FINITE ELEMENT POINTWISE RESULTS ON CONVEX POLYHEDRAL DOMAINS

DMITRIY LEYKEKHMAN[†] AND BORIS VEXLER[‡]

Abstract. The main goal of the paper is to establish that the L^1 norm of jumps of the normal derivative across element boundaries and the L^1 norm of the Laplacian of a piecewise polynomial finite element function can be controlled by corresponding weighted discrete H^2 norm on convex polyhedral domains. In the finite element literature such results are only available for piecewise linear elements in two dimensions and the extension to convex polyhedral domains is rather technical. As a consequence of this result, we establish almost pointwise stability of the Ritz projection and the discrete resolvent estimate in L^∞ norm.

Key words. elliptic problems, finite elements, maximum norm, error estimates, pointwise error estimates, resolvent

AMS subject classifications.

1. Introduction. As a simple model of a second order elliptic partial differential equation we consider,

$$
\begin{aligned}
-\Delta u(x) &= f(x), & x \in \Omega, \\
u(x) &= 0, & x \in \partial\Omega,
\end{aligned}
\tag{1.1}
$$

with a right-hand side $f \in L^r(\Omega)$, $r > \frac{3}{2}$ on a convex polyhedral domain $\Omega \subset \mathbb{R}^3$. Let u_h be the Ritz projection of u onto the space of continuous piecewise polynomial finite element functions V_h on the mesh $\mathcal{T}$ consisting of elements τ. The aim of this paper is to establish some important weighted and pointwise results for u_h in three space dimensions that are not available in the literature but required in a number of applications (cf. [22, 23, 24]).

Our main technical result is Lemma 2.6 that shows

$$
\sum_{\tau \in \mathcal{T}} \left(\|\Delta v_h\|_{L^1(\tau)} + \|[\![\partial_n v_h]\!]\|_{L^1(\partial\tau)} \right) \leq C|\ln h|^{\frac{1}{2}} \left(\|\sigma^{\frac{3}{2}} \Delta_h v_h\|_{L^2(\Omega)} + \|\sigma^{\frac{1}{2}} \nabla v_h\|_{L^2(\Omega)} \right), \quad \forall v_h \in V_h, \tag{1.2}
$$

where the weight σ describing an h-dependent regularized distance is introduced in (2.6) and Δ_h is the discrete Laplace operator defined in (2.4). This estimate says that the L^1 norm of jumps of the normal derivative across element boundary as well as the L^1 norm of the Laplacian of any piecewise polynomial function can be controlled by the properly weighted discrete H^2-norm. A corresponding result for piecewise linear functions was proved by Rannacher [31] in two dimensions and was used to establish pointwise stability of the semidiscrete and fully discrete backward Euler solution of parabolic problems on convex polygonal domains. However, in order to extend this result to three dimensions one has overcome some serious technical obstacles. To accomplish this, we require several additional technical lemmas. Some results are standard, however Lemma 2.5 is rather peculiar and can be thought as weighted Gagliardo-Nirenberg interpolation inequality. It shows that for any $w \in H_0^1(\Omega)$, any $\alpha, \beta \in \mathbb{R}$ with $\alpha \geq -\frac{1}{2}$ and any $1 \leq p \leq \infty$, there holds,

$$
\|\sigma^\alpha w\|_{L^2(\Omega)}^2 \leq C \|\sigma^{\alpha-\beta} w\|_{L^p(\Omega)} \|\sigma^{\alpha+1+\beta} \nabla w\|_{L^q(\Omega)}, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{1.3}
$$

This result is an extension and a generalization of Lemma 3.4 from [1] and provides one with a great flexibility in manipulating weighted spaces, especially for Galerkin finite element solutions. Thus, by choosing $\beta = 0$ and $p = q = 2$, we obtain

$$
\|\sigma^\alpha w\|_{L^2(\Omega)} \leq C \|\sigma^{\alpha+1} \nabla w\|_{L^2(\Omega)}, \quad \forall w \in H_0^1(\Omega), \quad \forall \alpha \geq -\frac{1}{2},
$$

i.e. the estimate allows, for example, to "trade" derivatives for weights. An estimate similar to (1.3) with a weight function $|x|$ instead of $\sigma(x)$ can be found in [6].

These two technical lemmas, having an independent interest, are powerful results and have a variety of applications. One application provided in this paper is almost stability of the Ritz projection in L^∞ norm, (cf. Theorem 3.1),

$$
\|u_h\|_{L^\infty(\Omega)} \leq C|\ln h| \|u\|_{L^\infty(\Omega)}. \tag{1.4}
$$

[†]Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA (leykekhman@math.uconn.edu). The author was partially supported by NSF grant DMS-1522555.

[‡]Chair of Optimal Control, Center for Mathematical Sciences, Technische Universität München, Boltzmannstraße 3, 85748 Garching b. Munich, Germany (vexler@ma.tum.de).

Due to the fact that the Ritz-Projection is an identity on V_h, the above stability result is equivalent to the almost best approximation property of the error

$$\|u - u_h\|_{L^\infty(\Omega)} \le C|\ln h| \inf_{\chi \in V_h} \|u - \chi\|_{L^\infty(\Omega)}. \tag{1.5}$$

We would like to mention that estimates (1.4) and (1.5) for piecewise linear elements are asymptotically sharp and $|\ln h|$ is necessary as was shown in the example of Haverkamp [20] in two dimensional setting, cf. also [16]. It is known, that the factor $|\ln h|$ can be removed for higher order finite elements on smooth domains, see, e.g., [35]. However, whether $|\ln h|$ is necessary for higher order elements on convex polyhedral domains is an open question.

The second application provided in this paper is the following discrete resolvent estimate (cf. Theorem 4.3),

$$\|(z + \Delta_h)^{-1}\chi\|_{L^\infty(\Omega)} \le \frac{C|\ln h|}{|z - \lambda|} \|\chi\|_{L^\infty(\Omega)}, \quad \text{for } z \in \mathbb{C}\backslash\Sigma_{\lambda,\gamma}, \quad \text{for all } \chi \in V_h, \tag{1.6}$$

where

$$\Sigma_{\lambda,\gamma} = \{z \in \mathbb{C} : |\arg(z - \lambda)| \le \gamma\}, \tag{1.7}$$

for any $\gamma \in (0, \frac{\pi}{2})$ and $\lambda \in [0, \lambda_0]$, where $\lambda_0 > 0$ is the smallest eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions. Such resolvent estimates are useful in treatment of certain fully discrete schemes in L^∞ norm (cf. [34], [43] chap. 9) and we require them to establish fully discrete maximal parabolic regularity in [22]. In two space dimensions, such resolvent estimates were established in [3], Lemma 6.1. For smooth bounded domains in $\mathbb{R}^N$ with $N \ge 2$ the logarithmic term can be removed [4], but the analysis there requires the following continuous resolvent estimate

$$\|(z + \Delta)^{-1}v\|_{W^{1,\infty}(\Omega)} \le \frac{C}{(1 + |z|)^{\frac{1}{2}}} \|v\|_{L^\infty(\Omega)},$$

which for convex polyhedral domains we were not able to locate in the literature. If such a resolvent estimate is valid on convex polyhedral domains then following the analysis similar to [4], the logarithmic term can be removed as well. However, for our applications in [22, 24] the above estimate with $|\ln h|$ is sufficient. Results (1.2)-(1.6) constitute the main results of the paper.

Pointwise error estimates in the finite element literature for the second order elliptic problems started in the works of Nitsche [28, 27], Natterer [26], Scott [41], and Frehse and Rannacher [15]. Since then a lot of work was done in various settings [11, 12, 30, 32, 33, 35, 36, 37, 38, 39]. Nevertheless, results in three dimensions are rather scarce, especially on non-smooth domains. Rannacher [30] and Schatz and Wahlbin [35] were the firsts who showed best approximation property in L^∞ and $W^{1,\infty}$ norms on smooth bounded domains in $\mathbb{R}^N$ with $N \ge 2$, however results in [35] targeted interior error estimates and the global results were just byproducts. Later in [38], the effect of a "skin" layer was analyzed. The first stability result for the Ritz projection in $W^{1,\infty}$ norm without a logarithmic term on non-smooth (convex polygonal) domains were obtained by Rannacher and Scott in [32]. Using a similar technique, such a result for three dimensional polyhedral domains was first provided in the book [5] with some additional geometrical restrictions beyond convexity. This restriction was removed in Guzmán et al. [19] and later extended to more general meshes (cf. Demlow et al. [8]). However, the estimate (1.4) is known only on smooth domains (cf. [38]) or two dimensional polygonal domains [33]. To cover the case of a three-dimensional convex polyhedral domain, we among other things exploited the idea from [19] and used the $C^{1,\lambda}$ regularity (cf. Lemma 2.9 and the proof of Lemma 2.10) instead of the $W^{2,p}(\Omega)$ regularity with $p > 3$, which would require additional geometrical restrictions, cf. [5].

Currently, in the finite element literature on pointwise error estimates on unstructured meshes, there exist two popular techniques, global weighted technique, due to J. Nitsche, and the technique based on local energy estimates due to Schatz and Wahlbin. Both techniques are natural from an analytical point of view and more or less equivalent, meaning that the same results can be established by either technique. Thus, adapting a certain technique for a proof is just a matter of taste. In this paper both techniques are used. Although, the proof of Lemma 2.6 is based on a weighted technique, the proof of Lemma 2.10 is based on local energy technique, which appeared more convenient for us in that particular proof. There is also a technique based on Campanato spaces due to Dolzmann [9], but it is technically more involved and has never caught up much support.

The rest of the paper is organized as follows. In Section 2 we introduce the notation, define the weight function and weighted norms. Then we continue with a series of lemmas including the key results Lemma 2.5 and Lemma 2.6. In Section 3, we establish (1.4) in Theorem 3.1 and in Section 4, we establish the resolvent estimate (1.6) in Theorem 4.3.

2. Notation and weighted norm estimates. Throughout the paper we use the usual notation for Lebesgue and Sobolev spaces. For any set D, we will denote the $L^2(D)$ norm by $\|\cdot\|_D$. Other norms we will write explicitly. We denote by $(\cdot, \cdot)$ the $L^2(\Omega)$ inner product and we will specify a subdomain by a subscript in the case it is not the whole Ω.

Let $\mathcal{T}$ denote a quasi-uniform triangulation of Ω with a mesh size h, i.e., $\mathcal{T} = \{\tau\}$ is a partition of Ω into tetrahedrons τ of diameter h_τ such that for $h = \max_\tau h_\tau$,

$$\text{diam}(\tau) \leq h \leq C|\tau|^{\frac{1}{3}}, \quad \forall \tau \in \mathcal{T}$$

hold. Let V_h be the set of all functions in $H_0^1(\Omega)$ that are polynomial of degree k, $k \geq 1$ on each τ, i.e. V_h is the usual space of Lagrangian finite elements of degree k. For the space V_h we will utilize the L^2-Projection $P_h: L^2(\Omega) \to V_h$ defined by

$$(P_h v, \chi)_\Omega = (v, \chi)_\Omega, \quad \forall \chi \in V_h, \tag{2.1}$$

the Ritz-Projection $R_h: H_0^1(\Omega) \to V_h$ defined by

$$(\nabla R_h v, \nabla \chi)_\Omega = (\nabla v, \nabla \chi)_\Omega, \quad \forall \chi \in V_h, \tag{2.2}$$

and the usual nodewise interpolant $I_h: C_0(\Omega) \to V_h$ with usual approximation properties (cf., e.g., [7, Theorem 3.1.5])

$$\|u - I_h u\|_{L^q(\Omega)} \leq C h^{2+3(\frac{1}{q} - \frac{1}{p})} \|u\|_{W^{2,p}(\Omega)}, \quad \text{for} \quad q \geq p > \frac{3}{2}. \tag{2.3}$$

Moreover we introduce the discrete Laplace operator $\Delta_h: V_h \to V_h$ by

$$(-\Delta_h v_h, \chi)_\Omega = (\nabla v_h, \nabla \chi)_\Omega, \quad \forall \chi \in V_h. \tag{2.4}$$

Let $x_0 \in \Omega$ be a fixed (but arbitrary) point. Associated to this point we introduce a smooth Delta function [44, Lemma 2.2], which we will denote by $\tilde{\delta} = \tilde{\delta}_{x_0}$, cf. also [40]. This function is supported in one cell, denoted by τ_0, and satisfies

$$(\chi, \tilde{\delta})_{\tau_0} = \chi(x_0), \quad \forall \chi \in \mathbb{P}^k(\tau_0).$$

In addition from [44, Lemma 2.2] we also have

$$\|\tilde{\delta}\|_{W^{s,p}(\Omega)} \leq C h^{-s-3(1-\frac{1}{p})}, \quad 1 \leq p \leq \infty, \quad s = 0, 1, 2. \tag{2.5}$$

Thus in particular $\|\tilde{\delta}\|_{L^1(\Omega)} \leq C$, $\|\tilde{\delta}\|_\Omega \leq C h^{-\frac{3}{2}}$, and $\|\tilde{\delta}\|_{L^\infty(\Omega)} \leq C h^{-3}$. Next we introduce a weight function

$$\sigma(x) = \sqrt{|x - x_0|^2 + K^2 h^2}, \tag{2.6}$$

where $K > 0$ is sufficiently large constant to be chosen later. One can easily check that σ satisfies the following properties, (cf., e.g., [7], Sec. 3.3),

$$\|\sigma^{-\frac{3}{2}}\|_\Omega \leq C|\ln h|^{\frac{1}{2}}, \tag{2.7a}$$

$$|\nabla^l \sigma| \leq C \sigma^{1-l}, \quad l = 1, 2, \ldots, \tag{2.7b}$$

$$\max_\tau \sigma \leq C \min_\tau \sigma, \quad \forall \tau \in \mathcal{T}. \tag{2.7c}$$

Next we require an estimate in weighted norms for the L^2-Projection P_h. For piecewise linear case in two dimensions this result is established in Lemma 7.1 in [13]. In the appendix we provide a proof for arbitrary polynomial order.

LEMMA 2.1. *There exist $\varepsilon > 0$ and a constant $C > 0$, such that for any positive function $\varphi: \Omega \to \mathbb{R}_+$ satisfying $h^l|\nabla^l \varphi| \leq \varepsilon \varphi$ for all $l = 1, 2, \ldots, k$ and any $v \in L^2(\Omega)$, we have*

$$\|\varphi P_h v\|_\Omega \leq C\|\varphi v\|_\Omega.$$

The next lemma provides error estimates for the projection and the interpolation errors in weighted norms. In what follows we will use the discrete L^2 norm $\|\cdot\|_{h,\Omega}$ for functions defined cellwise:

$$\|z\|_{h,\Omega}^2 = \sum_{\tau \in \mathcal{T}} \|z\|_{L^2(\tau)}^2.$$

LEMMA 2.2. *Let* $v \in C_0(\Omega) \cap H_0^1(\Omega)$ *be cellwise in* H^l, *i.e.* $v|_\tau \in H^l(\tau)$ *for all* $\tau \in \mathcal{T}$ *and some* $2 \le l \le k+1$. *Then the following estimates hold for any* $\alpha \in \mathbb{R}$ *and* K *large enough:*

$$\|\sigma^\alpha(v - I_h v)\|_\Omega + h\|\sigma^\alpha\nabla(v - I_h v)\|_\Omega + h^2\|\sigma^\alpha\nabla^2(v - I_h v)\|_{h,\Omega} \le ch^l\|\sigma^\alpha\nabla^l v\|_{h,\Omega}, \qquad (2.8)$$

$$\|\sigma^\alpha(v - P_h v)\|_\Omega + h\|\sigma^\alpha\nabla(v - P_h v)\|_\Omega + h^2\|\sigma^\alpha\nabla^2(v - P_h v)\|_{h,\Omega} \le ch^l\|\sigma^\alpha\nabla^l v\|_{h,\Omega}. \qquad (2.9)$$

Proof. The estimates (2.8) are straightforward due to the local nature of the nodewise interpolant I_h and due to estimate (2.7c). The estimate for the first term in (2.9) is standard for $\alpha = 0$. For $\alpha \ne 0$ we consider $\varphi = \sigma^\alpha$. There holds for any $l = 1, 2, \ldots, k$

$$|\nabla^l \varphi| = |\nabla^l(\sigma^\alpha)| \le C_\alpha \sigma^{\alpha - l} \le C_\alpha \sigma^{-l} \varphi \le C_\alpha K^{-l} h^{-l} \varphi,$$

where we used (2.7b) and the fact that $\sigma \ge Kh$. For K large enough this φ fulfills the condition of Lemma 2.1 and we get

$$\|\sigma^\alpha(P_h v - I_h v)\|_\Omega = \|\sigma^\alpha P_h(v - I_h v)\|_\Omega \le C\|\sigma^\alpha(v - I_h v)\|_\Omega.$$

Therefore

$$\|\sigma^\alpha(v - P_h v)\|_\Omega \le C\|\sigma^\alpha(v - I_h v)\|_\Omega$$

and we get the estimate of the first term (2.9) using (2.8). The estimate for the second term in (2.9) is obtained by the inverse inequality, which holds in weighted norms again due to (2.7c). $\square$

The next lemma is a superconvergence result in weighted norms.

LEMMA 2.3. *Let* $v_h \in V_h$. *Then the following estimates hold for any* $\alpha, \beta \in \mathbb{R}$ *and* K *large enough:*

$$\|\sigma^\alpha(\mathrm{Id} - I_h)(\sigma^\beta v_h)\|_\Omega + h\|\sigma^\alpha\nabla(\mathrm{Id} - I_h)(\sigma^\beta v_h)\|_\Omega \le ch\|\sigma^{\alpha+\beta-1} v_h\|_\Omega, \qquad (2.10)$$

$$\|\sigma^\alpha(\mathrm{Id} - P_h)(\sigma^\beta v_h)\|_\Omega + h\|\sigma^\alpha\nabla(\mathrm{Id} - P_h)(\sigma^\beta v_h)\|_\Omega \le ch\|\sigma^{\alpha+\beta-1} v_h\|_\Omega. \qquad (2.11)$$

Proof. We prove the estimate for the first term in (2.10). By the estimate (2.8) from the previous lemma we obtain,

$$\|\sigma^\alpha(\mathrm{Id} - I_h)(\sigma^\beta v_h)\|_\Omega \le ch^{k+1}\|\sigma^\alpha\nabla^{k+1}(\sigma^\beta v_h)\|_{h,\Omega}.$$

Using the multi-index notation with $\gamma \in \mathbb{N}^3$, we have

$$\|\sigma^\alpha\nabla^{k+1}(\sigma^\beta v_h)\|_{h,\Omega}^2 = \sum_{|\gamma|=k+1} \|\sigma^\alpha\partial^\gamma(\sigma^\beta v_h)\|_{h,\Omega}^2.$$

Using the Leibniz's formula for $\partial^\gamma(\sigma^\beta v_h)$ and noticing that $\partial^\gamma v_h = 0$ on a cell τ for $|\gamma| = k+1$ (since v_h is a polynomial of degree at most k on τ), for each $\tau \in \mathcal{T}$ we obtain:

$$\partial^\gamma(\sigma^\beta v_h) = \sum_{|\delta|>0, \delta \le \gamma} \binom{\gamma}{\delta} \partial^\delta(\sigma^\beta)\partial^{\gamma-\delta} v_h,$$

where the inequality $\delta \le \gamma$ for multi-indices is understood as $\delta_i \le \gamma_i$ for $i = 1, 2, 3$ and the binomial coefficient is defined as

$$\binom{\gamma}{\delta} = \binom{\gamma_1}{\delta_1}\binom{\gamma_2}{\delta_2}\binom{\gamma_3}{\delta_3}.$$

By the property of σ, namely (2.7b), we have

$$|\partial^\delta(\sigma^\beta)| \le c\sigma^{\beta-|\delta|}.$$

Therefore we get for any $\tau \in \mathcal{T}$ and any γ with $|\gamma| = k+1$,

$$h^{k+1}\|\sigma^\alpha \partial^\gamma(\sigma^\beta v_h)\|_\tau \le ch^{k+1} \sum_{|\delta|>0,\delta\le\gamma} \|\sigma^\alpha \partial^\delta(\sigma^\beta)\partial^{\gamma-\delta}v_h\|_\tau \le ch^{k+1} \sum_{|\delta|>0,\delta\le\gamma} \|\sigma^{\alpha+\beta-|\delta|}\partial^{\gamma-\delta}v_h\|_\tau.$$

Using property (2.7c) of σ and the inverse inequality $\|\partial^{\gamma-\delta}v_h\|_\tau \le ch^{|\delta|-k-1}\|v_h\|_\tau$, we obtain

$$h^{k+1} \sum_{|\delta|>0,\delta\le\gamma} \|\sigma^{\alpha+\beta-|\delta|}\partial^{\gamma-\delta}v_h\|_\tau \le C h \sum_{|\delta|>0,\delta\le\gamma} h^{|\delta|-1}\|\sigma^{\alpha+\beta-|\delta|}v_h\|_\tau.$$

Finally using $h \le \sigma$, we obtain the desired estimate.

The estimates for the second term in (2.10) and estimates (2.11) follow by similar arguments as in the proof of Lemma 2.2. □

The next lemma shows how the weight σ compensates the singularity of the regularized Delta function $\tilde{\delta}$.

LEMMA 2.4. *There holds*

$$\|\sigma^{\frac{3}{2}}\tilde{\delta}\|_\Omega + h\|\sigma^{\frac{3}{2}}\nabla\tilde{\delta}\|_\Omega + h^2\|\sigma^{\frac{3}{2}}\nabla^2\tilde{\delta}\|_\Omega + \|\sigma^{\frac{3}{2}}P_h\tilde{\delta}\|_\Omega \le C. \tag{2.12}$$

Proof. Using that the support of $\tilde{\delta}_{x_0}$ is in a single element τ_0 and using (2.5), we have

$$\|\sigma^{\frac{3}{2}}\tilde{\delta}\|_\Omega^2 = \int_{\tau_0} |\sigma^{\frac{3}{2}}\tilde{\delta}|^2 dx \le \|\tilde{\delta}\|_{L^\infty(\Omega)}^2 \int_{\tau_0} (|x-x_0|^2 + K^2h^2)^{\frac{3}{2}} dx \le Ch^{-6}h^3|\tau_0| \le C.$$

Similarly we get

$$\|\sigma^{\frac{3}{2}}\nabla\tilde{\delta}\|_\Omega^2 = \int_{\tau_0} |\sigma^{\frac{3}{2}}\nabla\tilde{\delta}|^2 dx \le \|\nabla\tilde{\delta}\|_{L^\infty(\Omega)}^2 \int_{\tau_0} (|x-x_0|^2 + K^2h^2)^{\frac{3}{2}} dx \le Ch^{-8}h^3|\tau_0| \le Ch^{-2}$$

and thus $\|\sigma^{\frac{3}{2}}\nabla\tilde{\delta}\|_\Omega \le ch^{-1}$. Analogously we obtain the estimate $\|\sigma^{\frac{3}{2}}\nabla^2\tilde{\delta}\|_\Omega \le ch^{-2}$. For the last term in (2.12) we obtain by (2.9)

$$\|\sigma^{\frac{3}{2}}P_h\tilde{\delta}\|_\Omega \le \|\sigma^{\frac{3}{2}}\tilde{\delta}\|_\Omega + \|\sigma^{\frac{3}{2}}(P_h-I)\tilde{\delta}\|_\Omega \le \|\sigma^{\frac{3}{2}}\tilde{\delta}\|_\Omega + ch^2\|\sigma^{\frac{3}{2}}\nabla^2\tilde{\delta}\|_\Omega \le C.$$

□

The idea of the next lemma comes from the two-dimensional argument used within the proof of Lemma 3.4 from [1]. However, the proof of the three-dimensional result is different. A similar estimate with the weight being the distance function $|x|$ instead of $\sigma(x)$ can be found in [6].

LEMMA 2.5. *There exists a constant C independent of K and h such that for any $f \in H_0^1(\Omega)$, any $\alpha, \beta \in \mathbb{R}$ with $\alpha \ge -\frac{1}{2}$ and any $1 \le p \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ holds:*

$$\|\sigma^\alpha f\|_\Omega^2 \le C\|\sigma^{\alpha-\beta} f\|_{L^p(\Omega)}\|\sigma^{\alpha+1+\beta}\nabla f\|_{L^q(\Omega)},$$

provided $\|\sigma^{\alpha-\beta} f\|_{L^p(\Omega)}$ and $\|\sigma^{\alpha+1+\beta}\nabla f\|_{L^q(\Omega)}$ are bounded.

Proof. We assume that the domains Ω is contained in a ball $B_R(x_0)$ centered in x_0 with the radius R sufficiently large and denote by $r = r(x) = |x-x_0|$ the distance to x_0. We define f on the whole $B_R(x_0)$ by extending it by 0 outside of Ω. In the following we will use spherical coordinates (r, θ, φ) and the notation

$$\omega = (\theta, \varphi) \in \mathcal{S} := (0, \pi) \times (0, 2\pi).$$

Moreover, we will use the convention $f = f(x) = f(r, \omega)$ as well as $\sigma = \sigma(x) = \sigma(r)$. Notice that

$$\frac{d}{dr}(\sigma(r)) = \frac{d}{dr}(\sigma(r) - \sigma(0)) = \frac{r}{\sigma}. \tag{2.13}$$

Then transforming the integral, we get

$$\|\sigma^\alpha f\|_\Omega^2 = \int_\Omega \sigma^{2\alpha} f^2 dx = \int_{\mathcal{S}} \int_0^R \sigma^{2\alpha}(r) f^2 r^2 dr d\omega. \tag{2.14}$$

Using (2.13) and integrating by parts, we obtain

$$\int_0^R \sigma^{2\alpha}(r) f^2 r^2 dr = \int_0^R \frac{d}{dr} \left(\sigma(r) - \sigma(0) \right) \sigma^{2\alpha+1} f^2 r dr = -\int_0^R \left(\sigma(r) - \sigma(0) \right) \frac{d}{dr} (\sigma^{2\alpha+1} f^2 r) dr$$

$$= -\int_0^R \left(\sigma(r) - \sigma(0) \right) \left((2\alpha+1)\sigma^{2\alpha} \frac{r}{\sigma} f^2 r + \sigma^{2\alpha+1} f^2 + 2\sigma^{2\alpha+1} f f_r r \right) dr.$$

Using the assumption $\alpha \geq -\frac{1}{2}$, we have

$$-\int_0^R \left(\sigma(r) - \sigma(0) \right) \left((2\alpha+1)\sigma^{2\alpha} \frac{r}{\sigma} f^2 r + \sigma^{2\alpha+1} f^2 \right) dr \leq 0$$

and as a result

$$\int_0^R \sigma^{2\alpha}(r) f^2 r^2 dr \leq -\int_0^R \left(\sigma(r) - \sigma(0) \right) 2\sigma^{2\alpha+1} f f_r r dr.$$

Using that

$$\frac{\sigma(r) - \sigma(0)}{r^2} = \frac{\sqrt{r^2 + K^2 h^2} - Kh}{r^2} = \frac{1}{\sqrt{r^2 + K^2 h^2} + Kh} \leq \sigma^{-1},$$

and applying the Cauchy-Schwarz inequality together with the fact that $r \leq \sigma$, for $1 < p < \infty$, we have

$$\|\sigma^\alpha f\|_\Omega^2 \leq C \int_{\mathcal{S}} \int_0^R \left| \frac{\sigma(r) - \sigma(0)}{r^2} \right| \sigma^{2\alpha+1} |f| |f_r| r^3 dr d\omega \leq C \int_{\mathcal{S}} \int_0^R \sigma^{2\alpha} |f| |f_r| r^3 dr d\omega$$

$$\leq C \int_{\mathcal{S}} \int_0^R \sigma^{2\alpha+1} |f| |f_r| r^2 dr d\omega$$

$$\leq C \int_{\mathcal{S}} \int_0^R \left(\sigma^{\alpha-\beta} |f| r^{\frac{2}{p}} \right) \left(\sigma^{\alpha+1+\beta} |f_r| r^{\frac{2}{q}} \right) dr d\omega$$

$$\leq C \left(\int_{\mathcal{S}} \int_0^R \sigma^{p(\alpha-\beta)} |f|^p r^2 dr d\omega \right)^{1/p} \left(\int_{\mathcal{S}} \int_0^R \sigma^{q(\alpha+1+\beta)} |f_r|^q r^2 dr d\omega \right)^{1/q}$$

$$\leq C \|\sigma^{\alpha-\beta} f\|_{L^p(\Omega)} \|\sigma^{\alpha+1+\beta} \nabla f\|_{L^q(\Omega)}.$$

For $p = 1$ or $p = \infty$ the estimate is similar. $\square$

The next lemma is a three dimensional version of Lemma 2.4 in [31]. The analysis in three dimensions is more involved than the corresponding analysis in two dimensions. The main difficulty lies in the fact that we need to deal with odd powers of the weight function σ and this causes serious technical difficulties. In the proof we mix two popular techniques from the finite element literature on pointwise estimates.

LEMMA 2.6. *There exists a constant $C > 0$ independent on h, such that for any $v_h \in V_h$,*

$$\sum_{\tau \in \mathcal{T}} \left(\|\Delta v_h\|_{L^1(\tau)} + \|[\![\partial_n v_h]\!]\|_{L^1(\partial\tau)} \right) \leq C |\ln h|^{\frac{1}{2}} \left(\|\sigma^{\frac{3}{2}} \Delta_h v_h\|_\Omega + \|\sigma^{\frac{1}{2}} \nabla v_h\|_\Omega \right).$$

Proof. We define v as the solution of

$$\begin{cases} -\Delta v = -\Delta_h v_h & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.15}$$

Thus, by construction v_h is the Ritz projection of v, i.e., $R_h v = v_h$ and $v \in H^2(\Omega) \cap H_0^1(\Omega)$ with

$$\|v\|_{H^2(\Omega)} \leq C \|\Delta v\|_\Omega = C \|\Delta_h v_h\|_\Omega.$$

Using the fact that jumps of v are zero, the trace inequality and the inverse inequality we have

$$\sum_\tau \|[\![\partial_n v_h]\!]\|_{L^1(\partial\tau)} = \sum_\tau \|[\![\partial_n[v_h - v]]\!]\|_{L^1(\partial\tau)}$$

$$\leq C\sum_\tau \left(h^{-1}\|\nabla(v - v_h)\|_{L^1(\tau)} + \|\nabla^2(v - v_h)\|_{L^1(\tau)}\right)$$

$$\leq C\sum_\tau \left(h^{-1}\|\nabla(v - v_h)\|_{L^1(\tau)} + \|\nabla^2(v - I_h v)\|_{L^1(\tau)} + \|\nabla^2(v_h - I_h v)\|_{L^1(\tau)}\right)$$

$$\leq C\sum_\tau \left(h^{-1}\|\nabla(v - v_h)\|_{L^1(\tau)} + \|\nabla^2(v - I_h v)\|_{L^1(\tau)} + h^{-1}\|\nabla(v_h - I_h v)\|_{L^1(\tau)}\right)$$

$$\leq C\sum_\tau \left(h^{-1}\|\nabla(v - v_h)\|_{L^1(\tau)} + h^{-1}\|\nabla(v - I_h v)\|_{L^1(\tau)} + \|\nabla^2(v - I_h v)\|_{L^1(\tau)}\right). \tag{2.16}$$

Similarly, we obtain for the Laplacian of v_h

$$\sum_\tau \|\Delta v_h\|_{L^1(\tau)} \leq \sum_\tau \left(\|\Delta v\|_{L^1(\tau)} + \|\Delta(v_h - v)\|_{L^1(\tau)}\right)$$

$$\leq C\sum_\tau \left(\|\nabla^2 v\|_{L^1(\tau)} + h^{-1}\|\nabla(v - v_h)\|_{L^1(\tau)} + h^{-1}\|\nabla(v - I_h v)\|_{L^1(\tau)} + \|\nabla^2(v - I_h v)\|_{L^1(\tau)}\right). \tag{2.17}$$

Combining (2.16) and (2.17), using the properties of σ and Lemma 2.2, we obtain

$$\sum_\tau \left(\|\Delta v_h\|_{L^1(\tau)} + \|[\![\partial_n[v_h - v]]\!]\|_{L^1(\partial\tau)}\right)$$

$$\leq C\sum_\tau \left(\|\nabla^2 v\|_{L^1(\tau)} + h^{-1}\|\nabla(v - v_h)\|_{L^1(\tau)} + h^{-1}\|\nabla(v - I_h v)\|_{L^1(\tau)} + \|\nabla^2(v - I_h v)\|_{L^1(\tau)}\right)$$

$$\leq C\|\sigma^{-\frac{3}{2}}\|_\Omega \left(\|\sigma^{\frac{3}{2}}\nabla^2 v\|_\Omega + h^{-1}\|\sigma^{\frac{3}{2}}\nabla(v - v_h)\|_\Omega + h^{-1}\|\sigma^{\frac{3}{2}}\nabla(v - I_h v)\|_\Omega + \|\sigma^{\frac{3}{2}}\nabla^2(v - I_h v)\|_{h,\Omega}\right)$$

$$\leq C|\ln h|^{\frac{1}{2}}\left(h^{-1}\|\sigma^{\frac{3}{2}}\nabla(v - v_h)\|_\Omega + \|\sigma^{\frac{3}{2}}\nabla^2 v\|_\Omega\right),$$

To conclude the proof, we need to establish that

$$h^{-1}\|\sigma^{\frac{3}{2}}\nabla(v - v_h)\|_\Omega + \|\sigma^{\frac{3}{2}}\nabla^2 v\|_\Omega \leq C\left(\|\sigma^{\frac{3}{2}}\Delta_h v_h\|_\Omega + \|\sigma^{\frac{1}{2}}\nabla v_h\|_\Omega\right),$$

which we will show in the next two lemmas separately. □

In the next lemma we will treat the $\|\sigma^{\frac{3}{2}}\nabla^2 v\|_\Omega$ term.

LEMMA 2.7. *There exists a constant $C > 0$ independent on h, such that for any $v_h \in V_h$ and the corresponding $v \in H^2(\Omega) \cap H_0^1(\Omega)$ defined by (2.15) the following estimate holds,*

$$\|\sigma^{\frac{3}{2}}\nabla^2 v\|_\Omega \leq C\left(\|\sigma^{\frac{3}{2}}\Delta_h v_h\|_\Omega + \|\sigma^{\frac{1}{2}}\nabla v_h\|_\Omega\right).$$

Proof. Since

$$\partial_{ij}^2(\sigma^{\frac{3}{2}}v) = \sigma^{\frac{3}{2}}\partial_{ij}^2 v + (\partial_i(\sigma^{\frac{3}{2}})\partial_j v + \partial_j(\sigma^{\frac{3}{2}})\partial_i v) + \partial_{ij}^2(\sigma^{\frac{3}{2}})v,$$

and by properties of σ, $|\nabla(\sigma^{\frac{3}{2}})| \leq C|\sigma^{\frac{1}{2}}|$ and $|\nabla^2(\sigma^{\frac{3}{2}})| \leq C\sigma^{-\frac{1}{2}}$, we obtain

$$\|\sigma^{\frac{3}{2}}\nabla^2 v\|_\Omega \leq \|\nabla^2(\sigma^{\frac{3}{2}}v)\|_\Omega + c\|\sigma^{\frac{1}{2}}\nabla v\|_\Omega + c\|\sigma^{-\frac{1}{2}}v\|_\Omega. \tag{2.18}$$

Using Lemma 2.5 with $\alpha = -\frac{1}{2}$, $\beta = 0$ and $p = 2$ we have

$$\|\sigma^{-\frac{1}{2}}v\|_\Omega \leq C\|\sigma^{\frac{1}{2}}\nabla v\|_\Omega. \tag{2.19}$$

By the global $H^2(\Omega)$ regularity, we have

$$\|\nabla^2(\sigma^{\frac{3}{2}}v)\|_\Omega \leq C\|\Delta(\sigma^{\frac{3}{2}}v)\|_\Omega.$$

For $\Delta(\sigma^{\frac{3}{2}}v)$ we obtain

$$\Delta(\sigma^{\frac{3}{2}}v) = \Delta(\sigma^{\frac{3}{2}})v + 3\sigma^{\frac{1}{2}}\nabla\sigma \cdot \nabla v + \sigma^{\frac{3}{2}}\Delta v$$

and thus by the properties of σ we get

$$\|\nabla^2(\sigma^{\frac{3}{2}}v)\|_{\Omega} \le C\|\sigma^{-\frac{1}{2}}v\|_{\Omega} + C\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega} + C\|\sigma^{\frac{3}{2}}\Delta v\|_{\Omega}$$

and by using (2.19) we obtain

$$\|\nabla^2(\sigma^{\frac{3}{2}}v)\|_{\Omega} \le C\left(\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega} + \|\sigma^{\frac{3}{2}}\Delta v\|_{\Omega}\right) = C\left(\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega} + \|\sigma^{\frac{3}{2}}\Delta_h v_h\|_{\Omega}\right), \tag{2.20}$$

where we applied the definition of v (2.15) in the last step. It remains to estimate $\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega}$. There holds

$$\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega}^2 = (\sigma\nabla v, \nabla v) = (\sigma\nabla v, \nabla(v - v_h)) + (\sigma^{\frac{1}{2}}\nabla v, \sigma^{\frac{1}{2}}\nabla v_h)$$

$$\le (\sigma\nabla v, \nabla(v - v_h)) + \frac{1}{2}\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega}^2 + \frac{1}{2}\|\sigma^{\frac{1}{2}}\nabla v_h\|_{\Omega}^2$$

and therefore

$$\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega}^2 \le 2(\sigma\nabla v, \nabla(v - v_h)) + \|\sigma^{\frac{1}{2}}\nabla v_h\|_{\Omega}^2.$$

For the first term we get

$$(\sigma\nabla v, \nabla(v - v_h)) = -(\nabla \cdot (\sigma\nabla v), v - v_h) = -(\nabla\sigma \cdot \nabla v + \sigma\Delta v, v - v_h)$$

$$= -(\sigma^{\frac{1}{2}}\nabla\sigma \cdot \nabla v + \sigma^{\frac{3}{2}}\Delta v, \sigma^{-\frac{1}{2}}(v - v_h))$$

$$\le \frac{1}{4}\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega}^2 + C\|\sigma^{\frac{3}{2}}\Delta v\|_{\Omega}^2 + Ch^{-1}\|v - v_h\|_{\Omega}^2$$

$$\le \frac{1}{4}\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega}^2 + C\|\sigma^{\frac{3}{2}}\Delta_h v_h\|_{\Omega}^2 + Ch^{-1}h^4\|v\|_{H^2(\Omega)}^2$$

$$\le \frac{1}{4}\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega}^2 + C\|\sigma^{\frac{3}{2}}\Delta_h v_h\|_{\Omega}^2 + Ch^3\|\Delta v\|_{\Omega}^2$$

$$\le \frac{1}{4}\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega}^2 + C\|\sigma^{\frac{3}{2}}\Delta_h v_h\|_{\Omega}^2,$$

where we used again (2.15) and that $h \le \sigma$. This results in

$$\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega}^2 \le C\|\sigma^{\frac{3}{2}}\Delta_h v_h\|_{\Omega}^2 + \|\sigma^{\frac{1}{2}}\nabla v_h\|_{\Omega}^2,$$

which together with (2.20) completes the proof. □

As the next step we have to estimate $h^{-1}\|\sigma^{\frac{3}{2}}\nabla(v - v_h)\|_{\Omega}$. In the proof of this estimate we will make a heavy use of pointwise estimates for the Green's function and its derivatives. The proof of the next lemma for a general second order elliptic equation can be found in [18].

LEMMA 2.8. *Let $\Omega \subset \mathbb{R}^3$ be a convex domain of polyhedral type. Let $G(x, y)$ denotes the elliptic Green's function of the Laplace operator on the domain Ω. Then the following estimates hold,*

$$|G(x, y)| \le C|x - y|^{-1}, \tag{2.21a}$$

$$|\nabla_x G(x, y)| \le C|x - y|^{-2}. \tag{2.21b}$$

Sharper Hölder type estimates for the Green's function are derived for three dimensional polyhedral domains in [19, Theorem 1]. We summarize them in the following lemma.

LEMMA 2.9. *Let $\Omega \subset \mathbb{R}^3$ be a convex domain of polyhedral type. There exist $0 < \lambda < 1$ that depends on geometry of Ω and a constant C such that the estimates*

$$\frac{|\partial_k G(x, \xi) - \partial_k G(y, \xi)|}{|x - y|^{\lambda}} \le C\left(|x - \xi|^{-2-\lambda} + |y - \xi|^{-2-\lambda}\right), \quad for \quad k = 1, 2, 3,$$

are satisfied for all $x, y, \xi \in \Omega$, $x \neq y$. Now we are ready to establish an estimate for $\|\sigma^{\frac{3}{2}}\nabla(v - v_h)\|_\Omega$.

LEMMA 2.10. *There exists a constant* $C > 0$ *independent on* h, *such that for any* $v_h \in V_h$ *and the corresponding* $v \in H^2(\Omega) \cap H_0^1(\Omega)$ *defined by (2.15) and for* K *sufficiently large, the following estimate holds*

$$\|\sigma^{\frac{3}{2}}\nabla(v - v_h)\|_\Omega \leq Ch\left(\|\sigma^{\frac{3}{2}}\Delta_h v_h\|_\Omega + \|\sigma^{\frac{1}{2}}\nabla v_h\|_\Omega\right).$$

Proof. To obtain the estimate we use a dyadic decomposition of Ω. Denote $d_j = 2^{-j}\operatorname{diam}(\Omega)$ for $j = 0, 1, \ldots$ and define innermost set

$$\Omega_* = \{\, x \in \Omega \mid |x - x_0| \leq C_* h \,\}$$

where the constant C_* to be determined later and

$$\Omega_j = \{\, x \in \Omega \mid d_{j+1} < |x - x_0| \leq d_j \,\}.$$

Let J be chosen such that $d_{J+1} \leq C_* h_* \leq d_J$. Note that by construction $\operatorname{diam}(\Omega_j) \leq d_j$, $J \leq C|\ln h|$, and $\sigma \leq \sqrt{d_j^2 + (Kh)^2} \leq d_j + Kh$ on Ω_j. Furthermore, for $k > j + 1$ so that $d_j > d_k$ there holds

$$\frac{1}{2}d_j \leq \operatorname{dist}(\Omega_k, \Omega_j) \leq d_j. \tag{2.22}$$

We have the decomposition

$$\Omega = \Omega_* \cup \bigcup_{j=0}^{J} \Omega_j.$$

Moreover, in the following analysis we will need the following sets

$$\Omega_j' = \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1} \quad \text{and} \quad \Omega_j'' = \Omega_{j-2} \cup \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1} \cup \Omega_{j+2}.$$

Denote by $e := v - v_h$. We have

$$\|\sigma^{\frac{3}{2}}\nabla(v - v_h)\|_\Omega^2 = \|\sigma^{\frac{3}{2}}\nabla e\|_\Omega^2 \leq (C_* + K)^3 h^3 \|\nabla e\|_{\Omega_*}^2 + C\sum_{j=0}^{J}(d_j + Kh)^3\|\nabla e\|_{\Omega_j}^2$$

$$\leq C(C_* + K)^3 h^3 \|\nabla e\|_\Omega^2 + C\sum_{j=0}^{J} d_j^3\|\nabla e\|_{\Omega_j}^2.$$

By global best approximation result, elliptic H^2-regularity, and $h \leq \frac{\sigma}{K}$, we have

$$h^{\frac{3}{2}}\|\nabla e\|_\Omega \leq Ch^{\frac{3}{2}}\|\nabla(v - I_h v)\|_\Omega \leq Ch^{\frac{3}{2}+1}\|\nabla^2 v\|_\Omega \leq Ch^{\frac{3}{2}+1}\|\Delta v\|_\Omega \leq CK^{-\frac{3}{2}}h\|\sigma^{\frac{3}{2}}\Delta_h v_h\|_\Omega.$$

Using local energy estimates [29], we have for any $\chi \in V_h$,

$$\|\nabla e\|_{\Omega_j}^2 \leq C\left(\|\nabla(v - \chi)\|_{\Omega_j'}^2 + d_j^{-2}\|v - \chi\|_{\Omega_j'}^2 + d_j^{-2}\|e\|_{\Omega_j'}^2\right). \tag{2.23}$$

Taking $\chi = I_h v$ and using the approximation theory, $h \leq d_j$ and that $d_j \leq c\sigma$ on Ω_j', we have

$$\sum_{j=0}^{J} d_j^3\left(\|\nabla(v - \chi)\|_{\Omega_j'}^2 + d_j^{-2}\|v - \chi\|_{\Omega_j'}^2\right) \leq Ch^2\sum_{j=0}^{J} d_j^3\|\nabla^2 v\|_{\Omega_j'}^2 \leq Ch^2\|\sigma^{\frac{3}{2}}\nabla^2 v\|_\Omega^2. \tag{2.24}$$

Later on, we will control the term on the right-hand side of (2.24) using Lemma 2.7. Thus, we need to estimate $\sum_{j=0}^{J} d_j\|e\|_{\Omega_j'}^2$. First we notice that $\sum_{j=0}^{J} d_j\|e\|_{\Omega_j'}^2 \leq \frac{7}{2}\sum_{j=0}^{J,*} d_j\|e\|_{\Omega_j}^2$, where the notation $\sum_{i=0}^{J,*}$ means that

the sum includes the innermost set Ω_*. For $j = *, J, J-1$ using that $d_{J-1} = 2d_J \le 4C_*h$ and the global approximation result, we obtain

$$\sum_{j=J-1}^{J,*} d_j \|e\|_{\Omega_j}^2 \le CC_* h^5 \|\nabla^2 v\|_\Omega^2 \le CC_* h^5 \|\Delta v\|_\Omega^2 = CC_* K^{-3} h^2 \|\sigma^{\frac{3}{2}} \Delta_h v\|_\Omega^2. \tag{2.25}$$

To estimate $\sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2$, we use a duality argument. Let w be the solution of the following problem

$$\begin{aligned} -\Delta w &= e \cdot \mathbb{1}_{\Omega_j}, & &\text{in } \Omega, \\ w &= 0, & &\text{on } \partial\Omega, \end{aligned} \tag{2.26}$$

where $\mathbb{1}_{\Omega_j}$ is the characteristic function of Ω_j. Then

$$\|e\|_{\Omega_j}^2 = (\nabla w, \nabla e)_\Omega = (\nabla(w - I_h w), \nabla e)_\Omega = \sum_{i=0}^{J,*} (\nabla(w - I_h w), \nabla e)_{\Omega_i}.$$

We now consider three cases:

Case 1: $|i - j| \le 2$. In this case $\frac{1}{2} d_i \le d_j \le 2d_i$ and $\Omega_i \subset \Omega_j'$. Using the Cauchy-Schwarz inequality and the H^2-regularity we obtain

$$\sum_{|i-j|\le 1} (\nabla(w - I_h w), \nabla e)_{\Omega_i} \le Ch \|\nabla^2 w\|_\Omega \|\nabla e\|_{\Omega_j'} \le Ch \|e\|_{\Omega_j} \|\nabla e\|_{\Omega_j'},$$

and as a result, we obtain

$$\sum_{j=0}^{J-2} d_j \sum_{|i-j|\le 1} (\nabla(w - I_h w), \nabla e)_{\Omega_i} \le Ch \sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j} \|\nabla e\|_{\Omega_j'} := S_1.$$

Hence,

$$\begin{aligned} S_1 &\le C \left(\sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}} \left(\sum_{j=0}^{J-1} d_j h^2 \|\nabla e\|_{\Omega_j}^2 \right)^{\frac{1}{2}} \\ &\le C \left(\sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}} K^{-1} \left(\sum_{j=0}^{J-1} \|\sigma^{\frac{3}{2}} \nabla e\|_{\Omega_j'}^2 \right)^{\frac{1}{2}} \\ &\le CK^{-1} \left(\sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}} \|\sigma^{\frac{3}{2}} \nabla e\|_\Omega, \end{aligned}$$

where we used that $\frac{h}{\sigma} \le K^{-1}$ and $d_j \le 2\sigma$ on Ω_j.

Case 2: $i < j - 2$. In this case $d_i > d_j$ and $\frac{1}{2} d_i \le \text{dist}(\Omega_j, \Omega_i) \le d_i$.

$$\sum_{i<j-2} (\nabla(w - I_h w), \nabla e)_{\Omega_i} \le \sum_{i<j-2} \|\nabla(w - I_h w)\|_{\Omega_i} \|\nabla e\|_{\Omega_i} \le Ch \sum_{i<j-2} \|\nabla^2 w\|_{\Omega_i} \|\nabla e\|_{\Omega_i}.$$

Using that w is harmonic on Ω_i and the Hölder inequality, we have

$$\|\nabla^2 w\|_{\Omega_i} \le Cd_i^{-1} \|\nabla w\|_{\Omega_i'} \le Cd_i^{\frac{1}{2}} \|\nabla w\|_{L^\infty(\Omega_i')}.$$

Using Green's function representation, Green's function estimate (2.21b) and that $\frac{1}{2} d_i \le \text{dist}(\Omega_j, \Omega_i') \le d_i$, for $x \in \Omega_i'$ we have

$$|\nabla w(x)| = \left| \int_{\Omega_j} \nabla_x G(x, y) e(y) dy \right| \le C \int_{\Omega_j} \frac{|e(y)|}{|x-y|^2} dy \le Cd_i^{-2} \|e\|_{L^1(\Omega_j)} \le Cd_i^{-2} d_j^{\frac{3}{2}} \|e\|_{\Omega_j}.$$

As a result

$$\|\nabla^2 w\|_{\Omega_i} \leq C d_i^{-\frac{3}{2}} d_j^{\frac{3}{2}} \|e\|_{\Omega_j},$$

and hence the total contribution of this term to the sum is

$$\sum_{j=0}^{J-2} d_j \sum_{i<j-2} (\nabla(w - I_h w), \nabla e)_{\Omega_i} \leq \sum_{j=0}^{J-2} d_j \sum_{i<j-2} h\|\nabla^2 w\|_{\Omega_i}\|\nabla e\|_{\Omega_i}$$

$$\leq Ch \sum_{j=0}^{J-2} d_j^{\frac{5}{2}} \|e\|_{\Omega_j} \sum_{i<j-2} d_i^{-\frac{3}{2}} \|\nabla e\|_{\Omega_i} := S_2.$$

Changing the order of summation we obtain

$$S_2 \leq Ch \sum_{i=0}^{J-2} d_i^{-\frac{3}{2}} \|\nabla e\|_{\Omega_i} \sum_{j=i+3}^{J-2} d_j^{\frac{5}{2}} \|e\|_{\Omega_j}.$$

Now using the properties of geometric series, we have

$$\sum_{j=i+3}^{J-2} d_j^{\frac{5}{2}} \|e\|_{\Omega_j'} \leq \left(\sum_{j=i+2}^{J} d_j^4 \right)^{\frac{1}{2}} \left(\sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}} \leq C d_i^2 \left(\sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}}.$$

As a result using that $d_i \leq 2\sigma$ on Ω_i

$$S_2 \leq Ch \sum_{i=0}^{J-2} d_i^{\frac{1}{2}} \|\nabla e\|_{\Omega_i} \left(\sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}}$$

$$\leq C \left(\sum_{i=0}^{J} \|\sigma^{\frac{3}{2}} \nabla e\|_{\Omega_i}^2 \right)^{\frac{1}{2}} \left(\sum_{i=0}^{J} \left(\frac{h}{d_i} \right)^2 \right)^{\frac{1}{2}} \left(\sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}} \qquad (2.27)$$

$$\leq C C_*^{-1} \|\sigma^{\frac{3}{2}} \nabla e\|_{\Omega} \left(\sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}},$$

where we used that $\sum_{i=0}^{J} \left(\frac{h}{d_i} \right)^2 \leq C C_*^{-2}$.

Case 3: $i > j + 2$. In this case $d_i < d_j$ and $\frac{1}{2} d_j \leq \text{dist}(\Omega_j, \Omega_i) \leq d_j$. In this case, using the Hölder inequality and the approximation theory, we have

$$\sum_{i=j+3}^{J,*} (\nabla(w - I_h w), \nabla e)_{\Omega_i} \leq \sum_{i=j+3}^{J,*} \|\nabla(w - I_h w)\|_{L^\infty(\Omega_i)} \|\nabla e\|_{L^1(\Omega_i)}$$

$$\leq Ch^\lambda \sum_{i=j+3}^{J} \|w\|_{C^{1,\lambda}(\Omega_i)} d_i^{\frac{3}{2}} \|\nabla e\|_{\Omega_i} + Ch^\lambda \|w\|_{C^{1,\lambda}(\Omega_*)} (C_* h)^{\frac{3}{2}} \|\nabla e\|_{\Omega_*},$$

where λ depends on the domain Ω and such that $w \in C^{1,\lambda}(\Omega)$ cf. [25]. Following [19], for $x, y \in \Omega_i$ by Lemma 2.9 for $k = 1, 2, 3$ we have

$$\frac{|\partial_k w(x) - \partial_k w(y)|}{|x - y|^\lambda} \leq \int_{\Omega_j} \frac{|\partial_{x_i} G(x, \xi) - \partial_{y_i} G(y, \xi)|}{|x - y|^\lambda} |e(\xi)| d\xi$$

$$\leq C \max_{\xi \in \Omega_j} (|x - \xi|^{-2-\lambda} + |y - \xi|^{-2-\lambda}) \int_{\Omega_j} |e(\xi)| d\xi$$

$$\leq C d_j^{-2-\lambda} d_j^{\frac{3}{2}} \|e\|_{\Omega_j} \leq C d_j^{-\frac{1}{2}-\lambda} \|e\|_{\Omega_j}.$$

Thus, we obtain

$$\|w\|_{C^{1,\lambda}(\Omega_i)} \le C d_j^{-\frac{1}{2}-\lambda} \|e\|_{\Omega_j}, \quad i = j+2, \dots, J, *.$$

As a result, the total contribution to the sum is

$$\sum_{j=0}^{J-2} d_j \sum_{i=j+3}^{J,*} (\nabla(w - I_h w), \nabla e)_{\Omega_i} \le C h^\lambda \sum_{j=0}^{J-2} d_j^{\frac{1}{2}-\lambda} \|e\|_{\Omega_j} \left(\sum_{i=j+3}^{J} d_i^{\frac{3}{2}} \|\nabla e\|_{\Omega_i} + (C_* h)^{\frac{3}{2}} \|\nabla e\|_{\Omega_*} \right) := S_3.$$

Changing the order of summation we obtain

$$S_3 \le C h^\lambda \left(\sum_{i=0}^{J} d_i^{\frac{3}{2}} \|\nabla e\|_{\Omega_i} + (C_* h)^{\frac{3}{2}} \|\nabla e\|_{\Omega_*} \right) \sum_{j=0}^{i-3} d_j^{\frac{1}{2}-\lambda} \|e\|_{\Omega_j}.$$

Using the properties of the geometric series we get

$$\sum_{j=0}^{i-3} d_j^{\frac{1}{2}-\lambda} \|e\|_{\Omega_j} \le \left(\sum_{j=0}^{i-3} d_j^{-2\lambda} \right)^{\frac{1}{2}} \left(\sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}} \le C d_i^{-\lambda} \left(\sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}}.$$

Hence,

$$\begin{aligned}
S_3 &\le C \left(\sum_{i=0}^{J} \left(\frac{h}{d_i} \right)^{2\lambda} + \left(\frac{C_*}{K} \right)^3 \right)^{\frac{1}{2}} \left(\sum_{i=0}^{J,*} \|\sigma^{\frac{3}{2}} \nabla e\|_{\Omega_i}^2 \right)^{\frac{1}{2}} \left(\sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}} \\
&\le C \left(C_*^{-\lambda} + \left(\frac{C_*}{K} \right)^{\frac{3}{2}} \right) \|\sigma^{\frac{3}{2}} \nabla e\|_\Omega \left(\sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}},
\end{aligned} \tag{2.28}$$

where we used that $\sum_{i=0}^{J} \left(\frac{h}{d_i} \right)^{2\lambda} \le C C_*^{-2\lambda}$ and $h \le \frac{\sigma}{K}$. Using (2.25) and combining the cases 1,2, and 3 and canceling by $\left(\sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}}$, we obtain

$$\sum_{j=0}^{J,*} d_j \|e\|_{\Omega_j}^2 \le C \left(K^{-2} + C_*^{-2} + C_*^{-2\lambda} + \left(\frac{C_*}{K} \right)^3 \right) \|\sigma^{\frac{3}{2}} \nabla e\|_\Omega^2 + C h^2 \|\sigma^{\frac{3}{2}} \Delta_h v_h\|_\Omega^2. \tag{2.29}$$

Combining the above estimate with (2.23) and (2.24), we obtain

$$\|\sigma^{\frac{3}{2}} \nabla e\|_\Omega^2 \le C h^2 \|\sigma^{\frac{3}{2}} \nabla^2 v\|_\Omega^2 + C \left(K^{-2} + C_*^{-2} + C_*^{-2\lambda} + \left(\frac{C_*}{K} \right)^3 \right) \|\sigma^{\frac{3}{2}} \nabla e\|_\Omega^2 + C h^2 \|\sigma^{\frac{3}{2}} \Delta_h v_h\|_\Omega^2.$$

Taking $K = C_*^2$ and selecting C_* sufficiently large we conclude that

$$\|\sigma^{\frac{3}{2}} \nabla e\|_\Omega^2 \le C_{C_*, K} h^2 \|\sigma^{\frac{3}{2}} \nabla^2 v\|_\Omega^2 + C h^2 \|\sigma^{\frac{3}{2}} \Delta_h v_h\|_\Omega^2.$$

Applying Lemma 2.7 to the first term on the right hand side and taking square root concludes the proof. □

The following Lemma provides a discrete analog of the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and will be used in Section 4.

LEMMA 2.11. *There exists a constant C independent of h such that*

$$\|v_h\|_{L^\infty(\Omega)} \le C \|\Delta_h v_h\|_\Omega, \quad \forall v_h \in V_h.$$

Proof. To establish this lemma, we similarly to the proof of Lemma 2.6 define v as the solution of

$$-\Delta v = -\Delta_h v_h \quad \text{in } \Omega,$$
$$v = 0 \qquad \text{on } \partial\Omega. \tag{2.30}$$

Thus, by construction v_h is the Ritz projection of v, i.e., $R_h v = v_h$ and $v \in H^2(\Omega) \cap H_0^1(\Omega)$ with

$$\|v\|_{H^2(\Omega)} \leq C\|\Delta v\|_\Omega = C\|\Delta_h v_h\|_\Omega.$$

By the triangle inequality,

$$\|v_h\|_{L^\infty(\Omega)} \leq \|P_h v\|_{L^\infty(\Omega)} + \|v_h - P_h v\|_{L^\infty(\Omega)}.$$

Using the stability of the L^2-projection in L^∞ norm, see, e.g., [10] and the Sobolev embedding theorem, we have

$$\|P_h v\|_{L^\infty(\Omega)} \leq C\|v\|_{L^\infty(\Omega)} \leq C\|v\|_{H^2(\Omega)} \leq C\|\Delta_h v_h\|_\Omega.$$

On the other hand, using the inverse inequality first, then the triangle inequality and the standard error estimates, we obtain

$$\|v_h - P_h v\|_{L^\infty(\Omega)} \leq Ch^{-\frac{3}{2}}\|v_h - P_h v\|_\Omega \leq Ch^{-\frac{3}{2}}\left(\|v - P_h v\|_\Omega + \|v - v_h\|_\Omega\right) \leq Ch^{\frac{1}{2}}\|\Delta_h v_h\|_\Omega.$$

This establishes the lemma. □

3. Stability of the Ritz projection in L^∞-norm. Using the technical Lemmas 2.5-2.6 we can establish several important results. The first result shows stability (modulo logarithm) of the Ritz projection in L^∞-norm. This result is known for smooth domains $\Omega \subset \mathbb{R}^N$ and for polygonal domains in $\mathbb{R}^2$. To the best of our knowledge this result is new for convex polyhedral domains $\Omega \subset \mathbb{R}^3$.

THEOREM 3.1. *There exists a constant C independent of h such that for the solution u of (1.1) and its Ritz projection $u_h = R_h u \in V_h$ holds*

$$\|u_h\|_{L^\infty(\Omega)} \leq C|\ln h|\|u\|_{L^\infty(\Omega)}.$$

Proof. Let $x_0 \in \bar\tau_0$ such that $\|u_h\|_{L^\infty(\Omega)} = |u_h(x_0)|$. For such x_0 define a regularized Green's function g, that satisfies

$$-\Delta g(x) = \tilde\delta(x), \quad x \in \Omega,$$
$$g(x) = 0, \qquad x \in \partial\Omega, \tag{3.1}$$

where $\tilde\delta(x) := \tilde\delta_{x_0}(x)$ is the regularized Delta function from (2.5). We define $g_h = R_h g \in V_h$, i.e.

$$(\nabla g_h, \nabla \chi) = (\tilde\delta, \chi) \quad \forall \chi \in V_h. \tag{3.2}$$

Then using the Galerkin orthogonality, integration by parts, and Lemma 2.6, we obtain

$$u_h(x_0) = (\nabla u_h, \nabla g_h) = (\nabla u, \nabla g_h) = \sum_{\tau \in \mathcal{T}} \left((u, [\![\partial_n g_h]\!])_{\partial\tau} + (u, -\Delta g_h)_\tau\right)$$

$$\leq \|u\|_{L^\infty(\Omega)} \sum_{\tau \in \mathcal{T}} \left(\|[\![\partial_n g_h]\!]\|_{L^1(\partial\tau)} + \|\Delta g_h\|_{L^1(\tau)}\right) \leq C|\ln h|^{\frac{1}{2}}\|u\|_{L^\infty(\Omega)} \left(\|\sigma^{\frac{3}{2}}\Delta_h g_h\|_\Omega + \|\sigma^{\frac{1}{2}}\nabla g_h\|_\Omega\right). \tag{3.3}$$

The equation (3.2) for g_h is equivalent to $-\Delta_h g_h = P_h\tilde\delta$, and then using Lemma 2.4 we easily obtain

$$\|\sigma^{\frac{3}{2}}\Delta_h g_h\|_\Omega = \|\sigma^{\frac{3}{2}}P_h\tilde\delta\|_\Omega \leq C. \tag{3.4}$$

To estimate $\|\sigma^{\frac{1}{2}}\nabla g_h\|_\Omega$, we set $\chi = P_h(\sigma g_h)$ in (3.2) and obtain

$$(\nabla g_h, \nabla P_h(\sigma g_h)) = (\tilde\delta, P_h(\sigma g_h)).$$

Using the fact that

$$(\nabla g_h, \nabla P_h(\sigma g_h)) = (\nabla g_h, \nabla(\sigma g_h)) + (\nabla g_h, \nabla(P_h(\sigma g_h) - \sigma g_h))$$
$$= \|\sigma^{\frac{1}{2}}\nabla g_h\|_\Omega^2 + (\nabla g_h, \nabla \sigma g_h) + (\nabla g_h, \nabla(P_h(\sigma g_h) - \sigma g_h)),$$

and

$$(\tilde{\delta}, P_h(\sigma g_h)) = (P_h\tilde{\delta}, \sigma g_h) = (\sigma^{\frac{3}{2}} P_h\tilde{\delta}, \sigma^{-\frac{1}{2}} g_h)$$

we have

$$\|\sigma^{\frac{1}{2}}\nabla g_h\|_\Omega^2 \leq \|\sigma^{\frac{1}{2}}\nabla g_h\|_\Omega \left(\|\sigma^{-\frac{1}{2}}\nabla \sigma g_h\|_\Omega + \|\sigma^{-\frac{1}{2}}\nabla(P_h(\sigma g_h) - \sigma g_h)\|_\Omega \right) + \|\sigma^{-\frac{1}{2}} g_h\|_\Omega \|\sigma^{\frac{3}{2}} P_h\tilde{\delta}\|_\Omega.$$

Using that $|\nabla \sigma| \leq C$, Lemma 2.3, and Lemma 2.4, and kicking back $\|\sigma^{\frac{1}{2}}\nabla g_h\|_\Omega^2$, we obtain

$$\|\sigma^{\frac{1}{2}}\nabla g_h\|_\Omega^2 \leq C \left(\|\sigma^{-\frac{1}{2}} g_h\|_\Omega^2 + 1 \right). \tag{3.5}$$

To estimate $\|\sigma^{-\frac{1}{2}} g_h\|_\Omega$ we use Lemma 2.5, with $\alpha = \beta = -\frac{1}{2}$ and $p = 3$, to obtain

$$\|\sigma^{-\frac{1}{2}} g_h\|_\Omega \leq C\|g_h\|_{L^3(\Omega)}^{\frac{1}{2}} \|\nabla g_h\|_{L^{\frac{3}{2}}(\Omega)}^{\frac{1}{2}}. \tag{3.6}$$

Using the inverse and the triangle inequalities,

$$\|g_h\|_{L^3(\Omega)} \leq \|g\|_{L^3(\Omega)} + \|g - g_h\|_{L^3(\Omega)} \leq \|g\|_{L^3(\Omega)} + \|I_h g - g_h\|_{L^3(\Omega)} + \|g - I_h g\|_{L^3(\Omega)}$$
$$\leq \|g\|_{L^3(\Omega)} + Ch^{-\frac{1}{2}}\|I_h g - g_h\|_\Omega + \|g - I_h g\|_{L^3(\Omega)}$$
$$\leq \|g\|_{L^3(\Omega)} + Ch^{-\frac{1}{2}}\|g - g_h\|_\Omega + Ch^{-\frac{1}{2}}\|g - I_h g\|_\Omega + \|g - I_h g\|_{L^3(\Omega)}.$$

Using the approximation theory (2.3), the standard L^2 estimate, and the properties of $\tilde{\delta}$ function, we have

$$h^{-\frac{1}{2}}\|g - g_h\|_\Omega + h^{-\frac{1}{2}}\|g - I_h g\|_\Omega + \|g - I_h g\|_{L^3(\Omega)} \leq Ch^{\frac{3}{2}}\|g\|_{H^2(\Omega)} \leq Ch^{\frac{3}{2}}\|\tilde{\delta}\|_\Omega \leq C. \tag{3.7}$$

Next we will show

$$\|g\|_{L^3(\Omega)} \leq C|\ln h|^{\frac{1}{3}}. \tag{3.8}$$

To establish that we use a Green's function representation

$$g(x) = \int_{\tau_0} G(x, y)\tilde{\delta}(y)dy.$$

Define $B_h = B_{3h}(x_0) \cap \Omega$ and $B_h^c = \Omega \backslash B_h$ and consider two cases: $x \in B_h$ and $x \in B_h^c$. In the case $x \in B_h$, we obtain using spherical coordinates centered at x as well as (2.5) and (2.21a)

$$|g(x)| \leq \|\tilde{\delta}\|_{L^\infty(\tau_0)} \int_{\tau_0} |G(x, y)|dy \leq Ch^{-3} \int_0^{ch} \frac{1}{\rho}\rho^2 d\rho \leq Ch^{-1}.$$

Hence,

$$\|g\|_{L^3(B_h)}^3 \leq Ch^{-3} \int_{B_h} dx \leq C. \tag{3.9}$$

In the case $x \in B_h^c$, we have for any $y \in \tau_0$ by the triangle inequality

$$|x - y| \geq |x - x_0| - |y - x_0| \geq |x - x_0| - h$$

and therefore again by (2.5) and (2.21a)

$$|g(x)| \leq \|\tilde{\delta}\|_{L^1(\tau_0)} \frac{C}{|x - x_0| - h} \leq \frac{C}{|x - x_0| - h}.$$

Hence,

$$\|g\|^3_{L^3(B^c_h)} \le C \int_{B^c_h} \frac{dx}{(|x - x_0| - h)^3} \le C|\ln h|.$$

Together with (3.9) that shows (3.8). Combining (3.7) and (3.8) we have established

$$\|g_h\|_{L^3(\Omega)} \le C|\ln h|^{\frac{1}{3}}. \tag{3.10}$$

To treat $\|\nabla g_h\|^{\frac{1}{2}}_{L^{\frac{3}{2}}(\Omega)}$ we use the Hölder's inequality and (2.7a). We have

$$\|\nabla g_h\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\Omega)} \le \|\sigma^{-\frac{3}{4}}\|_{L^4(\Omega)}\|\sigma^{\frac{3}{4}}|\nabla g_h|^{\frac{3}{2}}\|_{L^{\frac{4}{3}}(\Omega)} \le \|\sigma^{-\frac{3}{2}}\|^{\frac{1}{2}}_{\Omega}\|\sigma^{\frac{1}{2}}\nabla g_h\|^{\frac{3}{2}}_{\Omega} \le C|\ln h|^{\frac{1}{4}}\|\sigma^{\frac{1}{2}}\nabla g_h\|^{\frac{3}{2}}_{\Omega}.$$

Hence,

$$\|\nabla g_h\|_{L^{\frac{3}{2}}(\Omega)} \le C|\ln h|^{\frac{1}{6}}\|\sigma^{\frac{1}{2}}\nabla g_h\|_{\Omega}. \tag{3.11}$$

Thus, combining (3.5), (3.6), (3.10), and the above estimate, we have

$$\begin{aligned}
\|\sigma^{\frac{1}{2}}\nabla g_h\|^2_{\Omega} &\le C\left(\|\sigma^{-\frac{1}{2}}g_h\|^2_{\Omega} + 1\right) \le C\left(\|g_h\|_{L^3(\Omega)}\|\nabla g_h\|_{L^{\frac{3}{2}}(\Omega)} + 1\right) \\
&\le C\left(|\ln h|^{\frac{1}{3}+\frac{1}{6}}\|\sigma^{\frac{1}{2}}\nabla g_h\|_{\Omega} + 1\right) \\
&\le C|\ln h|^{\frac{1}{2}}\|\sigma^{\frac{1}{2}}\nabla g_h\|_{\Omega}.
\end{aligned} \tag{3.12}$$

Dividing both sides by $\|\sigma^{\frac{1}{2}}\nabla g_h\|_{\Omega}$, we finally obtain

$$\|\sigma^{\frac{1}{2}}\nabla g_h\|_{\Omega} \le C|\ln h|^{\frac{1}{2}},$$

which together with (3.3) and (3.4) establishes the theorem. □

4. Resolvent Estimates. In this section we establish some resolvent estimates. Since we will be dealing with complex valued function spaces, we need to modify the definition of the L^2-inner product as

$$(u, v)_\Omega = \int_\Omega u(x)\bar{v}(x)\, dx,$$

where $\bar{v}$ is the complex conjugate of v and the finite element space as $\mathbb{V}_h = V_h + iV_h$.

In the continuous case for Lipschitz domains the following result was shown in [42]: There exists a constant C such that

$$\|(z + \Delta)^{-1}v\|_{L^p(\Omega)} \le \frac{C}{1 + |z|}\|v\|_{L^p(\Omega)}, \quad z \in \mathbb{C} \setminus \Sigma_{0,\gamma}, \quad 1 \le p \le \infty, \quad v \in L^p(\Omega), \tag{4.1}$$

where $\Sigma_{0,\gamma}$ is defined in (1.7). Using the identity $\Delta(z + \Delta)^{-1} = \mathrm{Id} - z(z + \Delta)^{-1}$, one immediately obtains,

$$\|\Delta(z + \Delta)^{-1}v\|_{L^p(\Omega)} \le C\|v\|_{L^p(\Omega)}, \quad z \in C \setminus \Sigma_{0,\gamma}, \quad 1 \le p \le \infty, \quad v \in L^p(\Omega). \tag{4.2}$$

In the following analysis we will also require a Green's function estimate for the resolvent equation with a real parameter $s > 0$, i.e. for $(s - \Delta)^{-1}$.

LEMMA 4.1. *Let $s > 0$ and $\Gamma_s(x, y)$ be the Green's function for the operator $s - \Delta$ with zero Dirichlet boundary conditions. Then there exists a constant C independent of s such that*

$$\Gamma_s(x, y) \le \frac{C}{|x - y|}. \tag{4.3}$$

Proof. The Green's function $\Gamma_s(x, y)$ for $s - \Delta$ has a representation

$$\Gamma_s(x, y) = \int_0^\infty e^{-st}H(t, x, y)dt, \tag{4.4}$$

where $H(t, x, y)$ is the Green's function for the heat equation. Because of zero Dirichlet boundary data, $H(t, x, y)$ is bounded by the fundamental solution of the heat equation and satisfies, cf. [14, Chapter 2.3],

$$H(t, x, y) \leq \frac{C}{t^{\frac{3}{2}}} e^{-c\frac{|x-y|^2}{t}}.$$

The integral estimate (cf. [21, Appendix])

$$\int_0^\infty \frac{e^{-\frac{d^2}{t} - st}}{t^{\frac{3}{2}}} dt \leq \frac{C}{d}, \quad \text{for all } d > 0, \tag{4.5}$$

gives us the lemma. $\square$

First we prove a discrete resolvent estimate with respect to $L^2(\Omega)$ norm.

LEMMA 4.2. *For any $\gamma \in (0, \frac{\pi}{2})$ there exists a constant $C = C_\gamma$ independent of h and z such that for any $\lambda \in [0, \lambda_0]$ with $\lambda_0 > 0$ being the smallest eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions holds*

$$\|(z + \Delta_h)^{-1}\chi\|_\Omega \leq \frac{C_\gamma}{|z - \lambda|}\|\chi\|_\Omega, \quad \forall z \in \mathbb{C}\backslash\Sigma_{\lambda,\gamma}, \quad \forall \chi \in V_h,$$

where $\Sigma_{\lambda,\gamma}$ is defined in (1.7). The constant C_γ behaves like γ^{-1} for small values of γ.

Proof. For given $\chi \in V_h$ let $u_h \in \mathbb{V}_h$ be the solution of

$$-zu_h - \Delta_h u_h = \chi.$$

The existence and uniqueness of u_h (cf., e. g., [17]) follow by the fact that all eigenvalues of the discrete Laplacian Δ_h are real and positive and for the smallest discrete eigenvalue $\lambda_{0,h}$ there holds $\lambda_{0,h} \geq \lambda_0$. Testing this equation with $\bar{u}_h$, we obtain

$$-z\|u_h\|_\Omega^2 + \|\nabla u_h\|_\Omega^2 = (\chi, u_h)_\Omega$$

and therefore

$$(\lambda - z)\|u_h\|_\Omega^2 + \|\nabla u_h\|_\Omega^2 - \lambda\|u_h\|_\Omega^2 = (\chi, u_h)_\Omega. \tag{4.6}$$

By definition of λ_0 we have

$$\|\nabla u_h\|_\Omega^2 \geq \lambda_0\|u_h\|_\Omega^2$$

and hence

$$\delta := \|\nabla u_h\|_\Omega^2 - \lambda\|u_h\|_\Omega^2 \geq (\lambda_0 - \lambda)\|u_h\|_\Omega^2 \geq 0.$$

Thus the equation (4.6) can be rewritten as

$$|\lambda - z|\|u_h\|_\Omega^2 e^{i\phi} + \delta = (\chi, u_h)_\Omega,$$

with $|\phi| \leq \pi - \gamma$. Multiplying this equation with $e^{-\frac{i\phi}{2}}$, taking real part and exploiting $\delta \geq 0$ and $\cos\left(\frac{\phi}{2}\right) > 0$, we obtain

$$|\lambda - z|\|u_h\|_\Omega^2 \leq \frac{1}{\cos\left(\frac{\phi}{2}\right)}|(\chi, u_h)_\Omega| \leq \frac{1}{\sin\left(\frac{\gamma}{2}\right)}|(\chi, u_h)_\Omega| = C_\gamma|(\chi, u_h)_\Omega|.$$

This results in

$$\|u_h\|_\Omega \leq \frac{C_\gamma}{|\lambda - z|}\|\chi\|_\Omega.$$

This completes the proof. $\square$

Using the continuous resolvent results, (4.3), Lemma 4.2, and results from Section 2, we establish the discrete resolvent estimate for the L^∞ norm.

THEOREM 4.3. *For any $\gamma \in (0, \frac{\pi}{2})$, there exists a constant $C = C_\gamma$ independent of h and z such that*

$$\|(z + \Delta_h)^{-1}\chi\|_{L^\infty(\Omega)} \leq \frac{C|\ln h|}{|z - \lambda|}\|\chi\|_{L^\infty(\Omega)}, \quad \forall z \in \mathbb{C}\backslash\Sigma_{\lambda,\gamma}, \quad \forall \chi \in V_h, \quad \forall\lambda \in [0, \lambda_0],$$

where $\Sigma_{\lambda,\gamma}$ is defined in (1.6) and λ_0 is the smallest eigenvalue of $-\Delta$.

Proof. First, we establish the theorem with $\lambda = 0$, i.e.

$$\|(z + \Delta_h)^{-1}\chi\|_{L^\infty(\Omega)} \leq \frac{C|\ln h|}{|z|}\|\chi\|_{L^\infty(\Omega)}, \quad \forall z \in \mathbb{C}\backslash\Sigma_{0,\gamma}, \quad \forall\chi \in V_h, \tag{4.7}$$

and then following the argument of [3] at the end of Section 6, we establish the theorem with $\lambda \in [0, \lambda_0]$. To show (4.7), we follow ideas of [43], Thm. 6.5, and [2]. The argument in [43] is purely two-dimensional and we have to adapt it to our three-dimensional setting.

Let $x_0 \in \Omega$ be a fixed point and let $\tilde{\delta} = \tilde{\delta}^h_{x_0}$ be the smooth Delta function introduced in Section 2. Then,

$$|(z + \Delta_h)^{-1}\chi(x_0)| = |((z + \Delta_h)^{-1}\chi, P_h\tilde{\delta})| = |(\chi, (\bar{z} + \Delta_h)^{-1}P_h\tilde{\delta})|.$$

We define an adjoint regularized Green's function $G = G^{x_0}(x, \bar{z})$ by

$$G = G^{x_0}(x, \bar{z}) = (\bar{z} + \Delta)^{-1}\tilde{\delta}$$

and its discrete analog $G_h = G_h^{x_0}(x, \bar{z}) \in \mathbb{V}_h$ by

$$G_h = G_h^{x_0}(x, \bar{z}) = (\bar{z} + \Delta_h)^{-1}P_h\tilde{\delta},$$

which we can write in the weak form as

$$z(\varphi, G_h) - (\nabla\varphi, \nabla G_h) = (\varphi, \tilde{\delta}), \quad \forall\varphi \in \mathbb{V}_h. \tag{4.8}$$

Using (2.7a) we get

$$|(z + \Delta_h)^{-1}\chi(x_0)| = |(\chi, G_h)| \leq \|\sigma^{-\frac{3}{2}}\|_\Omega\|\chi\|_{L^\infty}\|\sigma^{\frac{3}{2}}G_h\|_\Omega \leq C|\ln h|^{\frac{1}{2}}\|\chi\|_{L^\infty}\|\sigma^{\frac{3}{2}}G_h\|_\Omega.$$

Thus we only need to establish

$$\|\sigma^{\frac{3}{2}}G_h\|_\Omega \leq C|\ln h|^{\frac{1}{2}}|z|^{-1}. \tag{4.9}$$

Consider the expression

$$-z\|\sigma^{\frac{3}{2}}G_h\|^2_\Omega + \|\sigma^{\frac{3}{2}}\nabla G_h\|^2_\Omega = -z(\sigma^3 G_h, G_h) + (\nabla(\sigma^3 G_h), \nabla G_h) - 3(\sigma^2\nabla\sigma G_h, \nabla G_h). \tag{4.10}$$

By taking $\chi = -P_h(\sigma^3 G_h)$ in (4.8) and subtracting it from (4.10), we obtain

$$-z\|\sigma^{\frac{3}{2}}G_h\|^2_\Omega + \|\sigma^{\frac{3}{2}}\nabla G_h\|^2_\Omega = F, \tag{4.11}$$

where

$$F = F_1 + F_2 + F_3 := -(P_h(\sigma^3 G_h), \tilde{\delta}) + (\nabla(\sigma^3 G_h - P_h(\sigma^3 G_h)), \nabla G_h) - 3(\sigma^2\nabla\sigma G_h, \nabla G_h).$$

Since $\gamma \leq |\arg z| \leq \pi$, this equation is of the form

$$e^{i\alpha}a + b = f, \quad \text{with} \quad a, b > 0, \quad 0 \leq |\alpha| \leq \pi - \gamma,$$

by multiplying it by $e^{-\frac{i\alpha}{2}}$ and taking real parts, we have

$$a + b \leq \left(\cos\left(\frac{\alpha}{2}\right)\right)^{-1}|f| \leq \left(\sin\left(\frac{\gamma}{2}\right)\right)^{-1}|f| = C_\gamma|f|.$$

From (4.11) we therefore conclude

$$|z|\|\sigma^{\frac{3}{2}}G_h\|_\Omega^2 + \|\sigma^{\frac{3}{2}}\nabla G_h\|_\Omega^2 \le C_\gamma|F|, \quad \text{for} \quad z \in \Sigma_{0,\gamma}.$$

Using the Cauchy-Schwarz inequality, Lemma 2.4, and the arithmetic-geometric mean inequality we obtain,

$$|F_1| = |(\sigma^3 G_h, P_h\tilde{\delta})| \le \|\sigma^{\frac{3}{2}}G_h\|_\Omega\|\sigma^{\frac{3}{2}}P_h\tilde{\delta}\|_\Omega \le C\|\sigma^{\frac{3}{2}}G_h\|_\Omega \le CC_\gamma|z|^{-1} + \frac{|z|}{2C_\gamma}\|\sigma^{\frac{3}{2}}G_h\|_\Omega^2.$$

To estimate F_2 we use Lemma 2.3 and the Cauchy-Schwarz and the arithmetic-geometric mean inequalities,

$$|F_2| \le \|\sigma^{-\frac{3}{2}}\nabla(\sigma^3 G_h - P_h(\sigma^3 G_h))\|_\Omega\|\sigma^{\frac{3}{2}}\nabla G_h\|_\Omega \le \frac{1}{4C_\gamma}\|\sigma^{\frac{3}{2}}\nabla G_h\|_\Omega^2 + CC_\gamma\|\sigma^{\frac{1}{2}}G_h\|_\Omega^2.$$

Finally, using the properties of σ, we obtain

$$|F_3| \le C\|\sigma^{\frac{1}{2}}G_h\|_\Omega\|\sigma^{\frac{3}{2}}\nabla G_h\|_\Omega \le \frac{1}{4C_\gamma}\|\sigma^{\frac{3}{2}}\nabla G_h\|_\Omega^2 + CC_\gamma\|\sigma^{\frac{1}{2}}G_h\|_\Omega^2.$$

Combining estimates for $F_i's$ and kicking back, we obtain

$$|z|\|\sigma^{\frac{3}{2}}G_h\|_\Omega^2 + \|\sigma^{\frac{3}{2}}\nabla G_h\|_\Omega^2 \le C\left(|z|^{-1} + \|\sigma^{\frac{1}{2}}G_h\|^2\right). \tag{4.12}$$

Thus, in order to establish (4.7), we need to show

$$\|\sigma^{\frac{1}{2}}G_h\|_\Omega^2 \le C|\ln h||z|^{-1}. \tag{4.13}$$

To accomplish that, we consider the expression

$$-z\|\sigma^{\frac{1}{2}}G_h\|_\Omega^2 + \|\sigma^{\frac{1}{2}}\nabla G_h\|_\Omega^2 = -z(G_h, \sigma G_h) + (\nabla G_h, \nabla(\sigma G_h)) - (\nabla G_h, \nabla\sigma G_h).$$

Testing (4.8) with $\varphi = P_h(\sigma G_h)$ we obtain similarly as above

$$|z|\|\sigma^{\frac{1}{2}}G_h\|_\Omega^2 + \|\sigma^{\frac{1}{2}}\nabla G_h\|_\Omega^2 \le C_\gamma|f|, \quad \text{for} \quad z \in \mathbb{C} \setminus \Sigma_{0,\gamma},$$

where

$$f = f_1 + f_2 + f_3 := -(P_h(\sigma G_h), \tilde{\delta}) + (\nabla(\sigma G_h - (P_h(\sigma G_h)), \nabla G_h) - (\nabla\sigma G_h, \nabla G_h).$$

Using the Cauchy-Schwarz inequality, Lemma 2.4, and the arithmetic-geometric mean inequality we obtain,

$$|f_1| = |(\sigma G_h, P_h\tilde{\delta})| \le \|\sigma^{-\frac{1}{2}}G_h\|_\Omega\|\sigma^{\frac{3}{2}}P_h\tilde{\delta}\|_\Omega \le C\|\sigma^{-\frac{1}{2}}G_h\|_\Omega.$$

To estimate f_2 we use Lemma 2.3 and the Cauchy-Schwarz and the arithmetic-geometric mean inequalities,

$$|f_2| \le \|\sigma^{-\frac{1}{2}}\nabla(\sigma G_h - (P_h(\sigma G_h)))\|_\Omega\|\sigma^{\frac{1}{2}}\nabla G_h\|_\Omega \le \frac{1}{4C_\gamma}\|\sigma^{\frac{1}{2}}\nabla G_h\|_\Omega^2 + CC_\gamma\|\sigma^{-\frac{1}{2}}G_h\|_\Omega^2.$$

Finally, using properties of σ, we obtain

$$|f_3| \le C\|\sigma^{-\frac{1}{2}}G_h\|_\Omega\|\sigma^{\frac{1}{2}}\nabla G_h\|_\Omega \le \frac{1}{4C_\gamma}\|\sigma^{\frac{1}{2}}\nabla G_h\|_\Omega^2 + CC_\gamma\|\sigma^{-\frac{1}{2}}G_h\|_\Omega^2.$$

Combining estimates for $f_i's$ and kicking back, we obtain

$$|z|\|\sigma^{\frac{1}{2}}G_h\|_\Omega^2 + \|\sigma^{\frac{1}{2}}\nabla G_h\|_\Omega^2 \le C\left(\|\sigma^{-\frac{1}{2}}G_h\|_\Omega + \|\sigma^{-\frac{1}{2}}G_h\|_\Omega^2\right) \le C\left(\|\sigma^{-\frac{1}{2}}G_h\|_\Omega^2 + 1\right). \tag{4.14}$$

Now we proceed as in the proof of Theorem 3.1. To estimate $\|\sigma^{-\frac{1}{2}}G_h\|_\Omega$ we use Lemma 2.5, with $\alpha = \beta = -\frac{1}{2}$ and $p = 3$, to obtain

$$\|\sigma^{-\frac{1}{2}}G_h\|_\Omega \le C\|G_h\|_{L^3(\Omega)}^{\frac{1}{2}}\|\nabla G_h\|_{L^{\frac{3}{2}}(\Omega)}^{\frac{1}{2}}. \tag{4.15}$$

Using the inverse and the triangle inequalities, we obtain

$$\|G_h\|_{L^3(\Omega)} \le \|G\|_{L^3(\Omega)} + Ch^{-\frac{1}{2}}\|G - G_h\|_{\Omega} + Ch^{-\frac{1}{2}}\|G - I_hG\|_{\Omega} + \|G - I_hG\|_{L^3(\Omega)}.$$

Using the approximation result (2.3), L^2-norm error estimates, H^2-regularity, and the properties of $\tilde{\delta}$ function, we have

$$h^{-\frac{1}{2}}\|G - G_h\|_{\Omega} + h^{-\frac{1}{2}}\|G - I_hG\|_{\Omega} + \|G - I_hG\|_{L^3(\Omega)} \le Ch^{\frac{3}{2}}\|G\|_{H^2(\Omega)} \le Ch^{\frac{3}{2}}\|\tilde{\delta}\|_{\Omega} \le C. \qquad (4.16)$$

The L^2-norm error estimates and H^2 regularity is shown for example in [17, Theorem 3.1] for convex polygonal domains, but the proof there works for convex polyhedral domains as well. Next we will show

$$\|G\|_{L^3(\Omega)} \le C|\ln h|^{\frac{1}{3}}. \qquad (4.17)$$

To establish that, we use

$$\|G\|_{L^3(\Omega)} = \|(\bar{z} + \Delta)^{-1}\tilde{\delta}\|_{L^3(\Omega)} = \|(|z| - \Delta)(\bar{z} + \Delta)^{-1}(|z| - \Delta)^{-1}\tilde{\delta}\|_{L^3(\Omega)}$$
$$\le (1 + 2|z|\|(\bar{z} + \Delta)^{-1}\|_{L^3 \to L^3})\|(|z| - \Delta)^{-1}\tilde{\delta}\|_{L^3(\Omega)} \le C\|(|z| - \Delta)^{-1}\tilde{\delta}\|_{L^3(\Omega)},$$

where we used continuous resolvent estimate (4.1) with $p = 3$ and

$$(|z| - \Delta)(\bar{z} + \Delta)^{-1} = (\bar{z} + |z| - \bar{z} - \Delta)(\bar{z} + \Delta)^{-1} = (\bar{z} + |z|)(\bar{z} + \Delta)^{-1} - \text{Id} .$$

Now to estimate $\|(|z| - \Delta)^{-1}\tilde{\delta}\|_{L^3}$ we use the Green's function representation

$$\|(|z| - \Delta)^{-1}\tilde{\delta}\|_{L^3(\Omega)} = \|(\Gamma_s(\cdot, y), \tilde{\delta}(y))_{\Omega}\|_{L^3(\Omega)},$$

and exactly as in the proof of Theorem 3.1 using Lemma 4.1, we obtain

$$\|(|z| - \Delta)^{-1}\tilde{\delta}\|_{L^3(\Omega)} \le C|\ln h|^{\frac{1}{3}}.$$

Combining, we have established

$$\|G_h\|_{L^3(\Omega)} \le C|\ln h|^{\frac{1}{3}}. \qquad (4.18)$$

Using same analysis as in Theorem 4.3, we get

$$\|\nabla G_h\|_{L^{\frac{3}{2}}(\Omega)} \le C|\ln h|^{\frac{1}{6}}\|\sigma^{\frac{1}{2}}\nabla G_h\|_{\Omega}. \qquad (4.19)$$

Thus, using (4.14) and the above estimates, we have

$$|z|\|\sigma^{\frac{1}{2}}G_h\|_{\Omega}^2 + \|\sigma^{\frac{1}{2}}\nabla G_h\|_{\Omega}^2 \le C\left(\|G_h\|_{L^3(\Omega)}\|\nabla G_h\|_{L^{\frac{3}{2}}(\Omega)} + 1\right)$$
$$\le C\left(|\ln h|^{\frac{1}{2}}\|\sigma^{\frac{1}{2}}\nabla G_h\|_{\Omega} + 1\right)$$
$$\le C|\ln h| + \frac{1}{2}\|\sigma^{\frac{1}{2}}\nabla G_h\|_{\Omega}^2.$$

Kicking back $\|\sigma^{\frac{1}{2}}\nabla G_h\|_{\Omega}^2$, we finally obtain

$$\|\sigma^{\frac{1}{2}}G_h\|_{\Omega}^2 \le C|\ln h||z|^{-1},$$

which shows (4.13) and hence the theorem for $\lambda = 0$.

To show the result with $\lambda \in [0, \lambda_0]$, we use the argument similar to [3, Lemma 6.1]. We decompose $\mathbb{C}\backslash\Sigma_{\lambda,\gamma}$ as $\mathbb{C}\backslash\Sigma_{\lambda,\gamma} = D_1 \cup D_2$, where

$$D_1 = \left\{ z \in \mathbb{C}\backslash\Sigma_{\lambda,\gamma} \;\middle|\; \frac{\gamma}{2} \le |\arg z| \le \pi \text{ and } |z| \ge \frac{\lambda}{2} \right\}$$

and

$$D_2 = \left\{ z \in \mathbb{C} \backslash \Sigma_{\lambda,\gamma} \,\middle|\, |\arg z| \le \frac{\gamma}{2} \right\} \cup \left\{ z \in \mathbb{C} \,\middle|\, |z| \le \frac{\lambda}{2} \right\}.$$

Since $|z - \lambda| \le 3|z|$ for $z \in D_1$, the theorem follows from (4.7). Thus, it remains to establish the bound for $z \in D_2$. Using $\Delta_h(z + \Delta_h)^{-1} = \mathrm{Id} - z(z + \Delta_h)^{-1}$, and Lemma 2.11, we have

$$\|(z + \Delta_h)^{-1}\chi\|_{L^\infty(\Omega)} \le C\|\Delta_h(z + \Delta_h)^{-1}\chi\|_\Omega \le C\left(1 + |z|\,\|(z + \Delta_h)^{-1}\|_{L^2 \to L^2}\right)\|\chi\|_\Omega. \qquad (4.20)$$

For $z \in D_2$ using Lemma 4.2, we obtain

$$\|(z + \Delta_h)^{-1}\|_{L^2 \to L^2} \le \frac{C}{|z - \lambda|}.$$

Inserting this in (4.20) we obtain

$$\|(z + \Delta_h)^{-1}\chi\|_{L^\infty(\Omega)} \le C\left(1 + \frac{|z|}{|z - \lambda|}\right)\|\chi\|_\Omega \le C\left(|z| + |z - \lambda|\right)\frac{\|\chi\|_{L^\infty(\Omega)}}{|z - \lambda|}.$$

To complete the proof of the theorem, we notice that the term $|z| + |z - \lambda|$ is uniformly bounded on D_2 by a constant depending only on γ and λ_0. $\square$

Appendix. Proof of Lemma 2.1. Adding and subtracting $I_h(\varphi^2 P_h v)$ and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|\varphi P_h v\|_\Omega^2 = (P_h v, \varphi^2 P_h v)_\Omega &= (P_h v, \varphi^2 P_h v - I_h(\varphi^2 P_h v))_\Omega + (v, I_h(\varphi^2 P_h v))_\Omega \\
&\le \|\varphi P_h v\|_\Omega \|\varphi^{-1}\left(\varphi^2 P_h v - I_h(\varphi^2 P_h v)\right)\|_\Omega + \|\varphi v\|_\Omega \|\varphi^{-1} I_h(\varphi^2 P_h v)\|_\Omega.
\end{aligned} \qquad (4.21)$$

Adapting the notation of [13], define

$$\underline{\varphi}_\tau = \min_\tau \varphi \quad \text{and} \quad \overline{\varphi}_\tau = \max_\tau \varphi$$

for each element τ. First we notice that $h|\nabla\varphi| \le \varepsilon\varphi$ implies

$$\overline{\varphi}_\tau \le C\underline{\varphi}_\tau, \qquad (4.22)$$

for ε sufficiently small with the constant C independent of τ. Indeed, the inequality

$$\overline{\varphi}_\tau \le \underline{\varphi}_\tau + h\|\nabla\varphi\|_{L^\infty(\tau)},$$

and the assumption

$$h\|\nabla\varphi\|_{L^\infty(\tau)} \le \varepsilon\overline{\varphi}_\tau,$$

imply (4.22) for ε sufficiently small.

Thus, using (4.22), the triangle inequality, and the property that for the nodal Lagrange interpolant holds

$$I_h(\varphi^2 P_h v) = I_h(I_h(\varphi^2) P_h v),$$

we have

$$\begin{aligned}
\|\varphi^{-1}\left(\varphi^2 P_h v - I_h(\varphi^2 P_h v)\right)\|_\tau &\le \underline{\varphi}_\tau^{-1}\|\varphi^2 P_h v - I_h(\varphi^2 P_h v)\|_\tau \\
&\le \underline{\varphi}_\tau^{-1}\|\left(\varphi^2 - I_h(\varphi^2)\right) P_h v\|_\tau + \underline{\varphi}_\tau^{-1}\|I_h(\varphi^2) P_h v - I_h(I_h(\varphi^2) P_h v)\|_\tau \\
&:= J_1 + J_2.
\end{aligned}$$

Using the approximation properties of I_h and the assumption that $h|\nabla\varphi| \le \varepsilon\varphi$, we obtain

$$J_1 \le C\underline{\varphi}_\tau^{-1} h\|\nabla(\varphi^2)\|_{L^\infty(\tau)}\|P_h v\|_\tau \le C\varepsilon\|\varphi P_h v\|_\tau. \qquad (4.23)$$

To estimate J_2, we first notice that on each element $\partial^\gamma(P_h v) = 0$ and $\partial^\gamma(I_h(\varphi^2)) = 0$ for all multi-indices $\gamma \in \mathbb{N}^3$ with $|\gamma| = k + 1$. Thus, using the approximations theory and the Leibniz's formula, we have

$$J_2 \leq C\underline{\varphi}_\tau^{-1} h^{k+1} |I_h(\varphi^2)P_h v|_{H^{k+1}(\tau)} \leq C\underline{\varphi}_\tau^{-1} h^{k+1} \sum_{l=1}^{k} |I_h(\varphi^2)|_{W_\infty^l(\tau)} |P_h v|_{H^{k+1-l}(\tau)}.$$

Using the stability of the interpolant in W_∞^l and the inverse inequality, we obtain

$$J_2 \leq C\underline{\varphi}_\tau^{-1} \|P_h v\|_\tau \sum_{l=1}^{k} h^l |\varphi^2|_{W_\infty^l(\tau)}.$$

Finally, using the assumption $h^l |\nabla^l \varphi| \leq \varepsilon\varphi$, we obtain

$$J_2 \leq C\varepsilon \|\varphi P_h v\|_\tau.$$

Thus combining the estimates for J_1 and J_2 and summing over the element, we have established that

$$\|\varphi^{-1} \left(\varphi^2 P_h v - I_h(\varphi^2 P_h v)\right)\|_\Omega \leq C\varepsilon \|\varphi P_h v\|_\Omega. \tag{4.24}$$

The above estimate by the triangle inequality also implies that

$$\|\varphi^{-1} I_h(\varphi^2 P_h v)\|_\Omega \leq C(\varepsilon + 1) \|\varphi P_h v\|_\Omega. \tag{4.25}$$

Inserting the estimates (4.24) and (4.25) into (4.21), we obtain

$$\|\varphi P_h v\|_\Omega^2 \leq C\varepsilon \|\varphi P_h v\|_\Omega^2 + C(\varepsilon + 1) \|\varphi v\|_\Omega \|\varphi P_h v\|_\Omega,$$

which for ε sufficiently small implies the lemma.

REFERENCES

[1] T. APEL, O. BENEDIX, D. SIRCH, AND B. VEXLER, *A priori mesh grading for an elliptic problem with Dirac right-hand side*, SIAM J. Numer. Anal., 49 (2011), pp. 992–1005.

[2] N. Y. BAKAEV, *Resolvent estimates of elliptic differential and finite-element operators in pairs of function spaces*, Int. J. Math. Sci., (2004), pp. 217–238.

[3] N. Y. BAKAEV, S. LARSSON, AND V. THOMÉE, *Long time behaviour of backward difference type methods for parabolic equations with memory in Banach space*, East-West J. Numer. Math., 6 (1998), pp. 185–206.

[4] N. Y. BAKAEV, V. THOMÉE, AND L. B. WAHLBIN, *Maximum-norm estimates for resolvents of elliptic finite element operators*, Math. Comp., 72 (2003), pp. 1597–1610 (electronic).

[5] S. C. BRENNER AND L. R. SCOTT, *The Mathematical Theory of Finite Element Methods*, Springer Verlag, Berlin, Heidelberg, New York, third ed., 2008.

[6] L. CAFFARELLI, R. KOHN, AND L. NIRENBERG, *First order interpolation inequalities with weights*, Compositio Math., 53 (1984), pp. 259–275.

[7] P. G. CIARLET, *The Finite Element Method for Elliptic Problems*, North–Holland, Amsterdam, New York, 1978.

[8] A. DEMLOW, D. LEYKEKHMAN, A. H. SCHATZ, AND L. B. WAHLBIN, *Best approximation property in the W_∞^1 norm for finite element methods on graded meshes*, Math. Comp., 81 (2012), pp. 743–764.

[9] G. DOLZMANN, *Optimal convergence for the finite element method in Campanato spaces*, Math. Comp., 68 (1999), pp. 1397–1427.

[10] J. DOUGLAS, JR., T. DUPONT, AND L. WAHLBIN, *The stability in L^q of the L^2-projection into finite element function spaces*, Numer. Math., 23 (1974/75), pp. 193–197.

[11] K. ERIKSSON, *Finite element methods of optimal order for problems with singular data*, Math. Comp., 44 (1985), pp. 345–360.

[12] ———, *An adaptive finite element method with efficient maximum norm error control for elliptic problems*, Math. Models Methods Appl. Sci., 4 (1994), pp. 313–329.

[13] K. ERIKSSON AND C. JOHNSON, *Adaptive finite element methods for parabolic problems. II. Optimal error estimates in $L_\infty L_2$ and $L_\infty L_\infty$*, SIAM J. Numer. Anal., 32 (1995), pp. 706–740.

[14] L. C. EVANS, *Partial Differential Equations*, American Mathematical Society, Providence, RI, 1998.

[15] J. FREHSE AND R. RANNACHER, *Eine L^1-Fehlerabschätzung für diskrete Grundlösungen in der Methode der finiten Elemente*, in Finite Elemente (Tagung, Univ. Bonn, Bonn, 1975), Inst. Angew. Math., Univ. Bonn, Bonn, 1976, pp. 92–114. Bonn. Math. Schrift., No. 89.

[16] I. FRIED, *On the optimality of the pointwise accuracy of the finite element solution*, Internat. J. Numer. Methods Engrg., 15 (1980), pp. 451–456.

[17] H. Fujita and A. Mizutani, *On the finite element method for parabolic equations. I. Approximation of holomorphic semi-groups*, J. Math. Soc. Japan, 28 (1976), pp. 749–771.

[18] M. Grüter and K.-O. Widman, *The Green function for uniformly elliptic equations*, Manuscripta Math., 37 (1982), pp. 303–342.

[19] J. Guzmán, D. Leykekhman, J. Rossmann, and A. H. Schatz, *Hölder estimates for Green's functions on convex polyhedral domains and their applications to finite element methods*, Numer. Math., 112 (2009), pp. 221–243.

[20] R. Haverkamp, *Eine Aussage zur L_∞-Stabilität und zur genauen Konvergenzordnung der H_0^1-Projektionen*, Numer. Math., 44 (1984), pp. 393–405.

[21] D. Leykekhman, *Uniform error estimates in the finite element method for a singularly perturbed reaction-diffusion problem*, Math. Comp., 77 (2008), pp. 21–39 (electronic).

[22] D. Leykekhman and B. Vexler, *Discrete maximal parabolic regularity for Galerkin finite element methods*, (2015). submitted, Preprint arXiv:1505.04808.

[23] ———, *Pointwise best approximation results for Galerkin finite element solutions of parabolic problems*, (2015). submitted, Preprint arXiv:1508.01165.

[24] ———, *A priori error estimates for three dimensional parabolic optimal control problems with pointwise control*, (2015). submitted.

[25] V. G. Maz'ya and J. Rossmann, *On the Agmon-Miranda maximum principle for solutions of elliptic equations in polyhedral and polygonal domains*, Ann. Global Anal. Geom., 9 (1991), pp. 253–303.

[26] F. Natterer, *Über die punktweise Konvergenz finiter Elemente*, Numer. Math., 25 (1975/76), pp. 67–77.

[27] J. Nitsche, *L_∞-convergence of finite element approximations*, in Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975), Springer, Berlin, 1977, pp. 261–274. Lecture Notes in Math., Vol. 606.

[28] J. A. Nitsche, *L_∞-convergence of finite element approximation*, in Journées "Éléments Finis" (Rennes, 1975), Univ. Rennes, Rennes, 1975, p. 18.

[29] J. A. Nitsche and A. H. Schatz, *Interior estimates for Ritz-Galerkin methods*, Math. Comp., 28 (1974), pp. 937–958.

[30] R. Rannacher, *Zur L^∞-Konvergenz linearer finiter Elemente beim Dirichlet-Problem*, Math. Z., 149 (1976), pp. 69–77.

[31] ———, *L^∞-stability estimates and asymptotic error expansion for parabolic finite element equations*, in Extrapolation and defect correction (1990), vol. 228 of Bonner Math. Schriften, Univ. Bonn, Bonn, 1991, pp. 74–94.

[32] R. Rannacher and R. Scott, *Some optimal error estimates for piecewise linear finite element approximations*, Math. Comp., 38 (1982), pp. 437–445.

[33] A. H. Schatz, *A weak discrete maximum principle and stability of the finite element method in L_∞ on plane polygonal domains. I*, Math. Comp., 34 (1980), pp. 77–91.

[34] A. H. Schatz, V. C. Thomée, and L. B. Wahlbin, *Maximum norm stability and error estimates in parabolic finite element equations*, Comm. Pure Appl. Math., 33 (1980), pp. 265–304.

[35] A. H. Schatz and L. B. Wahlbin, *Interior maximum norm estimates for finite element methods*, Math. Comp., 31 (1977), pp. 414–442.

[36] ———, *Maximum norm estimates in the finite element method on plane polygonal domains. I*, Math. Comp., 32 (1978), pp. 73–109.

[37] ———, *Maximum norm estimates in the finite element method on plane polygonal domains. II. Refinements*, Math. Comp., 33 (1979), pp. 465–492.

[38] ———, *On the quasi-optimality in L_∞ of the $\dot{H}^1$-projection into finite element spaces*, Math. Comp., 38 (1982), pp. 1–22.

[39] ———, *Interior maximum-norm estimates for finite element methods. II*, Math. Comp., 64 (1995), pp. 907–928.

[40] R. Scott, *Finite element convergence for singular data*, Numer. Math., 21 (1973/74), pp. 317–327.

[41] ———, *Optimal L^∞ estimates for the finite element method on irregular meshes*, Math. Comp., 30 (1976), pp. 681–697.

[42] Z. W. Shen, *Resolvent estimates in L^p for elliptic systems in Lipschitz domains*, J. Funct. Anal., 133 (1995), pp. 224–251.

[43] V. Thomée, *Galerkin finite element methods for parabolic problems*, vol. 25 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, second ed., 2006.

[44] V. Thomée and L. B. Wahlbin, *Stability and analyticity in maximum-norm for simplicial Lagrange finite element semidiscretizations of parabolic equations with Dirichlet boundary conditions*, Numer. Math., 87 (2000), pp. 373–389.