ in D. Similarly, $(p, a) \in J^{1-}\phi(x, t)$ if and only if

$$\phi(y, s) \ge \phi(x, t) + a(s - t) + p \cdot (y - x) + o\left(|s - t| + |y - x|\right)$$

for $(y, s) \to (x, t)$. $J^{1\pm}\phi(x, t)$ are called the *first-order parabolic semijets* of ϕ at (x, t). Note that $J^{1-}\phi(x, t)$ is often also called *subdifferential* of ϕ at (x, t).

Definition 2. Let F and ϕ_0 be given. We say that $\phi : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is a *viscosity subsolution* of (1) for the given data if ϕ is upper semi-continuous, $\phi(\cdot, 0) \le \phi_0$ on $\mathbb{R}^n$ and

$$a + F(x)\,|p| \le 0$$

for each $(x, t) \in D$ and $(a, p) \in J^{1+}\phi(x, t)$. Similarly, ϕ is a *viscosity supersolution* if ϕ is lower semi-continuous, $\phi(\cdot, 0) \ge \phi_0$ and $a + F(x)\,|p| \ge 0$ for all $(a, p) \in J^{1-}\phi(x, t)$.

The function ϕ solves (1) in the viscosity sense if it is both a viscosity sub- and supersolution. Note that this implies, in particular, that ϕ is continuous and that $\phi(x, 0) = \phi_0(x)$ for all $x \in \mathbb{R}^n$.

The following result, which states the existence of a unique viscosity solution to (1) as well as the so-called *comparison principle*, is well-known (see, for instance, [16]):

Theorem 1. *For given F and ϕ_0, there exists a unique viscosity solution $\phi : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ of (1).*

Furthermore, if ϕ_1 and ϕ_2 are viscosity sub- and supersolutions to (1), respectively, with $\phi_1(x, 0) \le \phi_2(x, 0)$ for all $x \in \mathbb{R}^n$, then $\phi_1 \le \phi_2$ pointwise on $\mathbb{R}^n \times [0, \infty)$.

We will now proceed to deduce several useful consequences of this solution concept for (1). As our first result, we show an intuitively trivial but still important fact: If the speed vanishes everywhere, then the solution of (1) is constant in time.

Lemma 1. *Let ϕ solve (1) for $F = 0$ and some initial function ϕ_0. Then $\phi(x, t) = \phi_0(x)$ for all $x \in \mathbb{R}^n$ and all times $t \ge 0$.*

Proof. We have to show that $\phi(x, t) = \phi_0(x)$ is indeed a solution of (1) with $F = 0$. It is clear that the initial condition is satisfied. Let $(p, a) \in J^{1+}\phi(x, t)$ for some $(x, t) \in D$. Then

$$\phi(y, s) - \phi(x, t) \le a(s - t) + p \cdot (y - x) + o\left(|s - t| + |y - x|\right) \tag{2}$$

for all $y \in \mathbb{R}^n$ and $s > 0$ by the definition of $J^{1+}\phi(x, t)$. Consider, in particular, $y = x$ and a sequence $s_k \to t$. The left-hand side of (2) vanishes since ϕ is constant in time, so that we can re-arrange the relation to read

$$0 \le a \cdot \frac{s_k - t}{|s_k - t|} + \frac{o\left(|s_k - t|\right)}{|s_k - t|}.$$

For s_k converging to t from above, this gives $0 \le a$ in the limit. For $s_k \to t$ from below, it follows that $0 \le -a$, thus $a = 0$ must necessarily hold. Hence $a + F(x)\,|p| = 0 \le 0$ is satisfied, and ϕ is indeed a viscosity subsolution of (1). In the same way, one can also show that it is a supersolution. □

The comparison principle (second part of Theorem 1) implies that the solution is monotone with respect to the speed:

Lemma 2. *Let $F_1 \ge F_2$ in $\mathbb{R}^n$, and let ϕ_1 and ϕ_2 be solutions to (1) for $F = F_1$ and $F = F_2$, respectively, with initial conditions $\phi_1(x, 0) = \phi_{1,0}(x)$ and $\phi_2(x, 0) = \phi_{2,0}(x)$. If $\phi_{1,0} \le \phi_{2,0}$ on $\mathbb{R}^n$, then $\phi_1 \le \phi_2$ on the whole of $\mathbb{R}^n \times [0, \infty)$.*

Proof. We shall show that ϕ_1 is a viscosity subsolution to (1) also with speed $F = F_2$. Then Theorem 1 implies the claim. Let $(p, a) \in J^{1+}\phi_1(x, t)$ for some $(x, t) \in D$. Then

$$a + F_2(x)\,|p| \leq a + F_1(x)\,|p| \leq 0$$

since ϕ_1 is a solution to the equation with $F = F_1$. Thus it is, indeed, also a subsolution with $F = F_2$. $\square$

Lemma 3. *Let $F \geq 0$ and ϕ solve (1). Then for each $x \in \mathbb{R}^n$, $\phi(x, \cdot)$ is decreasing on $[0, \infty)$. If $F \leq 0$ instead, then $\phi(x, \cdot)$ is increasing in time.*

Proof. Let $F \geq 0$ and $s \geq 0$ be given. We have to show $\phi(x, s) \geq \phi(x, t)$ for all $x \in \mathbb{R}^n$ and $t > s$. If $s > 0$, we can shift the initial time to s and use $\phi(\cdot, s)$ as initial data, so assume $s = 0$ without loss of generality. By Lemma 1, we know that $\tilde{\phi}(x, t) = \phi(x, 0)$ solves (1) with $\tilde{F} = 0$. Since $F \geq 0 = \tilde{F}$, Lemma 2 implies that $\phi(x, t) \leq \tilde{\phi}(x, t) = \phi(x, 0)$, which finishes the proof. For the case $F \leq 0$, a similar argument can be applied. $\square$

Note that, due to the concept of viscosity solutions we use, (1) has no time-reversal symmetry. However, there exists an important symmetry property with respect to sign changes in F and ϕ, which will be useful later:

Lemma 4. *Let ϕ be the solution of (1) for some F and ϕ_0. Then $-\phi$ solves the equation for $-F$ and with initial data $-\phi_0$.*

Proof. The initial condition is obviously satisfied. Let $(p, a) \in J^{1+}(-\phi)(x, t)$, and denote the generic error term for simplicity by $r = o\left(|s - t| + |y - x|\right)$. Then it holds that

$$\begin{aligned} (-\phi)(y, s) &\leq (-\phi)(x, t) + a(s - t) + p \cdot (y - x) + r, \\ \phi(y, s) &\geq \phi(x, t) - a(s - t) - p \cdot (y - x) + r. \end{aligned}$$

Hence, $(-p, -a) \in J^{1-}\phi(x, t)$. Since ϕ is a supersolution of (1), this implies that

$$-a + F(x)\,|-p| = -a + F(x)\,|p| \geq 0 \quad \Leftrightarrow \quad a - F(x)\,|p| \leq 0.$$

Thus $-\phi$ is a subsolution when the speed is $-F$. By the same argument, one can show that $-\phi$ is a supersolution in this case as well. $\square$

We are now able to show an interesting result that allows us to reduce the general problem to the case where $F \geq 0$ or $F \leq 0$ throughout the domain. The latter case can be reduced itself to $F \geq 0$ by Lemma 4. Hence, we can restrict ourselves to the consideration of $F \geq 0$ for most of the remainder of the paper.

Theorem 2. *For a general Lipschitz continuous $F : \mathbb{R}^n \to \mathbb{R}$, define*

$$F^+(x) = \max(F(x), 0), \quad F^-(x) = \min(F(x), 0)$$

together with the open sets $\Omega^+ = F^{-1}\left((0, \infty)\right)$ and $\Omega^- = F^{-1}\left((-\infty, 0)\right)$. Let $\phi^\pm$ be the solutions of (1) for $F^\pm$ with initial data ϕ_0. Then

$$\phi(x, t) = \left\{ \begin{array}{rl} \phi^+(x, t) & \textit{for } x \in \Omega^+, \\ \phi^-(x, t) & \textit{for } x \in \Omega^-, \\ \phi_0(x) & \textit{if } F(x) = 0 \end{array} \right. \tag{3}$$

solves (1) for F and with initial data ϕ_0.

Furthermore, $\phi^+ \leq \phi_0 \leq \phi^-$ throughout $\mathbb{R}^n \times [0, \infty)$, and $\phi^\pm(x, t) = \phi_0(x)$ for all $x \notin \Omega^\pm$ and $t \geq 0$.

Proof. The relation $\phi^+ \leq \phi_0 \leq \phi^-$ follows immediately from Lemma 2 since $F^- \leq 0 \leq F^+$ and ϕ_0 is the solution for $F = 0$ by Lemma 1. It is clear that ϕ, as defined in (3), satisfies the initial condition $\phi(x, 0) = \phi_0(x)$, since this condition is imposed on both of $\phi^\pm$.

The next step is to show $\phi^+(x, t) = \phi_0(x)$ for all $x \notin \Omega^+$ and $t > 0$. For this, define

$$\tilde{\phi}(x, t) = \left\{ \begin{array}{rl} \phi^+(x, t) & \text{for } x \in \Omega^+, \\ \phi_0(x) & \text{if } F(x) \leq 0. \end{array} \right.$$

Take note that $\phi^+ \leq \tilde{\phi} \leq \phi_0$ since $\phi^+ \leq \phi_0$. Because $\Omega^+ = F^{-1}((0,\infty))$ is open, it follows that $\tilde{\phi}$ is upper semi-continuous. To see this, let $(x,t) \in \mathbb{R}^n \times [0,\infty)$ be arbitrary and $(x_k,t_k) \to (x,t)$. If $x \in \Omega^+$, then $x_k \in \Omega^+$ if k is large enough. Consequently,

$$\limsup_{k\to\infty} \tilde{\phi}(x_k,t_k) = \limsup_{k\to\infty} \phi^+(x_k,t_k) = \phi^+(x,t) = \tilde{\phi}(x,t)$$

by continuity of ϕ^+. If, on the other hand, $x \notin \Omega^+$, then

$$\limsup_{k\to\infty} \tilde{\phi}(x_k,t_k) \leq \limsup_{k\to\infty} \phi_0(x_k,t_k) = \phi_0(x,t) = \tilde{\phi}(x,t)$$

since $\tilde{\phi} \leq \phi_0$ is always the case and ϕ_0 is continuous.

We proceed to show that $\tilde{\phi}$ is a subsolution of (1) with speed F^+, which will then imply $\tilde{\phi} \leq \phi^+$ by Theorem 1 and thus further $\phi^+ = \tilde{\phi}$. Let $(x,t) \in D$ and $(p,a) \in J^{1+}\tilde{\phi}(x,t)$. If $x \in \Omega^+$, then note that $\tilde{\phi} = \phi^+$ in a neighbourhood of (x,t) since Ω^+ is open. Thus (p,a) is also in $J^{1+}\phi^+(x,t)$, which implies $a + F^+(x)\,|p| = 0 \leq 0$ since ϕ^+ is the solution for F^+. Assume now $x \in \mathbb{R}^n \setminus \Omega^+$, i. e., $F(x) \leq 0$. This implies $F^+(x) = 0$ and also $\tilde{\phi}(x,t) = \phi_0(x)$ constantly in time. In this case, we can show that $a = 0$ with the same argument as in the proof of Lemma 1. Hence also $a + F^+(x)\,|p| = 0 \leq 0$, which shows that $\tilde{\phi}$ is, indeed, a subsolution of (1) with speed F^+. Similarly, one can show that $\phi^-(x,t) = \phi_0(x)$ for all $x \notin \Omega^-$ and $t > 0$.

It remains to verify that ϕ as defined in (3) is actually a solution of (1) with speed F. Take note that the considerations above imply that $\phi^+(x,t) = \phi^-(x,t) = \phi_0(x)$ whenever $F(x) = 0$. Thus, ϕ is continuous since $\phi^\pm$ as well as ϕ_0 are continuous. With the same argument that was used above for $\tilde{\phi}$, one can now also show that ϕ itself is both a sub- and supersolution of (1). □

3 Generalised Distances

As a preparation for the following results, in this section we will investigate the *Eikonal equation*

$$F(x)\,|\nabla d_y(x)| = 1, \;\; d_y(y) = 0 \tag{4}$$

for some fixed "source" $y \in \mathbb{R}^n$. For this stationary equation, viscosity solutions can be defined in a similar way to Definition 2. As discussed above, we assume that $F : \mathbb{R}^n \to [0,\infty)$ has compact support and is Lipschitz continuous with constant L_F. Consequently, it attains a maximal value, so that we can find $\overline{F} \in \mathbb{R}$ with $0 \leq F(x) \leq \overline{F}$ for all $x \in \mathbb{R}^n$. Dropping the requirement that F has compact support for the moment and assuming $F = 1$ on the whole space, one can show that $d_y(x) = |x - y|$ solves (4). In this simplified case, d_y is just the usual geometric distance to y. For more general F, the solution $d_y(x)$ of (4) yields a *generalised distance* instead. As we will see in Theorem 3, this distance corresponds to the shortest travel time from the source y to some arbitrary target x, with F defining the allowed speed of movement at each point in space. There exists a vast literature about (4), mainly for the case that F is uniformly bounded away from zero. See, for instance, Theorem 5.3 on page 132 of [23]. We will, however, not use this assumption here and deduce some relevant results for the more general case instead. As in the preceding section, let us define $\Omega^+ = F^{-1}((0,\infty))$ as the set where F is strictly positive. Ω^+ is clearly open and bounded, and each connected component of Ω^+ is also open and thus even path-connected. This, in particular, implies that the set $X_{\mathrm{ad}}(x,y)$ defined below is not empty.

Definition 3. Let $C \subset \Omega^+$ be a connected component and $x, y \in C$. A *path* connecting x with y is a function $\xi \in W^{1,\infty}([0,1],\mathbb{R}^n)$ with $\xi(0) = x$, $\xi(1) = y$ and $\xi(t) \in C$ for all $t \in [0,1]$. $X_{\mathrm{ad}}(x,y)$ is the set of all such paths connecting x with y.

For $\xi \in X_{\mathrm{ad}}(x,y)$, the *Euclidean arc length* is defined as

$$|\xi| = \int_0^1 |\xi'(t)|\,dt$$

in the usual way. We can also define the corresponding *F-induced length* of ξ as

$$l(\xi) = \int_0^1 \frac{|\xi'(t)|}{F(\xi(t))}\,dt. \tag{5}$$

By the Sobolev embedding theorem (see, for instance, Theorem 6 on page 270 of [14]), each path $\xi \in X_{\mathrm{ad}}(x, y)$ is continuous. Furthermore, by the continuity of F also $F \circ \xi$ is continuous, and thus this function attains a minimum on the compact interval $[0, 1]$. Since ξ maps into C, this minimum must actually be strictly positive. Thus, (5) is well-defined with $0 \leq l(\xi) < \infty$. Also note that using the fundamental theorem of calculus, one can easily show that the straight line

$$S_{xy}(t) = x + t(y - x)$$

between x and y has the shortest possible arc length: Let $\xi \in X_{\mathrm{ad}}(x, y)$, then

$$|\xi| = \int_0^1 |\xi'(t)|\, dt \geq \left| \int_0^1 \xi'(t)\, dt \right| = |\xi(1) - \xi(0)| = |x - y| = |S_{xy}| .$$

Another key observation is that one can always reparametrise a given path ξ such that $|\xi'(t)| = |\xi|$ is constant for all $t \in [0, 1]$. This corresponds to a reparametrisation by path length followed by a scaling of the time interval from $[0, |\xi|]$ back to $[0, 1]$. It is easy to see that this operation does not change any geometrical properties and leaves, in particular, $|\xi|$ and $l(\xi)$ invariant. In the following, we will most of the time assume without loss of generality that this is done for the considered paths. With this assumption, the path length (5) becomes

$$l(\xi) = \int_0^1 \frac{|\xi|}{F(\xi(t))}\, dt. \tag{6}$$

Finally, note that paths $\xi \in W^{1,\infty}([0, 1], \mathbb{R}^n)$ are Lipschitz continuous. We denote the optimal Lipschitz constant by

$$\operatorname{Lip}(\xi) = \sup_{t \neq s \in [0,1]} \left| \frac{\xi(t) - \xi(s)}{t - s} \right| .$$

For paths reparametrised by arc length in the way described, it follows easily that $\operatorname{Lip}(\xi) = |\xi|$.

Definition 4. For $x \in \mathbb{R}^n$, we set $d(x, x) = 0$. For $y \neq x$, if there exists a connected component $C \subset \Omega^+$ with $x, y \in C$, we define

$$d(x, y) = \inf_{\xi \in X_{\mathrm{ad}}(x,y)} l(\xi).$$

Otherwise, we set $d(x, y) = \infty$.

This defines a generalised distance $d(x, y) \in [0, \infty]$ between any two points x and y. This distance corresponds to the "shortest travel time" between the points under the speed field F. If the points are not in the same connected component of Ω^+, then each path between them must necessarily pass through intermediate points z with $F(z) = 0$, which justifies the definition of $d(x, y) = \infty$ in this case. This will be further clarified by Lemma 5 below.

Since the difference between our situation and the one handled commonly in the literature is that we allow $F(x) \to 0$ as $x \to \partial\Omega^+$ without a strictly positive lower bound, we have to pay special attention to this case. It turns out, however, that Lipschitz continuity of F ensures that paths with finite length can never actually reach $\partial\Omega^+$:

Lemma 5. *Let $C \subset \Omega^+$ be a connected component, $X \subset C$ be compact, and choose $M > 0$. Then there exists $\underline{F} > 0$ such that for all $x \in X$, $y \in C$ and $\xi \in X_{\mathrm{ad}}(x, y)$, the condition $F(\xi(t)) \leq \underline{F}$ for some $t \in [0, 1]$ necessarily implies $l(\xi) \geq M$.*

Proof. Let $\xi \in X_{\mathrm{ad}}(x, y)$ and assume that $F(\xi(t_0)) \leq \underline{F}$ for some $\underline{F} > 0$ and $t_0 \in [0, 1]$. From Lipschitz continuity, we get

$$F(\xi(t)) \leq \underline{F} + |\xi|\, L_F\, |t - t_0| .$$

But this also implies

$$\frac{|\xi|}{F(\xi(t))} \geq \frac{|\xi|}{\underline{F} + |\xi|\, L_F (t_0 - t)}$$

for all $t \in [0, t_0]$. Hence

$$l(\xi) \geq \int_0^{t_0} \frac{|\xi|}{F(\xi(t))}\, dt \geq |\xi| \int_0^{t_0} \frac{1}{\underline{F} + |\xi|\, L_F (t_0 - t)}\, dt = \frac{\log(|\xi|\, L_F t_0 + \underline{F}) - \log(\underline{F})}{L_F}.$$

Now assume $F(x) \geq F_0 > 0$ for all $x \in X$, which is possible by compactness of X, and pick $x \in X$. Again by Lipschitz continuity, we have

$$F_0 \leq F(x) = F(\xi(0)) \leq \underline{F} + |\xi|\, L_F t_0 \quad \Rightarrow \quad |\xi|\, L_F t_0 \geq F_0 - \underline{F} > \frac{F_0}{2}$$

if only $\underline{F}$ is chosen less than $F_0/2$. In particular, $|\xi|\, L_F t_0$ is bounded away from zero with a constant depending only on X. But this gives further

$$l(\xi) \geq \frac{\log(|\xi|\, L_F t_0) - \log(\underline{F})}{L_F} \geq \frac{\log(F_0/2) - \log(\underline{F})}{L_F},$$

which can be made arbitrarily large by choosing $\underline{F}$ small enough. Again, the choice of $\underline{F}$ for the claim to be satisfied only depends on X and F but not the particular x, y or ξ under consideration. □

Lemma 5 can be interpreted as a coercivity result: Since path lengths become infinite when approaching the boundary of Ω^+, such paths can never be relevant for the determination of *shortest paths* and, consequently, the distance $d\,(\cdot,\cdot)$. This allows us to restrict ourselves to *compact* subsets of Ω^+ in these situations. As a first application of this feature, let us show the existence of a path with minimal length if x and y are in the same connected component (i. e., $d\,(x,y) < \infty$). For this, we make use of the theorem of Arzelà-Ascoli (see C.7 in [14]) to get compactness as well as Lipschitz estimates to show lower semi-continuity. To be precise:

Lemma 6. *Let $x, y \in \mathbb{R}^n$ with $d\,(x,y) < \infty$. Let $\xi \in X_{\mathrm{ad}}\,(x,y)$, $(\xi_k) \subset X_{\mathrm{ad}}\,(x,y)$ and $L > 0$ be a uniform Lipschitz constant for all ξ_k. Assume $\xi_k \to \xi$ uniformly and that all ξ_k are parametrised by path length. (We do not assume this to be true for ξ.) Then also ξ is Lipschitz continuous and we have*

$$|\xi| = \int_0^1 |\xi'(t)|\, dt \leq \mathrm{Lip}\,(\xi) \leq \liminf_{k\to\infty} \mathrm{Lip}\,(\xi_k) \leq L.$$

Proof. For simplicity, assume that $\lim_{k\to\infty} \mathrm{Lip}\,(\xi_k)$ exists. (If that is not the case, choose a subsequence that converges to the limes inferior.) Fix $t \neq s$ and note that

$$\left| \frac{\xi(t) - \xi(s)}{t - s} \right| = \lim_{k\to\infty} \left| \frac{\xi_k(t) - \xi_k(s)}{t - s} \right| \leq \lim_{k\to\infty} \mathrm{Lip}\,(\xi_k)\,.$$

Since this is true for arbitrary t and s, it also holds in the supremum to give $\mathrm{Lip}\,(\xi) \leq \lim_{k\to\infty} \mathrm{Lip}\,(\xi_k)$. The remaining estimates follow immediately. □

Lemma 7. *Let $x, y \in \mathbb{R}^n$ and $d\,(x,y) < \infty$. Then there exists $\xi \in X_{\mathrm{ad}}\,(x,y)$ such that $d\,(x,y) = l(\xi)$.*

Proof. By definition of $d\,(x,y)$, we can find a *minimising sequence* (ξ_k) with $l(\xi_k) \to d\,(x,y)$. Assume that each ξ_k is parametrised by path length. This implies that there exists a uniform Lipschitz constant L for all ξ_k, since each has Lipschitz constant $\mathrm{Lip}\,(\xi_k) = |\xi_k|$. The arc lengths $|\xi_k|$, in turn, are bounded uniformly because the sequence minimises $l(\xi_k)$ and

$$\frac{\mathrm{Lip}\,(\xi_k)}{\overline{F}} \leq \int_0^1 \frac{|\xi_k|}{F(\xi_k(t))}\, dt = l(\xi_k).$$

By the theorem of Arzelà-Ascoli, there exists a continuous path ξ that is the uniform limit of a subsequence of (ξ_k). Without loss of generality, assume that the subsequence is (ξ_k) itself, so that $\xi_k \to \xi$ uniformly. Furthermore, also ξ is Lipschitz continuous with Lipschitz constant L by Lemma 6. Thus $\xi \in X_{\mathrm{ad}}\,(x,y)$. Note that, in particular, the image of ξ has to lie inside of $C \subset \Omega^+$. If this were not the case, then Lemma 5 would imply that the sequence (ξ_k) has unbounded path lengths. This would contradict the assumption that it is a minimising sequence. By definition of $d\,(x,y)$, it is clear that $l(\xi) \geq d\,(x,y)$. It remains to show that also $l(\xi) \leq d\,(x,y) = \lim_{k\to\infty} l(\xi_k)$ holds.

For this, define $g(\tau) = 1/F(\xi(\tau))$. Since the image of ξ lies inside of Ω^+, we know that $F \circ \xi$ is bounded away from zero. Hence, g is Lipschitz continuous with some constant L_g. Assume that we have some partition $\mathcal{I}$ of $[0,1]$ into intervals $I_i = [t_i, t_{i+1}]$ and that $\mathcal{I}$ has fineness $h = \sup_i (t_{i+1} - t_i)$. Choose

$\epsilon > 0$ arbitrary and pick $K \in \mathbb{N}$ such that $\operatorname{Lip}(\xi; I_i) \le \operatorname{Lip}(\xi_k; I_i) + \epsilon$ for all intervals $I_i \in \mathcal{I}$ and $k \ge K$. This is possible due to Lemma 6 (applied to the intervals I_i instead of $[0, 1]$).

Now, using again the estimates in Lemma 6, we get:

$$\int_0^1 g(\tau)\,|\xi'(\tau)|\;d\tau \le \sum_{I_i\in\mathcal{I}} \sup_{\tau\in I_i} g(\tau)\cdot(t_{i+1}-t_i)\cdot \operatorname{Lip}(\xi; I_i) \le \sum_{I_i\in\mathcal{I}} \sup_{\tau\in I_i} g(\tau)\cdot \left(\int_{I_i} |\xi_k'(\tau)|\;d\tau + (t_{i+1}-t_i)\epsilon\right)$$

Since g is Lipschitz continuous, we also know that

$$\sup_{\tau\in I_i} g(\tau) \le \inf_{\tau\in I_i} g(\tau) + hL_g.$$

Furthermore, clearly

$$\inf_{\tau\in I_i} g(\tau) \int_{I_i} |\xi_k'(\tau)|\;d\tau \le \int_{I_i} g(\tau)\,|\xi_k'(\tau)|\;d\tau.$$

All that taken together yields

$$\begin{aligned}\int_0^1 g(\tau)\,|\xi'(\tau)|\;d\tau &\le \sum_{I_i\in\mathcal{I}} \left(\int_{I_i} g(\tau)\,|\xi_k'(\tau)|\;d\tau + hL_g \int_{I_i} |\xi_k'(\tau)|\;d\tau + (t_{i+1}-t_i)\epsilon\,\|g\|_\infty\right) \\ &= \int_0^1 g(\tau)\,|\xi_k'(\tau)|\;d\tau + hL_g\,|\xi_k| + \epsilon\,\|g\|_\infty .\end{aligned}$$

Since both ϵ and $\mathcal{I}$ and thus h were arbitrary, this implies the claim. Note that $|\xi_k|$ is bounded for $k \to \infty$. □

We continue by deriving some fundamental properties of and estimates for these path lengths and the distance function $d(\cdot,\cdot)$:

Lemma 8. *Let $x, y \in \mathbb{R}^n$ be arbitrary. Then*

$$|x-y| \le \overline{F}\cdot d(x,y).$$

Proof. For $d(x,y) = \infty$ and for $x = y$, the claim is obvious. Thus assume that x and y are in the same connected component of Ω^+, and let $\xi \in X_{\mathrm{ad}}(x,y)$. Then

$$l(\xi) = \int_0^1 \frac{|\xi|}{F(\xi(x))}\;dt \ge \frac{|\xi|}{\overline{F}} \ge \frac{|x-y|}{\overline{F}},$$

where we have used the assumption of a reparametrised ξ and (6). □

Lemma 8 gives an important estimate relating $d(x,y)$ and $|x-y|$. If we use Lipschitz continuity of F in addition, also more precise estimates are possible especially for points close to each other. In particular, we get the following localised version of Lemma 8:

Lemma 9. *Let $x \in \Omega^+$ and $y \in \mathbb{R}^n$. Then we have:*

$$|x-y| \le \frac{F(x)}{L_F}\left(e^{L_F d(x,y)} - 1\right)$$

If furthermore $L_F\,|x-y| < F(x)$, then also

$$\begin{aligned} d(x,y) \le l(S_{xy}) &\le \frac{1}{L_F}\log\frac{F(x)}{F(x) - L_F\,|x-y|}, \\ |x-y| &\ge \frac{F(x)}{L_F}\left(1 - e^{-L_F d(x,y)}\right)\end{aligned}$$

hold. As before, $S_{xy} \in X_{\mathrm{ad}}(x,y)$ denotes the straight line between x and y.

Proof. If $d(x,y)=\infty$, the first claim is clear. For the remaining case, we choose $\xi \in X_{\text{ad}}(x,y)$ arbitrary. Let ξ be parametrised by its arc length $|\xi|$. Then $|\xi| \geq |x-y|$ and Lipschitz continuity of F implies

$$F(\xi(t)) \leq F(x) + L_F\,|\xi(t)-x| = F(x) + L_F \left| \int_0^t \xi'(\tau)\,d\tau \right| \leq F(x) + L_F t\,|\xi|$$

for all $t \in [0,1]$. This yields

$$l(\xi) = \int_0^1 \frac{|\xi|}{F(\xi(t))}\,dt \geq \int_0^1 \frac{|\xi|}{F(x)+L_F\,|\xi|\,t}\,dt = \frac{1}{L_F}\log\frac{F(x)+|\xi|\,L_F}{F(x)} \geq \frac{1}{L_F}\log\left(1+\frac{|x-y|\,L_F}{F(x)}\right),$$

which also holds in the infimum over all possible paths ξ. Thus

$$e^{L_F d(x,y)} \geq 1 + \frac{|x-y|\,L_F}{F(x)},$$

which further implies the first estimate.

For the second estimate, note that $d(x,y) \leq l(S_{xy})$ as well as $|S_{xy}| = |x-y|$. Lipschitz continuity tells us that

$$F(S_{xy}(t)) \geq F(x) - L_F\,|S_{xy}(t)-x| = F(x) - L_F\,|x-y|\,t,$$

and by our assumption this expression is guaranteed to be positive for all $t \in [0,1]$. This implies also, in particular, that S_{xy} lies entirely inside of Ω^+. Thus we find

$$d(x,y) \leq l(S_{xy}) = \int_0^1 \frac{|S_{xy}|}{F(S_{xy}(t))}\,dt \leq \int_0^1 \frac{|x-y|}{F(x)-L_F\,|x-y|\,t}\,dt = \frac{1}{L_F}\log\frac{F(x)}{F(x)-L_F\,|x-y|}.$$

The third estimate is just an equivalent reformulation of the second one. □

The following is a variant of Theorem 5.1 on page 117 of [23], adapted for our problem (4):

Lemma 10. *$d(\cdot,\cdot)$ defines a metric on each connected component of Ω^+.*

Proof. By Definition 4, $d(x,x)=0$ for each $x \in \mathbb{R}^n$. Let $C \subset \Omega^+$ be a connected component, $x,y \in C$ and $x \neq y$. For each $\xi \in X_{\text{ad}}(x,y)$, we can define $\xi_c(t) = \xi(1-t)$, which yields $\xi_c \in X_{\text{ad}}(y,x)$ with $l(\xi) = l(\xi_c)$. Hence $d(x,y) = d(y,x)$ holds also in this case. Finally, let $z \in C$ be given in addition. We use Lemma 7 to choose $\xi_1 \in X_{\text{ad}}(x,z)$ and $\xi_2 \in X_{\text{ad}}(z,y)$ with

$$l(\xi_1) + l(\xi_2) = d(x,z) + d(z,y).$$

We define ξ to be the concatenation of ξ_1 and ξ_2 with subsequent reparametrisation. Consequently, $\xi \in X_{\text{ad}}(x,y)$ and

$$d(x,y) \leq l(\xi) = l(\xi_1) + l(\xi_1) = d(x,z) + d(z,y).$$

It remains to verify that we also have non-degeneracy in the form of

$$d(x,y) = 0 \;\Rightarrow\; x = y.$$

This, however, follows directly from Lemma 8. □

Lemma 11. *Let $\Omega' \subset \Omega^+$ be compact and convex. Then*

$$d(x,y) \leq L\,|x-y|$$

for all $x,y \in \Omega'$, where $L = 1/\underline{F}$ and $\underline{F} > 0$ is the minimum of F over Ω'.

Proof. Let $\underline{F} > 0$ be the minimum of F on the compact set Ω' and choose $x,y \in \Omega'$. We consider the straight line S_{xy} as particular path in $X_{\text{ad}}(x,y)$. Convexity ensures that $S_{xy}(t) \in \Omega'$ for all $t \in [0,1]$. The claim follows with

$$d(x,y) \leq l(S_{xy}) = \int_0^1 \frac{|S_{xy}|}{F(S_{xy}(t))}\,dt \leq \frac{|x-y|}{\underline{F}}.$$

□

Lemma 12. *$d(\cdot,\cdot)$ is continuous in both arguments on each connected component of Ω^+.*

Proof. Let $C \subset \Omega^+$ be a connected component and $x \in C$. By symmetry according to Lemma 10 it is sufficient to show that $d_x(\cdot) = d(\cdot, x)$ is continuous on C. First, we show that d_x is continuous at x itself. For this, let $(x_k) \subset C$ with $x_k \to x$ as $k \to \infty$ and pick $\delta > 0$ such that $\overline{B_\delta(x)} \subset C$. This is possible since C is open. Assume for simplicity and without loss of generality that $|x_k - x| < \delta$ for all $k \in \mathbb{N}$. Since $\overline{B_\delta(x)}$ is compact and convex, we can apply Lemma 11 in order to deduce

$$0 \le d_x(x_k) = d(x_k, x) \le L\,|x_k - x|$$

for some constant L. Since the right-hand side vanishes with $x_k \to x$, we find that also $d_x(x_k) \to 0 = d_x(x)$. This shows continuity of d_x at x.

Now let $x, y \in C$ be arbitrary and pick $\epsilon > 0$. From the previous argument, we know that there exists $\delta > 0$ such that $d(x, x') < \epsilon$ if only $|x' - x| < \delta$. Applying the triangle inequality and the reverse triangle inequality from Lemma 10 for some intermediate point $x' \in B_\delta(x)$, we get

$$d(x,y) \le d(x,x') + d(x',y) \quad \Rightarrow \quad d(x',y) \ge d(x,y) - d(x,x') \ge d(x,y) - \epsilon$$

and

$$d(x',y) \le d(x',x) + d(x,y) \le d(x,y) + \epsilon.$$

Hence we find $|d(x',y) - d(x,y)| < \epsilon$ whenever $|x' - x| < \delta$, which is the claimed continuity. ☐

After this quite general discussion, let us get back to the task of solving (4). For the case of a strictly positive, uniform lower bound $F \ge \underline{F} > 0$, this is well-understood. In order to reduce our more general problem to the known results, we will make use of Lemma 5. As a first step, let us introduce a sequence of *cut-off speeds*:

Definition 5. For given $\underline{F} > 0$, we define $\tilde{F}(x) = \max(F(x), \underline{F})$. The notation $l_{\underline{F}}(\xi)$ and $d_{\underline{F}}(x,y)$ will be used for path lengths and distances according to Definition 3 and Definition 4, respectively, based on $\tilde{F}$ instead of the original F.

Note that $\tilde{F}$ does no longer have compact support, but this is no problem since we only used compact support above in order to guarantee a strictly positive minimum of F over certain subsets. In the case of $\tilde{F}$, this is already fulfilled by definition, and thus all arguments go through correspondingly. The distance $d_{\underline{F}}(x,y)$ is equivalent to $L(x,y)$ given in (47) on page 116 of [23] when using $n(x) = 1/\tilde{F}(x)$, and Theorem 5.1 on page 117 of [23] will be the main tool on which we build our results. There, it is shown that $d_{\underline{F}}(\cdot, y)$ is a viscosity solution of (4) if the speed F is replaced by $\tilde{F}$. We will now work on reducing our situation to the case where the result of [23] is applicable.

Lemma 13. *Let $C \subset \Omega^+$ be a connected component and $X \subset C$ be compact. Then there exists $\underline{F} > 0$ such that $d(x,y) = d_{\underline{F}'}(x,y)$ for all $\underline{F}' \in (0, \underline{F}]$ and $x, y \in X$.*

Proof. Let $\tilde{F}$ be the cut-off speed for some $\underline{F} > 0$. Then clearly $F(x) \le \tilde{F}(x)$ and thus for an arbitrary path we always have $l_{\underline{F}}(\xi) \le l(\xi)$. Thus also $d_{\underline{F}}(x,y) \le d(x,y)$ is always the case. It remains to show that our assumptions actually imply equality. Furthermore, if we show the result for a single $\underline{F}$ as cut-off threshold, it holds also for all smaller thresholds. This is the case because $\underline{F}' \le \underline{F}$ implies $d(x,y) = d_{\underline{F}}(x,y) \le d_{\underline{F}'}(x,y) \le d(x,y)$.

Set $M = \max_{x,y \in X} d(x,y)$, which is well-defined and finite because of Lemma 12 since X is compact. We now apply Lemma 5 for this M to get a corresponding threshold $\underline{F}$. We can assure $F(x) > \underline{F}$ for all $x \in X$ by decreasing $\underline{F}$ further as necessary. Now, assume that $d_{\underline{F}}(x,y) < d(x,y)$ for some $x, y \in X$. Choose a minimising path $\xi \in X_{\mathrm{ad}}(x,y)$ with $l_{\underline{F}}(\xi) = d_{\underline{F}}(x,y)$. Consequently,

$$l_{\underline{F}}(\xi) < d(x,y) \le l(\xi).$$

This, however, implies that there exists $t \in [0,1]$ with $F(\xi(t)) \le \underline{F}$. If that would not be the case, then $F(\xi(t)) = \tilde{F}(\xi(t))$ for all $t \in [0,1]$ and thus the lengths would have to coincide. Define now

$$t_0 = \inf\{t \in [0,1] \mid F(\xi(t)) \le \underline{F}\} = \min\{t \in [0,1] \mid F(\xi(t)) \le \underline{F}\},$$

which is well-defined because the set is non-empty and F is continuous. Furthermore, for all $t \in [0, t_0]$ we have $F(\xi(t)) \geq \underline{F}$ and consequently $F(\xi(t)) = \tilde{F}(\xi(t))$. Also, $t_0 > 0$ because $x \in X$ and thus $F(\xi(0)) = F(x) > \underline{F}$. Now define a new path $\xi_1(t) = \xi(t/t_0)$. We have $\xi_1 \in X_{\text{ad}}(x, \xi(t_0))$, and since F equals $\tilde{F}$ along the path, also $l(\xi_1) = l_{\underline{F}}(\xi_1)$. Since $F(\xi_1(1)) = \underline{F}$, we can apply Lemma 5 now to ξ_1 to deduce $l(\xi_1) \geq M$. This is a contradiction, since

$$l_{\underline{F}}(\xi) \geq l_{\underline{F}}(\xi_1) = l(\xi_1) \geq M$$

and we know $l_{\underline{F}}(\xi) < d(x, y) \leq M$. □

Theorem 3. *Let $C \subset \Omega^+$ be a connected component and $y \in C$ be fixed. Then $d_y(\cdot) = d(\cdot, y)$ is a viscosity solution of (4) in $C \setminus \{y\}$.*

Proof. d_y is continuous on C by Lemma 12, and the boundary condition is trivially satisfied because we have $d_y(y) = d(y, y) = 0$. It remains to show that it satisfies $F(x) |\nabla d_y(x)| = 1$ in the viscosity sense. For this, let $x \in C \setminus \{y\}$ be fixed and $p \in J^{1+} d_y(x)$ be in the superdifferential. Now choose $Y \subset C$ compact, connected, with smooth boundary and such that $x, y \in Y^\circ$. Choose $\underline{F}$ for Y according to Lemma 13 and such that $\underline{F} \leq \min_{z \in Y} F(z)$ in addition. Then $\tilde{F} = F$ on Y and $d_y(x') = d(x', y) = d_{\underline{F}}(x', y)$ for all $x' \in Y$. The latter property holds, in particular, also in a neighbourhood of x so that $p \in J^{1+} d_{\underline{F}}(x, y)$ must be true. Since $d_{\underline{F}}(\cdot, y)$ solves (4) with $\tilde{F}$ in $Y^\circ \setminus \{y\}$ in the viscosity sense as noted above according to Theorem 5.1 on page 117 of [23], this implies $F(x) |p| = \tilde{F}(x) |p| \leq 0$. This, however, is all we need to show that d_y is a viscosity subsolution of (4) on the whole of $C \setminus \{y\}$. By a symmetric argument one can also show that it is a viscosity supersolution. □

We conclude this section with a final auxiliary result that will be useful later:

Lemma 14. *Let $X \subset \mathbb{R}^n$ be closed, $x \in X$ and assume that $d(x, y) \geq t$ for some $t \geq 0$ and all $y \in \partial X$. Then $d(x, y) > t$ for all $y \notin X$. More precisely: If $y \notin X$ and $\delta > 0$ are such that $B_\delta(y) \cap X = \emptyset$, then $d(x, y) \geq t + \delta / \overline{F}$.*

Proof. Let $y \notin X$, then there exists $\delta > 0$ such that $B_\delta(y) \subset \mathbb{R}^n \setminus X$ since $\mathbb{R}^n \setminus X$ is open. Consequently, $|x' - y| \geq \delta > 0$ for all $x' \in X$. Clearly, $y \neq x$ and if $x \notin \Omega^+$, $y \notin \Omega^+$ or they are not in the same connected component of Ω^+, then $d(x, y) = \infty > t$ holds. So assume that $x, y \in \Omega^+$ are in the same connected component and choose $\xi \in X_{\text{ad}}(x, y)$. Since ξ is continuous, the set $\xi^{-1}(X) \subset [0, 1]$ is closed and since it is also bounded, it is compact. Define thus

$$t_0 = \max \{ t \in [0, 1] \mid \xi(t) \in X \} \text{ and } y_0 = \xi(t_0).$$

Then $y_0 \in X$ and furthermore $y_0 \in \partial X$ since every sequence $(\xi(t_k))$ with $t_k \to t_0$ from above is in $\mathbb{R}^n \setminus X$ and converges to y_0. Denote the two "pieces" of ξ up to and starting at y_0 by ξ_1 and ξ_2, respectively. Then $\xi_1 \in X_{\text{ad}}(x, y_0)$, $\xi_2 \in X_{\text{ad}}(y_0, y)$ and $l(\xi) = l(\xi_1) + l(\xi_2)$. Since $y_0 \in \partial X$, we know by our assumption that $l(\xi_1) \geq d(x, y_0) \geq t$. Furthermore, Lemma 8 implies that

$$l(\xi_2) \geq d(y_0, y) \geq \frac{|y_0 - y|}{\overline{F}} \geq \frac{\delta}{\overline{F}}.$$

Both estimates together imply $l(\xi) \geq t + \delta / \overline{F}$. Since ξ was arbitrary, also $d(x, y) \geq t + \delta / \overline{F} > t$ follows. □

Note that Lemma 14 can be interpreted as a variant of the classical *minimum principle*: If $x \in \mathbb{R}^n$ is fixed and $Y \subset \mathbb{R}^n$ open with $x \notin Y$, then $d(x, \cdot)$ attains its minimum over $\overline{Y}$ at ∂Y. A corresponding maximum principle does, however, not hold in general: If $F(y)$ is very small or even zero, then all paths connecting x to y may have a larger length than paths "circling around" y. In this case, $d(x, y)$ can, indeed, have a local maximum at y. See also [2], where a more general result is derived for viscosity solutions. The Eikonal equation, in particular, is covered by example 5. A discussion of the classical minimum and maximum principles for harmonic functions can be found in section 2.5 of [18].

4 The Hopf-Lax Formula

We now turn our attention from the stationary problem (4) considered in Section 3 back to the time-dependent level-set equation (1). As before, we assume that F is Lipschitz continuous with constant L_F and has compact support, and that also ϕ_0 is Lipschitz continuous with constant L_{ϕ_0}. Since we are particularly interested in the evolving geometries described by the zero level set of $\phi(\cdot, t)$, we also introduce the notation

$$\Omega_t = \phi(\cdot, t)^{-1}\left((-\infty, 0)\right), \quad \Gamma_t = \phi(\cdot, t)^{-1}\left(\{0\}\right).$$

Clearly, Ω_t is open for all $t \geq 0$. Similarly, the set $\Gamma_t \cup \Omega_t = \phi(\cdot, t)^{-1}\left((-\infty, 0]\right)$ is closed since ϕ is continuous. Note, though, that Γ_t need not be the topological boundary $\partial\Omega_t$ of Ω_t. If ϕ is "degenerate", Γ_t may contain interior points. This effect is called *fattening* and will be discussed in Subsection 5.1 below.

From classical optimal-control theory, it is well-known that the level-set equation (1) can be related to a *Mayer problem*. For a thorough discussion, see section III.3 of [1]. Below, we state the main arguments only briefly. For this, let us consider $F \geq 0$ for a moment. We define

$$S_t(x) = \left\{\xi \in W^{1,\infty}([0, t], \mathbb{R}^n) \mid \xi(0) = x,\ |\xi'(\tau)| \leq F(\xi(\tau)) \text{ for all } \tau \in [0, t]\right\}.$$

This is the set of paths starting in x with length at most t. While it is similar in spirit to the set $X_{\mathrm{ad}}(x, y)$ of Definition 3 used above, there is a slight difference: Before, we fixed both end points of the path and were interested in the path length. Now, we fix the starting point and the length, and consider possible end points. These paths can be used to define the *reachable set* from x in time t as

$$R_t(x) = \{\xi(t) \mid \xi \in S_t(x)\}.$$

We are now interested in the following minimisation problem:

$$\phi(x, t) = \inf_{\xi \in S_t(x)} \phi_0(\xi(t)) = \inf_{y \in R_t(x)} \phi_0(y) \tag{7}$$

Standard arguments show that the value function ϕ of this problem is a viscosity solution of the corresponding Hamilton-Jacobi-Bellman equation. This equation, in turn, is nothing else than the level-set equation (1). Furthermore, one can also relate the reachable set to the distances discussed in the previous Section 3. This yields

$$R_t(x) = \{y \in \mathbb{R}^n \mid d(x, y) \leq t\}.$$

Note that this set is compact, so that the infimum in (7) is actually a minimum. Thus, we have established the following *Hopf-Lax formula* for the level-set equation:

Theorem 4. *Let $F \geq 0$. The Hopf-Lax formula*

$$\phi(x, t) = \min\{\phi_0(y) \mid y \in \mathbb{R}^n,\ d(x, y) \leq t\} \tag{8}$$

gives the viscosity solution of the level-set equation (1) in $\mathbb{R}^n \times [0, \infty)$.

This formula can also be derived for the case $F \geq \underline{F} > 0$ from Theorem 3.1 on page 140 of [6]. To remove the required lower bound and show Theorem 4, one can then proceed with cut-off arguments as in the proof of Theorem 3. However, we believe that the derivation based directly on the Mayer problem is the most straight-forward argument.

Based on this representation formula (8) shown for the level-set function ϕ, we will now proceed to derive corresponding formulas describing the evolving sets Ω_t, Γ_t and $\Gamma_t \cup \Omega_t$ themselves. The crucial ingredient in those formulas is the distance between a point x and the initial set (not just a single point as given by $d(x, \cdot)$). This distance corresponds to the time it takes the evolving front to arrive at x:

Definition 6. Let $F \geq 0$ and denote by $d(\cdot, \cdot)$ the distance introduced in Definition 4. For $x \in \mathbb{R}^n$, we set

$$d_0(x) = \inf_{y \in \Gamma_0 \cup \Omega_0} d(x, y), \quad d_0'(x) = \inf_{y \in \Omega_0} d(x, y).$$

If C is a connected component of Ω^+, then d_0 is finite on the whole of C if and only if C contains a part of the initial domain $\Gamma_0 \cup \Omega_0$. If this is not the case, then the distance is infinity on the whole of C. For d_0', a corresponding statement is true.

Take note that it follows immediately from Definition 6 that $d_0(x) \le d_0'(x)$ holds for all $x \in \mathbb{R}^n$. We will show now that strict inequality can only be the case if Γ_0 has interior. This is an unusual situation in applications, although we have not excluded it so far. Also note that the infimum is actually a minimum if we take it over a closed set.

Lemma 15. *For each $x \in \mathbb{R}^n$, there exist $y \in \Gamma_0 \cup \Omega_0$ and $y' \in \overline{\Omega_0}$ with $d_0(x) = d(x, y)$ and $d_0'(x) = d(x, y')$. Furthermore, if $\overline{\Omega_0} = \Gamma_0 \cup \Omega_0$ and either $x \in \Omega^+$ or $x \notin \Gamma_0$, then $d_0(x) = d_0'(x)$.*

Proof. Let $x \in \mathbb{R}^n$ be given. Consider the case $x \notin \Omega^+$ first. If $x \in \Gamma_0 \cup \Omega_0$, then $d_0(x) = d(x, x) = 0$ and the claim is true. If this is not the case, then $d_0(x) = d(x, y) = \infty$ for all $y \ne x$, and the claim also holds. The same argument can be used for Ω_0 and d_0'. For the second claim, we can assume $x \notin \Gamma_0$ since $x \notin \Omega^+$ by assumption. But then either $d_0(x) = d_0'(x) = 0$ if $x \in \Omega_0$, or otherwise $d_0(x) = d_0'(x) = \infty$ since then $x \notin \Gamma_0 \cup \Omega_0$.

Now, assume that $x \in C$ where $C \subset \Omega^+$ is a connected component. If $d_0(x) = \infty$, then $C \cap (\Gamma_0 \cup \Omega_0) = \emptyset$ and we can choose y to be any element of $\Gamma_0 \cup \Omega_0$. The same if $d_0'(x) = \infty$. So assume from now on that $d_0(x)$ and $d_0'(x)$ are both finite. This together with Lemma 5 implies that there exists a compact set $X \subset C$ such that

$$d_0(x) = \inf_{y \in X \cap (\Gamma_0 \cup \Omega_0)} d(x, y), \quad d_0'(x) = \inf_{y \in X \cap \Omega_0} d(x, y). \tag{9}$$

To see this, choose $y \in C$ arbitrarily for a moment. Then $d(x, y) < \infty$. According to Lemma 5, there exists $\underline{F} > 0$ such that $d(x, y') > d(x, y)$ for all y' with $F(y') < \underline{F}$. Thus, choosing $X = F^{-1}\left(\left[\underline{F}, \overline{F}\right]\right)$ ensures (9).

Note that $d(x, \cdot)$ is finite and continuous when restricted to X. By taking a minimising sequence and using this continuity as well as compactness of the sets $X \cap (\Gamma_0 \cup \Omega_0)$ and $X \cap \overline{\Omega_0}$, we see that the infima in (9) are actually minima. If $\overline{\Omega} = \Gamma_0 \cup \Omega_0$ and $y \in \Gamma_0 \cup \Omega_0$ is chosen with $d_0(x) = d(x, y)$, then

$$d_0(x) = d(x, y) \ge d_0'(x) \ge d_0(x),$$

showing equality between $d_0(x)$ and $d_0'(x)$. □

Theorem 5. *Let $F \ge 0$. Then the evolving sets can be represented as*

$$\begin{aligned} \Gamma_t \cup \Omega_t &= \{x \in \mathbb{R}^n \mid d_0(x) \le t\}, \\ \Omega_t &= \{x \in \mathbb{R}^n \mid d_0'(x) < t\}, \\ \Gamma_t &= \{x \in \mathbb{R}^n \mid d_0(x) \le t \le d_0'(x)\} \end{aligned}$$

for all $t > 0$. If $\overline{\Omega_0} = \Gamma_0 \cup \Omega_0$, then the last relation states that

$$\Gamma_t = \{x \in \mathbb{R}^n \mid d_0(x) = t\} \cup (\Gamma_0 \setminus \Omega^+). \tag{10}$$

Proof. We use Theorem 4 to express ϕ by (8). Let $x \in \Gamma_t \cup \Omega_t$. By definition, this means $\phi(x, t) \le 0$. Hence (8) implies that there exists $y \in \Gamma_0 \cup \Omega_0$ with $d(x, y) \le t$. This, in turn, yields $d_0(x) \le d(x, y) \le t$. The other way round, let $d_0(x) \le t$. By Lemma 15, there exists $y \in \Gamma_0 \cup \Omega_0$ with $d(x, y) = d_0(x) \le t$, such that $\phi(x, t) \le \phi_0(y) \le 0$ by (8) and thus $x \in \Gamma_t \cup \Omega_t$.

Now assume that $d_0'(x) < t$. Applying Lemma 15 again, we find that there exists $y \in \overline{\Omega_0}$ with $d(x, y) = d_0'(x) < t$, and thus by continuity of $d(x, \cdot)$, there also exists $y' \in \Omega_0$ with $d(x, y') < t$. Hence (8) implies that $\phi(x, t) \le \phi_0(y') < 0$ and thus $x \in \Omega_t$. If, on the other hand, $x \in \Omega_t$ and thus $\phi(x, t) < 0$, there exists $y \in \Omega_0$ with $d(x, y) \le t$. This implies at least $d_0'(x) \le d(x, y) \le t$. However, assume that $d_0'(x) = t$. In this case, $d(x, y) = t$ must hold, and also $d(x, y') \ge t$ must be the case for all $y' \in \Omega_0$. Since Ω_0 is open, there exists a small radius $\delta > 0$ such that $B_\delta(y) \subset \Omega_0$. Define $X = \mathbb{R}^n \setminus B_\delta(y)$, which is closed, and note that $x \in X$ because otherwise $x \in \Omega_0$ and this would lead to a contradiction with $0 = d(x, x) \ge t > 0$. Because $\partial X = \partial B_\delta(y) \subset \Omega_0$, we know that $d(x, y') \ge t$ for all $y' \in \partial X$. This, however, implies $d(x, y) > t$ with Lemma 14, which is a contradiction. Thus we have shown that $d_0'(x) < t$ must be the case.

For the third equality, note that Γ_t and Ω_t are clearly disjoint, so that the relation

$$\Gamma_t = (\Gamma_t \cup \Omega_t) \setminus \Omega_t = \{x \in \mathbb{R}^n \mid d_0(x) \le t \text{ and not } d_0'(x) < t\} = \{x \in \mathbb{R}^n \mid d_0(x) \le t \le d_0'(x)\}$$

holds. Finally, assume that we know $\overline{\Omega_0} = \Gamma_0 \cup \Omega_0$ in addition. Consider $x \in \mathbb{R}^n$. If $x \in \Gamma_0 \setminus \Omega^+$, then $\phi(x,t) = \phi_0(x) = 0$ and thus $x \in \Gamma_t$ by Theorem 2. This shows that $\Gamma_0 \setminus \Omega^+$ is always a subset of both sides of (10). Consider now the case $x \notin \Gamma_0 \setminus \Omega^+$. For these x, Lemma 15 implies that $d_0(x) = d_0'(x)$, and thus also (10) holds. □

To conclude this section, we will now combine Theorem 5 with Theorem 2 to give formulas for the case of arbitrary signs of F. For this, we define $\Omega^- = F^{-1}((-\infty, 0))$ and $\Omega^z = F^{-1}(\{0\})$. We also introduce the notation

$$\Omega_0' = \mathbb{R}^n \setminus (\Gamma_0 \cup \Omega_0) = {\phi_0}^{-1}((0, \infty))$$

and assume for simplicity that we are in the case $\overline{\Omega_0} = \Gamma_0 \cup \Omega_0$. Then $\mathbb{R}^n = \Omega_0 \cup \Gamma_0 \cup \Omega_0'$ is a disjoint decomposition into two open sets and the interface Γ_0 between them, which has empty interior. Next, we define for the extended situation a modified version of d_0 from Definition 6:

$$D(x) = \begin{cases} \inf_{y \in \Gamma_0 \cup \Omega_0} d(x,y) & \text{for } x \in \Omega^+, \\ -\inf_{y \in \Gamma_0 \cup \Omega_0'} d(x,y) & \text{for } x \in \Omega^- \end{cases} \tag{11}$$

Here, $d(\cdot,\cdot)$ is according to the metric discussed in Section 3 for the speed chosen as $|F| \ge 0$. On Ω^+, where F is positive, the front moves outwards. In this case, D gives the time it takes the front to reach points outside the initial geometry. For Ω^- with negative F, the front moves inward and D is negative *inside* the original geometry. There, $-D$ is the time until an originally interior point is hit by the front and later no longer part of Ω_t at all. This convention for the sign of D gives it somewhat the characteristics of a signed distance function of the initial geometry (although with respect to the metric $d(\cdot,\cdot)$ induced by $|F|$ instead of the usual Euclidean distance). Take note that $D = d_0$ on Ω^+ by its definition in (11). On Ω^-, the function D is defined in a similar way. This implies that most of the local properties of d_0 that we will derive in the following (e. g., Lemma 16) carry over to D.

Corollary 1. *Let F be Lipschitz continuous and have compact support. Assume that $\overline{\Omega_0} = \Gamma_0 \cup \Omega_0$ and use the notation above. Then*

$$\begin{aligned} \Omega_t &= \{x \in \Omega^+ \mid D(x) < t\} \cup (\Omega_0 \cap \Omega^z) \cup \{x \in \Omega^- \mid D(x) < -t\}, \\ \Gamma_t &= \{x \in \Omega^+ \mid D(x) = t\} \cup (\Gamma_0 \cap \Omega^z) \cup \{x \in \Omega^- \mid D(x) = -t\}, \\ \Gamma_t \cup \Omega_t &= \{x \in \Omega^+ \mid D(x) \le t\} \cup ((\Gamma_0 \cup \Omega_0) \cap \Omega^z) \cup \{x \in \Omega^- \mid D(x) \le -t\} \end{aligned}$$

for all $t > 0$. Note that we no longer require that $F \ge 0$ or $F \le 0$ throughout $\mathbb{R}^n$.

Proof. Since according to Theorem 2 the level-set function $\phi(x,\cdot)$ is constant in time for each $x \in \Omega^z$, it is clear that the formulas hold for those x and we only have to consider $x \in \Omega^\pm$. Let $x \in \Omega^+$. Then Theorem 2 tells us that $\phi(x,t) = \phi^+(x,t)$ where ϕ^+ solves (1) with $F^+ = \max(F,0) \ge 0$. In particular, the evolving sets are the same as those generated by ϕ^+ when we are inside of Ω^+. Take note that $D(x) = d_0(x)$ in this case, since $d(x,\cdot)$ induced by $|F|$ is the same as $d(x,\cdot)$ induced by F^+ using arguments based on Lemma 5. Hence,

$$\Omega_t \cap \Omega^+ = \{x \in \Omega^+ \mid d_0(x) < t\} = \{x \in \Omega^+ \mid D(x) < t\}$$

follows by Theorem 5 applied to F^+. The other relations follow in the same way.

It remains to consider the case $x \in \Omega^-$. In this case, according to Theorem 2 and Lemma 4, $\phi(x,t) = -\phi^-(x,t)$ where ϕ^- solves (1) with $F^- = -\min(F,0) \ge 0$ and the initial data given by $-\phi_0$. Since this initial data corresponds to the initial geometry Ω_0', the distance for applying Theorem 5 is now $-D(x)$ in this case. This yields

$$x \notin \Omega_t \;\Leftrightarrow\; \phi(x,t) \ge 0 \;\Leftrightarrow\; \phi^-(x,t) \le 0 \;\Leftrightarrow\; -D(x) \le t \;\Leftrightarrow\; D(x) \ge -t.$$

In other words,

$$\Omega_t \cap \Omega^- = \{x \in \Omega^- \mid D(x) < -t\}$$

when taking the complement. The same can be done for the other sets as well. □

The Hopf-Lax formula derived above in Corollary 1 will be used in the following section to draw some conclusions about the shape evolution. Besides these theoretical purposes, it can also be directly employed for the *numerical computation* of evolved domains. The distance function D defined in (11) can be computed efficiently using a Fast-Marching Method (see [27] and chapter 8 of [28]). Since one has to apply Fast Marching twice to handle arbitrary signs of the speed field, this yields a *Composite Fast-Marching* method. Once this is done, the evolved domains can be assembled very cheaply for arbitrary times just by using Corollary 1. This property is useful, for instance, in the context of shape optimisation with a line search strategy: With a single computation of D, the evolved domain can be computed for various "trial step lengths" t. We employ this method successfully, for instance in [21]. The implementation is freely available in [22].

5 Applications

Let us now discuss some important conclusions from and applications of our main results shown above in Theorem 5 and Corollary 1. The powerful Hopf-Lax formula for the evolved domain Ω_t allows us to derive new results about non-fattening, shape calculus, Lipschitz continuity of the evolved level-set function ϕ and the effect of perturbations in the speed field or initial domain. While we believe that all of them are very interesting and important, particularly the shape calculus of Subsection 5.2 can be employed directly for *level-set based shape optimisation.*

5.1 Measure-Theoretic Non-Fattening

When the level-set approach is used to describe geometries, the set Γ_t as defined above is usually thought of as the "boundary" of the geometry one is interested in. With this interpretation, one definitely does not want Γ_t to become "fat" in any way (for instance, developing interior points, or having non-zero measure). A classical result showing *non-fattening* in the former, topological sense under certain conditions is [3]. We are not aware of any results with respect to the latter, measure-theoretic notion of non-fattening. Based on the representation of the evolving sets derived in Theorem 5, the issue of non-fattening can now be investigated with relative ease.

Lemma 16. *Let $\Omega^+ = F^{-1}((0,\infty))$ as before, let $C \subset \Omega^+$ be a connected component and assume that d_0 is finite on C. Then d_0 is locally Lipschitz continuous on C and, in particular, differentiable almost everywhere in C. The same holds for d_0'.*

Proof. We can assume $F \geq 0$ throughout $\mathbb{R}^n$ without loss of generality, as we only consider Ω^+ anyway. We also only consider d_0 here; the same arguments can be applied for d_0'. Note that if local Lipschitz continuity is shown, differentiability almost everywhere follows by Rademacher's theorem (see Theorem 2 on page 81 of [15]).

Let $\Omega' \subset C$ be compact and convex. Since F is continuous, we can introduce $\underline{F} > 0$ as the minimum of F over Ω'. We will show now that d_0 has the Lipschitz constant $L = 1/\underline{F}$ on Ω'. For this, let $x, y \in \Omega'$ be given. We can choose $x_0 \in \Gamma_0 \cup \Omega_0$ with $d_0(x) = d(x, x_0)$ by Lemma 15. Note also that according to our assumption $d_0(x) < \infty$, and that $d(x, y) \leq L|x - y|$ as shown in Lemma 11. Thus by the triangle inequality we have

$$d_0(y) \leq d(y, x_0) \leq d(y, x) + d(x, x_0) \leq d_0(x) + L|x - y|.$$

If we exchange the roles of x and y, the same argument can be applied to derive the estimate the other way round. Taking both inequalities together, we get $|d_0(x) - d_0(y)| \leq L|x - y|$. □

As a next step, we consider again the Eikonal equation

$$F(x)|\nabla d(x)| = 1 \text{ in } C \setminus (\Gamma_0 \cup \Omega_0), \quad d(x) = 0 \text{ on } \Gamma_0 \cup \Omega_0, \tag{12}$$

where $C \subset \Omega^+$ is a connected component on which d_0 is finite. Intuitively, it makes sense that the distance d_0 of Definition 6 should solve (12) in some sense. This will be investigated in the following, because it will be a useful tool for the proof of our non-fattening result Theorem 6. Of course, corresponding properties always also hold for d_0' when $\overline{\Omega_0}$ is used instead of $\Gamma_0 \cup \Omega_0$.

Lemma 17. *d_0 is a viscosity supersolution to (12).*

Proof. Recall that

$$d_0 = \inf_{y \in \Gamma_0 \cup \Omega_0} d_y$$

is defined as pointwise infimum of a family of functions $d_y(\cdot) = d(\cdot, y)$. Each d_y is a viscosity solution to (4) according to Theorem 3. Thus their infimum is also at least a viscosity supersolution to the equation. (See, for instance, Theorem 2.4.5 in [16] for this well-known property of viscosity solutions.) Since $d_0(x) \geq 0$ is fulfilled for all $x \in \mathbb{R}^n$ anyway, it holds in particular for $x \in \Gamma_0 \cup \Omega_0$ so that the boundary condition is also satisfied. □

Lemma 18. *d_0 solves (12) almost everywhere. In particular, $F(x)\,|\nabla d_0(x)| = 1$ for all $x \in \Omega^+ \setminus (\Gamma_0 \cup \Omega_0)$ at which d_0 is differentiable.*

Proof. Fix $x \in \Omega^+ \setminus (\Gamma_0 \cup \Omega_0)$ such that $d_0(x) < \infty$ and $\nabla d_0(x)$ exists. Note that according to Lemma 17, $F(x)\,|\nabla d_0(x)| \geq 1$ and thus also, in particular, $\nabla d_0(x) \neq 0$. We have to show $F(x)\,|\nabla d_0(x)| \leq 1$. Define $p_0 = \nabla d_0(x) / |\nabla d_0(x)|$ and note that $|\nabla d_0(x)| = \nabla d_0(x) \cdot p_0$, which is the directional derivative of d_0 in direction p_0. For $\epsilon > 0$, consider $\overline{B_\epsilon(x)}$. If ϵ is small enough, this is a compact and convex subset of Ω^+, so that Lemma 16 yields that d_0 is Lipschitz continuous on $\overline{B_\epsilon(x)}$. The Lipschitz constant is $L_\epsilon = 1/\underline{F}_\epsilon$, where $\underline{F}_\epsilon = \min_{y \in \overline{B_\epsilon(x)}} F(y)$. By continuity of F, $L_\epsilon \to 1/F(x)$ as $\epsilon \to 0$. Hence

$$|\nabla d_0(x)| = \nabla d_0(x) \cdot p_0 = \lim_{\epsilon \to 0^+} \frac{d_0(x + \epsilon p_0) - d_0(x)}{\epsilon} \leq \lim_{\epsilon \to 0^+} \frac{|d_0(x + \epsilon p_0) - d_0(x)|}{\epsilon} \leq \lim_{\epsilon \to 0^+} L_\epsilon = \frac{1}{F(x)},$$

which completes the proof. □

Lemma 19. *Let $\Omega \subset \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be Lipschitz continuous. Then*

$$\left|f^{-1}(\{0\})\right| = |\{x \in \Omega \mid f(x) = 0 \text{ and } f \text{ is differentiable at } x \text{ and } \nabla f(x) = 0\}|, \tag{13}$$

where $|\cdot|$ denotes the n-dimensional Lebesgue measure of the preimage sets.

Proof. This follows immediately from Lemma 7.7 on page 152 of [17]. □

Lemma 19 shows that if a level-set of some Lipschitz continuous function "fattens" in measure, then there must also exist a set of positive measure on which its gradient exists and vanishes. This can not happen for our case of d_0, since we know that it solves (12) almost everywhere. This is the central argument in the proof of our main non-fattening result:

Theorem 6. *Let $|\Gamma_0| = 0$. Then $|\Gamma_t| = 0$ for all $t \geq 0$.*

Proof. Note first that $|\Gamma_0| = 0$ implies, in particular, $\overline{\Omega_0} = \Gamma_0 \cup \Omega_0$. If this were not the case, then Γ_0 would have interior points and thus non-zero measure. We want to apply Corollary 1 for Γ_t to calculate its measure. Note that the part $\Gamma_0 \cap \Omega^z$ can be ignored, since it has zero measure by assumption. Thus, consider $\Gamma_t \cap \Omega^+ = D^{-1}(\{t\})$ first. Since $D = d_0$ in Ω^+, Lemma 16 and Lemma 18 apply. These results together imply that $\nabla D(x) \neq 0$ for almost all $x \in \Gamma_t \cap \Omega^+$. Hence Lemma 19 implies $|\Gamma_t \cap \Omega^+| = 0$. The same argument can also be used for $|\Gamma_t \cap \Omega^-|$, thus finally $|\Gamma_t| = 0$ follows. □

We conclude this subsection by using our representation formula to show non-fattening also in a topological sense. This result is similar to the classical result of [3]. Note, though, that our result concerns the sets for each instant in time separately, while the result of [3] considers the topological properties of the evolving sets in space-time.

Theorem 7. *Let $F \geq 0$ and assume that $\overline{\Omega_0} = \Gamma_0 \cup \Omega_0$. Then $\overline{\Omega_t} = \Gamma_t \cup \Omega_t$ for all $t \geq 0$.*

Proof. Observe that we only have to consider points with $F(x) > 0$, since all sets are stationary in time at points x with $F(x) = 0$. Thus we will only consider $x \in \Omega^+$ below. We use Theorem 5, and take note that $d_0'(x) = d_0(x)$ for the situation considered here and all $x \in \Omega^+$ according to Lemma 15. Let first $x \in \overline{\Omega_t} \cap \Omega^+$. This means that there exists a sequence $(x_k) \subset \Omega_t$ with $x_k \to x$ as $k \to \infty$. Since $F(x) > 0$ and F is continuous, we can, without loss of generality, assume that also $(x_k) \subset \Omega^+$. Theorem 5 applied to $x_k \in \Omega_t$ implies that $d_0'(x_k) = d_0(x_k) < t$, and thus using continuity of d_0, we obtain $d_0(x) \leq t$. This further implies $x \in \Gamma_t \cup \Omega_t$ when we use Theorem 5 a second time.

For the reverse inclusion, assume now $x \in (\Gamma_t \cup \Omega_t) \cap \Omega^+$. Since nothing is to be shown if $x \in \Omega_t$, assume that $x \in \Gamma_t$. Thus $d_0(x) = t$ by Theorem 5. Lemma 15 implies that there exists $x_0 \in \Gamma_0 \cup \Omega_0$ with $d_0(x) = d(x, x_0) = t$. Assume that $x \notin \overline{\Omega_t}$, which means that there exists $\delta > 0$ with $\overline{B_\delta(x)} \subset \Omega^+ \setminus \Omega_t$. In other words, $d_0(y) \geq t$ for all $y \in \overline{B_\delta(x)}$. Note that this also implies $d(x_0, y) \geq t$ for all those y, since $d_0(y) \leq d(x_0, y)$. Consider now the closed set $X = \mathbb{R}^n \setminus B_\delta(x)$, for which we know $x_0 \in X$ and $d(x_0, y) \geq t$ for all $y \in \partial X = \partial B_\delta(x)$. Thus Lemma 14 implies $d(x_0, x) > t$, which is a contradiction. Hence $x \in \overline{\Omega_t}$. □

For $F \leq 0$, the "reverse" statement of Theorem 7 holds (by using Lemma 4 and taking complements of all sets involved):

$$(\Gamma_0 \cup \Omega_0)^\circ = \Omega_0 \quad \Rightarrow \quad (\Gamma_t \cup \Omega_t)^\circ = \Omega_t$$

It is, however, not possible to get both results at the same time, and also not to get one of them for changing sign of F. This is demonstrated by the following example:

Example 1. Let $\phi_0(x) = |x| - 1$, such that $\Omega_0 = B_1(0)$ and $\Gamma_0 = \partial\Omega_0$. Choose $F \leq 0$ with compact support and Lipschitz continuous such that $F(x) = -1$ for all $x \in B_{1+\epsilon}(0)$ with some $\epsilon > 0$. Then $\Omega_t = \{x \in \mathbb{R}^n \mid |x| < 1 - t\}$ is a shrinking circle that disappears for $t \geq 1$ entirely. Hence for $t = 1$, we have $\Gamma_1 \cup \Omega_1 = \{0\} \neq \overline{\Omega_1} = \emptyset$.

Similarly, if we choose Ω_0 to be $B_2(0) \setminus \overline{B_1(0)}$, for instance, and $F \geq 0$ with $\operatorname{supp}(F) \subset B_2(0)$ and $F(x) = 1$ for all $x \in B_{1+\epsilon}(0)$, then $\Omega_t = B_2(0) \setminus \{x \in \mathbb{R}^n \mid |x| < 1 - t\}$ and at $t = 1$, the hole disappears. In this case, $(\Gamma_1 \cup \Omega_1)^\circ = B_2(0) \neq \Omega_1 = B_2(0) \setminus \{0\}$.

5.2 Shape Sensitivity of Domain Functionals

If one considers a functional depending on the evolving sets, one is often also interested in its derivative with respect to time in the shape propagation. This leads to *shape derivatives*, which form the foundation for level-set based schemes for shape optimisation. For instance, [26], [29], [12] and other applications depend on such a shape calculus. In the applied literature, this is not always rigorously justified or it relies on smoothness assumptions on the domain which may not be fulfilled in practice. Based on our representation formula in Corollary 1, we are able to rigorously derive such a shape derivative for an important class of domain functionals. This result can be applied to the mentioned and other problems. In particular, our shape calculus requires *no regularity assumptions* on the domain Ω besides being an open set. We are not aware of any other result that has this feature.

In this subsection, we will always assume that $|\Gamma_t| = 0$ holds for all times as per Theorem 6. For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we define the domain functional

$$j(t) = \int_{\Omega_t} f \, dx.$$

With the help of the co-area formula (Theorem 2 on page 117 of [15]) and Corollary 1, the functional $j(t)$ can be expressed in terms of D:

Theorem 8. *If Corollary 1 holds for Ω_t, then*

$$j(t) = j(0) + \int_0^t \int_{\Omega^+ \cap D^{-1}(\{s\})} F f \, d\sigma \, ds + \int_0^t \int_{\Omega^- \cap D^{-1}(\{-s\})} F f \, d\sigma \, ds \tag{14}$$

for all $t \geq 0$. Based on (11), this expression can also be written more compactly as

$$j(t) = j(0) + \int_{-t}^t \int_{D^{-1}(\{s\})} F f \, d\sigma \, ds.$$

Proof. For $t = 0$, the claim is clear. So assume $t > 0$ fixed now. We use the decomposition of Ω_t that is given in Corollary 1 as well as the representation

$$j(t) = j(0) + \int_{\Omega_t \setminus \Omega_0} f \, dx - \int_{\Omega_0 \setminus \Omega_t} f \, dx. \tag{15}$$

Note further that

$$\Omega_t \setminus \Omega_0 = \Omega^+ \cap D^{-1}\left((0,t)\right)$$

is the part that an outward moving boundary has created over time, while

$$\Omega_0 \setminus \Omega_t = \Omega^- \cap D^{-1}\left((-t,0)\right)$$

is the part that an inward moving boundary has removed from Ω_0.

Consider the first of those sets now and recall that D is locally Lipschitz continuous on $\Omega^+ \cap D^{-1}\left((0,t)\right)$ according to Lemma 16. Furthermore, $|\nabla D(x)| = 1/F(x)$ holds for almost all $x \in \Omega^+ \cap D^{-1}\left((0,t)\right)$ because of Lemma 18. Let (A_k) be a sequence of compact subsets of $\Omega^+ \cap D^{-1}\left((0,t)\right)$ converging in measure to $\Omega^+ \cap D^{-1}\left((0,t)\right)$ as $k \to \infty$. Such a sequence exists by regularity of the Lebesgue measure (Theorem 2.20 on page 50 of [25]). Since these sets are compact, D is Lipschitz continuous when restricted to each A_k. We define χ_k to be the characteristic function of A_k, χ that of $\Omega^+ \cap D^{-1}\left((0,t)\right)$ and set $g_k = \chi_k F f$. Then $g_k \in L^1(\mathbb{R}^n)$ for each $k \in \mathbb{N}$, since F has compact support. Also, $\chi_k \to \chi$ as $k \to \infty$ in $L^1(\mathbb{R}^n)$. Hence the co-area formula yields

$$\int_{A_k} f\,dx = \int_{\mathbb{R}^n} |\nabla D|\, g_k\,dx = \int_{\mathbb{R}} \int_{D^{-1}(\{s\})} \chi_k F f\,d\sigma\,ds = \int_0^t \int_{A_k \cap D^{-1}(\{s\})} F f\,d\sigma\,ds.$$

Using Lebesgue's dominated convergence theorem, we can pass the limit $k \to \infty$ to obtain

$$\int_{\Omega^+ \cap D^{-1}((0,t))} f\,dx = \int_0^t \int_{\Omega^+ \cap D^{-1}(\{s\})} F f\,d\sigma\,ds.$$

For the part $\Omega^- \cap D^{-1}\left((-t,0)\right)$, basically the same argument can be applied when we take the correct signs into account. As above, we proceed assuming that D is Lipschitz continuous by using suitable compact cut-off sets and the dominated convergence theorem. Here, if χ is the characteristic function of $\Omega^- \cap D^{-1}\left((-t,0)\right)$ and we define $g = \chi\,|F|\,f = -\chi F f$. Then $|\nabla D(x)| = 1/\,|F(x)| = -1/F(x)$ for almost all $x \in \Omega^- \cap D^{-1}\left((-t,0)\right)$, since $-D$ is the solution for speed $|F| = -F$ in this part of the domain according to (11). Hence, again using the co-area formula, we get:

$$\begin{aligned}
\int_{\Omega^- \cap D^{-1}((-t,0))} f\,dx &= \int_{\mathbb{R}^n} |\nabla D|\, g\,dx = \int_{\mathbb{R}} \int_{D^{-1}(\{s\})} \chi\,|F|\,f\,d\sigma\,ds \\
&= -\int_{-t}^0 \int_{\Omega^- \cap D^{-1}(\{s\})} F f\,d\sigma\,ds = -\int_0^t \int_{\Omega^- \cap D^{-1}(\{-s\})} F f\,d\sigma\,ds
\end{aligned}$$

Using this now in (15) gives the correct term of (14). ☐

As an immediate corollary of Theorem 8, the *shape derivative* of j can be calculated in direction of a particular deformation described by a speed field F:

Corollary 2. *j is differentiable for almost all $t \geq 0$ and the derivative is given by*

$$j'(t) = \int_{\Omega^+ \cap D^{-1}(\{t\})} F f\,d\sigma + \int_{\Omega^- \cap D^{-1}(\{-t\})} F f\,d\sigma. \tag{16}$$

Proof. This follows by using the Lebesgue differentiation theorem (Theorem 13.15 in [30]) on j in the form of (14), where the dependence on t is only in the upper bound of the one-dimensional outer integral. The co-area formula guarantees that the integrand is really a function of $L^1(\mathbb{R})$ as is required for the differentiation theorem. ☐

Note that the argument employed by the proof of Corollary 2 unfortunately only implies differentiability for *almost all* times and not full differentiability at *every* t. For this, one would have to show in addition that the derivative given in (16) can be continuously extended to all $t \geq 0$. We believe that this is, indeed, the case under reasonable assumptions. This question is the focus of ongoing research at the moment, but we cannot present a further result here. For shape optimisation based on a gradient-descent scheme, particularly $j'(0)$ would be interesting. It is not clear by Corollary 2 alone, though, that this derivative exists. Hence, our further analysis will be based on Theorem 8 instead of Corollary 2, so that we can formulate results that hold without an "almost all" qualification. These results will state absolute continuity of the shape functionals. This, in turn, allows to deduce the existence of a weak "almost everywhere" derivative in the same way as Corollary 2.

For the remainder of this subsection, we assume for simplicity that $F \geq 0$ is the case. It is straightforward to apply Theorem 8 in full generality in order to generalise the results to arbitrary signs of F. For a fixed speed field F, let Ω_t and Γ_t describe the evolved domain as per Theorem 5. We consider now a more general shape functional

$$J(t) = J(\Omega_t) = \int_{\Omega_t} f(x, \Omega_t)\, dx. \tag{17}$$

The integrand $f(\cdot, \Omega)$ is assumed to be integrable for any fixed domain Ω. Furthermore, let us, for now, assume that it has a weak shape derivative f' in the sense that

$$f(x, \Omega_t) = f(x, \Omega_0) + \int_0^t f'(x, \Omega_s)\, ds \tag{18}$$

holds for all $x \in \mathbb{R}^n$ and $t \geq 0$. The function $f'(\cdot, \Omega)$ must also be integrable for all fixed domains Ω. Under these assumptions, we can derive a *total shape differential*:

Corollary 3. *Let J and f be as above. Then J is absolutely continuous, i. e.,*

$$J(t) = J(0) + \int_0^t J'(s)\, ds = J(0) + \int_0^t \left(\int_{\Gamma_s} F f(x, \Omega_s)\, d\sigma + \int_{\Omega_s} f'(x, \Omega_s)\, dx \right) ds. \tag{19}$$

Proof. By integrating (18) over Ω_t, we find

$$J(\Omega_t) = \int_{\Omega_t} f(x, \Omega_0)\, dx + \int_{\Omega_t} \int_0^t f'(x, \Omega_s)\, ds\, dx.$$

Applying Theorem 8 to the first term (where Ω_0 is now fixed) and Fubini's theorem to the second, this further yields

$$J(\Omega_t) = J(\Omega_0) + \int_0^t \int_{\Gamma_s} F f(x, \Omega_0)\, d\sigma\, ds + \int_0^t \int_{\Omega_t} f'(x, \Omega_s)\, dx\, ds.$$

Note that this result already looks *almost* like the claimed (19). However, it has Ω_0 instead of Ω_s in the middle term and Ω_t instead of Ω_s in the last one. Consequently, it remains to show that

$$\int_0^t \int_{\Gamma_s} F \left(f(x, \Omega_s) - f(x, \Omega_0) \right) d\sigma\, ds = \int_0^t \left(\int_{\Omega_t} f'(x, \Omega_s)\, dx - \int_{\Omega_s} f'(x, \Omega_s)\, dx \right) ds.$$

With the corresponding shape derivatives for the differences, we can turn this equation into

$$\int_0^t \int_{\Gamma_s} F \int_0^s f'(x, \Omega_\tau)\, d\tau\, d\sigma\, ds = \int_0^t \int_s^t \int_{\Gamma_\tau} F f'(x, \Omega_s)\, d\sigma\, d\tau\, ds. \tag{20}$$

Using Fubini's theorem again on the left-hand side and renaming s and τ on the right-hand side, this is further equal to

$$\int_0^t \int_0^s \int_{\Gamma_s} F f'(x, \Omega_\tau)\, d\sigma\, d\tau\, ds = \int_0^t \int_\tau^t \int_{\Gamma_s} F f'(x, \Omega_\tau)\, d\sigma\, ds\, d\tau.$$

Since both sides of this equation only express different ways to integrate over the same right triangle in the (s, τ)-plane, this shows that (20) and thus the claim are true. □

In the final part of this subsection, we will consider when (18) holds for a special class of shape-dependent integrands. For this, we first need a general-purpose chain rule for absolutely continuous functions:

Lemma 20. *Let $f : \mathbb{R}^k \to \mathbb{R}$ be continuously differentiable and $g_1, \ldots, g_k : \mathbb{R} \to \mathbb{R}$ be absolutely continuous. We consider*

$$h : \mathbb{R} \to \mathbb{R}, \ \ h(t) = f(g_1(t), \ldots, g_k(t)).$$

Then h is also absolutely continuous, and

$$h'(t) = \sum_{i=1}^{k} \partial_i f(g_1(t), \ldots, g_k(t)) \cdot g_i'(t). \tag{21}$$

Proof. This follows from part (ii) of Theorem 4 on page 129 of [15]. □

In applications, it is common that the shape-dependence of the integrand f is due to some number of shape-dependent quantities. For instance, the integrand may depend on the volume $|\Omega|$ of the current domain or other, related values. For these integrands, we can use the results above to derive their shape derivatives as well. In particular, we are interested in integrands of the form

$$f(x, \Omega) = f(x, G_1(\Omega), \ldots, G_k(\Omega)). \tag{22}$$

If the G's have shape derivatives themselves, Lemma 20 can be used together with Corollary 3. In this situation, J is again absolutely continuous with respect to the parameter t and we get

$$J(t) = J(0) + \int_0^t \left(\int_{\Gamma_s} F f \, d\sigma + \sum_{i=1}^{k} \int_{\Omega_s} \partial_i f \cdot G' \, dx \right) ds. \tag{23}$$

Thus, if all the G's are domain functionals of the form (17), (22) themselves, we can recursively apply (23) to find shape derivatives. As long as there are no circular dependencies among the various shape-dependent quantities (i. e., the dependency graph is a *tree*), this process will work fine.

5.3 Lipschitz Continuity with Optimal Constants

It is a well-known fact that viscosity solutions of an initial-value problem (like the level-set equation (1)) often preserve Lipschitz continuity of the initial function ϕ_0. Usually, this property is deduced from the comparison principle. See, for instance, Theorem 3.5.1 in [16] or the related result in [20] for bounded domains. Following a slightly different route, we can also use our representation formula (8) to show Lipschitz continuity of ϕ both in time (see Theorem 9) and spatially (in Theorem 11). Based on the construction given in Example 2, we can even demonstrate that our results are sharp.

Before we can show Lipschitz continuity of ϕ in time, we have to consider how admissible points in the minimum of (8) change if the upper bound t is changed.

Lemma 21. *Let $F(x) \geq 0$ for some $x \in \mathbb{R}^n$. Assume that $|F| \leq \overline{F}$, $x^t \in \mathbb{R}^n$ and $s, t \geq 0$ with $d(x, x^t) \leq t$. Then there exists $x^s \in \mathbb{R}^n$ with $d(x, x^s) \leq s$ and $|x^s - x^t| \leq \overline{F}\,|t - s|$.*

Proof. Consider first the trivial case $s = 0$: We can pick $x^s = x$ and know by Lemma 8 that

$$\left|x^s - x^t\right| = \left|x - x^t\right| \leq \overline{F} \cdot d\left(x, x^t\right) \leq \overline{F} \cdot t = \overline{F}\,|t - 0|\,.$$

Thus, assume $s > 0$. Similarly the case $s \geq d(x, x^t)$ is also trivial, as one can choose $x^s = x^t$. Consequently, we assume further $s < d(x, x^t) \leq t$ from now on. Choose $\xi \in X_{\mathrm{ad}}(x, x^t)$ with $l(\xi) = d(x, x^t)$. Since ξ and d are continuous, we know that $\tau \mapsto d(x, \xi(\tau))$ is continuous and ranges from 0 at $\tau = 0$ to $d(x, x^t) > s$ at $\tau = 1$. Thus the intermediate value theorem yields τ_0 and $x^s = \xi(\tau_0)$ with $d(x, x^s) = s$. Denote by $\tilde{\xi}$ the part of ξ between x^s and x^t, i. e., for times in $[\tau_0, 1]$. Then we get

$$d(x, x^s) + l(\tilde{\xi}) = s + l(\tilde{\xi}) \leq l(\xi) \leq t$$

since $d(x, x^s)$ is the *shortest* distance between x and x^s, while the initial part of ξ is just a *particular* path. Hence we get also

$$l(\tilde{\xi}) \leq t - s = |t - s|\,,$$

which further implies that

$$\left|x^s - x^t\right| \leq \overline{F} \cdot l(\tilde{\xi}) \leq \overline{F}\,|t - s|\,.$$

□

Theorem 9. *Let $|F| \le \overline{F}$ on $\mathbb{R}^n$. Then*

$$|\phi(x,s) - \phi(x,t)| \le L_{\phi_0}\overline{F} \cdot |t-s|$$

for all $x \in \mathbb{R}^n$ and $s, t \ge 0$.

Proof. Let $x \in \mathbb{R}^n$ and $s, t \ge 0$. If $F(x) = 0$, then $\phi(x,t) = \phi(x,s) = \phi_0(x)$, thus this case is trivial. If $F(x) < 0$, we can use Lemma 4 to reduce the situation to the case of $F(x) > 0$. Thus, assume $F(x) > 0$ without loss of generality now. Pick x^t as minimiser of (8). I. e., $\phi(x,t) = \phi_0(x^t)$ and $d(x, x^t) \le t$. Using Lemma 21, we can define x^s corresponding to our s. Then

$$\phi(x,s) \le \phi_0(x^s) \le \phi_0\left(x^t\right) + L_{\phi_0}\left|x^s - x^t\right| \le \phi(x,t) + L_{\phi_0}\overline{F} \cdot |s-t|\,.$$

Using a symmetric argument with s and t exchanged completes the proof. □

Next, we can show spatial Lipschitz continuity *in terms of the distance d*. This is a consequence of (8) and Lipschitz continuity in time. Note that d is itself Lipschitz continuous where F is bounded away from zero. Consequently, Theorem 10 actually gives a Lipschitz constant that is *uniform for all times* in these cases. However, where F may become zero or change its sign, this result makes no statement and the subsequent Theorem 11 must be applied instead.

Theorem 10. *Let $x, y \in \mathbb{R}^n$, $t \ge 0$ and $|F| \le \overline{F}$. Denote by $L = L_{\phi_0}\overline{F}$ the temporal Lipschitz constant of ϕ according to Theorem 9. Then*

$$|\phi(x,t) - \phi(y,t)| \le L d(x,y)\,.$$

Proof. If $x = y$, then the claim is trivial, so assume $x \ne y$. Hence, if $F(x) = 0$, $F(y) = 0$ or they have differing signs, then $d(x,y) = \infty$ and nothing is to show. The case $F(x), F(y) < 0$ can be reduced to $F(x), F(y) > 0$ with the help of Lemma 4, so assume $F(x), F(y) > 0$ from now on. For the trivial case of $t = 0$ we get

$$|\phi(x,0) - \phi(y,0)| = |\phi_0(x) - \phi_0(y)| \le L_{\phi_0}|x-y| \le L_{\phi_0}\overline{F} d(x,y)\,,$$

where the last estimate is due to Lemma 8.

Consider now $t > 0$ and note that ϕ is given by (8). If we choose x^t and y^t as minimisers for $\phi(x,t)$ and $\phi(y,t)$, respectively, we get

$$\phi(x,t) = \phi_0\left(x^t\right), \quad \phi(y,t) = \phi_0\left(y^t\right), \quad \max\left(d\left(x,x^t\right), d\left(y,y^t\right)\right) \le t.$$

Denote $d(x, y^t) = s$ and note that our assumption of Lipschitz continuity of ϕ in time gives

$$\phi(x,t) - L|t-s| \le \phi(x,s) \le \phi_0\left(y^t\right) = \phi(y,t),$$

so that further

$$\phi(x,t) \le \phi(y,t) + L\left|t - d\left(x,y^t\right)\right|.$$

Consider first the case $d(x, y^t) \ge t$. Then

$$\left|t - d\left(x,y^t\right)\right| = d\left(x,y^t\right) - t \le d(x,y) + d\left(y,y^t\right) - t \le d(x,y)\,,$$

which gives

$$\phi(x,t) \le \phi(y,t) + L d(x,y)\,.$$

In the second case of $d(x, y^t) < t$, we get

$$\phi(x,t) \le \phi_0\left(y^t\right) = \phi(y,t) \le \phi(y,t) + L d(x,y)$$

since y^t is admissible also for x in (8). If we repeat this argument now with x and y exchanged, the claimed Lipschitz continuity follows. □

We will continue with the final goal to show spatial Lipschitz constants with respect to the usual Euclidean distance $|x-y|$. As a first step towards this result, we can show that this is the case if $F(x)=0$ at least for one of the two points involved. This is a very important piece of information, as it complements the earlier result in Theorem 10, which handles the situation within the support of F.

Lemma 22. *Let $x,y\in\mathbb{R}^n$, $t\geq 0$ and assume $F(x)=0$. If $y^t\in\mathbb{R}^n$ realises the minimum in (8) for $\phi(y,t)$, then*

$$\left|y^t-y\right|\leq\left(e^{L_F t}-1\right)|x-y|$$

and furthermore

$$|\phi(x,t)-\phi(y,t)|\leq L_{\phi_0}e^{L_F t}\,|x-y|\,.$$

Proof. If $F(y)=0$, then $y^t=y$ is the minimiser of (8), which makes the first estimate trivial. The same is true if $t=0$. For $F(y)<0$, we can use Lemma 4 to convert the situation to the remaining case of $F(y)>0$ as before. Note that $F(y)\leq F(x)+L_F\,|x-y|=L_F\,|x-y|$. Combining this with the first estimate of Lemma 9 yields

$$\left|y-y^t\right|\leq\frac{F(y)}{L_F}\left(e^{L_F t}-1\right)\leq\left(e^{L_F t}-1\right)|x-y|\,.$$

For the second part, we use this result in combination with (8) to get

$$\begin{aligned}|\phi(x,t)-\phi(y,t)| &= \left|\phi_0(x)-\phi_0\left(y^t\right)\right|\leq L_{\phi_0}\left(|x-y|+\left|y-y^t\right|\right)\\ &\leq L_{\phi_0}\left(|x-y|+\left(e^{L_F t}-1\right)|x-y|\right)=L_{\phi_0}e^{L_F t}\,|x-y|\,.\end{aligned}$$

□

Lemma 23. *Let $x,y\in\Omega^+$ and $t>0$. Assume that $\phi(y,t)=\phi_0\left(y^t\right)$ with $d\left(y,y^t\right)\leq t$. Then there exists $x'\in\mathbb{R}^n$ with $d\left(x,x'\right)\leq t$ and $\left|x'-y^t\right|\leq e^{L_F t}\,|x-y|$.*

Proof. If $d\left(x,y^t\right)\leq t$, we can choose $x'=y^t$. Also, if $F(y)\leq L_F\,|x-y|$, we can use $x'=x$. In this situation, the first estimate in Lemma 9 gives

$$\left|x'-y^t\right|\leq|x-y|+\left|y-y^t\right|\leq|x-y|+\frac{F(y)}{L_F}\left(e^{L_F t}-1\right)\leq|x-y|\cdot\left(1+e^{L_F t}-1\right)=e^{L_F t}\,|x-y|\,.$$

Thus consider now the case $d\left(x,y^t\right)>t$ and $F(y)>L_F\,|x-y|$. Let $s=l(S_{xy})$ denote the path length of the straight line S_{xy} from x to y. The third estimate in Lemma 9 implies that

$$|x-y|\geq\frac{F(y)}{L_F}\left(1-e^{-sL_F}\right)\;\Leftrightarrow\;\left(1-e^{-sL_F}\right)\leq\frac{L_F\,|x-y|}{F(y)}. \tag{24}$$

Apply Lemma 7 to choose $\xi_y\in X_{\mathrm{ad}}\left(y,y^t\right)$ with $l(\xi_y)=d\left(y,y^t\right)\leq t$. We will construct x' on the path ξ that is formed by first following S_{xy} from x to y and then moving along ξ_y from y to y^t. Note that S_{xy} is entirely inside of Ω^+ since $F(y)>L_F\,|x-y|$ and

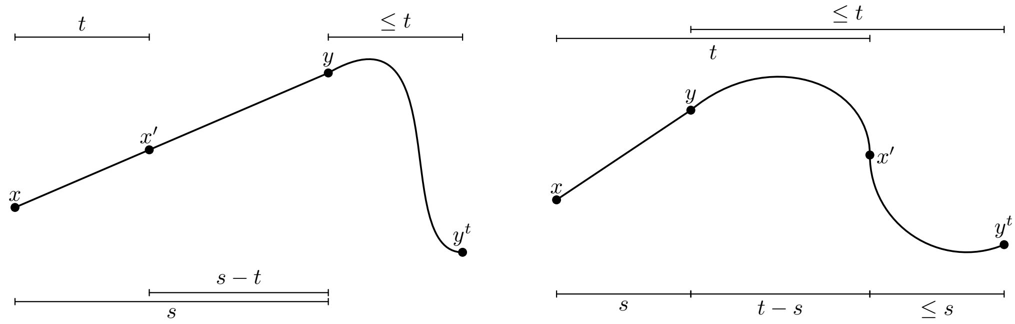
$$F(S_{xy}(\tau))=F(x+\tau(y-x))\geq F(y)-L_F(1-\tau)\,|x-y|>0$$

for arbitrary $\tau\in[0,1]$. This path ξ can be expressed explicitly as

$$\xi(\tau)=\begin{cases}S_{xy}(2\tau) & \text{for } \tau\in[0,1/2],\\ \xi_y(2\tau-1) & \text{for } \tau\in[1/2,1].\end{cases}$$

Denote for a moment the length of ξ restricted to $[0,\tau]$ by $\lambda(\tau)$ and note that λ is continuous. Since $\lambda(1)=l(\xi)\geq d\left(x,y^t\right)>t$ and $\lambda(0)=0<t$, we can find $\tau_0\in(0,1)$ with $\lambda(\tau_0)=t$. Choose $x'=\xi(\tau_0)$, so that $d\left(x,x'\right)\leq\lambda(\tau_0)=t$. It remains to show $\left|x'-y^t\right|\leq e^{L_F t}\,|x-y|$. For this, we have to consider two cases depending on which segment of ξ the point x' comes to lie on. The path ξ is sketched for both situations in Figure 2.

If $\tau_0\leq 1/2$, then x' is still part of the straight initial piece of ξ as shown in Figure 2a. This means that $t\leq s$ as well as $|x-y|=|x-x'|+|x'-y|$. Equality holds here because x, x' and y are colinear. Since



(a) The case $t < s$.

(b) The case $t > s$.

Figure 2: Sketches for the situations in the proof of Lemma 23. Indicated is always the path length according to l in Definition 3.

the path length from x' to y on S_{xy} is the remaining $s-t$ and thus also, in particular, $d(x',y) \le s-t$, we can again employ Lemma 9 to find

$$|x'-y| \le \frac{F(x')}{L_F}\left(e^{L_F d(x',y)}-1\right) \le \frac{F(x')}{L_F}\left(e^{(s-t)L_F}-1\right).$$

Together with Lipschitz continuity, this yields

$$F(y) \le F(x') + L_F|x'-y| \le F(x')\left(1+e^{(s-t)L_F}-1\right) = F(x')e^{(s-t)L_F} \quad \Rightarrow \quad F(x') \ge F(y)e^{(t-s)L_F}.$$

Furthermore, since

$$F(x') \ge F(y) - L_F|x'-y| > L_F(|x-y|-|x'-y|) = L_F|x'-x|,$$

Lemma 9 is applicable again and gives

$$|x'-x| \ge \frac{F(x')}{L_F}\left(1-e^{-tL_F}\right) \ge \frac{F(y)}{L_F}\left(e^{(t-s)L_F}-e^{-sL_F}\right) = \frac{F(y)}{L_F}\left(e^{L_Ft}-1\right)e^{-sL_F}.$$

All together, we have

$$\begin{aligned} |x'-y^t| &\le |x'-y| + |y-y^t| = |x-y| + |y-y^t| - |x-x'| \\ &\le |x-y| + \frac{F(y)}{L_F}\left(e^{L_Ft}-1\right) - \frac{F(y)}{L_F}\left(e^{L_Ft}-1\right)e^{-sL_F} \\ &= |x-y| + \frac{F(y)}{L_F}\left(e^{L_Ft}-1\right)\left(1-e^{-sL_F}\right) \\ &\le |x-y| + \frac{F(y)}{L_F}\left(e^{L_Ft}-1\right)\frac{L_F|x-y|}{F(y)} = e^{L_Ft}|x-y|, \end{aligned}$$

which finishes the proof for this case. The last estimate is due to (24).

Now consider $\tau_0 \ge 1/2$, which means that x' lies on ξ_y between y and y^t, and that $t \ge s$. Take a look at Figure 2b. Consequently, if we pick the piece of ξ_y between y and x' (for times in $[1/2, \tau_0]$), its path length is $t-s \ge 0$. Since $y^t = \xi_y(1)$ and $l(\xi_y) \le t$, we know that the length of the remaining piece of ξ_y between x' and y^t is at most s. Thus

$$|x'-y^t| \le \frac{F(x')}{L_F}\left(e^{L_Fs}-1\right) = \frac{F(x')}{L_F}e^{sL_F}\left(1-e^{-sL_F}\right)$$

by Lemma 9. Using (24), this yields

$$|x'-y^t| \le \frac{F(x')}{L_F}e^{sL_F}\frac{L_F|x-y|}{F(y)} = \frac{F(x')}{F(y)}e^{sL_F}|x-y|. \tag{25}$$

Similarly to the last case, we can combine Lemma 9 and the Lipschitz continuity of F to obtain

$$F(x') \le F(y) + L_F\,|x'-y| \le F(y)e^{(t-s)L_F},$$

which allows us to rewrite (25) to

$$\left|x'-y^t\right| \le e^{tL_F}\,|x-y|\,.$$

□

Now we have everything together to show spatial Lipschitz continuity:

Theorem 11. *For all $x, y \in \mathbb{R}^n$ and $t \ge 0$, we have the Lipschitz estimate*

$$|\phi(x,t)-\phi(y,t)| \le L_{\phi_0}e^{L_F t}\cdot|x-y|\,.$$

Proof. If $F(x)=0$ or $F(y)=0$, the result follows from Lemma 22. If $F(x)$ and $F(y)$ have different signs, we can split the straight line S_{xy} between x and y with some point z that has $F(z)=0$, use Theorem 2 and apply Lemma 22 twice to get the claimed Lipschitz continuity. Also, if $t=0$, the result follows since $\phi(\cdot,0)=\phi_0$ is Lipschitz continuous. Thus it remains to consider, without loss of generality, the case $F(x), F(y) > 0$ and $t>0$. Let $y^t \in \mathbb{R}^n$ with $d\left(y,y^t\right) \le t$ and $\phi(y,t)=\phi_0\left(y^t\right)$ be a minimiser of (8). Using Lemma 23, we get $x' \in \mathbb{R}^n$ with $d\left(x,x'\right) \le t$ and $|x'-y^t| \le e^{L_F t}\,|x-y|$. It follows that

$$\phi(x,t) \le \phi_0\left(x'\right) \le \phi_0\left(y^t\right) + L_{\phi_0}\left|x'-y^t\right| \le \phi(y,t) + L_{\phi_0}e^{L_F t}\,|x-y|\,,$$

which gives the claimed result when the same argument is applied again with x and y exchanged. □

We will now conclude this subsection with an example that demonstrates that the constants shown in Theorem 9 and Theorem 11 are sharp:

Example 2. Let $L_{\phi_0}, L_F, a > 0$ be given. We define $\phi_0 : \mathbb{R} \to \mathbb{R}$ by

$$\phi_0(x) = \begin{cases} 0 & \text{if } x \le 0, \\ -L_{\phi_0}x & \text{for } x \in [0,2a], \\ -2aL_{\phi_0} & \text{if } x \ge 2a \end{cases}$$

as well as $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \begin{cases} L_F x & \text{for } x \in [0,a], \\ L_F(2a-x) & \text{for } x \in [a,2a], \\ 0 & \text{else.} \end{cases}$$

This situation is shown in Figure 3. Note that ϕ_0 and F are Lipschitz continuous with Lipschitz constants L_{ϕ_0} and L_F, respectively, F has compact support on $[0,2a]$ and that $F \ge 0$. Furthermore,

$$|F(x)| \le \overline{F} = aL_F$$

for all $x \in \mathbb{R}$. I. e., the parameter a can be used to choose the maximal value $\overline{F}$ of F independently of the Lipschitz constants. Thus all quantities that appear in the proven Lipschitz constants can be influenced by the parameters in this example. This situation fulfils all assumptions we have made for the theoretical considerations above, so that our results apply here. If we denote the solution to (1) by ϕ as usual, Theorem 4 holds and thus ϕ is given by (8).

If x or y are not in $(0,2a)$, then clearly $d\left(x,y\right)=\infty$ if $x \ne y$ and $d\left(x,y\right)=0$ otherwise. For $x, y \in (0,2a)$, we have

$$d\left(x,y\right) = \left|\int_x^y \frac{1}{F(\xi)}\,d\xi\right|.$$

Note that there is no possible "choice" for different paths in one dimension. The absolute value ensures that the expression is correct also for $y < x$, when the integral itself is negative. Note that $d\left(x,y\right) \to \infty$ for $y \to 2a$ and that ϕ_0 is strictly decreasing on $[0,2a]$. This implies that for $x \in (0,2a)$ and $t \ge 0$, the minimiser of (8) is always the unique $x^t \in [x,2a)$ with $d\left(x,x^t\right)=t$. Assume $x \in (0,a]$ and $x' \in [x,a]$. Then $F(\xi)=L_F\xi$ for $\xi \in [x,x']$ and we can solve the integral to get

$$d\left(x,x'\right) = \int_x^{x'} \frac{1}{L_F\xi}\,d\xi = \frac{\log x' - \log x}{L_F}. \tag{26}$$

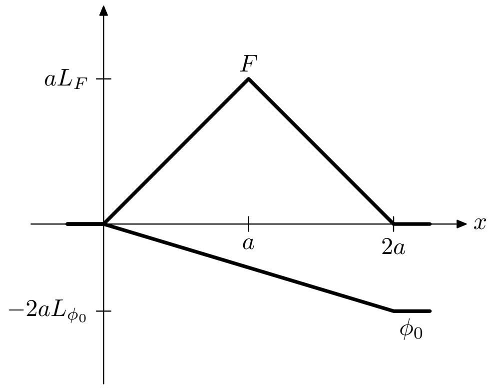


Figure 3: The situation of Example 2.

Thus if $t \leq d(x,a)$, we know that $x^t \in [x,a]$ can be found by solving $d(x,x^t) = t$ together with (26) for the unknown x^t. It is trivial to see that the result is $x^t = xe^{L_F t}$. Thus we have shown that for every $x \in (0,a)$ and $t \geq 0$ small enough, the solution is given by

$$\phi(x,t) = \phi_0\left(x^t\right) = \phi_0\left(xe^{L_F t}\right) = -L_{\phi_0} x e^{L_F t}. \tag{27}$$

We will now take derivatives of (27) in order to verify that this solution does, indeed, realise the Lipschitz constants we have shown. Note that for arbitrarily large t, there always exists $x \in (0,a)$ with $t < d(x,a)$ so that (27) can be applied. Taking the derivative with respect to x shows that the maximum Lipschitz constant according to Theorem 11 is, indeed, tested with this example. For the time derivative, we get

$$\left|\frac{\partial \phi}{\partial t}(x,t)\right| = L_{\phi_0} L_F x e^{L_F t}$$

as lower bound on the Lipschitz constant, which is valid at least for every $x \in (0,a)$ and $t \geq 0$ with $t \leq d(x,a)$. Clearly the largest bound is achieved if x is as large as possible, which means just so large that $t = d(x,a)$. By taking (26) into account, this is at $x = ae^{-L_F t}$. Consequently, the temporal Lipschitz constant must be at least

$$L_{\phi_0} L_F a e^{-L_F t} e^{L_F t} = a L_F L_{\phi_0} = \overline{F} L_{\phi_0}.$$

This matches the result from Theorem 9.

5.4 Propagation Speed of Perturbations

Since F in the level-set equation (1) describes a speed of movement, it is intuitive to assume that the maximal speed $\overline{F}$ is also the maximal speed with which perturbations in the initial geometry and/or the speed field itself propagate. With the help of the representation formula (8), this result can be easily proven. We will assume here that a perturbation happens on some set $A \subset \mathbb{R}^n$ and that we consider a point $x \notin A$ with Euclidean distance $\delta > 0$ to A, i. e.,

$$\delta = \inf_{y \in A} |x - y| > 0. \tag{28}$$

The first result concerns perturbations in the initial geometry:

Lemma 24. *Assume $F \geq 0$ and $|F| \leq \overline{F}$. Let ϕ_0 and $\tilde{\phi}_0$ be two initial level-set functions and ϕ as well as $\tilde{\phi}$ the corresponding solutions of (1) for the same F in both cases. Assume that $\phi_0(x) = \tilde{\phi}_0(x)$ for all $x \notin A$, and that $x \in \mathbb{R}^n \setminus \overline{A}$ is given with $\delta > 0$ defined according to (28). Then*

$$\phi(x,t) = \tilde{\phi}(x,t)$$

for all $t < \delta/\overline{F}$.

Proof. We may assume that $F(x) > 0$ because $\phi(x,t) = \phi_0(x) = \tilde{\phi}_0(x) = \tilde{\phi}(x,t)$ otherwise, which makes the statement is trivial. From Lemma 8 we know that $|x-y| \leq \overline{F} \cdot d(x,y)$ for all $y \in \mathbb{R}^n$. In particular, this implies for all $y \in A$:

$$\delta \leq |x-y| \leq \overline{F} \cdot d(x,y) \quad \Rightarrow \quad d(x,y) \geq \frac{\delta}{\overline{F}}$$

Choose now $t < \delta/\overline{F}$ and y with $d(x,y) \leq t$. It follows that $y \notin A$ and thus $\phi_0(y) = \tilde{\phi}_0(y)$. The claim follows now using the form (8) for the solutions as implied by Theorem 4. □

Next, we consider what happens if the same initial geometry propagates with two different speed fields F and $\tilde{F}$:

Lemma 25. *Let $F, \tilde{F} \geq 0$ be two different speed functions with $|F|, \left|\tilde{F}\right| \leq \overline{F}$. Assume furthermore that $F(x) = \tilde{F}(x)$ for all $x \notin A$, and let ϕ_0 be some initial geometry. We denote by d and $\tilde{d}$ the distances induced by F and $\tilde{F}$, respectively, and by ϕ and $\tilde{\phi}$ the solutions to (1) for both speed fields with the same initial data ϕ_0. For $x \in \mathbb{R}^n \setminus \overline{A}$, let $\delta > 0$ be as in (28). Then*

$$\phi(x,t) = \tilde{\phi}(x,t)$$

for all $t < \delta/\overline{F}$.

Proof. If $F(x) = 0$ the claim is clear, so assume $F(x) > 0$. Since $x \notin A$, this also implies $\tilde{F}(x) = F(x) > 0$. We want to show that

$$\{y \in \mathbb{R}^n \mid d(x,y) \leq t\} = \left\{y \in \mathbb{R}^n \mid \tilde{d}(x,y) \leq t\right\}, \tag{29}$$

which then implies the claim via (8) and Theorem 4. So let $y \in \mathbb{R}^n$ with $d(x,y) \leq t < \delta/\overline{F}$. Let $\xi \in X_{\mathrm{ad}}(x,y)$ be some admissible path with $l(\xi) < \delta/\overline{F}$. Assume there exists $t_0 \in [0,1]$ with $z = \xi(t_0) \in A$. But then $l(\xi) \geq d(x,z) + d(z,y)$ and

$$d(x,z) \geq \frac{|x-z|}{\overline{F}} \geq \frac{\delta}{\overline{F}}$$

by Lemma 8, which is a contradiction. Thus ξ never touches A and consequently $l(\xi) = \tilde{l}(\xi)$. This implies that every (short enough) path in the infimum for $d(x,y)$ is also admissible for $\tilde{d}(x,y)$ with the same length. Hence $\tilde{d}(x,y) \leq d(x,y)$, showing inclusion from left to right in (29). The inclusion from right to left works just the same. □

For a final result, we can combine Lemma 24 and Lemma 25 into a single theorem:

Theorem 12. *Let $|F|, \left|\tilde{F}\right| \leq \overline{F}$ and ϕ_0, $\tilde{\phi}_0$ be two initial level-set functions. Denote the corresponding solutions of (1) by ϕ and $\tilde{\phi}$, respectively. Assume that $F(x) = \tilde{F}(x)$ and $\phi_0(x) = \tilde{\phi}_0(x)$ for all $x \notin A$. Then for each $x \in \mathbb{R}^n \setminus A$ with $\delta > 0$ defined as per (28), we have*

$$\phi(x,t) = \tilde{\phi}(x,t)$$

for all $t \leq \delta/\overline{F}$.

Proof. It is enough to consider $x \in \mathbb{R}^n \setminus \overline{A}$ and $t < \delta/\overline{F}$ since ϕ and $\tilde{\phi}$ are continuous. Thus let $x \in \mathbb{R}^n \setminus \overline{A}$ and $t < \delta/\overline{F}$. Note that $F(x) = \tilde{F}(x)$, and that we can reduce the general case to that of $F, \tilde{F} \geq 0$ by using Theorem 2. We introduce an "intermediate solution" $\hat{\phi}$ as the solution of (1) with F and $\tilde{\phi}_0$. Lemma 24 implies that $\hat{\phi}(x,t) = \phi(x,t)$. Furthermore, Lemma 25 implies also $\hat{\phi}(x,t) = \tilde{\phi}(x,t)$, so that the claim is shown. □

Note also that the upper bound $\delta/\overline{F}$ can be further improved if necessary: Instead of estimating F and $\tilde{F}$ very roughly by $\overline{F}$, we can define

$$d(x,A) = \inf_{y \in A} d(x,y) = \inf_{y \in A} \tilde{d}(x,y).$$

Equality between the definition with d and that with $\tilde{d}$ is due to Lemma 14, which implies that the shortest paths must be outside of A. Following the proof of Theorem 12 closely, one can see that it remains true for all $t \leq d(x,A)$ even though $d(x,A) < \delta/\overline{F}$ in general.

Acknowledgements

The author would like to thank Wolfgang Ring of the University of Graz for fruitful discussions about this topic and thorough proof-reading of the manuscript. This work is supported by the Austrian Science Fund (FWF) and the International Research Training Group IGDK 1754.

References

[1] Martino Bardi and Itali Capuzzo-Dolcetta. *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*. Systems & Control: Foundations & Applications. Birkhäuser, Boston / Basel / Berlin, 1997.

[2] Martino Bardi and Francesca Da Lio. On the strong maximum principle for fully nonlinear degenerate elliptic equations. *Archiv der Mathematik*, 73:276–285, 1999.

[3] G. Barles, H. M. Soner, and P. E. Souganidis. Front propagation and phase field theory. *SIAM J. Control and Optimization*, 31(2):439–469, March 1993.

[4] Martin Burger. A framework for the construction of level set methods for shape optimization and reconstruction. *Interfaces and Free Boundaries*, 5:301–329, 2003.

[5] Martin Burger, Norayr Matevosyan, and Marie-Therese Wolfram. A Level Set Based Shape Optimization Method for an Elliptic Obstacle Problem. *Mathematical Models and Methods in Applied Sciences*, 21(4):619–649, 2011.

[6] I. Capuzzo-Dolcetta. A generalized hopf-lax formula: Analytical and approximations aspects. In Fabio Ancona, editor, *Geometric Control and Nonsmooth Analysis*, volume 76 of *Series on Advances in Mathematics for Applied Sciences*, pages 136–150. World Scientific, 2008.

[7] E. Carlini, M. Falcone, N. Forcadel, and R. Monneau. Convergence of a generalized fast-marching method for an eikonal equation with a velocity-changing sign. *SIAM Journal of Numerical Analysis*, 46(6):2920–2952, 2008.

[8] Vicent Caselles, Francine Catté, Tomeu Coll, and Françoise Dibos. A geometric model for active contours in image processing. *Numerische Mathematik*, 66:1–31, 1993.

[9] M. G. Crandall. Viscosity solutions: a primer. In *Viscosity Solutions and Applications*, Lecture Notes in Mathematics. Springer, 1995.

[10] Michael G. Crandall, Hitoshi Ishii, and Pierre-Louis Lions. User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American Mathematical Society*, 27(1):1–67, July 1992.

[11] M. C. Delfour and J.-P. Zolésio. *Shapes and Geometries*. Advances in Design and Control. SIAM, second edition, 2001.

[12] Marc Droske and Wolfgang Ring. A Mumford-Shah Level-Set Approach For Geometric Image Registration. *SIAM Journal on Applied Mathematics*, 66(6):2127–2148, 2006.

[13] John W. Eaton, David Bateman, and Søren Hauberg. *GNU Octave version 3.0.1 manual: a high-level interactive language for numerical computations*. CreateSpace Independent Publishing Platform, 2009. ISBN 1441413006.

[14] Lawrence C. Evans. *Partial Differential Equations*. Graduate Studies in Mathematics. American Mathematical Society, 1999.

[15] Lawrence C. Evans and Ronald F. Gariepy. *Measure Theory and Fine Properties of Functions*. Studies in Advanced Mathematics. CRC Press, 1992.

[16] Yoshikazu Giga. *Surface Evolution Equations: a level set approach*. Monographs in mathematics. Birkhäuser, 2006.

[17] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order*, volume 224 of *Grundlehren der mathematischen Wissenschaften*. Springer, Berlin, second edition, 2001.

[18] Lester L. Helms. *Potential Theory*. Springer, second edition, 2014.

[19] Antoine Henrot and Michel Pierre. *Variation et optimisation de formes: Une analyse géométrique*, volume 48 of *Mathématiques & Applications*. Springer, 2005.

[20] Bernhard Kawohl and Nikolay Kutev. Comparison Principle and Lipschitz Regularity for Viscosity Solutions of Some Classes of Nonlinear Partial Differential Equations. *Funkcialaj Ekvacioj*, 43:241–253, 2000.

[21] Daniel Kraft. A Hopf-Lax Formula for the Level-Set Equation and Applications to PDE-Constrained Shape Optimisation. In *Proceedings of the 19th International Conference on Methods and Models in Automation and Robotics*, pages 498–503, 2014.

[22] Daniel Kraft. The `level-set` package for GNU Octave. Octave Forge, 2014–2015. `http://octave.sourceforge.net/level-set/`.

[23] Pierre-Louis Lions. *Generalized solutions of Hamilton-Jacobi equations*. Research Notes in Mathematics. Pitman Advanced Publishing Program, 1982.

[24] S. Osher and J. A. Sethian. Fronts propagating with curvature-dependent speed: Algorithms based on hamilton-jacobi formulations. *Journal of Computational Physics*, 79:12–49, 1988.

[25] Walter Rudin. *Real and Complex Analysis*. McGraw-Hill, third edition, 1987.

[26] Fadil Santosa. A Level-Set Approach for Inverse Problems Involving Obstacles. *ESIAM: Control, Optimisation and Calculus of Variations*, 1:17–33, 1996.

[27] J. A. Sethian. A Fast Marching Level Set Method for Monotonically Advancing Fronts. *Proceedings of the National Academy of Sciences*, 93(4):1591–1595, 1996.

[28] J. A. Sethian. *Level Set Methods and Fast Marching Methods: Evolving interfaces in computational geometry, fluid mechanics, computer vision, and materials science*. Cambridge University Press, Cambridge, second edition, 1999.

[29] B. C. Vemuri, J. Ye, Y. Chen, and C. M. Leonard. Image registration via level-set motion: Applications to atlas-based segmentation. *Medical Image Analysis*, 7:1–20, 2003.

[30] J. Yeh. *Real Analysis: Theory of Measure and Integration*. World Scientific, Singapore, second edition, 2006.

[31] Laurent Younes. *Shapes and Diffeomorphisms*, volume 171 of *Applied Mathematical Sciences*. Springer, 2010.