Dirichlet control of elliptic state constrained problems

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We study a state constrained Dirichlet optimal control problem and derive a priori error estimates for its finite element discretization. Additional control constraints may or may not be included in the formulation. The pointwise state constraints are prescribed in the interior of a convex polygonal domain. We obtain a priori error estimates for the $L^2(\Gamma)$-norm of order $h^{1-1/p}$ for pure state constraints and $h^{3/4-1/(2p)}$ when additional control constraints are present. Here, p is a real number that depends on the largest interior angle of the domain. Unlike in e.g. distributed or Neumann control problems, the state functions associated with L^2-Dirichlet control have very low regularity, i.e. they are elements of $H^{1/2}(\Omega)$. By considering the state constraints in the interior we make use of higher interior regularity and separate the regularity limiting influences of the boundary on the one-hand, and the measure in the right-hand-side of the adjoint equation associated with the state constraints

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on the other hand. We note in passing that in case of control constraints, these may be interpreted as state constraints on the boundary.

1 Introduction

PDE constrained optimal control problems with pointwise state constraints are known to cause certain theoretical and numerical difficulties. Some progress has recently been made regarding the numerical analysis of such problems. A priori discretization error estimates and convergence results are available for different classes of problems, including linear-quadratic distributed control problems [7, 10, 15, 17, 23, 24], problems with Neumann boundary control [18], problems with finitely many state constraints [8, 22, 19], or problems with finitely many control parameters [21, 22]. In [21] the control parameters may influence a linear combination of nonhomogeneous Dirichlet boundary data with high-regularity. In this work, we are concerned with a Dirichlet boundary control problem, which admits less regularity for L^2-control functions than for instance Neumann boundary control problems. We will focus on presenting a priori error estimates for the finite element discretization of such linear-quadratic problems with pointwise state constraints in the interior of the domain.

We prove an error rate for the $L^2(\Gamma)$-norm of the control of $\mathcal{O}(h^{1-1/p})$ for problems without control constraints (cf. Theorem 6.4), which seems to be optimal regarding the existing results in the literature. The error rate will be limited by the effects of the boundary term to $h^{1-1/p}$ (cf. [11, 20]) and the effects of having a measure as the Lagrange multiplier associated to the state constraints. If we include also control constraints in our analysis, we obtain an order of convergence of $\mathcal{O}(h^{3/4-1/(2p)})$ (cf. Theorem 6.11). To the authors' knowledge, results on discretization error estimates for state-constrained problems in the literature deal with distributed or Neumann boundary control problems, only. The order of almost $\mathcal{O}(h)$ obtained by Deckelnick and Hinze [15] or by Meyer [23] for distributed controls is for domains with smooth boundary. In [23] a comment about convex polygons is made, and an order $\mathcal{O}(h^{1/2})$ is obtained. The estimate of order $\mathcal{O}(h|\log h|)$ obtained by Casas, Mateos and Vexler in [10] is based on the fact that, for the problem treated in that work, an enhanced regularity of the Lagrange multiplier can be proven under mild assumptions. The same order is obtained in [17, Corollary 3.3] under the assumption of uniform boundedness of the distributed controls in the $L^\infty(\Omega)$-norm.

In [17, Remark 3.4] it is noticed that for both distributed and Neumann boundary state-constrained control problems (and using a variational discretization) an easy argument can be used to show that in the presence of control constraints the same proof made for the pure state-constrained case applies also for the control-and-state-constrained case. This can be done due to the high regularity of both the control and the state. Unfortunately, such argument cannot be transferred to our problem due to the low regularity of the involved functions. Therefore, we must use two completely different methods of proof for the two cases.

Let us present an outline of the paper. In the next section we introduce the problem and the notation that will be used throughout the work. In Section 3 we collect and prove the regularity results we are going to need. Section 4 is devoted to the derivation of optimality conditions, as well as the regularity properties of the optimal solution that can be derived from these. Next, in Section 5, we discretize the problem using finite elements. Our main results are presented and proven in Section 6. After introducing a technical assumption on the mesh and proving an approximation result for the normal derivative of the adjoint state, we split the presentation and use different techniques of proof for the no-control-constrained and the control-constrained cases. The proofs are presented in Subsections 6.1 and 6.2, respectively, and the aforementioned orders of convergence of $\mathcal{O}(h^{1-1/p})$ and $\mathcal{O}(h^{3/4-1/(2p)})$ are obtained in each setting. Finally, we remark that if we use the technique of Subsection 6.1 to the control-constrained case or the technique of Subsection 6.2 to the no-control-constrained case, we get worse orders of convergence in both cases.

2 The control problem

Throughout the article, we are dealing with the following linear-quadratic optimal control problem:

$$\left.\begin{aligned} &\min J(u) = \frac{1}{2}\|y_u - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)} \\ &-\Delta y_u = 0 \text{ in } \Omega,\ y_u = u \text{ on } \Gamma \\ &a(x) \le y_u(x) \le b(x) \text{ for a.e. } x \in \bar\Omega_1 \\ &\alpha(x) \le u(x) \le \beta(x) \text{ on } \Gamma. \end{aligned}\right\} \tag{$\mathbb{P}$}$$

Here, $\Omega \subset \mathbb{R}^2$ is an bounded convex domain with polygonal boundary Γ and $\Omega_1 \subset\subset \Omega$ is an open set. With this notation, we mean that the closure of Ω_1 is included in Ω: $\bar\Omega_1 \subset \Omega$.

We denote by $\pi/3 \le \omega < \pi$ the largest interior angle of Γ, and by

$$p_\Omega = 2/(2 - \pi/\max\{\omega, \pi/2\}) > 2 \quad \text{and} \quad s_\Omega = 1 + \pi/\omega \in (2, 4]$$

the exponents giving the maximal elliptic regularity in $W^{2,p}(\Omega)$ for $p < p_\Omega$ (cf. [16, Theorem 4.4.3.7]) and $H^s(\Omega)$ for $s < s_\Omega$ (cf. [16, Theorem 5.1.1.4]). We consider a target state y_d regular enough, i.e., we will assume $y_d \in L^p(\Omega) \cap H^{s-2}(\Omega)$ for all $p < p_\Omega$ and all $s < s_\Omega$.

Moreover, for the state constraints, we consider two functions $a, b \in C(\bar\Omega_1)$ such that $a(x) < b(x)$ on $\bar\Omega_1$ and, for the control constraints, two functions $\alpha, \beta \in W^{1-1/p_\Omega, p_\Omega}(\Gamma)$, such that $\alpha(x) < \beta(x)$ on Γ. With an abuse of notation, we will include in our formulation the absence of one or several constraints allowing the cases $a \equiv -\infty$, $b \equiv \infty$, $\alpha \equiv -\infty$ or $\beta \equiv \infty$. Further assumptions on the regularity of the state constraints will be made in order to obtain error estimates. Finally, consider $\nu > 0$ a regularization parameter.

To end this section, let us introduce some short notation. As usual, $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, $(\cdot,\cdot)_\Gamma$ is the inner product in $L^2(\Gamma)$ and $\langle\cdot,\cdot\rangle$ is the duality product between $C(\bar\Omega_1)$ and its dual $\mathcal{M}(\bar\Omega_1)$, the space of regular Borel measures on $\bar\Omega_1$. To handle the constraints, we will use the sets

$$K = \{y \in C(\bar\Omega_1)\colon a(x) \le y(x) \le b(x)\ \forall x \in \bar\Omega_1\},$$

$$U_{\alpha,\beta} = \{u \in L^2(\Gamma)\colon \alpha(x) \le u(x) \le \beta(x) \text{ for a.e. } x \in \Gamma\},$$

and

$$U_{ad} = \{u \in U_{\alpha,\beta}\colon y_u \in K\}.$$

We will denote by $\mathrm{Proj}_{[a,b]}(c) = \min\{b, \max\{a, c\}\}$ for any real numbers a, b, c the projection of c onto the interval $[a, b]$. Finally, we will denote by $\{\chi_j\}_{j=1}^m$ the vertices of Γ counted counterclockwise, with $\chi_{m+1} = \chi_1$, and by Γ_j the part of Γ joining vertices χ_j and χ_{j+1}.

3 Some regularity results for the related PDEs

It is well known that in a polygonal domain, for any $u \in L^2(\Gamma)$ there exists a unique $y_u \in H^{1/2}(\Omega)$ solving the state equation in the transposition sense:

$$\int_\Omega y_u \Delta z dx = \int_\Gamma u \partial_n z ds\ \forall z \in H^2(\Omega) \cap H^1_0(\Omega).$$

Moreover, the estimate

$$\|y_u\|_{H^{1/2}(\Omega)} \le C\|u\|_{L^2(\Gamma)} \tag{1}$$

holds; see [1, Theorem 2.4] for a proof even in non-convex polygonal domains. This defines a linear and continuous control-to-state operator

$$S\colon L^2(\Gamma) \to H^{1/2}(\Omega).$$

In this section we will collect some regularity results for the state and the adjoint state equation that will be needed in the rest of the work.

Lemma 3.1. *The control-to-state mapping is continuous from $H^{s-3/2}(\Gamma)$ to $H^{s-1}(\Omega)$ for all $s < s_\Omega$.*

Proof. Consider $u \in H^{s-3/2}(\Gamma)$: from the trace theorem [16, Theorem 1.5.2.8] we know that there exists $U \in H^{s-1}(\Omega)$ such that $\mathrm{trace}(U) = u$. Consider $z = U - y_u$. This function satisfies

$$-\Delta z = -\Delta U \text{ in } \Omega,\ z = 0 \text{ on } \Gamma.$$

Since $-\Delta U \in H^{s-3}(\Omega)$ and $s - 3 < s_\Omega - 2$, then [16, Theorem 5.1.1.4] implies that $z \in H^{s-1}(\Omega)$ and consequently y_u belongs to $H^{s-1}(\Omega)$ as well. □

Lemma 3.2. *For any open set $\Omega' \subset\subset \Omega$, the control-to-state mapping $S\colon L^2(\Gamma) \to H^{1/2}(\Omega)$, $Su = y_u$ is continuous*

1. from $L^2(\Gamma) \to C(\bar\Omega')$*;*

2. from $W^{1-1/p,p}(\Gamma)$ *to* $W^{2,p}(\Omega')$ *for all* $p < p_\Omega$*;*

3. and from $H^{s-3/2}(\Gamma)$ *to* $H^s(\Omega')$ *for all* $s < s_\Omega$*.*

Proof. The proof follows the usual techniques for interior regularity results. We will prove in detail the first statement.

1. Since y_u is a harmonic function, and hence continuous in Ω, we have that $y_u \in C(\bar\Omega')$ and T is well defined from $L^2(\Gamma) \to C(\bar\Omega')$. In $\mathbb{R}^2$, we have that $H^{1/2}(\Omega) \hookrightarrow L^4(\Omega)$, and using (1) we can write

$$\|y_u\|_{L^4(\Omega)} + \|\nabla y_u\|_{W^{-1,4}(\Omega)} \leq C\|u\|_{L^2(\Gamma)}. \tag{2}$$

Consider now a cut-off function $\eta \in \mathcal{D}(\Omega)$, $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $\bar\Omega'$, as well as $\eta \equiv 0$ on $\Omega \setminus \Omega''$, with some subdomain Ω'' satisfying $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. Taking into account that $\Delta y_u = 0$, we have that ηy_u satisfies the equation

$$-\Delta(\eta y_u) = -y\Delta\eta - 2\nabla\eta \cdot \nabla y_u \text{ in } \Omega, \quad \eta y_u = 0 \text{ on } \Gamma. \tag{3}$$

Since $4 > 2$, using the classical estimate by Stampacchia [26], we obtain

$$\|y_u\|_{L^\infty(\Omega')} \leq \|\eta y_u\|_{L^\infty(\Omega)} \leq \|y\Delta\eta + \nabla\eta \cdot \nabla y_u\|_{W^{-1,4}(\Omega)}$$

and the first result follows from this inequality and (2)

2. We make use of $y_u \in W^{1,p}(\Omega)$ (see [1, Lemma 2.3]), and hence $\nabla y_u \in L^p(\Omega)$. We repeat the process from Step 1, taking into account the $W^{2,p}(\Omega)$ regularity of ηy_u, which follows from [16, Theorem 4.4.3.7].

3. From Lemma 3.1 we have $y_u \in H^{s-1}(\Omega)$, and therefore $\nabla y_u \in H^{s-2}(\Omega)$. Since $s < s_\Omega$, we can apply [16, Theorem 5.1.1.4] to (3) to obtain $\eta y_u \in H^s(\Omega)$, and hence $y_u \in H^s(\Omega')$. □

To eventually obtain regularity results for the control via the optimality system, we proceed by discussing regularity of adjoint equations. For $u \in L^2(\Gamma)$ and $\mu \in \mathcal{M}(\bar\Omega_1)$ we define $\varphi_{\rm r}(u) \in H_0^1(\Omega)$ and $\varphi_{\rm s}(\mu) \in W_0^{1,t}(\Omega)$ for all $t < 2$ the unique solutions of

$$\begin{aligned} -\Delta\varphi_{\rm r}(u) &= y_u - y_d \text{ in } \Omega,\ \varphi_{\rm r}(u) = 0 \text{ on } \Gamma,\\ -\Delta\varphi_{\rm s}(\mu) &= \mu \qquad\quad \text{in } \Omega,\ \varphi_{\rm s}(\mu) = 0 \text{ on } \Gamma, \end{aligned}$$

where the last equation must be understood in the transposition sense:

$$(\varphi_{\rm s}(\mu), -\Delta z) = \langle \mu, z\rangle \ \forall z \in H_0^1(\Omega) \text{ s.t. } \Delta z \in L^2(\Omega). \tag{4}$$

Notice that if $\Delta z \in L^2(\Omega)$, then $z \in H^2_{loc}(\Omega)$ and hence $z \in C(\bar\Omega_1)$, so the definition is meaningful.

Lemma 3.3. *If $u \in L^2(\Gamma)$, then*

$$\varphi_{\mathrm{r}}(u) \in W^{2,q}(\Omega),\ \partial_n\varphi_{\mathrm{r}}(u) \in W^{1-1/q,q}(\Gamma)\ \forall q \le 4,\ q < p_\Omega. \tag{5}$$

If, further, $u \in H^{1/2}(\Gamma)$, then we also have that

$$\varphi_{\mathrm{r}}(u) \in W^{2,p}(\Omega),\ \partial_n\varphi_{\mathrm{r}}(u) \in W^{1-1/p,p}(\Gamma)\ \forall p < p_\Omega, \tag{6}$$

$$\varphi_{\mathrm{r}}(u) \in H^s(\Omega),\ \partial_n\varphi_{\mathrm{r}}(u) \in \prod_{j=1}^m H^{s-3/2}(\Gamma_j)\ \forall s \le 3,\ s < s_\Omega, \tag{7}$$

and

$$\partial_n\varphi_{\mathrm{r}}(u) \in H^{s-3/2}(\Gamma)\ \forall s < \min\{3, s_\Omega\}. \tag{8}$$

Proof. Suppose $u \in L^2(\Gamma)$. Then $y_u \in H^{1/2}(\Omega) \hookrightarrow L^4(\Omega)$ and usual regularity results (cf. [16, Theorem 4.4.3.7]) will give us that $\varphi_{\mathrm{r}}(u) \in W^{2,q}(\Omega)$ for $q \le 4$, $q < p_\Omega$. The trace theorem (e.g. [16, Theorem 1.6.1.5]) states then that

$$\partial_n\varphi_{\mathrm{r}}(u) \in \prod_{j=1}^m W^{1-1/q,q}(\Gamma_j)\ \forall q \le 4, q < p_\Omega.$$

Since $\varphi_{\mathrm{r}}(u) = 0$ on Γ, we have that $\partial_n\varphi_{\mathrm{r}}(u)(\chi_j) = 0$ (see [9, Lemma A.2] and [6, §4]) and $\partial_n\varphi_{\mathrm{r}}(u) \in C(\Gamma)$. This compatibility condition is enough (cf. [16, Theorem 1.5.2.3(b)] to obtain the global regularity in Γ.

If $u \in H^{1/2}(\Gamma)$, then $y_u \in H^1(\Omega) \subset L^p(\Omega)$ for all $p < p_\Omega$. Relations in (6) follow now in the same way as we proved (5). The regularity result [16, Theorem 5.1.1.4] gives us $\varphi_{\mathrm{r}}(u) \in H^s(\Omega)$ for all $s \le 3$, $s < s_\Omega$ and the trace theorem hence yields

$$\partial_n\varphi_{\mathrm{r}}(u) \in \prod_{j=1}^m H^{s-3/2}(\Gamma_j)\ \forall s \le 3, s < s_\Omega.$$

If $s_\Omega \le 5/2$ (i.e., for $\omega \ge 2\pi/3$), the already mentioned global continuity of $\partial_n\varphi_{\mathrm{r}}$ is enough to obtain the desired global regularity on the boundary. If $5/2 < s_\Omega < 3$ (this is, for angles $\pi/2 < \omega < 2\pi/3$) this continuity condition gives us also that $\partial_n\varphi_{\mathrm{r}}(u) \in H^1(\Gamma)$; on the other hand, the definition of the Sobolev space $H^{s-3/2}(\Gamma_j)$ for $s > 5/2$ gives that

$$\partial_\tau\partial_n\varphi_{\mathrm{r}}(u) \in \prod_{j=1}^m H^{s-5/2}(\Gamma_j).$$

Since $s < 3$, it is known (cf. [16, Theorem 1.5.2.3(a)]) that no compatibility condition is required at the corners to have

$$\prod_{j=1}^m H^{s-5/2}(\Gamma_j) = H^{s-5/2}(\Gamma). \tag{9}$$

All together, we obtain that $\partial_n\varphi_{\mathrm{r}}(u) \in H^1(\Gamma)$ and its derivative satisfies $\partial_\tau\partial_n\varphi_{\mathrm{r}}(u) \in H^{s-5/2}(\Gamma)$. These are precisely the conditions that define the space $H^{s-3/2}(\Gamma)$ (for $5/2 < s < 7/2$), and therefore $\partial_n\varphi_{\mathrm{r}}(u) \in H^{s-3/2}(\Gamma)$ by definition.

For $s_\Omega \ge 3$ and $s = 3$, (9) is no longer true in general. □

Lemma 3.4. *For every open set* Ω_2 *with smooth boundary* Γ_2 *such that* $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ *and every* $\mu \in \mathcal{M}(\bar{\Omega}_1)$

$$\varphi_{\rm s}(\mu) \in W_0^{1,t}(\Omega) \cap W^{2,p}(\Omega \setminus \bar{\Omega}_2) \cap H^s(\Omega \setminus \bar{\Omega}_2)\ \forall t < 2,\ p < p_\Omega,\ s < s_\Omega, \tag{10}$$

$$\partial_n \varphi_{\rm s}(\mu) \in W^{1-1/p,p}(\Gamma) \cap \prod_{j=1}^{m} H^{s-3/2}(\Gamma_j)\ \forall p < p_\Omega,\ s < s_\Omega, \tag{11}$$

and

$$\partial_n \varphi_{\rm s}(\mu) \in H^{s-3/2}(\Gamma)\ \forall s < \min\{3, s_\Omega\}. \tag{12}$$

Proof. Since $\varphi_{\rm s}(\mu)$ is harmonic in $\Omega \setminus \bar{\Omega}_1$, we have that $\varphi_{\rm s}(\mu) \in C^\infty_{loc}(\Omega \setminus \bar{\Omega}_1)$.

For any open set Ω_2 with smooth boundary Γ_2 such that $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$, $\varphi_{\rm s}(\mu)$ is the solution of the following boundary value problem:

$$-\Delta \varphi_{\rm s}(\mu) = 0 \text{ in } \Omega \setminus \bar{\Omega}_2,\ \varphi_{\rm s}(\mu) = 0 \text{ on } \Gamma, \varphi_{\rm s}(\mu) = g \text{ on } \Gamma_2, \tag{13}$$

where g is the trace of $\varphi_{\rm s}(\mu)$ on Γ_2 and is a $C^\infty(\Gamma_2)$ function. Therefore, using [16, Theorems 4.4.3.7 and 5.1.1.4] we obtain (10). Notice that now we do not have the restriction $s \leq 3$, since the right hand side of (13) is zero.

The regularity of its normal derivative is proven using the trace theory as in Lemma 3.3.

□

Some further interior regularity will also be useful later.

Lemma 3.5. *For any open sets* Ω_2 *and* Ω_3 *such that* $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega$

$$\varphi_{\rm s}(\mu) \in W^{2,\infty}(\Omega_3 \setminus \bar{\Omega}_2)$$

and

$$\|\varphi_{\rm s}(\mu)\|_{W^{2,\infty}(\Omega_3 \setminus \bar{\Omega}_2)} \leq C \|\mu\|_{\mathcal{M}(\bar{\Omega}_1)},$$

where C *depends on the distance from* $\bar{\Omega}_1$ *to* $\Omega_3 \setminus \bar{\Omega}_2$.

Proof. The first statement is obvious since $\varphi_{\rm s}(\mu)$ is harmonic $\Omega \setminus \bar{\Omega}_1$ and $\Omega_3 \setminus \bar{\Omega}_2 \subset\subset \Omega \setminus \bar{\Omega}_1$.

The proof of the continuity estimate is that of Lemma 3.2. Here we need to use a bootstrapping argument with two open sets Ω' and Ω'' such that $\Omega_3 \setminus \bar{\Omega}_2 \subset\subset \Omega'' \subset\subset \Omega' \subset\subset \Omega \setminus \bar{\Omega}_1 \subset\subset \Omega$ to obtain the intermediate results

$$\begin{aligned} \|\varphi_{\rm s}(\mu)\|_{W^{2,\infty}(\Omega_3 \setminus \bar{\Omega}_2)} &\leq \|\varphi_{\rm s}(\mu)\|_{W^{4,t}(\Omega_3 \setminus \bar{\Omega}_2)} \leq C_1 \|\varphi_{\rm s}(\mu)\|_{W^{3,t}(\Omega'')} \\ &\leq C_2 \|\varphi_{\rm s}(\mu)\|_{W^{2,t}(\Omega')} \leq C_3 \|\varphi_{\rm s}(\mu)\|_{W^{1,t}(\Omega)} \leq C \|\mu\|_{\mathcal{M}(\bar{\Omega}_1)}. \end{aligned}$$

□

4 Optimality conditions and regularity of the solution

Define the Lagrangian of the problem, $\mathcal{L}\colon L^2(\Gamma)\times\mathcal{M}(\bar\Omega_1)\times\mathcal{M}(\bar\Omega_1)\to\mathbb{R}$ as

$$\mathcal{L}(u,\mu^+,\mu^-) = J(u) + \langle \mu^+, y_u - b\rangle + \langle \mu^-, a - y_u\rangle.$$

We have that for any $u,v\in L^2(\Gamma)$ and $\mu^+,\mu^-\in\mathcal{M}(\bar\Omega_1)$, with $\mu=\mu^+-\mu^-$ the first derivatives are given by the expressions (see [11])

$$J'(u)v = (-\partial_n\varphi_{\mathrm{r}}(u)+\nu u, v)_\Gamma$$
$$\partial_u\mathcal{L}(u,\mu^+,\mu^-)v = (-\partial_n\varphi_{\mathrm{r}}(u) - \partial_n\varphi_{\mathrm{s}}(\mu)+\nu u, v)_\Gamma$$

and the second derivatives are independent of u, μ^+, and μ^- since the problem is quadratic and the constraints are linear:

$$J''(u)v^2 = \partial^2_{uu}\mathcal{L}(u,\mu^+,\mu^-)v^2 = \|y_v\|^2_{L^2(\Omega)} + \nu\|v\|^2_{L^2(\Gamma)}.$$

Definition 4.1. *We will say that u is a feasible point for $(\mathbb{P})$ if $u\in U_{ad}$. We will say that $u_0\in U_{ad}$ is a feasible Slater point for $(\mathbb{P})$ if there exist $\delta>0$ and $\varepsilon>0$ such that*

$$\alpha(x)+\delta\le u_0(x)\le\beta(x)-\delta \quad \text{for a.e } x\in\Gamma,$$
$$a(x)+\varepsilon\le y_{u_0}(x)\le b(x)-\varepsilon \text{ for a.e. } x\in\bar\Omega_1.$$

Theorem 4.2. *Suppose problem $(\mathbb{P})$ has a feasible point. Then it has a unique solution $\bar u\in U_{ad}$ with related state $\bar y = y_{\bar u}\in K$. If, further, $(\mathbb{P})$ has a feasible Slater point, then there exist two nonnegative measures $\bar\mu^+,\bar\mu^-\in\mathcal{M}(\bar\Omega_1)$ such that*

$$-\Delta\bar y = 0 \text{ in } \Omega,\ \bar y = \bar u \text{ on } \Gamma \tag{14a}$$
$$-\Delta\bar\varphi = \bar y - y_d + \bar\mu \text{ in } \Omega,\ \bar\varphi = 0 \text{ on } \Gamma \tag{14b}$$
$$\bar u(x) = \mathrm{Proj}_{[\alpha(x),\beta(x)]}\left(\frac{1}{\nu}\partial_n\bar\varphi(x)\right) \text{ on } \Gamma \tag{14c}$$
$$\langle\bar\mu, y-\bar y\rangle\le 0\ \forall y\in K \tag{14d}$$

and

$$\mathrm{supp}\,\bar\mu^+\subset\{x\in\bar\Omega_1\colon \bar y(x) = b(x)\} \tag{15a}$$
$$\mathrm{supp}\,\bar\mu^-\subset\{x\in\bar\Omega_1\colon \bar y(x) = a(x)\} \tag{15b}$$

where $\bar\mu = \bar\mu^+-\bar\mu^-$ and $\bar\varphi = \varphi_{\mathrm{r}}(\bar u)+\varphi_{\mathrm{s}}(\bar\mu)$.

Proof. Since Problem $(\mathbb{P})$ is strictly convex and we are supposing the existence of a feasible point, existence and uniqueness of a solution $\bar u\in L^2(\Gamma)$ is immediate.

Thanks to Lemma 3.2 and our assumption on the existence of a Slater point, from the expression of the derivative of the Lagrangian, we obtain (see, e.g., [5]) the existence of two nonnegative measures $\bar\mu^+$ and $\bar\mu^-$ such that (14d) holds and

$$\partial_u\mathcal{L}(\bar u,\bar\mu^+,\bar\mu^-)(u-\bar u)\ge 0\ \forall u\in U_{\alpha,\beta},$$

which in our case means

$$(-\partial_n\bar{\varphi} + \nu\bar{u}, u - \bar{u})_\Gamma \geq 0 \ \forall u \in U_{\alpha,\beta}, \tag{16}$$

that leads directly to the projection formula (14c). Relations like (15a) and (15b) are well known in the context of state constrained problems. See e.g. [10] for a proof for non-constant constraints. □

Remark 4.3. *The Lagrange multiplier $\bar{\mu}$ and the adjoint state $\bar{\varphi}$ need not be unique. Consider the following one-dimensional problem. $\Omega = (-1,1)$, $\Omega_1 = (-1/2, 1/2)$, $y_d \equiv -1/2$, $\nu = 1$, $b \equiv -1/2$. Then $\bar{y} \equiv -1/2$, $\bar{u} \equiv -1/2$ is the unique solution of the problem. But both pairs*

$$\bar{\varphi}_1 = \frac{1}{2}(1 - |x|), \ \bar{\mu}_1 = \delta_0$$

and

$$\bar{\varphi}_2 = \begin{cases} (1-|x|)/2 & \text{if } |x| > 1/2 \\ -x^2/2 + 3/8 & \text{if } |x| < 1/2, \end{cases} \quad \bar{\mu}_2 = \chi_{\Omega_1}$$

satisfy the optimality system.

Remark 4.4. *It is also possible to state first order necessary optimality conditions without the use of measures. Due to the convexity of U_{ad} and the expression for the derivative of J, we have that*

$$(-\partial_n\varphi_{\mathrm{r}}(\bar{u}) + \nu\bar{u}, u - \bar{u}) \geq 0 \ \forall u \in U_{ad}.$$

This would lead to the expression

$$\bar{u} = \mathrm{Proj}_{U_{ad}}\left(\frac{1}{\nu}\partial_n\varphi_{\mathrm{r}}(\bar{u})\right) \text{ in the sense of } L^2(\Gamma).$$

This strategy is used in [24].

Corollary 4.5. *If* $(\mathbb{P})$ *has a feasible Slater point, then*

$$\bar{u} \in W^{1-1/p,p}(\Gamma), \ \bar{y} \in W^{1,p}(\Omega) \ \forall p < p_\Omega. \tag{17}$$

Moreover, if $\alpha(x) < \bar{u}(x) < \beta(x)$ for all $x \in \Gamma$, then we also have

$$\bar{u} \in \prod_{i=1}^{m} H^{s-3/2}(\Gamma_i) \ \forall s \leq 3, \ s < s_\Omega \tag{18}$$

and

$$\bar{u} \in H^{s-3/2}(\Gamma), \bar{y} \in H^{s-1}(\Omega) \ \forall s < \min\{3, s_\Omega\}. \tag{19}$$

Proof. On the one hand, using Lemma 3.4 we know that $\partial_n\varphi_{\mathrm{s}}(\bar{\mu}) \in W^{1-1/p,p}(\Gamma)$ for all $p < p_\Omega$.

To consider $\partial_n\varphi_{\mathrm{r}}$ on the other hand, note that Equation (5) implies that $\partial_n\varphi_{\mathrm{r}}(\bar{u}) \in W^{1-1/q,q}(\Gamma)$ for all $q \leq 4$, $q < p_\Omega$. So the projection relation (14c) gives us that $\bar{u} \in W^{1-1/q,q}(\Gamma) \subset H^{1/2}(\Gamma)$. We can now use a bootstrap argument using the relations

(6) to obtain (17). The regularity of the state is an immediate consequence of the trace theorem; see [1, Lemma 2.3] for details.

Let us check (18) and (19) now. Since the control constraints are not attained, (14c) now reads

$$\bar{u}(x) = \frac{1}{\nu}\partial_n\bar{\varphi}(x) \text{ on } \Gamma.$$

Thanks to (17), already proved, we have that $\bar{u} \in H^{1/2}(\Gamma)$ and we can also apply (7) and (11) to deduce (18). Relation (19) follows from (8) and (12). The regularity of the state follows directly from Lemma 3.1. □

5 Discretization

Let $\{\mathcal{T}_h\}_h$ be a quasi-uniform family of triangulations of $\bar{\Omega}$. For the discretization of the state and the adjoint state we use the space of linear finite elements $Y_h \subset H^1(\Omega)$,

$$Y_h = \{y \in C(\bar{\Omega})\colon y_h \in P^1(T)\ \forall T \in \mathcal{T}_h\}.$$

As usual, we will abbreviate $Y_{h0} = Y_h \cap H^1_0(\Omega)$. For the control we use the space U_h of continuous piecewise linear functions that are the trace of some element of Y_h. We define the set of boundary nodes $\mathcal{B}_h = \{j\colon x_j \in \Gamma\}$ for later use. Finally, for the discrete Lagrange multiplier we use the space $\mathcal{M}_h \subset \mathcal{M}(\bar{\Omega}_1)$ which is spanned by Dirac measures corresponding to the nodes $\{x_j\}_{j\in\mathcal{I}_{1h}}$ of the finite element mesh that are elements of $\bar{\Omega}_1$.

For any function $y \in C(\bar{\Omega})$ (resp. $u \in C(\Gamma)$) we denote by $I_h y \in Y_h$ (resp. $I_h u \in U_h$) its nodal interpolator and for any function $u \in L^2(\Gamma)$, we will denote by $\Pi_h u \in U_h$ its projection onto U_h in the $L^2(\Gamma)$ sense, i.e.,

$$(\Pi_h u, v_h)_\Gamma = (u, v_h)_\Gamma\ \forall v_h \in U_h.$$

Notice that for $u_h \in U_h$, $\Pi_h u_h = u_h$. It is known (see [3, Eq. (2.20)], [11, Eq. (4.1)] or [14, Eq. (3.8)]) that if $u \in H^t(\Gamma)$, $0 \le t \le 2$

$$\|u - \Pi_h u\|_{L^2(\Gamma)} \le \|u\|_{H^t(\Gamma)} \text{ for } 0 \le t \le 2. \tag{20}$$

We will also use the space

$$Y_h^\Gamma = \{y_h \in Y_h\colon y_h(x_j) = 0 \text{ if } x_j \notin \Gamma\}.$$

We discretize the state equation without penalization, (using variational crime) (see [2, Theorem 5.2]): for any $u \in L^2(\Gamma)$, $y_h(u) \in Y_h$ is the solution of

$$(\nabla y_h(u), \nabla z_h) = 0\ \forall z_h \in Y_{h0},\ (y_h(u), v_h)_\Gamma = (u, v_h)_\Gamma\ \forall v_h \in U_h.$$

It is customary to say that $y_h(u)$ is the discrete harmonic extension of u. Notice that $y_h(u) \equiv \Pi_h u$ on Γ and hence, if $u_h \in U_h$, $y_h(u_h) \equiv u_h$ on Γ.

The discrete objective functional is defined as

$$J_h(u) = \frac{1}{2}\|y_h(u) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)}.$$

We will denote by

$$U_{\alpha,\beta,h} = \{u_h \in U_h : \alpha(x_j) \le u_h(x_j) \le \beta(x_j)\ \forall j \in \mathcal{B}_h\},$$

$$K_h = \{y_h \in Y_h : a(x_j) \le y_h(x_j) \le b(x_j)\ \forall x_j \in \bar\Omega_1\},$$

and

$$U_{ad,h} = \{u_h \in U_{\alpha,\beta,h} : y_h(u_h) \in K_h\}.$$

Our discrete control problem then reads as

$$\left.\begin{array}{l}\min J_h(u_h)\\ u_h \in U_{ad,h}.\end{array}\right\} \qquad (\mathbb{P}_h)$$

We will discuss some properties of problem $(\mathbb{P}_h)$ similar to those of problem $(\mathbb{P})$.

Definition 5.1. *We will say that u_h is a feasible point for $(\mathbb{P}_h)$ if $u_h \in U_{ad,h}$. We will call $u_{h0} \in U_{ad,h}$ a feasible Slater point for $(\mathbb{P}_h)$ if there exist $\delta_h > 0$ and $\varepsilon_h > 0$ such that*

$$\alpha(x_j) + \delta_h \le u_{h0}(x_j) \le \beta(x_j) - \delta_h\ \forall j \in \mathcal{B}_h,$$
$$a(x_j) + \varepsilon_h \le y_h(u_0)(x_j) \le b(x_j) - \varepsilon_h \forall x_j \in \bar\Omega_1.$$

Theorem 5.2. *Suppose that $(\mathbb{P})$ has a regular feasible Slater point $u_0 \in W^{1-1/p,p}(\Gamma)$, for some $p > 2$. Then there exists $h_0 > 0$ such that for all $0 < h < h_0$ the discrete problem $(\mathbb{P}_h)$ has a Slater feasible point $u_{h0} = \Pi_h u_0$.*

Moreover, the quantities δ_h and ε_h can be taken independent of h for h small enough.

Remark 5.3. *Different assumptions on the regularity of the Slater point are not rare in the related literature on control problems with both control and state constraints. See e.g [23, Assumption 6.2], [8, Remark 3.8] or [24, Assumption 2.1].*

Proof. Let $u_0 \in U_{ad}$ be the feasible Slater point for problem $(\mathbb{P})$, and define $u_{h0} = \Pi_h u_0$. With an inverse inequality, usual interpolation error estimates, and estimate (20) we obtain

$$\begin{aligned}
&\|u_0 - \Pi_h u_0\|_{L^\infty(\Gamma)}\\
&\quad\le \|u_0 - I_h u_0\|_{L^\infty(\Gamma)} + \|I_h u_0 - \Pi_h u_0\|_{L^\infty(\Gamma)}\\
&\quad\le Ch^{1-1/p}\|u_0\|_{W^{1-1/p,p}(\Gamma)} + Ch^{-1/2}\|I_h u_0 - \Pi_h u_0\|_{L^2(\Gamma)}\\
&\quad\le Ch^{1-1/p}\|u_0\|_{W^{1-1/p,p}(\Gamma)}\\
&\qquad + Ch^{-1/2}(\|I_h u_0 - u_0\|_{L^2(\Gamma)} + \|u_0 - \Pi_h u_0\|_{L^2(\Gamma)})\\
&\quad\le Ch^{1-1/p}\|u_0\|_{W^{1-1/p,p}(\Gamma)} + Ch^{-1/2}(h^{1-1/p}\|u_0\|_{H^{1-1/p}(\Gamma)}) \le Ch^{1/2-1/p}.
\end{aligned}$$

From this uniform convergence, and the fact that $\alpha(x) < u_0 < \beta(x)$ for all $x \in \Gamma$, we deduce the existence of some $h_0 > 0$ such that for all $0 < h < h_0$, $\alpha(x_j) < u_{h0}(x_j) < \beta(x_j)$ holds for all $x_j \in \Gamma$.

Since $u_{h0} \to u_0$ in $L^2(\Gamma)$, Lemma 3.2 allows to deduce that

$$\lim_{h\to 0} \|y_{u_0} - y_{u_{h0}}\|_{L^\infty(\Omega_1)} = 0. \tag{21}$$

On the other hand, using the interior error estimate from [25, Theorem 5.1] we have that for some open set Ω_2 such that $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ the estimate

$$\begin{aligned}&\|y_{u_{h0}} - y_h(u_{h0})\|_{L^\infty(\Omega_1)}\\ &\quad\le C(|\log h| \|y_{u_{h0}} - I_h y_{u_{h0}}\|_{L^\infty(\Omega_2)} + \|y_{u_{h0}} - y_h(u_{h0})\|_{L^2(\Omega)})\end{aligned}$$

holds. The first addend in this expression converges to zero since $y_{u_{h0}}$ is harmonic in Ω, and the second one as a consequence of [2, Theorem 5.5]. So we obtain

$$\lim_{h\to 0} \|y_{u_{h0}} - y_h(u_{h0})\|_{L^\infty(\Omega_1)} = 0. \tag{22}$$

From the triangle inequality, (21), and (22), we conclude $y_h(u_{h0}) \to y_{u_0}$ in $L^\infty(\Omega_1)$. Since $a(x) < y_{u_0}(x) < b(x)$ for all $x \in \bar\Omega_1$, there exists $h_0 > 0$ such that $a(x_j) < y_h(u_{h0})(x_j) < b(x_j)$ holds for all $0 < h < h_0$ and all $x_j \in \bar\Omega_1$. Hence, u_{h0} is a Slater point.

The independence of δ_h and ε_h with respect to h is clear from the definition of the Slater point u_0 and the proven uniform convergences. □

For any $u \in L^2(\Gamma)$ and $\mu \in \mathcal{M}(\bar\Omega_1)$, we define $\varphi_{\mathrm{r},h}(u), \varphi_{\mathrm{s},h}(\mu) \in Y_{h0}$ to be the unique solutions of

$$\begin{aligned}(\nabla\varphi_{\mathrm{r},h}(u), \nabla z_h) &= (y_h(u) - y_d, z_h) \ \forall z_h \in Y_{h0}\\ (\nabla\varphi_{\mathrm{s},h}(\mu), \nabla z_h) &= \langle \mu, z_h\rangle \qquad\qquad \forall z_h \in Y_{h0}.\end{aligned}$$

Let us now introduce the discrete variational normal derivative. For any linear operator $T_h \colon Y_h \to \mathbb{R}$, let $\varphi_h \in Y_{h0}$ be the solution of

$$(\nabla z_h, \nabla\varphi_h) = T_h(z_h) \ \forall z_h \in Y_{h0}.$$

Then its discrete variational normal derivative $\partial_n^h \varphi_h \in U_h$ is the unique solution of

$$(\partial_n^h \varphi_h, z_h)_\Gamma = (\nabla z_h, \nabla\varphi_h) - T_h(z_h) \ \forall z_h \in Y_h^\Gamma.$$

The Lagrangian $\mathcal{L}_h \colon L^2(\Gamma) \times \mathcal{M}_h \times \mathcal{M}_h \to \mathbb{R}$ of $(\mathbb{P}_h)$ is defined by

$$\mathcal{L}_h(u, \mu_h^+, \mu_h^-) = J(u) + \langle \mu_h^+, y_h(u) - b\rangle + \langle \mu_h^-, a - y_h(u)\rangle.$$

We have that for any $u, v \in L^2(\Gamma)$ and any $\mu_h^+, \mu_h^- \in \mathcal{M}_h$, with $\mu_h = \mu_h^+ - \mu_h^-$, the first derivatives of J_h and $\mathcal{L}_h$ are given by the expression

$$\begin{aligned}J_h'(u)v &= (-\partial_n^h \varphi_{\mathrm{r},h}(u) + \nu u, v)_\Gamma\\ \partial_u \mathcal{L}_h(u, \mu_h^+, \mu_h^-)v &= (-\partial_n^h \varphi_{\mathrm{r},h}(u) - \partial_n^h \varphi_{\mathrm{s},h}(\mu_h) + \nu u, v)_\Gamma,\end{aligned}$$

and again the second derivatives are independent of u, μ_h^+, and μ_h^- since the problem is quadratic and the constraints are linear:

$$J_h''(u)v^2 = \partial_{uu}^2 \mathcal{L}_h(u, \mu_h^+, \mu_h^-)v^2 = \|y_h(v)\|_{L^2(\Omega)}^2 + \nu\|v\|_{L^2(\Gamma)}^2.$$

Corollary 5.4. *If* ($\mathbb{P}$) *has a regular feasible Slater point, then there exists* $h_0 > 0$ *such that for all* $0 < h < h_0$ *the discrete problem* ($\mathbb{P}_h$) *has a unique solution* $\bar{u}_h \in U_{ad,h}$ *with related discrete state* $\bar{y}_h \in K_h$*. Moreover, there exist nonnegative measures* $\bar{\mu}_h^+, \bar{\mu}_h^- \in \mathcal{M}_h$ *such that*

$$(\nabla \bar{y}_h, \nabla z_h) = 0 \ \forall z_h \in Y_{h0}, \ (\bar{y}_h, v_h)_\Gamma = (\bar{u}, v_h)_\Gamma \ \forall v_h \in U_h \tag{23a}$$
$$(\nabla z_h, \nabla \bar{\varphi}_h) = (\bar{y}_h - y_d, z_h) + \langle \bar{\mu}_h, z_h \rangle \ \forall z_h \in Y_{h0} \tag{23b}$$
$$\langle \bar{\mu}_h, y_h - \bar{y}_h \rangle \le 0 \ \forall y_h \in K_h \tag{23c}$$
$$(\nu \bar{u}_h - \partial_n^h \bar{\varphi}_h, u_h - \bar{u}_h) \ge 0 \ \forall u_h \in U_{\alpha,\beta,h} \tag{23d}$$

where $\bar{\mu}_h = \bar{\mu}_h^+ - \bar{\mu}_h^-$ *and* $\bar{\varphi}_h = \varphi_{\mathrm{r},h}(\bar{u}_h) + \varphi_{\mathrm{s},h}(\bar{\mu}_h)$*.*

Proof. Problem ($\mathbb{P}_h$) is a finite dimensional strictly convex optimization problem whose feasible set is not empty due to Theorem 5.2, so it has a unique solution $\bar{u}_h \in U_{ad,h}$.

The optimality system is immediately obtained from the expression for the first derivative of the discrete Lagrangian. □

Lemma 5.5. *Under the assumptions of Corollary 5.4, the discrete Lagrange multipliers are bounded independently of* h*.*

Proof. Consider u_{h0} the sequence of feasible Slater points for problems ($\mathbb{P}_h$) found in Theorem 5.2. Since $u_{h0} \to u_0$ in $L^2(\Gamma)$, it is a bounded sequence, and the continuity of the solution operator from $L^2(\Gamma)$ to $L^2(\Omega)$, together with [2, Theorem 5.5], implies that $y_h(u_{h0})$ is also bounded in $L^2(\Omega)$. So we may deduce the existence of $C > 0$ such that

$$\|\bar{u}_h\|_{L^2(\Gamma)} \le \frac{2}{\nu} J_h(\bar{u}_h) \le \frac{2}{\nu} J_h(u_{h0}) \le C.$$

With the same reasoning made for the discrete states related to the Slater points, we deduce that the sequence of discrete optimal states $\bar{y}_h$ is also bounded in $L^2(\Omega)$.

Since u_{h0} is a Slater point for problem ($\mathbb{P}_h$), there exists $\rho > 0$ such that

$$a(x_j) \le y_h(u_{h0})(x_j) - \rho < y_h(u_{h0})(x_j) + \rho \le b(x_j) \ \forall x_j \in \bar{\Omega}_1.$$

Notice that $\bar{\mu}_h = \bar{\mu}_h^+ - \bar{\mu}_h^- \in \mathcal{M}_h$, and hence it is a combination of Dirac deltas centered at the nodes of the mesh. There exist $\bar{\lambda}_j \in \mathbb{R}$ for all j such that $x_j \in \bar{\Omega}_1$ such that

$$\bar{\mu}_h = \sum \bar{\lambda}_j \delta_{x_j}.$$

Define $z_h \in Y_h$ as

$$z_h(x_j) = \begin{cases} \rho & \text{if } \bar{\lambda}_j \ge 0 \\ -\rho & \text{if } \bar{\lambda}_j < 0 \\ 0 & \text{if } x_j \notin \bar{\Omega}_1 \end{cases}$$

Clearly, $y_{h0} + z_h \in K_h$, and using (23c) we have

$$\langle \bar{\mu}_h, y_h(u_{h0}) + z_h - \bar{y}_h \rangle \le 0.$$

So we have, using the definition of the discrete normal derivative of $\varphi_{s,h}(\bar\mu_h)$, the fact that $\varphi_{s,h}(\bar\mu_h) \in Y_{h0}$ together with the definition of discrete state, the discrete Euler-Lagrange condition (23d) together with the boundary conditions satisfied by the discrete states, the definition of discrete normal derivative of $\varphi_{r,h}(\bar u_h)$, the fact that $\varphi_{r,h}(\bar u_h) \in Y_{h0}$ together with the definition of discrete state and the already proved boundedness in $L^2(\Gamma)$ of the discrete optimal controls and the discrete Slater controls and in $L^2(\Omega)$ of its related states:

$$\begin{aligned}
\rho\|\bar\mu_h\|_{\mathcal{M}(\bar\Omega_1)} &= \rho\sum|\lambda_j| = \langle\bar\mu_h, z_h\rangle \\
&\le \langle\bar\mu_h, \bar y_h - y_h(u_{h0})\rangle \\
&= (\nabla(\bar y_h - y_h(u_{h0})), \nabla\varphi_{s,h}(\bar\mu_h)) - (\partial_n^h\varphi_{s,h}(\bar\mu_h), \bar y_h - y_h(u_{h0}))_\Gamma \\
&\le (\nu\bar u_h - \partial_n^h\varphi_{r,h}(\bar u), \bar u_h - u_{h0})_\Gamma \\
&= (\nu\bar u_h, \bar u_h - u_{h0})_\Gamma \\
&\quad + (\nabla(\bar y_h - y_h(u_{h0})), \nabla\varphi_{r,h}(\bar u_h)) - (\bar y_h - y_d, \bar y_h - y_h(u_{h0})) \\
&= (\nu\bar u_h, \bar u_h - u_{h0})_\Gamma \le C.
\end{aligned}$$

Hence the assertion is proven. □

6 Error estimates

To obtain error estimates, we will make the following technical assumption on the triangulation, which is not difficult to fulfill in practice:

Assumption (H) *There exists some $\bar h > 0$ and an open set $\Omega_{2,\bar h}$ such that $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_{2,\bar h} \subset\subset \Omega$ for some open set Ω_2 with smooth boundary Γ_2 such that for all $0 < h < \bar h$*

$$\bar\Omega_{2,\bar h} = \cup\{T \in \mathcal{T}_h\colon \text{ s.t. } x_j \in \bar\Omega_{2,\bar h}\ \forall x_j \text{ vertex of } T\}.$$

Notice that for every $T \in \mathcal{T}_h$, either $\operatorname{int} T \in \Omega_{2,\bar h}$ or $\operatorname{int} T \in \Omega\setminus\bar\Omega_{2,\bar h}$ and $\{\mathcal{T}_h\}_{h<\bar h}$ induces a quasi-uniform family of triangulations $\{\mathcal{T}_{2,h}\}_{h<\bar h}$ on $\Omega\setminus\bar\Omega_{2,\bar h}$. We define

$$\tilde Y_h = \{y_h \in C(\bar\Omega\setminus\Omega_{2,\bar h})\colon y_h \in P^1(T)\ \forall T \in \mathcal{T}_{2,h}\}$$

and

$$\tilde Y_{h,0} = \tilde Y_h \cap H_0^1(\Omega\setminus\bar\Omega_{2,\bar h}).$$

We will denote by $(\cdot,\cdot)_{\Omega\setminus\bar\Omega_{2,\bar h}}$ the inner product in $L^2(\Omega\setminus\bar\Omega_{2,\bar h})$. We will also use the space $\tilde U_h$ of the traces of the elements of $\tilde Y_h$ on $\Gamma_{2,\bar h}$, the boundary of $\Omega_{2,\bar h}$.

We can also define a variational discrete normal derivative on $\Gamma_{2,\bar h}$. For any $e_h \in U_h$, and $T_h\colon \tilde Y_h \to \mathbb{R}$ linear, let $\phi_h \in \tilde Y_h$ be the unique solution of

$$(\nabla\phi_h, \nabla z_h)_{\Omega\setminus\bar\Omega_{2,\bar h}} = T_h(z_h) \text{ for all } z_h \in \tilde Y_{h,0},\ \phi_h = e_h \text{ on } \Gamma,\ \phi_h = 0 \text{ on } \Gamma_{2,\bar h}.$$

Then it can be shown as in [11] that there exists a unique $\partial_n^h \phi_h \in \tilde{U}_h$ such that

$$(\partial_n^h \phi_h, z_h)_{\Gamma_{2,\bar{h}}} = (\nabla \phi_h, \nabla z_h)_{\Omega \setminus \bar{\Omega}_{2,\bar{h}}} - \ T_h(z_h) \quad \forall z_h \in \tilde{Y}_h. \tag{24}$$

We have the following relation between the boundary data on Γ and the discrete normal derivative on $\Gamma_{2,\bar{h}}$.

Lemma 6.1. *Suppose that Assumption (H) is satisfied, consider $e_h \in U_h$ and let $\phi_h \in \tilde{Y}_h$ be the unique solution of*

$$(\nabla \phi_h, \nabla z_h)_{\Omega \setminus \bar{\Omega}_{2,\bar{h}}} = 0 \ \forall z_h \in \tilde{Y}_{h,0}, \ \ \phi_h = e_h \ \text{ on } \Gamma, \ \ \phi_h = 0 \ \ \text{ on } \Gamma_{2,\bar{h}}.$$

Then, there exist $h_0 > 0$ and $C > 0$ such that for all $0 < h < h_0$

$$\|\partial_n^h \phi_h\|_{L^2(\Gamma_{2,\bar{h}})} \leq \frac{C}{h} \|e_h\|_{L^2(\Gamma)}$$

is satisfied.

Proof. Take any $v_h \in \tilde{U}_h$ and let $\eta_h \in \tilde{Y}_h$ be the unique solution of

$$(\nabla \eta_h, \nabla z_h)_{\Omega \setminus \bar{\Omega}_{2,\bar{h}}} = 0 \ \forall z_h \in \tilde{Y}_{h,0}, \ \ \eta_h = 0 \text{ on } \Gamma, \eta_h = v_h \text{ on } \Gamma_{2,\bar{h}}.$$

Then, with (24) and using the appropriate inverse inequality (cf. [4, Theorem (4.5.11)], we obtain

$$\begin{aligned}(\partial_n^h \phi_h, v_h)_{\Gamma_{2,\bar{h}}} &= (\nabla \phi_h, \nabla \eta_h)_{\Omega \setminus \bar{\Omega}_{2,\bar{h}}} \leq \|\nabla \phi_h\|_{L^2(\Omega \setminus \bar{\Omega}_{2,\bar{h}})} \|\nabla \eta_h\|_{L^2(\Omega \setminus \bar{\Omega}_{2,\bar{h}})} \\ &\leq C \|e_h\|_{H^{1/2}(\Gamma)} \|v_h\|_{H^{1/2}(\Gamma_{2,\bar{h}})} \\ &\leq C \frac{1}{h^{1/2}} \|e_h\|_{L^2(\Gamma)} \frac{1}{h^{1/2}} \|v_h\|_{L^2(\Gamma_{2,\bar{h}})}\end{aligned}$$

and the result follows. □

Lemma 6.2. *For any $u \in W^{1-1/p,p}(\Gamma)$ there exists some $h_1 > 0$ and some $C > 0$ independent of u such that for all $0 < h < h_1$ the following estimate holds*

$$\|\partial_n \varphi_{\rm r}(u) - \partial_n^h \varphi_{{\rm r},h}(u)\|_{L^2(\Gamma)} \leq C h^{1-1/p} \|u\|_{W^{1-1/p,p}(\Gamma)} \ \forall p < p_\Omega. \tag{25}$$

Suppose further that assumption (H) is satisfied. Then, for any $\mu \in \mathcal{M}(\bar{\Omega}_1)$, there exist some $h_2 > 0$ and $C > 0$ independent of μ such that for all $0 < h < h_2$ the following estimate holds

$$\|\partial_n \varphi_{\rm s}(\mu) - \partial_n^h \varphi_{{\rm s},h}(\mu)\|_{L^2(\Gamma)} \leq C h^{1-1/p} \|\mu\|_{\mathcal{M}(\bar{\Omega}_1)} \ \forall p < p_\Omega. \tag{26}$$

Proof. The proof of the estimates for both the regular part and the singular part is similar. We will write the details for the singular part, since it requires some more tricks. We will drop the dependence on μ in the following lines. First we write

$$\|\partial_n \varphi_{\rm s} - \partial_n^h \varphi_{{\rm s},h}\|^2_{L^2(\Gamma)} = \|\partial_n \varphi_{\rm s} - \Pi_h \partial_n \varphi_{\rm s}\|^2_{L^2(\Gamma)} + \|\Pi_h \partial_n \varphi_{\rm s} - \partial_n^h \varphi_{{\rm s},h}\|^2_{L^2(\Gamma)}.$$

From Lemma 3.4 and estimate (20) it follows that

$$\|\partial_n \varphi_{\rm s} - \Pi_h \partial_n \varphi_{\rm s}\|_{L^2(\Gamma)} \leq Ch^s \|\mu\|_{\mathcal{M}(\bar\Omega_1)}\ \forall s < \min\{3/2, s_\Omega - 3/2\}.$$

For the second addend, denote by $e_h = \Pi_h \partial_n \varphi_{\rm s} - \partial_n^h \varphi_{{\rm s},h}$ and define $\phi_h \in \tilde Y_h$ as the unique solution of

$$(\nabla \phi_h, \nabla z_h)_{\Omega\setminus\bar\Omega_{2,\bar h}} = 0 \text{ for all } z_h \in \tilde Y_{h,0},\ \phi_h = e_h \text{ on } \Gamma,\ \phi_h = 0\ \text{ on } \Gamma_{2,\bar h}.$$

We use the definition of Π_h and the value of ϕ_h on Γ to write

$$\begin{aligned}\|e_h\|^2_{L^2(\Gamma)} &= \|\Pi_h \partial_n \varphi_{\rm s} - \partial_n^h \varphi_{{\rm s},h}\|^2_{L^2(\Gamma)} \\ &= (\Pi_h \partial_n \varphi_{\rm s} - \partial_n^h \varphi_{{\rm s},h}, \Pi_h \partial_n \varphi_{\rm s} - \partial_n^h \varphi_{{\rm s},h})_\Gamma \\ &= (\partial_n \varphi_{\rm s} - \partial_n^h \varphi_{{\rm s},h}, \Pi_h \partial_n \varphi_{\rm s} - \partial_n^h \varphi_{{\rm s},h})_\Gamma \\ &= (\partial_n \varphi_{\rm s}, \phi_h)_\Gamma - (\partial_n^h \varphi_{{\rm s},h}, \phi_h)_\Gamma. \end{aligned} \tag{27}$$

Since $\phi_h = 0$ on $\Gamma_{2,\bar h}$, the extension of ϕ_h to $\Omega_{2,\bar h}$ by 0 is an element of Y_h. With an abuse of notation we well also refer to this extension as ϕ_h. Now we can use that $\phi_h \in H^1(\Omega)$ and apply Green's formula to obtain

$$(\partial_n \varphi_{\rm s}, \phi_h)_\Gamma = -(\phi_h, \mu) + (\nabla \phi_h, \nabla \varphi_{\rm s}) = (\nabla \phi_h, \nabla \varphi_{\rm s}) = (\nabla \phi_h, \nabla \varphi_{\rm s})_{\Omega\setminus\bar\Omega_{2,\bar h}}, \tag{28}$$

where we have used that $\operatorname{supp} \bar\mu \subset \bar\Omega_1 \subset \Omega_2 \subset\subset \Omega_{2,\bar h}$ and $\phi_h \equiv 0$ in $\Omega_{2,\bar h}$. In the same way we use that $\phi_h \in Y_h$ and the definition of the discrete normal derivative to obtain

$$\begin{aligned}(\partial_n^h \varphi_{{\rm s},h}, \phi_h)_\Gamma &= -(\phi_h, \mu) + (\nabla \phi_h, \nabla \varphi_{{\rm s},h}) \\ &= (\nabla \phi_h, \nabla \varphi_{{\rm s},h}) = (\nabla \phi_h, \nabla \varphi_{{\rm s},h})_{\Omega\setminus\bar\Omega_{2,\bar h}}. \end{aligned} \tag{29}$$

Now we use (27)–(29) and insert the zero $\pm(\nabla \phi_h, \nabla I_h \varphi_{\rm s})_{\Omega\setminus\bar\Omega_{2,\bar h}}$ to write

$$\|e_h\|^2_{L^2(\Gamma)} = (\nabla \phi_h, \nabla \varphi_{\rm s} - \nabla I_h \varphi_{\rm s})_{\Omega\setminus\bar\Omega_{2,\bar h}} + (\nabla \phi_h, \nabla I_h \varphi_{\rm s} - \nabla \varphi_{{\rm s},h})_{\Omega\setminus\bar\Omega_{2,\bar h}}. \tag{30}$$

Let us discuss the second term of (30). For any $z_h \in \tilde Y_h$ such that $z_h = 0$ on Γ, using the definition of discrete normal derivative, we have

$$(\nabla \phi_h, \nabla z_h)_{\Omega\setminus\bar\Omega_{2,\bar h}} = (\partial_n^h \phi_h, z_h)_{\Gamma_{2,\bar h}}$$

and therefore

$$\begin{aligned}(\nabla \phi_h, \nabla I_h \varphi_{\rm s} - \nabla \varphi_{{\rm s},h})_{\Omega\setminus\bar\Omega_{2,\bar h}} &= (\partial_n^h \phi_h, I_h \varphi_{\rm s} - \varphi_{{\rm s},h})_{\Gamma_{2,\bar h}} \\ &= (\partial_n^h \phi_h, I_h \varphi_{\rm s} - \varphi_{\rm s})_{\Gamma_{2,\bar h}} + (\partial_n^h \phi_h, \varphi_{\rm s} - \varphi_{{\rm s},h})_{\Gamma_{2,\bar h}}. \end{aligned}$$

From Lemma 3.5, we know that $\varphi_{\rm s}$ is regular in $\Omega_3 \setminus \bar\Omega_2$ for some $\Omega_3 \subset\subset \Omega$ such that $\Gamma_{2,\bar h} \subset \Omega_3 \setminus \bar\Omega_2$, so we use interpolation error estimates (see e.g. [13, Theorem 17.2]) and Lemma 6.1 for the first term. For the second one we also use Lemma 6.1 and the uniform estimate for Green functions [25, Theorem 6.1(i)]. This result is proved for Dirac measures, but the proof is the same (with the obvious changes) for any measure with compact support. We obtain

$$(\nabla\phi_h, \nabla I_h\varphi_{\rm s} - \nabla\varphi_{{\rm s},h})_{\Omega\setminus\bar\Omega_{2,\bar h}} \leq Ch|\log h|\|e_h\|_{L^2(\Gamma)}\|\mu\|_{\mathcal{M}(\bar\Omega_1)}.$$

For the first term in (30)

$$\begin{aligned}(\nabla\phi_h, \nabla\varphi_{\rm s} - \nabla I_h\varphi_{\rm s})_{\Omega\setminus\bar\Omega_{2,\bar h}} &\leq \|\phi_h\|_{W^{1,p'}(\Omega\setminus\bar\Omega_{2,\bar h})}\|\varphi_{\rm s} - I_h\varphi_{\rm s}\|_{W^{1,p}(\Omega\setminus\bar\Omega_{2,\bar h})}\\ &\leq C\|e_h\|_{W^{1-1/p',p'}(\Gamma)}h\|\varphi_s\|_{W^{2,p}(\Omega\setminus\Omega_{2,\bar h})}\\ &\leq C\|e_h\|_{H^{1-1/p'}(\Gamma)}h\|\mu\|_{\mathcal{M}(\bar\Omega_1)} \qquad (31)\\ &\leq Ch^{1/p'-1}\|e_h\|_{L^2(\Gamma)}h\|\mu\|_{\mathcal{M}(\bar\Omega_1)}\\ &= Ch^{1-1/p}\|e_h\|_{L^2(\Gamma)}\|\mu\|_{\mathcal{M}(\bar\Omega_1)}.\end{aligned}$$

Collecting all the estimates the proof is complete. For the regular part, it is easier, since we can define $\phi_h \in Y_h$ and we avoid the second term in (30). □

To obtain error estimates, we will follow two different methods of proof for problems with pure state constraints and problems with additional control constraints. We discuss the main differences of these methods along with the advantages and disadvantages of each of them at the end of the paper.

6.1 No control constraints

The main result of this part is the error estimate proved in Theorem 6.4. A technical lemma necessary for the proof is provided first.

Lemma 6.3. *Suppose that* ($\mathbb{P}$) *has a regular feasible Slater point,* $a, b \in W^{2,p}(\Omega_1)$ *for all* $p < p_\Omega$ *and* $\alpha(x) < \bar u(x) < \beta(x)$ *for all* $x \in \Gamma$. *Let* $\bar u$ *and* $\bar u_h$ *be the solutions of problems* ($\mathbb{P}$) *and* ($\mathbb{P}_h$)*, respectively, and* $\bar\mu$ *and* $\bar\mu_h$ *Lagrange multipliers associated to these solutions. Then*

$$(\partial_n^h\varphi_{{\rm s},h}(\bar\mu) - \partial_n^h\varphi_{{\rm s},h}(\bar\mu_h), \bar u - \bar u_h)_\Gamma \leq Ch^{2(1-1/p)}\ \forall p < p_\Omega.$$

Proof. Using the definition of the $L^2(\Gamma)$ projection, the definition of the discrete normal derivative, the equalities $y_h(\bar u) \equiv \Pi_h\bar u$, $\bar y_h \equiv \bar u_h$ on Γ, the fact that both $\varphi_{{\rm s},h}(\bar\mu_h), \varphi_{{\rm s},h}(\bar\mu) \in Y_{h0}$ and the discrete state equation, we obtain

$$\begin{aligned}(\partial_n^h\varphi_{{\rm s},h}(\bar\mu) - \partial_n^h\varphi_{{\rm s},h}(\bar\mu_h), \bar u - \bar u_h)_\Gamma &= (\partial_n^h\varphi_{{\rm s},h}(\bar\mu) - \partial_n^h\varphi_{{\rm s},h}(\bar\mu_h), \Pi_h\bar u - \bar u_h)_\Gamma\\ &= (\nabla(\varphi_{{\rm s},h}(\bar\mu) - \varphi_{{\rm s},h}(\bar\mu_h)), \nabla(y_h(\bar u) - \bar y_h))\\ &\quad - \langle\bar\mu - \bar\mu_h, y_h(\bar u) - \bar y_h\rangle\\ &= \langle\bar\mu_h - \bar\mu, y_h(\bar u) - \bar y_h\rangle\\ &= \langle\bar\mu^+, \bar y_h - y_h(\bar u)\rangle - \langle\bar\mu^-, \bar y_h - y_h(\bar u)\rangle\\ &\quad + \langle\bar\mu_h^+, y_h(\bar u) - \bar y_h\rangle - \langle\bar\mu_h^-, y_h(\bar u) - \bar y_h\rangle.\end{aligned}$$

For the first two addends we use that $\bar{y} = b$ on $\operatorname{supp}\mu^+$, $\bar{y} = a$ on $\operatorname{supp}\mu^-$, $I_h a \le \bar{y}_h \le I_h b$ and the estimates for the interpolation error to obtain

$$\begin{aligned}
&\langle \bar{\mu}^+, \bar{y}_h - y_h(\bar{u})\rangle - \langle \bar{\mu}^-, \bar{y}_h - y_h(\bar{u})\rangle \\
&\quad \le \langle \bar{\mu}^+, b - y_h(\bar{u})\rangle + \langle \bar{\mu}^+, I_h b - b\rangle + \langle \bar{\mu}^-, -a + y_h(\bar{u})\rangle + \langle \bar{\mu}^-, a - I_h a\rangle \\
&\quad = \langle \bar{\mu}^+, b - \bar{y}\rangle - \langle \bar{\mu}^-, a - \bar{y}\rangle + \langle \bar{\mu}^+ - \bar{\mu}^-, \bar{y} - y_h(\bar{u})\rangle \\
&\qquad + \langle \bar{\mu}^+, I_h b - b\rangle + \langle \bar{\mu}^-, a - I_h a\rangle \\
&\quad = \langle \bar{\mu}, \bar{y} - y_h(\bar{u})\rangle \ + \langle \bar{\mu}^+, I_h b - b\rangle + \langle \bar{\mu}^-, a - I_h a\rangle \\
&\quad \le \|\bar{\mu}\|_{\mathcal{M}(\bar{\Omega}_1)} \Big(\|\bar{y} - y_h(\bar{u})\|_{L^\infty(\Omega_1)} \\
&\qquad\qquad + C h^{2-2/p} (\|a\|_{W^{2,p}(\Omega_1)} + \|b\|_{W^{2,p}(\Omega_1)}) \Big) \ \forall p < p_\Omega.
\end{aligned}$$

To finish, we use that $\bar{y}_h = b$ on $\operatorname{supp}\bar{\mu}_h^+$, $\bar{y} - b \le 0$, $\bar{y}_h = a$ on $\operatorname{supp}\bar{\mu}_h^-$, and $\bar{y} - a \ge 0$ to obtain

$$\begin{aligned}
&\langle \bar{\mu}_h^+, y_h(\bar{u}) - \bar{y}_h\rangle - \langle \bar{\mu}_h^-, y_h(\bar{u}) - \bar{y}_h\rangle \\
&\quad = \langle \bar{\mu}_h^+, y_h(\bar{u}) - b\rangle - \langle \bar{\mu}_h^-, y_h(\bar{u}) - a\rangle \\
&\quad = \langle \bar{\mu}_h^+, y_h(\bar{u}) - \bar{y}\rangle + \langle \bar{\mu}_h^+, \bar{y} - b\rangle - \langle \bar{\mu}_h^-, y_h(\bar{u}) - \bar{y}\rangle - \langle \bar{\mu}_h^-, \bar{y} - a\rangle \\
&\quad \le \langle \bar{\mu}_h, y_h(\bar{u}) - \bar{y}\rangle \\
&\quad \le \|\bar{\mu}_h\|_{\mathcal{M}(\bar{\Omega}_1)} \|\bar{y} - y_h(\bar{u})\|_{L^\infty(\Omega_1)}.
\end{aligned}$$

All together we arrive at

$$\begin{aligned}
&(\partial_n^h \varphi_{s,h}(\bar{\mu}) - \partial_n^h \varphi_{s,h}(\bar{\mu}_h), \bar{u} - \bar{u}_h)_\Gamma \\
&\quad \le (\|\bar{\mu}\|_{\mathcal{M}(\Omega_1)} + \|\bar{\mu}_h\|_{\mathcal{M}(\Omega_1)}) \|\bar{y} - y_h(\bar{u})\|_{L^\infty(\Omega_1)} \\
&\qquad + C h^{2-2/p} (\|a\|_{W^{2,p}(\Omega_1)} + \|b\|_{W^{2,p}(\Omega_1)}) \qquad \forall p < p_\Omega.
\end{aligned}$$

Thanks to the boundedness of $\bar{\mu}_h$ proved in Lemma 5.5, it only remains to estimate $\|\bar{y} - y_h(\bar{u})\|_{L^\infty(\Omega_1)}$. We use the interior error estimates [25, Theorem 5.1], interpolation error estimates, finite element error for non-regular problems (cf. [4, Theorem (12.3.5)], the interior regularity results of Lemma 3.2, Lemma 3.1 and the regularity of the optimal control state of Corollary 4.5. For any open set Ω_2 such that $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$, all $p < p_\Omega$ and $s < \min\{3, s_\Omega\}$ we obtain

$$\begin{aligned}
\|\bar{y} - y_h(\bar{u})\|_{L^\infty(\Omega_1)} &\le C\big(|\log h| \|\bar{y} - I_h \bar{y}\|_{L^\infty(\Omega_2)} + \|\bar{y} - y_h(\bar{u})\|_{L^2(\Omega_2)} \big) \\
&\le C\big(|\log h| h^{2-2/p} \|\bar{y}\|_{W^{2,p}(\Omega_2)} + h^{s-1} \|\bar{y}\|_{H^{s-1}(\Omega)} \big) \\
&\le C\big(|\log h| h^{2-2/p} \|\bar{u}\|_{W^{1-1/p,p}(\Gamma)} + h^{s-1} \|\bar{u}\|_{H^{s-3/2}(\Gamma)} \big). \qquad (32)
\end{aligned}$$

Choosing $s = 3 - 2/p$ (which is smaller than 3 and s_Ω), the proof is complete. Since the result is valid for all $p < p_\Omega$, the $|\log h|$ term can be neglected. □

Theorem 6.4. *Let $\bar{u}$ and $\bar{u}_h$ be the solutions of problems ($\mathbb{P}$) and ($\mathbb{P}_h$), respectively, and suppose that ($\mathbb{P}$) has a regular feasible Slater point, Assumption (H) is satisfied,*

$a, b \in W^{2,p}(\Omega_1)$ for all $p < p_\Omega$ and $\alpha(x) < \bar{u}(x) < \beta(x)$ for all $x \in \Gamma$. Then there exists some $h_0 > 0$ and $C > 0$ such that for all $0 < h < h_0$

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \leq Ch^{1-1/p} \; \forall p < p_\Omega.$$

Proof. Since J is quadratic, we can write

$$\begin{aligned}\frac{\nu}{2}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} &\leq J_h''(u_\xi)(\bar{u} - \bar{u}_h)^2 = J_h'(\bar{u})(\bar{u} - \bar{u}_h) - J_h'(\bar{u}_h)(\bar{u} - \bar{u}_h) \\ &= (-\partial_n^h \varphi_{\mathrm{r},h}(\bar{u}) + \nu\bar{u}, \bar{u} - \bar{u}_h)_\Gamma - (-\partial_n^h \varphi_{\mathrm{r},h}(\bar{u}_h) + \nu\bar{u}_h, \bar{u} - \bar{u}_h)_\Gamma\end{aligned}$$

with some $u_\xi = \bar{u}_h + \xi(\bar{u} - \bar{u}_h)$, $0 \leq \xi \leq 1$. Inserting the term $\pm\partial_n\varphi_{\mathrm{r}}(\bar{u})$ and taking into account that in the absence of control constraints first order optimality conditions read like

$$\nu\bar{u} - \partial_n\varphi_{\mathrm{r}}(\bar{u}) - \partial_n\varphi_{\mathrm{s}}(\bar{\mu}) = 0 \tag{33}$$

$$\nu\bar{u}_h - \partial_n^h\varphi_{\mathrm{r},h}(\bar{u}_h) - \partial_n^h\varphi_{\mathrm{s},h}(\bar{\mu}_h) = 0, \tag{34}$$

we get to

$$\begin{aligned}\nu\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} &\leq (\partial_n\varphi_{\mathrm{s}}(\bar{\mu}) - \partial_n^h\varphi_{\mathrm{s},h}(\bar{\mu}_h), \bar{u} - \bar{u}_h)_\Gamma \\ &\quad + (\partial_n\varphi_{\mathrm{r}}(\bar{u}) - \partial_n^h\varphi_{\mathrm{r},h}(\bar{u}), \bar{u} - \bar{u}_h)_\Gamma \\ &= (\partial_n\varphi_{\mathrm{s}}(\bar{\mu}) - \partial_n^h\varphi_{\mathrm{s},h}(\bar{\mu}), \bar{u} - \bar{u}_h)_\Gamma \\ &\quad + (\partial_n^h\varphi_{\mathrm{s},h}(\bar{\mu}) - \partial_n^h\varphi_{\mathrm{s},h}(\bar{\mu}_h), \bar{u} - \bar{u}_h)_\Gamma \\ &\quad + (\partial_n\varphi_{\mathrm{r}}(\bar{u}) - \partial_n^h\varphi_{\mathrm{r},h}(\bar{u}), \bar{u} - \bar{u}_h)_\Gamma.\end{aligned} \tag{35}$$

The result then follows from (35) and Lemmas 6.2 and 6.3. □

6.2 Control constrained case.

We provide hence a different proof from the one done for the no-control-constrained case, where we use a technique similar to that followed by Meyer in [23] or Rösch and Steinig [24], where we show an order of convergence of $\mathcal{O}(h^{3/4-1/(2p)})$. Before stating and proving the main result of this section, we will collect some auxiliary results. We begin with the error estimates for the $L^2(\Gamma)$ projections.

Lemma 6.5. *The L^2-projection Π_h fulfills the projection error estimates*

$$\|u - \Pi_h u\|_{L^2(\Gamma)} \leq Ch^{1-1/p}\|u\|_{W^{1-1/p,p}(\Gamma)} \tag{36}$$

as well as

$$\|u - \Pi_h u\|_{H^{-1/2}(\Gamma)} \leq Ch^{3/2-1/p}\|u\|_{W^{1-1/p,p}(\Gamma)} \tag{37}$$

for all $u \in W^{1-1/p,p}(\Gamma)$ and all $p < +\infty$.

Proof. Estimate (36) follows from [12, Theorem 2.1] and usual interpolation error estimates. The proof of estimate (37) is a bit more delicate. It involves a duality argument that relies on the approximation property (20).

To shorten notation let us define $\mathcal{F} = \{v \in H^{1/2}(\Gamma)\colon \|v\|_{H^{1/2}(\Gamma)} = 1\}$. Using the definition of Π_h and of the dual norm we may write

$$\begin{aligned}
\|u - \Pi_h u\|_{H^{-1/2}(\Gamma)} &= \sup_{v\in\mathcal{F}} \langle u - \Pi_h u, v\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \\
&= \sup_{v\in\mathcal{F}} (u - \Pi_h u, v) \\
&= \sup_{v\in\mathcal{F}} (u - \Pi_h u, v - \Pi_h v) \\
&\le \sup_{v\in\mathcal{F}} \|u - \Pi_h u\|_{L^2(\Gamma)} \|v - \Pi_h v\|_{L^2(\Gamma)} \\
&\le \sup_{v\in\mathcal{F}} c h^{1-1/p} \|u\|_{W^{1-1/p,p}(\Gamma)} h^{1/2} \|v\|_{H^{1/2}(\Gamma)} \\
&= c h^{3/2-1/p} \|u\|_{W^{1-1/p,p}(\Gamma)},
\end{aligned}$$

and the proof is complete. □

The reader may compare (37) with [23, Eq. (4.2)] or [24, Eq. (3.4)] and wonder why we have not used the norm of $W^{1-1/p,p}(\Gamma)^*$ instead of the norm in $H^{-1/2}(\Gamma)$, which would have lead to an estimate of order $h^{2-2/p}$. The reason is that we will need the continuity of the solution operator into $L^2(\Omega)$ in (44), and this is not possible for data in $W^{1-1/p,p}(\Gamma)^*$.

Le us now state properly the meaning of the state equation for data $u \in H^{-1/2}(\Gamma)$. We will say that $y = Su$ if

$$\int_\Omega y \Delta z dx = \langle u, \partial_n z\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \ \forall z \in H^2(\Omega) \cap H^1_0(\Gamma).$$

Since $z = 0$ on Γ, $\partial_n z \in H^{1/2}(\Gamma)$ and the definition makes sense (see [9, Lemma A.2]).

Lemma 6.6. *The control-to-state-mapping $Su = y_u$ is well defined and continuous from $H^{-1/2}(\Gamma)$ to $L^2(\Omega)$. For any open set $\Omega' \subset\subset \Omega$, it is also continuous from $H^{-1/2}(\Gamma)$ to $C(\bar\Omega')$.*

Proof. The proof of the first part follows the usual duality argument. To shorten notation, let us denote $\mathcal{F} = \{f \in L^2(\Omega)\colon \|f\|_{L^2(\Omega)} = 1\}$ and for every $f \in L^2(\Omega)$, let z be the unique element in $H^2(\Omega) \cap H^1_0(\Omega)$ such that $-\Delta z = f$ in Ω. Then

$$\begin{aligned}
\|y\|_{L^2(\Omega)} &= \sup_{f\in\mathcal{F}} \int_\Omega y f dx = \sup_{f\in\mathcal{F}} -\langle u, \partial_n z\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \\
&\le \sup_{f\in\mathcal{F}} \|u\|_{H^{-1/2}(\Gamma)} \|\partial_n z\|_{H^{1/2}(\Gamma)} \\
&\le \sup_{f\in\mathcal{F}} C\|u\|_{H^{-1/2}(\Gamma)} \|f\|_{L^2(\Omega)} = C\|u\|_{H^{-1/2}(\Gamma)}.
\end{aligned}$$

The interior regularity can be proven using a similar discussion to that of Lemma 3.3 and a bootstrap argument like in Lemma 3.5 □

Lemma 6.7. *Suppose that* $(\mathbb{P})$ *has a regular feasible Slater point and that Assumption (H) is satisfied. Then the sequence of discrete optimal controls* $\bar{u}_h$ *of Problem* $(\mathbb{P}_h)$ *is bounded in the* $W^{1-1/p,p}(\Gamma)$*-norm independently of* h *for all* $p < p_\Omega$*.*

Proof. For the proof we refer to [11, Theorem 6.2]. This proof is based on the stability of the $L^2(\Gamma)$-projections in $W^{1-1/p,p}(\Gamma)$ stated in [12] and can be adapted with the obvious changes starting with our Lemma 6.2. □

Lemma 6.8. *Suppose that* $(\mathbb{P})$ *has a regular feasible Slater point. Let* $\bar{u}_h$ *be the optimal control of* $(\mathbb{P}_h)$*. There exists a sequence* $u^* = u^*(h)$ *of controls, uniformly bounded in* $W^{1-1/p,p}(\Gamma)$ *for all* $p < p_\Omega$*, that are feasible for* $(\mathbb{P})$*, and a constant* $C > 0$ *independent of* h *such that*

$$\|\bar{u}_h - u^*(h)\|_{H^{-1/2}(\Gamma)} \leq Ch^{3/2-1/p} \;\forall p < p_\Omega. \tag{38}$$

Proof. For $h > 0$ consider $u_{h0} = \Pi_h u_0$ the discrete Slater point introduced in Theorem 5.2. For $\kappa = \kappa(h)$ to be determined define the auxiliary control

$$u^* = \bar{u}_h + \kappa(u_{h0} - \bar{u}_h).$$

The boundedness of the sequences u^* follows directly from Lemma 6.7 and the stability of Π_h stated in [12, Theorem 2.3]. Then, clearly the error $\|\bar{u}_h - u^*\|_{H^{-1/2}(\Gamma)}$ is determinded by $\kappa(h)$. Notice, for instance,

$$u^* = (1-\kappa)\bar{u}_h + \kappa u_{h0} \leq (1-\kappa)\beta + \kappa(\beta - \delta_h) = \beta - \kappa\delta_h \leq \beta,$$

where δ_h is introduced in Definition 5.1. Repeating these calculations for the lower bound results in feasibility of u^* with respect to the control constraints. To check feasibility regarding the state constraints, observe that in $\bar{\Omega}_1$ we have

$$\begin{aligned} y(u^*) &= y_h(u^*) + y(u^*) - y_h(u^*) \\ &\leq (1-\kappa)y_h(\bar{u}_h) + \kappa y_h(u_{h0}) + \|y(u^*) - y_h(u^*)\|_{L^\infty(\Omega_1)}. \end{aligned}$$

Similar to (32), we obtain with [11, Theorem 5.4]

$$\begin{aligned} &\|y(u^*) - y_h(u^*)\|_{L^\infty(\Omega_1)} \\ &\quad\leq C\big(|\log h| \|y(u^*) - I_h y(u^*)\|_{L^\infty(\Omega_2)} + \|y(u^*) - y_h(u^*)\|_{L^2(\Omega)}\big) \\ &\quad\leq C\big(|\log h| h^{2-2/p} \|y(u^*)\|_{W^{2,p}(\Omega_2)} + h^{3/2-1/p}\|u^*\|_{W^{1-1/p,p}(\Gamma)}\big) \\ &\quad\leq C\big(|\log h| h^{2-2/p} \|u^*\|_{W^{1-1/p,p}(\Gamma)} + h^{3/2-1/p}\|u^*\|_{W^{1-1/p,p}(\Gamma)}\big) \\ &\quad\leq Ch^{3/2-1/p}\|u^*\|_{W^{1-1/p,p}(\Gamma)}. \end{aligned} \tag{39}$$

Taking into account that $u^*(h)$ is bounded in $W^{1-1/p,p}(\Gamma)$, all the estimates yield

$$\begin{aligned} y(u^*) &= y_h(u^*) + y(u^*) - y_h(u^*) \\ &\leq (1-\kappa)b + \kappa(b - \varepsilon_h) + Ch^{3/2-1/p} \\ &\leq b - \kappa\varepsilon_h + Ch^{3/2-1/p}. \end{aligned}$$

Noting that for h small enough, $\varepsilon_h > 0$ is independent of h (cf. Theorem 5.2), we obtain for $\kappa = Ch^{3/2-1/p}/\varepsilon_h$ feasibility with respect to the upper bound. Analogous calculations for the lower bound and the definition of $\kappa = \kappa(h)$ yields the assertion including the required error estimate. □

Lemma 6.9. *Suppose that* $(\mathbb{P})$ *has a regular feasible Slater point. Let* $\bar{u}$ *be the optimal control of* $(\mathbb{P})$. *There exists a sequence* u_h^* *of controls, uniformly bounded in* $W^{1-1/p,p}(\Gamma)$ *for all* $p < p_\Omega$, *that are feasible for* $(\mathbb{P}_h)$ *and a constant* $C > 0$ *independent of* h *such that*

$$\|\bar{u} - u_h^*\|_{L^2(\Gamma)} \le Ch^{1-1/p} \ \forall p < p_\Omega, \tag{40}$$

$$\|\bar{u} - u_h^*\|_{H^{-1/2}(\Gamma)} \le Ch^{3/2-1/p} \ \forall p < p_\Omega. \tag{41}$$

Proof. The proof is similar to the one of Lemma 6.8. Define

$$u_h^* = \Pi_h \bar{u} + \kappa(u_{h0} - \Pi_h \bar{u}),$$

and note that $u_{h0} = \Pi_h u_0$. The boundedness of the sequence u_h^* follows again directly from the stability of Π_h stated in [12, Theorem 2.3]. Obviously, for κ sufficiently small, $u_h^* \in U_{\alpha,\beta,h}$ is instantly verified. To discuss the state constraints in the interior of Ω, by means of the projection error estimate from Lemma 6.5 together with the interior regularity result in Lemma 6.6 and estimate (37), and the interior L^∞-error estimate for the state which is obtained as in the proof of the previous lemma, see (39), we obtain

$$\begin{aligned} y_h(u_h^*) &= y(u_h^*) + y_h(u_h^*) - y(u_h^*) \\ &= (1-\kappa)\bar{y} + \kappa y(u_0) + (1-\kappa) y(\Pi_h \bar{u} - \bar{u}) \\ &\quad + \kappa y(\Pi_h u_0 - u_0) + y_h(u_h^*) - y(u_h^*) \\ &\le (1-\kappa) b + \kappa(b - \varepsilon) + C(1-\kappa) h^{3/2-1/p} \|\bar{u}\|_{W^{1-1/p,p}(\Gamma)} \\ &\quad + C\kappa h^{3/2-1/p} \|u_0\|_{W^{1-1/p,p}(\Gamma)} + Ch^{3/2-1/p} \|u_h^*\|_{W^{1-1/p,p}(\Gamma)} \\ &\le b - \kappa\varepsilon + Ch^{3/2-1/p}, \end{aligned} \tag{42}$$

and thus we may choose $\kappa = Ch^{3/2-1/p}/\varepsilon$. To obtain the estimates (40) and (41), we use that $\kappa \le Ch^{3/2-1/p}$ and (36) and (37), respectively.

$$\|u_h^* - \bar{u}\|_{L^2(\Gamma)} \le \|\Pi_h \bar{u} - \bar{u}\|_{L^2(\Gamma)} + Ch^{3/2-1/p} \|u_{h0} - \Pi_h \bar{u}\|_{L^2(\Gamma)} \le Ch^{1-1/p}$$

as well as

$$\begin{aligned} \|u_h^* - \bar{u}\|_{H^{-1/2}(\Gamma)} &\le \|\Pi_h \bar{u} - \bar{u}\|_{H^{-1/2}(\Gamma)} + Ch^{3/2-1/p} \|u_{h0} - \Pi_h \bar{u}\|_{H^{-1/2}(\Gamma)} \\ &\le Ch^{3/2-1/p}. \end{aligned}$$

□

Lemma 6.10. *There exists* $C > 0$ *such that the following estimate holds:*

$$\|y(\bar{u}_h) - y_h(\bar{u}_h)\|_{L^2(\Omega)} + \|y(u_h^*) - y_h(u_h^*)\|_{L^2(\Omega)} \le Ch^{3/2-1/p} \ \forall p < p_\Omega. \tag{43}$$

Proof. The assertion follows from the error estimate for semilinear equations in [11, Theorem 5.4] and the uniform bounds stated in Lemmas 6.7 and 6.9.

$$
\begin{aligned}
\|y(\bar u_h) - y_h(\bar u_h)\|_{L^2(\Omega)} &+ \|y(u_h^*) - y_h(u_h^*)\|_{L^2(\Omega)} \\
&\le C h^{3/2-1/p}\left(\|\bar u_h\|_{W^{1-1/p,p}(\Gamma)} + \|u_h^*\|_{W^{1-1/p,p}(\Gamma)}\right) \\
&\le C h^{3/2-1/p}.
\end{aligned}
$$

□

With the preceding results, we are now in the position to prove our error estimates in the control-constrained case.

Theorem 6.11. *Let $\bar u$ and $\bar u_h$ be the solutions of problems $(\mathbb{P})$ and $(\mathbb{P}_h)$, respectively, and suppose that $(\mathbb{P})$ has a regular feasible Slater point and Assumption (H) is satisfied. Then there exists some $h_0 > 0$ and $C > 0$ such that for all $0 < h < h_0$*

$$
\|\bar u - \bar u_h\|_{L^2(\Gamma)} \le C h^{\frac{3}{4} - \frac{1}{2p}} \ \forall p < p_\Omega.
$$

Proof. We follow closely the technique of proof in [23], Lemma 7 and Theorem 3. We use the auxiliary controls u^* and u_h^* from Lemmas 6.8 and 6.9, that are feasible for Problems $(\mathbb{P})$ and $(\mathbb{P}_h)$, respectively, to test the variational inequalities for $(\mathbb{P})$ and $(\mathbb{P}_h)$. This leads to

$$
\begin{aligned}
0 &\le (\nu \bar u - \partial_n \bar\varphi, u^* - \bar u)_\Gamma &&= \nu(\bar u, u^* - \bar u)_\Gamma + (\bar y - y_d, y(u^* - \bar u)) \\
0 &\le (\nu \bar u_h - \partial_n^h \bar\varphi_h, u_h^* - \bar u_h)_\Gamma &&= \nu(\bar u_h, u_h^* - \bar u_h)_\Gamma + (\bar y_h - y_d, y_h(u_h^* - \bar u_h)),
\end{aligned}
$$

where the Lagrange multiplier terms disappear because of feasibility of u^* and u_h^* with respect to the state constraints. Then, adding both inequalities and straight forward computations lead to

$$
\begin{aligned}
0 \le &- \nu\|\bar u - \bar u_h\|_{L^2(\Gamma)}^2 + \nu\left((\bar u, u^* - \bar u_h)_\Gamma + (\bar u, u_h^* - \bar u)_\Gamma + (\bar u_h - \bar u, u_h^* - \bar u)_\Gamma\right) \\
&- \|\bar y_h - \bar y\|_{L^2(\Omega)}^2 + (y_h(\bar u_h) - y(\bar u), y_h(u_h^*) - y(u_h^*) + y(u_h^*) - y(\bar u)) \\
&+ (y(\bar u) - y_d, y(u^*) - y(\bar u_h) + y(u_h^*) - y(\bar u) \\
&+ y(\bar u_h) - y_h(\bar u_h) + y_h(u_h^*) - y(u_h^*)).
\end{aligned}
$$

Rearranging terms and estimating the right-hand-side of the last inequality further, we arrive at

$$
\begin{aligned}
&\frac{\nu}{2}\|\bar u-\bar u_h\|^2_{L^2(\Gamma)}+\frac{1}{2}\|\bar y-\bar y_h\|^2_{L^2(\Omega)}\\
&\quad\le \nu\|\bar u\|_{H^{1/2}(\Gamma)}\Big(\|u^*-\bar u_h\|_{H^{-1/2}(\Gamma)}+\|u_h^*-\bar u\|_{H^{-1/2}(\Gamma)}\Big)\\
&\qquad+\frac{\nu}{2}\|u_h^*-\bar u\|^2_{L^2(\Gamma)}+\|y_h(u_h^*)-y(u_h^*)\|^2_{L^2(\Omega)}+\|y(u_h^*)-y(\bar u)\|^2_{L^2(\Omega)}\\
&\qquad+\|y(\bar u)-y_d\|_{L^2(\Omega)}\big(\|y(u^*)-y(\bar u_h)\|_{L^2(\Omega)}+\|y(u_h^*)-y(\bar u)\|_{L^2(\Omega)}\big)\\
&\qquad+\|y(\bar u)-y_d\|_{L^2(\Omega)}\big(\|y(\bar u_h)-y_h(\bar u_h)\|_{L^2(\Omega)}+\|y_h(u_h^*)-y(u_h^*)\|_{L^2(\Omega)}\big)\\
&\quad\le \frac{\nu}{2}\|u_h^*-\bar u\|^2_{L^2(\Gamma)}+C\|u_h^*-\bar u\|^2_{H^{-1/2}(\Gamma)}+\|y(u_h^*)-y_h(u_h^*)\|^2_{L^2(\Omega)}\\
&\qquad+\Big(\nu\|\bar u\|_{H^{1/2}(\Gamma)}+C\|y(\bar u)-y_d\|_{L^2(\Omega)}\Big)\\
&\qquad\qquad\Big(\|u^*-\bar u_h\|_{H^{-1/2}(\Gamma)}+\|u_h^*-\bar u\|_{H^{-1/2}(\Gamma)}\Big)\\
&\qquad+\|y(\bar u)-y_d\|_{L^2(\Omega)}\big(\|y(\bar u_h)-y_h(\bar u_h)\|_{L^2(\Omega)}+\|y_h(u_h^*)-y(u_h^*)\|_{L^2(\Omega)}\big)\,,
\end{aligned}
$$

where we applied in particular Young's inequality, the Cauchy-Schwarz inequality, as well as the estimate

$$
\|y(u_h^*)-y(\bar u)\|_{L^2(\Omega)}\le C\|u_h^*-\bar u\|_{H^{-1/2}(\Gamma)}. \tag{44}
$$

which follows from Lemma 6.6.

We now use estimates (38), (40), (41) and (43). Collecting all estimates yields the assertion after taking the square root. □

6.3 Comparison between the two methods of proof

Let us end this manuscript with a short comment on the different methods of proof in Sections 6.1 and 6.2. If we try to write the proof of Subsection 6.2 for the non-control-constrained case, and we want to get an order $\mathcal{O}(h^{1-1/p})$ as we obtained in Subsection 6.1, somehow we should use the norm in $H^{s-3/2}(\Gamma)$ $(s<3, s<s_\Omega)$ instead of the norm in $W^{1-1/p,p}(\Gamma)$. Indeed, the optimal control has that regularity, which would improve the error for the L^2-projection, estimate (37). But to improve the FEM estimates (39), (42), and (43), using the same technique as in (32), we would need the norm in $H^{s-3/2}(\Gamma)$ of the discrete optimal controls to be bounded, as we state in Lemma 6.7 for the norm in $W^{1-1/p,p}(\Gamma)$. To have that bound, we would have to prove stability of Π_h in $H^{s-3/2}(\Gamma)$, (this is not proved in [12] but it can be proven with the same technique used therein) and an error estimate for the approximation of the adjoint state analogous to that of Lemma 6.2. The key point is that we are not able to improve the order of convergence $\mathcal{O}(h^{1-1/p})$ in (31), so the subsequent argument in the proof of Theorem 6.2 in [11], which eventually uses an inverse estimate, would not lead to the desired result.

On the other hand, to adapt the method of Section 6.1 to the control constrained case, we have to use the inequality form of the first order necessary conditions (16) and (23d) instead of (33) and (34). One idea to compare both inequalities is to use the interpolate introduced by Casas and Raymond cf. [11, Equation (7.9)] as test function, but this only leads to an order of $\mathcal{O}(h^{1/2-1/(2p)})$. The main reason for this is that in the analogous of Lemma 6.3, we would find the term $\|\bar{y} - y_h(u_h^{\rm CR})\|_{L^\infty(\Omega_1)}$, where $u_h^{\rm CR}$ is the afore-mentioned interpolate, which will be bounded by the finite element error estimate plus the interpolation error $\|\bar{u} - u_h^{\rm CR}\|_X$ in some appropriate norm. The finite element error is of order $\mathcal{O}(h^{3/2-1/p})$ (in contrast to the unconstrained case, where it is $\mathcal{O}(h^{2-2/p})$ due to the higher regularity of the control as shown in (32)), but the interpolation error $\|\bar{u} - u_h^{\rm CR}\|_{L^2(\Omega)}$ is of order $\mathcal{O}(h^{1-1/p})$ (cf. [11, Lemma 7.5]). To obtain a final order of $\mathcal{O}(h^{3/4-1/(2p)})$, it would be enough to prove that $\|\bar{u} - u_h^{\rm CR}\|_{H^{-1/2}(\Gamma)} \leq Ch^{3/2-1/p}$, but we have not been able to prove such an estimate. The key difference is that with the technique used in Section 6.2 we are able to use the L^2-projection instead of the Casas and Raymond interpolate, and we obtain an interpolation error in $H^{-1/2}(\Gamma)$ of order $\mathcal{O}(h^{3/2-1/p})$ (see eq. (37)). Notice also that we do not need to assume $a, b \in W^{2,p}(\Omega_1)$ to obtain the final error estimate in Theorem 6.11.

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