NONLINEAR AND LINEAR HYPERBOLIC SYSTEMS WITH DYNAMIC BOUNDARY CONDITIONS

Gilbert Peralta[†,*] and Georg Propst[*]

[†]Department of Mathematics and Computer Science
University of the Philippines Baguio
Governor Pack Road, Baguio City 2600, Philippines

[*]Institut für Mathematik und Wissenschaftliches Rechnen,
NAWI Graz, Karl-Franzens-Universität Graz
Heinrichstraße 36, A-8010 Graz, Austria

Abstract. We consider first order hyperbolic systems on an interval with dynamic boundary conditions. The well-posedness for linear systems is established using a variational method. The linear theory is used to analyze the local-in-time well-posedness for nonlinear systems. The results are applied to a model describing the flow of an incompressible fluid inside an elastic tube whose ends are attached to tanks. Global existence and stability for data that are smooth enough and close to the steady state are obtained using energy and entropy methods.

1. Introduction

The paper is concerned with first order hyperbolic systems on a bounded interval with dynamic boundary conditions that have the following form

$$\begin{cases} u_t(t,x) + A(u(t,x))u_x(t,x) = f(u(t,x)), & t > 0,\ 0 < x < 1, \\ B_0 u(t,0) = b_0(t,h(t)), & t > 0, \\ B_1 u(t,0) = b_1(t,h(t)), & t > 0, \\ h'(t) = H(t,h(t),u(t,0),u(t,1)), & t > 0, \\ u(0,x) = u_0(x), & 0 < x < 1, \\ h(0) = h_0. \end{cases} \tag{1.1}$$

This system occurs when the dynamics at the boundary interact with the waves in the interior. If H is independent of h then (1.1) includes system of balance laws with nonlocal boundary conditions. The dimensions of the constant boundary matrices B_0 and B_1 are $p \times n$ and $(n-p) \times n$, respectively, with $0 \le p \le n$ an integer to be specified below. For the functions $A : \mathcal{U} \to \mathbb{R}^{n\times n}$, $f : \mathcal{U} \to \mathbb{R}^n$, $b_0 : \mathbb{R} \times \mathcal{H} \to \mathbb{R}^p$, $b_1 : \mathbb{R} \times \mathcal{H} \to \mathbb{R}^{n-p}$ and $H : \mathbb{R} \times \mathcal{H} \times \mathbb{R}^{2n} \to \mathbb{R}^d$, where $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{H} \subset \mathbb{R}^d$ are open and convex, we assume that they are infinitely differentiable. We are interested in the well-posedness of the system (1.1) in a Sobolev space H^m for an integer $m \ge 3$ under suitable assumptions.

With regards to the PDE part, we assume the following standard hypotheses for hyperbolic systems, see [**2, 12**].

Friedrichs Symmetrizability. There exists a symmetric positive-definite matrix-valued function $S \in \mathscr{C}^\infty(\mathcal{U};\mathbb{R}^{n\times n})$, called the *Friedrichs symmetrizer*, that is bounded as well as its derivatives, $S(w)A(w)$ is symmetric for all $w \in \mathcal{U}$, and there exists $\alpha > 0$ such that $S(w) \geq \alpha I_n$ for all $w \in \mathcal{U}$.

Diagonalizability. For each $w \in \mathcal{U}$, $A(w)$ is diagonalizable with p positive eigenvalues and $n - p$ negative eigenvalues. In particular, $A(w)$ is invertible and has n independent eigenvectors.

Uniform Kreiss-Lopatinskiĭ Condition. There exists $C > 0$ such that for all $w \in \mathcal{U}$, there holds $\|V\| \leq C\|B_0 V\|$ for all $V \in E^u(A(w))$ and $\|W\| \leq C\|B_1 W\|$ for all $W \in E^s(A(w))$. Here, $E^u(A)$ and $E^s(A)$ denote the unstable and stable subspaces of a matrix A, respectively.

We note that due to the diagonalizability, the system is non-characteristic. For initial-boundary value problems associated with hyperbolic equations, care should be taken in imposing the boundary conditions in order for the problem not to be underdetermined or overdetermined. For diagonal systems, the number of boundary conditions should be equal to the number of incoming characteristics and the states corresponding to these should be imposed. For systems that are not diagonal, the Uniform Kreiss-Lopatinskiĭ Condition stated above provides the appropriate form of the boundary conditions. In the case of half-space, the UKL condition implies the decay at infinity $x \to +\infty$ for solutions of linear hyperbolic systems of the form $e^{\lambda t}U(x)$ where λ has a positive real part, refer to [**4**].

The well-posedness of (1.1) is established by linearizing the system and using an iterative scheme. There are several ways to perform the linearization. A successful approach is to freeze the states u and h appearing in A, f and H, while retaining the coupling on the boundary conditions. In line with this, we will discuss a linear version of (1.2), namely,

$$\begin{cases} u_t(t,x) + A(v(t,x))u_x(t,x) + R(t,x)u(t,x) = f(t,x), \\ B_0 u(t,0) = g_0(t) + Q_0(t)h(t), \\ B_1 u(t,1) = g_1(t) + Q_1(t)h(t), \\ h'(t) = H(t)h(t) + G_0(t)u(t,0) + G_1(t)u(t,1) + S(t), \\ u(0,x) = u_0(x), \\ h(0) = h_0, \end{cases} \tag{1.2}$$

for $0 < t < T$ and $0 < x < 1$ and appropriate matrices A, R, Q_i, S_i and H. The coefficient v is assumed to be at least Lipschitz. The goal is to prove the existence and uniqueness of weak solutions of (1.2) in L^2. Writing the system in variational form, so as to apply Friedrichs method, a problem occurs in eliminating the traces $u_{|x=0}$ and $u_{|x=1}$ in the differential equation for h due to possible limited regularity of G_0 and G_1. In fact, we only consider the case where they are bounded. This will be done by considering test functions in a certain graph space.

In Section 2, we will define weak solutions of the linear system (1.2) and sketch the proof of well-posedness and trace regularity. The nonlinear system (1.1) will be the focus of Section 3. We apply the results to a model describing the flow of a fluid in an elastic tube and outline the proof of global existence and stability.

2. Linear Hyperbolic PDE-ODE Systems

For the linear system (1.2) we assume that $v \in W^{1,\infty}(Q_T)^n$, $R \in L^\infty(Q_T)^{n\times n}$, $Q_0 \in L^\infty(0,T)^{p\times d}$, $Q_1 \in L^\infty(0,T)^{(n-p)\times d}$, $H \in L^\infty(0,T)^{d\times d}$, $S \in L^2(0,T)^d$ and $G_0, G_1 \in L^\infty(0,T)^{d\times n}$ where $Q_T = (0,T) \times (0,1)$ is the time-space domain.

The two main ingredients in writing the linear system into a variational form is the choice of test functions and an appropriate decomposition of the flux matrix in terms of the boundary matrices. For the latter, we note that there exist $N_0 \in \mathbb{R}^{(n-p)\times n}$, $N_1 \in \mathbb{R}^{p\times n}$, $C_0, M_1 \in \mathscr{C}^\infty(\mathcal{U};\mathbb{R}^{(n-p)\times n})$ and $C_1, M_0 \in \mathscr{C}^\infty(\mathcal{U};\mathbb{R}^{p\times n})$ such that

$$A(w) = M_y(w)^T B_y + C_y(w)^T N_y, \quad \text{for all } (w,y) \in \mathcal{U} \times \{0,1\}.$$

This is due to the hypothesis that B_0 and B_1 have full ranks, see [**4**]. In fact, N_0 is chosen so that $\binom{B_0}{N_0}$ is invertible with inverse $(Y_0\ D_0)$ where $Y_0 \in \mathbb{R}^{n\times p}$ and $D_0 \in \mathbb{R}^{n\times(n-p)}$ and one can take $M_0 = (AY_0)^T$ and $C_0 = (AD_0)^T$.

Let $L = \partial_t + A(v)\partial_x + R$ and $E(Q_T) = \{u \in L^2(Q_T)^n : Lu \in L^2(Q_T)^n\}$. Define $\tilde{A} = \nu_x + A^{-T}\nu_t$ where $\nu = (\nu_t, \nu_x)$ is the unit outward normal to ∂Q_T. The linear map $u \mapsto \tilde{A}u_{|\partial Q_T}$ from $H^1(Q_T)^n$ into $L^2(\partial Q_T)^n$ can be extended uniquely into a bounded linear operator from the graph space $E(Q_T)$ into $H^{-\frac{1}{2}}(\partial Q_T)$, see [**1, 9**]. Consider the subspace $\mathcal{E}(Q_T)$ of $E(Q_T)$ defined as the closure of $H^1(Q_T)^n$ with respect to the norm

$$\|u\|^2_{\mathcal{E}(Q_T)} = \|u\|^2_{L^2(Q_T)^n} + \|Lu\|^2_{L^2(Q_T)^n} + \|u_{|\partial Q_T}\|^2_{L^2(\partial Q_T)^n}.$$

It follows immediately from the definition that $u_{|\partial Q_T} \in L^2(Q_T)^n$ for every $u \in \mathcal{E}(Q_T)$. The spaces $E^*(Q_T)$ and $\mathcal{E}^*(Q_T)$ are defined analogously where L is replaced by the formal adjoint $L^* = -\partial_t - A^T\partial_x + R^T$ of L.

Given $f \in L^2(Q_T)^n$, $g_0 \in L^2(0,T)^p$, $g_1 \in L^2(0,T)^{n-p}$, $S \in L^2(0,T)^d$, $u_0 \in L^2(0,1)^n$ and $h_0 \in \mathbb{R}^d$, a pair of functions $(u,h) \in L^2(Q_T)^n \times L^2(0,T)^d$ is called a *weak solution* of the system (1.2) if the variational equation

$$\begin{aligned}
&\int_0^T\int_0^1 u(t,x)\cdot L^*\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t - \int_0^1 u_0(x)\cdot\varphi(0,x)\,\mathrm{d}x + h_0\cdot\eta(0)\\
&+\int_0^T h(t)\cdot(\eta'(t) + \tilde{H}(t)\eta(t) + Q_1(t)^T M_1(t)\varphi(t,1) - Q_0(t)^T M_0(t)\varphi(t,0))\,\mathrm{d}t\\
&= \int_0^T\int_0^1 f(t,x)\cdot\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T g_1(t)\cdot(M_1(t)\varphi(t,1) + (G_1(t)Y_1)^T\eta(t))\,\mathrm{d}t\\
&+\int_0^T g_0(t)\cdot(M_0(t)\varphi(t,0) - (G_0(t)Y_0)^T\eta(t))\,\mathrm{d}t - \int_0^T S(t)\cdot\eta(t)\,\mathrm{d}t \qquad (2.1)
\end{aligned}$$

where $\tilde{H} = (H + G_1Y_1Q_1 + G_0Y_0Q_0)^T$, holds for all $\varphi \in \mathcal{E}^*(Q_T)$ and for all $\eta \in H^1(0,T)^d$ such that $\varphi(T,\cdot) = 0$, $\eta(T) = 0$, $C_1\varphi_{|x=1} = -(G_1D_1)^T\eta$ and $C_0\varphi_{|x=0} = (G_0D_0)^T\eta$. This variational form is obtained by multiplying the differential equations by the appropriate test functions and then integrating by parts.

Since G_0 and G_1 are in L^∞, the functions $(G_1D_1)^T\eta$ and $(G_0D_0)^T\eta$ may be only in L^2 even for $\eta \in H^1(0,T)^d$. In order for the compatibility conditions $C_1\varphi_{|x=1} = -(G_1D_1)^T\eta$ and $C_0\varphi_{|x=0} = (G_0D_0)^T\eta$ to be meaningful, we take the space $\mathcal{E}^*(Q_T)$ to be the space for the first component instead of the space $H^1(Q_T)^n$ used in hyperbolic systems.

The existence of a weak solution is obtained from the following result in [**14**] generalizing the one given in [**8**], see also [**2, 5, 10**]. The proof is based on the Hahn-Banach and Riesz representation theorems.

Theorem 2.1. *Let X and Z be Hilbert spaces, Y be a subspace of X and $\Lambda : Y \to X$, $\Psi : Y \to Z$, $\Phi : Y \to Z$ be linear operators. Suppose that $W = \ker(\Phi)$ and $\Lambda(W)$ are nontrivial. If there exist $\gamma > 0$ and $C > 0$ such that*

$$\gamma\|w\|_X^2 + \|\Psi w\|_Z^2 \leq C(\gamma^{-1}\|\Lambda w\|_X^2 + \|\Phi w\|_Z^2), \quad \text{for all } w \in Y,$$

then the variational equation

$$(u, \Lambda w)_X = (F, w)_X + (G, \Psi)_Z, \quad \text{for all } w \in W, \tag{2.2}$$

for a given $(F, G) \in X \times Z$ has a solution $u \in X$. In addition, the solution is unique if and only if $\Lambda(W)$ is dense in X.

Introducing the weighted-in-time spaces $X = e^{-\gamma t}L^2(Q_T)^n \times e^{-\gamma t}L^2(0,T)^d$, $Y = \mathcal{E}^*(Q_T) \times H^1(0,T)^d$ and $Z = e^{-\gamma t}L^2(0,T)^{n-p} \times e^{-\gamma t}L^2(0,T)^p \times L^2(0,1)^n \times \mathbb{R}^d$, it is not hard to see that (2.1) can be written in the form (2.2). Therefore the first step in establishing well-posedness is to derive a priori estimates. For the ODE part, we have the following Poincaré-type inequality in [**14**]. Given $H \in L^\infty(0,T)^{d\times d}$ there are constants $C > 0$ and $\gamma_0 \geq 1$, both depending only on the L^∞-norm of H, such that

$$|\eta(0)|^2 + \gamma\|e^{\gamma t}\eta\|_{L^2(0,T)^d}^2 \leq C\left(\frac{1}{\gamma}\|\eta' + H\eta\|_{L^2(0,T)^d}^2 + e^{2\gamma T}|\eta(T)|^2\right) \tag{2.3}$$

holds for every $\eta \in H^1(0,T)^d$ and $\gamma \geq \gamma_0$.

On the other hand, for the PDE part, due to the assumptions stated in the introduction, there exist $C > 0$ and $\gamma_0 \geq 1$, both depending only on the $W^{1,\infty}$-norm and range of v and the L^∞-norm of R, such that

$$\begin{aligned}\|w_{|t=0}\|_{L^2(0,1)^n}^2 + \gamma\|e^{\gamma t}w\|_{L^2(Q_T)^n}^2 + \|e^{\gamma t}w_{|x=0}\|_{L^2(0,T)^n}^2 + \|e^{\gamma t}w_{|x=1}\|_{L^2(0,T)^n}^2 \\ \leq C\bigg(e^{2\gamma T}\|w_{|t=T}\|_{L^2(0,1)^n}^2 + \frac{1}{\gamma}\|e^{\gamma t}L^*w\|_{L^2(Q_T)^n}^2 \\ + \|e^{\gamma t}C_0(v)w_{|x=0}\|_{L^2(0,T)^{n-p}}^2 + \|e^{\gamma t}C_1(v)w_{|x=1}\|_{L^2(0,T)^p}^2\bigg)\end{aligned} \tag{2.4}$$

holds for all $w \in \mathcal{E}^*(Q_T)$ and $\gamma \geq \gamma_0$. The proof of this a priori estimate can be found in [**2, 5, 10**] in the case of where $w \in H^1(Q_T)^n$. The fact that it holds also on the space $\mathcal{E}^*(Q_T)$ follows by a density argument.

Combining the a priori estimates (2.3) and (2.4) with an absorption argument, Theorem 2.1 implies the following result. The interior-point trace regularity can be shown using standard multiplier techniques. For the proof, we refer to [**14**].

Theorem 2.2. *The system (1.2) has a unique weak solution $(u, h) \in L^2(Q_T)^n \times L^2(0,T)^d$. Furthermore, $(u, h) \in [C([0,T], L^2(0,1)^n) \cap \mathcal{E}(Q_T)] \times H^1(0,T)^d$ and $u_{|x=\xi} \in L^2(0,T)^n$ for every $\xi \in [0,1]$. The weak solution satisfies the estimate*

$$\begin{aligned}&e^{-2\gamma T}\|u\|_{C([0,T],L^2(0,1)^n)}^2 + \gamma\|e^{-\gamma t}u\|_{L^2(Q_T)^n}^2 + \|e^{-\gamma t}u_{|x=0}\|_{L^2(0,T)^n}^2 \\ &+ \|e^{-\gamma t}u_{|x=1}\|_{L^2(0,T)^n}^2 + \gamma\|e^{-\gamma t}h\|_{L^2(0,T)^d}^2 \leq C\bigg(\|u_0\|_{L^2(0,1)^n}^2 + |h_0|^2 \\ &+ \frac{1}{\gamma}\|e^{-\gamma t}f\|_{L^2(Q_T)^n}^2 + \|e^{-\gamma t}g_0\|_{L^2(0,T)^p}^2 + \|e^{-\gamma t}g_1\|_{L^2(0,T)^{n-p}}^2 + \frac{1}{\gamma}\|e^{-\gamma t}S\|_{L^2(0,T)^d}^2\bigg)\end{aligned}$$

for all $\gamma \geq \gamma_0$ for some $C > 0$ and $\gamma_0 \geq 1$.

For the constant coefficient case, it can be shown that the weak solution introduced above is equivalent to the one given by semigroup theory. However, proving that the associated differential operator generates a strongly continuous semigroup is a difficult task, see [**14**] for the discussion.

3. Nonlinear Systems and Application

For the nonlinear system (1.1), the main result is the local-in-time well-posedness and a blow-up criterion in finite time. The proof is technically long and for this reason we only outline the main ideas and refer to [**12**] for the details. In the following, we use the notation $CH^m(Q_T) = \bigcap_{j=0}^m C^j([0,T], H^{m-j}(0,1)^n)$.

Theorem 3.1. *Let $m \geq 3$ be an integer and $(u_0, h_0) \in H^m(0,1)^n \times \mathcal{H}$ satisfy appropriate compatibility conditions. Suppose that the range of u_0 lies in $\mathcal{K}_1 \subset \mathcal{U}$, $h_0 \in \mathcal{G}_1 \subset \mathcal{H}$, where $\mathcal{K}_1$ and $\mathcal{G}_1$ are compact and convex, and $\|u_0\|_{H^m(0,1)^n} \leq M$. Then there exists $T > 0$ depending only on $(\mathcal{K}_1, \mathcal{G}_1, M)$ such that (1.1) has a unique solution $(u,h) \in CH^m(Q_T) \times H^{m+1}(0,T)^d$ with traces $u_{|x=0}, u_{|x=1} \in H^m(0,T)^n$.*

If the maximal time of existence T^* is finite then the range of $(u(t), h(t))$ leaves every compact subset of $\mathcal{U} \times \mathcal{H}$ as $t \uparrow T^*$ or

$$\lim_{t \uparrow T^*} \|\partial_x u(t)\|_{L^\infty(0,1)^n} = \infty.$$

Sketch of Proof. The first step is to determine an invariant set for the iteration. Given $R, T, K > 0$ denote by $V^m_{T,K,R}$ the set of all elements $(v,g) \in CH^m(Q_T) \times H^m(0,T)^d$ with the following properties: $\partial_t^j v_{|t=0} = \partial_t^j u_{|t=0}$ and $\partial_t^j g(0) = \partial_t^j h(0)$ for $1 \leq j < m$ where the right hand sides can be written in terms of u_0 and h_0 by formal differentiation of the PDE and ODE, the range of (v,g) lies in $\mathcal{K}_1 \times \mathcal{G}_1$,

$$\|v\|_{W^{1,\infty}(Q_T)^n} + \|g\|_{W^{1,\infty}(0,T)^d} \leq K,$$

and

$$\|v\|_{H^m(Q_T)^n} + \|v_{|x=0}\|_{H^m(0,T)^n} + \|v_{|x=1}\|_{H^m(0,T)^n} + \|g\|_{H^m(0,T)^d} \leq R.$$

Consider the map $\mathscr{T} : V^m_{T,K,R} \to V^m_{T,K,R}$ defined as follows: Given $(v,g) \in V^m_{T,K,R}$, let $\mathscr{T}(v,g) =: (u,g)$ be the solution of the system

$$\begin{cases} u_t + A(v)u_x = f(v), & \text{in } (0,T) \times (0,1), \\ B_0 u_{|x=0} = b_0(t, h(t)), & \text{in } (0,T), \\ B_1 u_{|x=1} = b_1(t, h(t)), & \text{in } (0,T), \\ h'(t) = H(t, g(t), v(t,0), v(t,1)), & \text{in } (0,T), \\ u_{|t=0} = u_0, & \text{in } (0,1), \\ h_{|t=0} = h_0. & \end{cases}$$

The theory for linear hyperbolic systems with variable coefficients in [**2**] can be applied for the existence of a solution for this system. With additional a priori estimates in terms of the Sobolev norms, it can be shown that there are constants $T, K, R > 0$ such that $\mathscr{T}(v,g) \in V^m_{K,T,R}$ whenever $(v,g) \in V^m_{T,K,R}$, that is, $V^m_{T,K,R}$ is invariant under $\mathscr{T}$.

This introduces a sequence of elements $(u^n, h^n) \in V^m_{T,K,R}$ with $(u^n, h^n) = \mathscr{T}(u^{n-1}, h^{n-1})$ for $n \geq 1$ where $(u^0, h^0) \in V^m_{T,K,R}$ is a fixed element. By reducing $T > 0$ if necessary, it can be shown that the map $\mathscr{T}$ is contractive with respect to the norm of $L^2(Q_T)^n \times L^2(0,T)^d$ and therefore the above sequence converges in this space. The boundedness of the sequence in $H^m(Q_T)^n \times H^m(0,T)^d$ and interpolation theory show that the limit of the sequence described above is the solution of the nonlinear system. The additional regularity in time and the regularity of the traces follows from the regularity theory for linear hyperbolic systems with variable coefficients.

The proof of the blow-up criterion is standard. Indeed, one shows that if the conditions are not satisfied then it is possible to extend the solution on a larger time interval. We refer the details to the reference mentioned above. □

Now we consider the following system modeling the velocity u of a fluid contained in a horizontal elastic tube of length ℓ, vertical cross section A, that at each end is attached to tanks with horizontal cross section A_T:

$$\begin{cases} A_t + uA_x + Au_x = 0, & t > 0,\ 0 < x < \ell, \\ u_t + \kappa^2 A^{-\frac{1}{2}} A_x + uu_x = -\beta u, & t > 0,\ 0 < x < \ell, \\ A_T h_0'(t) = -A(t,0)u(t,0), & t > 0, \\ A_T h_\ell'(t) = A(t,\ell)u(t,\ell), & t > 0, \\ A(t,0) = (a_0 + bh_0(t))^2, & t > 0, \\ A(t,\ell) = (a_\ell + bh_\ell(t))^2, & t > 0, \\ A(0,x) = A^0(x), \quad u(0,x) = u^0(x), & 0 < x < \ell, \\ h_0(0) = h_0^0, \quad h_\ell = h_\ell^0. \end{cases} \tag{3.1}$$

Here, h_0 and h_ℓ are the level heights of the fluid in the left and right tanks, respectively, while the constants $\kappa, a_0, a_\ell, b > 0$ are related to the material properties of the tube and the physical properties of the fluid. The constant $\beta \geq 0$ is the damping coefficient which is related to the viscosity of the fluid. For more details in this model and the precise formulas for the parameters we refer to [**13**]. The system (3.1) is similar to the one considered in [**3, 11, 16**] in the context of valveless pumping.

With regards to the local-in-time existence and blow-up criterion, the result of Theorem 3.1 is applicable to (3.1). The next question is the existence of global solutions. The answer is affirmative provided that $\beta > 0$ and the initial data are smooth and close to the constant steady state $(A_e, 0, h_{0e}, h_{\ell e})$. This steady state is unique as long as the total volume of the fluid in the tube and in the tanks is constant. The proof of the global existence is based on the energy estimates obtained from appropriate entropy-entropy flux pairs for the hyperbolic system. Define the energy functional N_k by

$$\begin{aligned} N_k^2(t) &= \sup_{s\in[0,t]} (\|u(s)\|_{H^k}^2 + \|A^{\frac{1}{4}}(s) - A_e^{\frac{1}{4}}\|_{H^k}^2 + |h_0(s) - h_{0e}|^2 + |h_\ell(s) - h_{\ell e}|^2) \\ &\quad + \int_0^t \|u(s)\|_{H^k}^2 + k\|(A^{\frac{1}{4}})_x(s)\|_{H^{k-1}}^2 \, \mathrm{d}s \end{aligned}$$

for $k = 0, 1, 2$.

Theorem 3.2. *Suppose that $\beta > 0$. There exists $\delta_0 > 0$ such that if $N_2(0) \leq \delta_0$ then (3.1) has a unique global solution such that $A, u \in C([0,\infty), H^2(0,\ell)) \cap C^1([0,\infty), H^1(0,\ell))$, $h_0, h_\ell \in C^2[0,\infty)$ and $N_2(t) \leq CN_2(0)$ for all $t \geq 0$. Moreover, we have the asymptotic stability in $H^1 \times H^1 \times \mathbb{R}^2$*

$$\lim_{t\to\infty} (\|A(t) - A_e\|_{H^1(0,\ell)} + \|u(t)\|_{H^1(0,\ell)} + |h_0(t) - h_{0e}| + |h_\ell(t) - h_{\ell e}|) = 0$$

and the exponential stability in $L^2 \times L^2 \times \mathbb{R}^2$

$$\|A(t) - A_e\|_{L^2(0,\ell)} + \|u(t)\|_{L^2(0,\ell)} + |h_0(t) - h_{0e}| + |h_\ell(t) - h_{\ell e}| \le C(1 + t^k)e^{-\sigma t}$$

for some constants $C \ge 1$, $\sigma > 0$, $k \in \{0, 1\}$ *and for all* $t \ge 0$.

Proof. We only give the main ideas and refer to [**15**] for the complete proofs. For the global existence, the main goal is to prove the existence of a constant $\delta > 0$ such that $N_2(T) \le \delta$ implies $N_2(T) \le C(\delta)N_2(0)$ for some $C(\delta) > 0$ independent of T. Lower order estimates can be obtained by utilizing the following relative entropy and its corresponding relative entropy flux

$$\eta_0(A, u) = \frac{1}{2}Au^2 + \frac{4}{3}\kappa^2(A^{\frac{3}{2}} - A_e^{\frac{3}{2}}) - 2\kappa^2 A_e^{\frac{1}{2}}(A - A_e),$$
$$q_0(A, u) = \frac{1}{2}Au^3 + 2\kappa^2(A^{\frac{1}{2}} - A_e^{\frac{1}{2}})uA.$$

On the other hand, higher order estimates can be obtained by introducing suitable entropy-entropy flux pairs for the diagonalized system as in [**18**]. Once this a priori estimate is proved, a standard continuation argument shows that the local solution can be extended into a global one.

The asymptotic stability in $H^1 \times H^1 \times \mathbb{R}^2$ is a consequence of the uniform boundedness of the energy functional N_2 with respect to time. The exponential stability with respect to $X := L^2 \times L^2 \times \mathbb{R}^2$ is obtained by linearizing the system around the steady state. This gives us a linear evolution equation

$$\dot{Z}(t) = \mathcal{A}Z(t), \qquad \mathcal{A}\begin{pmatrix} B \\ v \\ g_0 \\ g_\ell \end{pmatrix} = \begin{pmatrix} -A_e v_x \\ -\alpha B_x - \beta v \\ -\frac{A_e}{A_T}v(0) \\ \frac{A_e}{A_T}v(\ell) \end{pmatrix},$$

with $\mathcal{A}$ a linear operator on X with domain $\mathcal{D}(\mathcal{A}) = \{(B, v, g_0, g_\ell) \in H^1(0,\ell)^2 \times \mathbb{R}^2 : B(0) = \gamma g_0, B(\ell) = \gamma g_\ell\}$ and the constants are given explicitly by

$$\alpha = \frac{\kappa^2}{\sqrt{A_e}}, \qquad \gamma = 2b(a_0 + bh_{0e}) = 2b(a_\ell + bh_{\ell e}).$$

The state for this equation is $Z = (A, u, h_0, h_\ell) - (A_e, 0, h_{0e}, h_{\ell e})$, the deviation from the equilibrium.

The linear operator $\mathcal{A}$ generates a strongly continuous group $(e^{tA})_{t\ge 0}$ on X and by the methods of nonharmonic Fourier analysis we have

$$\|e^{tA}Z_0\|_X \le C(1 + t^k)e^{-\sigma t}\|Z_0\|_X$$

for every $Z_0 \in \ker(\mathcal{A})^\perp$, where $\sigma = -\sup\{\mathrm{Re}\ \lambda : \lambda \in \sigma(\mathcal{A})\}$, $\sigma(\mathcal{A})$ is the spectrum of $\mathcal{A}$ and k is either 0 or 1 depending on the value of β, see [**13**] for the precise formulas.

Notice that this stability is only possible once the kernel of $\mathcal{A}$ is factored out. The elements of $\ker(\mathcal{A})$ are also steady states of the nonlinear system (3.1), however, they correspond to a different volume of the fluid. There is no reason for $Z(t)$ to be in the domain of $\mathcal{A}$. In fact, the compatibility conditions on the boundary stated in the domain of $\mathcal{A}$ is not satisfied in general. For this reason, we consider the new state variable $Y = Z - (\phi, 0, 0, 0)$ where

$$\phi(t, x) = \frac{\ell - x}{\ell}b^2(h_0(t) - h_{0e})^2 + \frac{x}{\ell}b^2(h_\ell(t) - h_{\ell e})^2.$$

One can see that $Y(t) \in D(\mathcal{A})$ for all $t \geq 0$ and the non-homogeneous system

$$\dot{Y}(t) = \mathcal{A}Y(t) + F(Y(t))$$

holds for some source term F.

The final step is to decompose $Y = Y_1 + Y_2$ in such a way that $Y_1(t) \in \ker(\mathcal{A})^\perp$ and $Y_2(t) \in \ker(\mathcal{A})$ for every $t \geq 0$. For Y_1 we apply the exponential stability of the semigroup generated by $\mathcal{A}$ and for Y_2 we prove that its norm in X is small provided that $N_2(0)$ is small. The exponential stability of the nonlinear system can now be obtained from these together with interpolation estimates and the Gronwall-type Lemma in [**6**]. □

References

[1] N. Antonić and K. Burazin, *Graph spaces of first-order linear partial differential operators*, Math. Commun. **14** (2009), 135–155.

[2] S. Benzoni-Gavage and D. Serre, Multidimensional Hyperbolic Partial Differential Equations, Oxford University Press, 2007.

[3] A. Borzì and G. Propst, *Numerical investigation of the Liebau phenomenon*, Z. Angew. Math. Phys. **54** (2003), 1050–1072.

[4] J. Chazarain and A. Piriou, Introduction to the Theory of Linear Partial Differential Equations, North-Holland Publishing Co., Amsterdam 1982.

[5] J.-F. Coulombel, Stabilité multidimensionnelle d'interfaces dynamiques; application aux transitions de phase liquide-vapeur, PhD thesis, École normale supérieure de Lyon, 2002.

[6] K. Fellner and G. Raoul, *Stability of stationary states of non-local interaction equations*, Mathematical and Computer Modelling, **53** (2011), pp. 2267–2291.

[7] M. Fernández, V. Milišić, and A. Quarteroni, *Analysis of a geometrical multiscale blood flow model based on the coupling of ODEs and hyperbolic PDEs*, Multiscale Model. Simul. **4** (2005), 215–236.

[8] K. O. Friedrichs, *Symmetric positive linear differential equations*, Comm. Pure Appl. Math. **11** (1958), 333–418.

[9] M. Jensen, Discontinuous Galerkin methods for Friedrichs systems with irregular solutions, PhD thesis, University of Oxford, 2004.

[10] G. Métivier, *Stability of multidimensional shocks*, Advances in the Theory of Shock Waves, ed. H. Freistühler and A. Szepessy pp. 25–103, Birkhäuser, Boston, 2001.

[11] J. T. Ottesen, *Valveless pumping in a fluid-filled closed elastic tube-system: one-dimensional theory with experimental validation*, J. Math. Biol. **46** (2003), 309–332.

[12] G. Peralta and G. Propst, *Local well-posedness of a class of hyperbolic PDE-ODE systems on a bounded interval*, to appear in Journal of Hyperbolic Differential Equations.

[13] G. Peralta and G. Propst, *Stability and boundary controllability of a linearized model of flow in an elastic tube*, to appear in ESAIM: COCV.

[14] G. Peralta and G. Propst, *Well-Posedness and regularity of linear hyperbolic systems with dynamic boundary conditions*, submitted.

[15] G. Peralta and G. Propst, *Global smooth solution to a hyperbolic system on an interval with dynamic boundary conditions*, submitted.

[16] H. J. Rath and I. Teipel, *Der Fördereffekt in ventillosen, elastischen Leitungen*, Z. Angew. Math. Phys. **29** (1978), 123–133.

[17] J. B. Rauch and F. J. Massey, *Differentiability of solutions to hyperbolic initial-boundary value problems*, Trans. Amer. Math. Soc. **189** (1974), 303–318.

[18] W. H. Ruan, M. E. Clark, M. Zhao, and A. Curcio, *Global solution to a hyperbolic problem arising in the modeling of blood flow in circulatory systems*, J. Math. Anal. Appl. **331** (2007), pp. 1068–1092.