UNIFORM EXPONENTIAL STABILITY OF A FLUID-PLATE INTERACTION MODEL DUE TO THERMAL EFFECTS

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Abstract. In this paper, we consider a coupled fluid-thermoelastic plate interaction model. The fluid velocity is modeled by the linearized 3D Navier-Stokes equation while the plate dynamics is described by a thermoelastic Kirchoff system. It is not assumed that the plate is necessarily flat. By eliminating the pressure term, the system is reformulated as an abstract evolution problem and its well-posedness is proved by semigroup methods. The dissipation in the system is due to the diffusion of the fluid and heat components. Uniform stability of the coupled system is established through multipliers and the energy method. The multipliers used for thermoelastic plate models in the literature are modified in accordance to the applicability of a certain Stokes map. The modification is similar to the one given by Haraux for damped wave equations.

1. Introduction

Consider an incompressible fluid occupying a bounded domain $\Omega \subset \mathbb{R}^3$ with sufficiently smooth boundary Γ. Suppose that part of the boundary is enclosed by a solid wall while the remaining part is enclosed by a thin elastic plate. Let Γ_1 and Γ_0 be the regions where the solid wall and the plate are located, respectively. Here, Γ_0 and Γ_1 are nonempty, $\overline{\Gamma}_1 \cup \overline{\Gamma}_0 = \Gamma$ and $\Gamma_1 \cap \Gamma_0 = \emptyset$. In this paper, we assume that the boundary Σ_0 of Γ_0 is nonempty and smooth enough. The fluid is modeled by the linearized 3D Navier-Stokes equation

$$\begin{cases} u_t - \mu\Delta_\Omega u + \nabla p = 0, & \text{in } (0,\infty)\times\Omega, \\ \operatorname{div} u = 0, & \text{in } (0,\infty)\times\Omega, \\ u = 0, & \text{on } (0,\infty)\times\Gamma_1, \\ u = \varphi_t\nu, & \text{on } (0,\infty)\times\Gamma_0. \end{cases} \tag{1.1}$$

In (1.1), u is the velocity vector field of the fluid, $\mu > 0$ is the viscosity of the fluid, φ is the transversal displacement of the plate and Δ_Ω is the Laplace operator in Ω. The coupling between the fluid and the plate is attained by matching the corresponding velocities on Γ_0. For the problem (1.1) to be well-posed one must impose the compatibility condition

$$\int_{\Gamma_0} \varphi_t \,\mathrm{d}\Gamma_0 = \int_{\Gamma_0} u\cdot\nu \,\mathrm{d}\Gamma_0 = 0, \quad \text{for } t \in (0,\infty). \tag{1.2}$$

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Thus, the average plate velocity should be zero for all times.

Suppose that the transversal displacement φ of the plate is described by Kirchoff's equation and is subjected to thermal effects given by the heat equation

$$\begin{cases} \varphi_{tt} - \gamma\Delta_{\Gamma_0}\varphi_{tt} + \Delta^2_{\Gamma_0}\varphi + \alpha\Delta_{\Gamma_0}\theta = p - \mu\nu\cdot\dfrac{\partial u}{\partial \nu}, & \text{in } (0,T)\times\Gamma_0, \\ \beta\theta_t - \eta\Delta_{\Gamma_0}\theta + \sigma\theta - \alpha\Delta_{\Gamma_0}\varphi_t = 0, & \text{in } (0,T)\times\Gamma_0, \end{cases} \tag{1.3}$$

where $\alpha, \beta, \eta > 0$, $\gamma, \sigma \geq 0$ and Δ_{Γ_0} is the Laplace-Beltrami operator on Γ_0 considered as a Riemannian manifold with boundary. The constant γ is proportional to the thickness of the plate. The case $\gamma = 0$, that is, plate's thickness is negligible, corresponds to the Euler-Bernoulli beam while $\gamma > 0$ is the Kirchoff's model. Without fluid interaction, it is well-known that the case $\gamma > 0$ is of parabolic-type at least for certain boundary conditions [17, 22]. More precisely, the corresponding system generates an analytic semigroup and hence *a fortiori* the uniform stability of the solutions. On the other hand, if $\gamma = 0$ then the system is of mixed hyperbolic-parabolic type.

We consider the case where the edge of the plate is clamped

$$\varphi = \frac{\partial\varphi}{\partial\nu} = 0, \quad \text{on } (0,\infty)\times\Sigma_0, \tag{1.4}$$

and the temperature at the boundary satisfies

$$\eta\lambda_1\frac{\partial\theta}{\partial\nu} + \lambda_2\theta = 0, \quad \text{on } (0,\infty)\times\Sigma_0, \tag{1.5}$$

where $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 > 0$. The case $\lambda_1, \lambda_2 > 0$ is Newton's law of cooling and the cases $\lambda_1 = 0$ and $\lambda_2 = 0$ mean that the temperature and the temperature flux at the boundary are zero, respectively. Finally, we supply the following initial data

$$u(0) = u^0, \quad \text{in } \Omega, \tag{1.6}$$

$$\varphi(0) = \varphi^0, \quad \varphi_t(0) = \varphi^1, \quad \theta(0) = \theta^0, \quad \text{in } \Gamma_0. \tag{1.7}$$

All throughout this paper we will assume that $\gamma > 0$ and $\sigma + \lambda_2 > 0$ and for the sake of simplicity we set without loss of generality that $\alpha = \beta = \eta = \mu = 1$. We will use the same notation for the Laplace operators in Ω and Γ_0, that is, Δ_Ω and Δ_{Γ_0} will be both denoted by Δ.

The energy at time $t \geq 0$ of the system (1.1)–(1.7) is defined by

$$E(t) = \frac{1}{2}\left(\int_\Omega |u(t)|^2\,\mathrm{d}\Omega + \int_{\Gamma_0} |\Delta\varphi(t)|^2 + |\varphi_t(t)|^2 + \gamma|\nabla\varphi_t(t)|^2 + |\theta(t)|^2\,\mathrm{d}\Gamma_0\right).$$

Formally differentiating the energy E and using the differential equations and boundary conditions in (1.1)–(1.5) we obtain

$$E'(t) = -\int_\Omega |\nabla u|^2\,\mathrm{d}\Omega - \int_{\Gamma_0}(\sigma|\theta|^2 + |\nabla\theta|^2)\,\mathrm{d}\Gamma_0 - \kappa\int_{\Sigma_0}|\theta|^2\,\mathrm{d}\Sigma_0$$

where

$$\kappa = \begin{cases} 0, & \text{if } \lambda_1 = 0, \\ \lambda_2/\lambda_1, & \text{if } \lambda_1 > 0. \end{cases} \tag{1.8}$$

This means that the dissipation of the system is due to the diffusion in the fluid and heat components.

Without thermal effects and with hinged boundary conditions $\varphi|_{\Sigma_0} = \Delta\varphi|_{\Sigma_0} = 0$ for the plate, the approximate controllability of the associated system was established by Lions and Zuazua [21] using duality and variational techniques. The system is somewhat similar to those presented in [11, 18]. In the absence of rotational forces ($\gamma = 0$) and thermal effects, the well-posedness and exponential stability the system has been established in [1, 3, 10] and in [9] accounting only for longitudinal displacement of the plate. To show the stability property, the authors in [1] used the frequency domain approach which is done by showing the uniform boundedness of the resolvents on the imaginary axis. On the other hand, the authors of [10] make use of an appropriate Lyapunov functional. It has been shown by Avalos and Bucci [2] that the system is stable even for $\gamma > 0$, but now the decay rate is rational.

In this work, we will exhibit the exponential decay of the energy for the solutions of the system (1.1)–(1.7) using appropriate multipliers in the time-domain space. One hindrance in using the multiplier method is on how to eliminate the terms arising from $p - \nu \cdot \frac{\partial u}{\partial \nu}$ in the plate equation. We can view the latter term as an (unbounded) feedback interconnection of the fluid and plate components. In fact, using the Agmon-Douglis-Nirenberg Theorem and a standard trace estimate one can majorize the L^2-norm of $p - \nu \cdot \frac{\partial u}{\partial \nu}$ in terms of the $H^{\frac{3}{2}}$-norm of φ_t, an estimate incompatible to the state space associated with φ_t. Alternatively, we shall eliminate this term using the properties of the Stokes map introduced by Chuesov and Ryzhkova [10], which is an improvement of the classical results in [27]. However, a problem arises in applying this map, it only applies to functions in Γ_0 that have zero average. To circumvent this problem we shall enforce the multipliers to have zero average through a simple but efficient construction.

2. Spaces and Operators for the Abstract Formulation

In this section we introduce the relevant spaces and operators necessary in the abstract formulation and analysis of the system (1.1)–(1.7). For the fluid component, the state space is given by

$$H = \{u \in [L^2(\Omega)]^3 : \operatorname{div} u = 0 \text{ in } \Omega \text{ and } u \cdot \nu = 0 \text{ on } \Gamma_1\}$$

and it is endowed with the L^2-norm. Recall that an element $u \in [L^2(\Omega)]^3$ with L^2-distributional divergence admits a generalized trace $u \cdot \nu \in H^{-\frac{1}{2}}(\Gamma)$. In fact, if

$$L^2_{\text{div}}(\Omega) = \{u \in [L^2(\Omega)]^3 : \operatorname{div} u \in L^2(\Omega)\}$$

is equipped with the graph norm then $u \mapsto u \cdot \nu \in \mathcal{L}(L^2_{\text{div}}(\Omega), H^{-\frac{1}{2}}(\Gamma))$. One can also localize the generalized trace $u \cdot \nu$ on Γ_0 and Γ_1. Consider the space

$$V = \{u \in [H^1(\Omega)]^3 : \operatorname{div} u = 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \Gamma_1\}$$

with the norm $\|\nabla \cdot\|_{L^2(\Omega)}$, which is equivalent to the full norm in $[H^1(\Omega)]^3$ according to the Poincaré inequality. If $C^\infty_{\sigma,\Gamma_0}(\Omega)$ is the space of all infinitely differentiable vector-valued functions on Ω that are divergence-free and vanish on a neighborhood of Γ_1, then H and V are the completions of $C^\infty_{\sigma,\Gamma_0}(\Omega)$ with respect to the L^2-norm and H^1-norm, respectively.

Next, we define the bi-Laplace and Laplace operators on various domains. Define the bi-Laplacian operator $A : D(A) \subset L^2(\Gamma_0) \to L^2(\Gamma_0)$, where

$$D(A) = H^4(\Gamma_0) \cap H^2_0(\Gamma_0),$$

by
$$A\varphi = \Delta^2\varphi.$$
It is known that A is a positive self-adjoint operator on $L^2(\Gamma_0)$ and hence the fractional powers A^α of A are well-defined on suitable domains for $\alpha \in \mathbb{R}$. In particular, $D(A^{\frac{1}{2}}) = H_0^2(\Gamma_0)$ and
$$\|\varphi\|_{D(A^{\frac{1}{2}})} = \|\Delta\varphi\|_{L^2(\Gamma_0)}, \quad \forall\, \varphi \in H_0^2(\Gamma_0).$$
Moreover, A can be extended to a bounded linear operator
$$A \in \mathcal{L}(D(A^{\frac{1}{2}}), D(A^{\frac{1}{2}})')$$
where $D(A^{\frac{1}{2}})'$ is the dual of $D(A^{\frac{1}{2}})$ with respect to the pivot space $L^2(\Gamma_0)$, and it holds that
$$\langle A\varphi, \tilde{\varphi}\rangle_{D(A^{\frac{1}{2}})'\times D(A^{\frac{1}{2}})} = (\Delta\varphi, \Delta\tilde{\varphi})_{L^2(\Gamma_0)}, \quad \forall\, \varphi, \tilde{\varphi} \in H_0^2(\Gamma_0). \tag{2.1}$$
Denote by $A_D : D(A_D) \subset L^2(\Gamma_0) \to L^2(\Gamma_0)$ the Dirichlet Laplacian
$$A_D\varphi = -\Delta\varphi$$
with domain $D(A_D) = H^2(\Gamma_0) \cap H_0^1(\Gamma_0)$. By the Poincaré inequality and standard elliptic results it can be seen that the norm $\|\cdot\|_{D(A_D)} := \|A_D\cdot\|_{L^2(\Gamma_0)}$ on $D(A_D)$ is equivalent to the induced norm of $D(A_D)$ as a subspace of $H^2(\Gamma_0)$. The operator A_D is a positive selfadjoint operator on $L^2(\Gamma_0)$ and
$$A_D^{-1} \in \mathcal{L}(L^2(\Gamma_0), H^2(\Gamma_0) \cap H_0^1(\Gamma_0)). \tag{2.2}$$
Therefore, the *pseudo-differential* operator A_D^{-1} is regularizing.

The operator $P_\gamma : D(A_D) \subset L^2(\Gamma_0) \to L^2(\Gamma_0)$ defined by
$$P_\gamma = I + \gamma A_D$$
is also positive self-adjoint operator with square root $P_\gamma^{\frac{1}{2}} : D(A_D^{\frac{1}{2}}) = H_0^1(\Gamma_0) \subset L^2(\Gamma_0) \to L^2(\Gamma_0)$ and there holds
$$(P_\gamma^{\frac{1}{2}}\varphi, P_\gamma^{\frac{1}{2}}\tilde{\varphi})_{L^2(\Gamma_0)} = (\varphi, \tilde{\varphi})_{L^2(\Gamma_0)} + \gamma(\nabla\varphi, \nabla\tilde{\varphi})_{L^2(\Gamma_0)}, \quad \forall\, \varphi, \tilde{\varphi} \in H_0^1(\Gamma_0).$$
A typical extension procedure shows that P_γ admits an extension
$$P_\gamma \in \mathcal{L}(H_0^1(\Gamma_0), H^{-1}(\Gamma_0))$$
which has a bounded inverse. In fact, if we equipped the space $H_0^1(\Gamma_0)$ with the norm $\|P_\gamma^{\frac{1}{2}}\cdot\|_{L^2(\Gamma_0)}$ and $H^{-1}(\Gamma_0)$ is regarded as the dual of $H_0^1(\Gamma_0)$ with respect to the pivot space $L^2(\Gamma_0)$, then P_γ becomes a unitary operator, see [28, Corollary 3.4.6].

Finally, we define $A_R : D(A_R) \subset L^2(\Gamma_0) \to L^2(\Gamma_0)$ by
$$A_R\theta = -\Delta\theta + \sigma\theta$$
where $D(A_R) = \{u \in H^2(\Gamma_0) : \lambda_1\frac{\partial\theta}{\partial\nu} + \lambda_2\theta = 0\}$. Note that if $\sigma = \lambda_1 = 0$ and $\lambda_2 > 0$ then $A_R = A_D$. Let V_R be the space $H^1(\Gamma_0)$ endowed with the inner product
$$(\theta, \tilde{\theta})_{V_R} = \sigma(\theta, \tilde{\theta})^2_{L^2(\Gamma_0)} + (\nabla\theta, \nabla\tilde{\theta})^2_{L^2(\Gamma_0)} + \kappa(\theta, \tilde{\theta})_{L^2(\Sigma_0)}$$
where κ is the constant defined in (1.8). Because $\sigma + \lambda_2 > 0$, it follows that V_R is equivalent to $H^1(\Gamma_0)$.

It will be needed to rewrite the heat equation in terms of the Dirichlet Laplacian so as to apply the operator A_D^{-1}. This process is sometimes called *homogenization.* For this reason we define the Dirichlet map D

$$h = Dg \iff \begin{cases} \Delta h = 0, & \text{in } \Gamma_0, \\ h = g, & \text{on } \Sigma_0, \end{cases}$$

The notations γ_0 and γ_1 for the zero and first order traces will be also used. The Robin map A_R can now be expressed in terms of the operators A_D, D and γ_0 as follows

$$A_R = A_D(I - D\gamma_0) + \sigma I. \tag{2.3}$$

In stuyding the system (1.1)–(1.7), the pressure term p will be eliminated as in [3, 4, 7]. This is done by observing that the pressure satisfies an elliptic boundary value problem with mixed Neumann and Robin type-boundary conditions. The suitable mixed Neumann-Robin maps R_0 and R_1 are as follows [2, 3]

$$p = R_0 q \iff \begin{cases} \Delta p = 0, \text{ in } \Omega, \\ \dfrac{\partial p}{\partial \nu} = 0, \text{ on } \Gamma_1, \\ \dfrac{\partial p}{\partial \nu} + P_\gamma^{-1} p = q, \text{ on } \Gamma_0, \end{cases}$$

$$p = R_1 q \iff \begin{cases} \Delta p = 0, \text{ in } \Omega, \\ \dfrac{\partial p}{\partial \nu} = q, \text{ on } \Gamma_1, \\ \dfrac{\partial p}{\partial \nu} + P_\gamma^{-1} p = 0, \text{ on } \Gamma_0. \end{cases}$$

The classic elliptic regularity results in [20, p. 152] give us

$$D \in \mathcal{L}(H^s(\Sigma_0), H^{s+\frac{1}{2}}(\Gamma_0)) \quad \text{and} \quad R_i \in \mathcal{L}(H^s(\Gamma_i), H^{s+\frac{3}{2}}(\Omega)) \tag{2.4}$$

for $s \in \mathbb{R}$ and $i = 0, 1$.

According to the compatibility condition (1.2), one also need to introduce functions on Γ_0 that have average zero. Let

$$\widehat{L}^2(\Gamma_0) = \{\varphi \in L^2(\Gamma_0) : \int_{\Gamma_0} \varphi \, \mathrm{d}\Gamma_0 = 0\}$$

viewed as a subspace of $L^2(\Gamma_0)$. For every $s \geq 0$ we define

$$\widehat{H}_0^s(\Gamma_0) = H_0^s(\Gamma_0) \cap \widehat{L}^2(\Gamma_0).$$

Clearly, $\widehat{H}_0^s(\Gamma_0)$ is a closed subspace of $H_0^s(\Gamma_0)$.

In dealing with the energy estimates, we will frequently use the following result in [10, Proposition 2.2]. This has been shown under the hypothesis that Γ_0 is flat. However, the proof can be adapted to a *curved* Γ_0. In fact, the flatness of Γ_0 was not explicitly used in the proof.

Theorem 2.1. *Let S be the Stokes map defined in the following way*

$$u = S\varphi \iff \begin{cases} -\Delta u + \nabla p = 0, & \text{in } \Omega, \\ \operatorname{div} u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_1, \\ u = \varphi\nu, & \text{on } \Gamma_0. \end{cases} \tag{2.5}$$

Then it holds that $S \in \mathcal{L}(\widehat{L}^2(\Gamma_0), [H^{\frac{1}{2}}(\Omega)]^3 \cap H) \cap \mathcal{L}(\widehat{H}_0^1(\Gamma_0), [H^{\frac{3}{2}}(\Omega)]^3 \cap H)$.

We can think of the Stokes map S as a lifting operator for functions in Γ_0 to functions in Ω. This information will allow us to eliminate the terms involving $p - \nu \cdot \frac{\partial u}{\partial \nu}$ in the thermoelastic system (1.3) when using the multiplier method. However, the drawback in applying the Stokes map is that the multiplier should have zero average. This means that we need to modify the multipliers introduced in the literature in order for them to be applicable in the present case. To do this, we subtract the original multiplier by a *suitable regularized version* with respect to space.

3. Abstract Formulation and Main Results

As noted in the earlier works [7, 8], one cannot apply the usual Leray projection method to deal with the Stokes equation (1.1) due to the Neumann trace boundary condition on Γ_0. An alternative and novel approach presented in these papers to solve this problem is to eliminate the pressure by writing it in terms of the fluid velocity and the state variables associated with the structure. The same idea has been used to compute numerical approximations for the Stokes equation using pressure-matrix methods, see [26, Section 9.6.1] and the references therein. The procedure will also enable us to recover the pressure for the strong solutions of the system. In our case, we need to express p as a function of the fluid velocity u, the plate displacement φ and temperature θ. With this representation it is possible to express the system (1.1)–(1.7) in an abstract form and semigroup methods are amenable to establish its well-posedness.

3.1. **Resolution of the Pressure.** For smooth solutions it can be checked that for each time t the pressure is a harmonic function in Ω with mixed Neumann-Robin boundary conditions

$$\begin{cases} \Delta p(t) = 0, \text{ in } \Omega, \\ \dfrac{\partial p(t)}{\partial \nu} = \Delta u(t) \cdot \nu, \text{ on } \Gamma_1, \\ \dfrac{\partial p(t)}{\partial \nu} + P_\gamma^{-1} p(t) = P_\gamma^{-1}\left(\Delta^2 \varphi(t) + \Delta\theta(t) + \nu \cdot \dfrac{\partial u(t)}{\partial \nu}\right) + \Delta u(t) \cdot \nu, \text{ on } \Gamma_0, \end{cases}$$

See [3] or the proof of Theorem 3.1 to verify this claim. Introducing the following maps

$$\begin{aligned} G_1 u &= R_1(\Delta u \cdot \nu|_{\Gamma_1}), \\ G_2 u &= R_0(\Delta u \cdot \nu|_{\Gamma_0} + P_\gamma^{-1}(\nu \cdot \gamma_1 u)), \\ G_3 \varphi &= R_0(P_\gamma^{-1} \Delta^2 \varphi), \\ G_4 \theta &= R_0(P_\gamma^{-1} \Delta\theta), \end{aligned}$$

the pressure term can be written in terms of (u, φ, θ) as

$$p = G(u, \varphi, \theta) := G_1 u + G_2 u + G_3 \varphi + G_4 \theta. \tag{3.1}$$

If $u \in [H^2(\Omega)]^3 \cap H$ then $\Delta u \in [L^2(\Omega)]^3$ and $\operatorname{div} \Delta u = \Delta \operatorname{div} u = 0$ in Ω so that $\Delta u \cdot \nu|_{\Gamma_i}$ is well-defined in $H^{-\frac{1}{2}}(\Gamma_i)$ for $i = 0, 1$. From (2.4) it follows that if

$$(u, \varphi, \theta) \in ([H^2(\Omega)]^3 \cap H) \times H^3(\Gamma_0) \times H^2(\Gamma_0)$$

then the maps G_i are well-defined and p given by (3.1) lies in $H^1(\Omega)$.

3.2. **Abstract Formulation.** As noted in the literature, we shall also need to factor the constant functions associated with the state space for displacement of the plate. Indeed, by integrating the compatibility condition (1.2) from 0 to t one obtains

$$\int_{\Gamma_0} \varphi(t)\, \mathrm{d}\Gamma_0 = \int_{\Gamma_0} \varphi(0)\, \mathrm{d}\Gamma_0.$$

With this consideration, we shall take the state space

$$\begin{aligned}\mathcal{H} = \{(u, \varphi_1, \varphi_2, \theta) \in H \times D(A^{\frac{1}{2}}) \times D(P_\gamma^{\frac{1}{2}}) \times L^2(\Gamma_0) \\ : \varphi_1, \varphi_2 \in \widehat{L}^2(\Gamma_0),\, u \cdot \nu|_{\Gamma_0} = \varphi_2\}\end{aligned}$$

which is a Hilbert space under the inner product

$$\begin{aligned}((u, \varphi_1, \varphi_2, \theta), (\tilde{u}, \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\theta}))_{\mathcal{H}} \;\; = \;\; & (u, \tilde{u})_{[L^2(\Omega)]^3} + (A^{\frac{1}{2}} \varphi_1, A^{\frac{1}{2}} \tilde{\varphi}_1)_{L^2(\Gamma_0)} \\ & + (P_\gamma^{\frac{1}{2}} \varphi_1, P_\gamma^{\frac{1}{2}} \tilde{\varphi}_2)_{L^2(\Gamma_0)} + (\theta, \tilde{\theta})_{L^2(\Gamma_0)}\end{aligned}$$

Define the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{A} = \begin{pmatrix} \Delta - \nabla G_1 - \nabla G_2 & -\nabla G_3 & 0 & -\nabla G_4 \\ 0 & 0 & I & 0 \\ P_\gamma^{-1}(G_1 + G_2 - \nu \cdot \gamma_1) & P_\gamma^{-1}(-A + G_3) & 0 & P_\gamma^{-1}(A_R - \sigma I + G_4) \\ 0 & 0 & -A_D & -A_R \end{pmatrix}$$

with domain $D(\mathcal{A})$ consisting of all elements $(u, \varphi_1, \varphi_2, \theta) \in \mathcal{H}$ satisfying all of the following conditions:

(i) $(u, \varphi_1, \varphi_2, \theta) \in ([H^2(\Omega)]^3 \cap V) \times H^3(\Gamma_0) \times D(A^{\frac{1}{2}}) \times D(A_R)$,
(ii) $-\Delta u + \nabla p \in H$ where $p = G(u, \varphi_1, \theta)$,
(iii) $u = \varphi_2 \nu$ on Γ_0,
(iv) $P_\gamma^{-1}(A\varphi_1 - (A_R - \sigma)\theta + \nu \cdot \gamma_1 u - p) \in \widehat{H}_0^1(\Gamma_0)$.

The system can now be written as an abstract Cauchy problem on $\mathcal{H}$

$$\begin{cases} \dot{U}(t) = \mathcal{A} U(t), \quad t > 0, \\ U(0) = U^0, \end{cases} \tag{3.2}$$

where $U = (u, \varphi_1, \varphi_2, \theta)$ and $U^0 = (u^0, \varphi^0, \varphi^1, \theta^0)$.

In the succeeding theorem, we show that $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions on $\mathcal{H}$ by invoking the Lumer-Philipps Theorem. This requires to prove the dissipativity of $\mathcal{A}$ and the range condition $R(\lambda I - \mathcal{A}) = \mathcal{H}$ for some $\lambda > 0$. Without thermal effects and flat Γ_0 this range condition was demonstrated in [3] using a nonstandard variational mixed method, specifically, the Babuška-Brezzi Theorem. This mixed formulation can be utilized in the numerical analysis for the approximations of steady-state solutions. However, in our case it is enough to prove the range condition by showing that $0 \in \rho(\mathcal{A})$. Indeed, if this is the case then from the fact that the resolvent set $\rho(\mathcal{A})$ is open, there exists $\lambda > 0$ such that $\lambda \in \rho(\mathcal{A})$ and thus $R(\lambda I - \mathcal{A}) = \mathcal{H}$. The latter method has its advantage. The system of equations that are needed to solve are now weakly coupled and one may solve them successively.

Theorem 3.1. *The operator $\mathcal{A}$ generates a strongly continuous semigroup of contractions on $\mathcal{H}$. In particular, for every initial data $U^0 \in \mathcal{H}$ the Cauchy problem admits a unique mild solution $U \in C([0,\infty), \mathcal{H})$. Moreover, the components u and θ of the solution satisfy*

$$(u, \theta) \in L^2(0, \infty; V \times V_R). \tag{3.3}$$

Proof. First we show the dissipativity of $\mathcal{A}$. Let $U = (u, \varphi_1, \varphi_2, \theta) \in D(\mathcal{A})$. Applying Green's formula gives us

$$\int_\Omega (\Delta u - \nabla p) \cdot u \,\mathrm{d}\Omega = \int_{\Gamma_0} \Big(\nu \cdot \frac{\partial u}{\partial \nu} - p\Big)\varphi_2 \,\mathrm{d}\Gamma_0 - \int_\Omega |\nabla u|^2 \,\mathrm{d}\Omega \tag{3.4}$$

after using $\operatorname{div} u = 0$ on Ω and the boundary conditions $u = \varphi_2 \nu$ on Γ_0 and $u = 0$ on Γ_1. On the other hand, from the duality pairing (2.1) we have

$$\begin{aligned}
&\int_{\Gamma_0} P_\gamma^{-\frac{1}{2}} \Big(-A\varphi_1 + (A_R - \sigma)\theta - \nu \cdot \frac{\partial u}{\partial \nu} + p\Big) P_\gamma^{\frac{1}{2}} \varphi_2 \,\mathrm{d}\Gamma_0 \\
&= -\left\langle A\varphi_1, \varphi_2 \right\rangle_{D(A^{\frac{1}{2}})' \times D(A^{\frac{1}{2}})} - \int_{\Gamma_0} (\Delta\theta)\varphi_2 \,\mathrm{d}\Gamma_0 - \int_{\Gamma_0} \Big(\nu \cdot \frac{\partial u}{\partial \nu} - p\Big)\varphi_2 \,\mathrm{d}\Gamma_0 \\
&= -\int_{\Gamma_0} \Delta\varphi_1 \Delta\varphi_2 \,\mathrm{d}\Gamma_0 + \int_{\Gamma_0} \nabla\theta \cdot \nabla\varphi_2 \,\mathrm{d}\Gamma_0 - \int_{\Gamma_0} \Big(\nu \cdot \frac{\partial u}{\partial \nu} - p\Big)\varphi_2 \,\mathrm{d}\Gamma_0
\end{aligned} \tag{3.5}$$

since $\varphi_2 = 0$ on Σ_0. For the heat component it holds that

$$\begin{aligned}
\int_{\Gamma_0} (\Delta\theta - \sigma\theta + \Delta\varphi_2)\theta \,\mathrm{d}\Gamma_0 \;\; = \;\; & -\sigma \int_{\Gamma_0} |\theta|^2 \,\mathrm{d}\Gamma_0 - \int_{\Gamma_0} |\nabla\theta|^2 \,\mathrm{d}\Gamma_0 - \kappa \int_{\Sigma_0} |\theta|^2 \,\mathrm{d}\Sigma_0 \\
& - \int_{\Gamma_0} \nabla\varphi_2 \cdot \nabla\theta \,\mathrm{d}\Gamma_0.
\end{aligned} \tag{3.6}$$

Taking the sum of (3.4)–(3.6) and getting the real part yield

$$\operatorname{Re}(\mathcal{A}U, U)_{\mathcal{H}} = -\int_\Omega |\nabla u|^2 \,\mathrm{d}\Omega - \int_{\Gamma_0} (\sigma|\theta|^2 + |\nabla\theta|^2) \,\mathrm{d}\Gamma_0 - \kappa \int_{\Sigma_0} |\theta|^2 \,\mathrm{d}\Sigma_0$$

and this shows that $\mathcal{A}$ is dissipative.

Next, we prove that 0 lies in the resolvent set of $\mathcal{A}$. Given $(u^*, \varphi_1^*, \varphi_2^*, \theta^*) \in \mathcal{H}$ we need to find a unique $(u, \varphi_1, \varphi_2, \theta) \in D(\mathcal{A})$ such that $\mathcal{A}(u, \varphi_1, \varphi_2, \theta) = (u^*, \varphi_1^*, \varphi_2^*, \theta^*)$ and

$$\|(u, \varphi_1, \varphi_2, \theta)\|_{\mathcal{H}} \le C\|(u^*, \varphi_1^*, \varphi_2^*, \theta^*)\|_{\mathcal{H}} \tag{3.7}$$

for some constant $C > 0$ independent on $(u, \varphi_1, \varphi_2, \theta)$ and $(u^*, \varphi_1^*, \varphi_2^*, \theta^*)$. The equation to be solved is equivalent in solving the Stokes equation

$$\begin{cases}
-\Delta u + \nabla p = -u^* \in H, & \text{in } \Omega, \\
\operatorname{div} u = 0, & \text{in } \Omega, \\
u = 0, & \text{on } \Gamma_1, \\
u = \varphi_1^* \nu \in \widehat{H}_0^2(\Gamma_0), & \text{on } \Gamma_0,
\end{cases} \tag{3.8}$$

the biharmonic equation

$$\begin{cases} \Delta^2\varphi_1 = -\Delta\theta + p - \nu \cdot \dfrac{\partial u}{\partial \nu} - (I-\gamma\Delta)\varphi_2^* \in H^{-1}(\Gamma_0), & \text{in } \Gamma_0, \\ \varphi_1 = \dfrac{\partial \varphi_1}{\partial \nu} = 0, & \text{on } \Sigma_0. \end{cases} \tag{3.9}$$

and the heat equation

$$\begin{cases} -\Delta\theta + \sigma\theta = \Delta\varphi_1^* - \theta^* \in L^2(\Gamma_0), & \text{in } \Gamma_0, \\ \lambda_1\theta + \lambda_2 \dfrac{\partial\theta}{\partial\nu} = 0, & \text{on } \Sigma_0. \end{cases} \tag{3.10}$$

where we used $\varphi_2 = \varphi_1^*$. Notice that we can solve first the Stokes and heat equations and then use the solution to solve the plate equation.

From [27, Theorem 2.4], the Stokes equation (3.8) admits a unique solution $(u, \tilde{p}) \in ([H^2(\Omega)]^3 \cap V) \times (H^1(\Omega)/\mathbb{R})$ and according to the Agmon-Douglis-Nirenberg Theorem and Poincaré inequality we deduce

$$\|u\|_{[H^2(\Omega)]^3} + \|\tilde{p}\|_{H^1(\Omega)/\mathbb{R}} \le C(\|\Delta\varphi_1^*\|_{L^2(\Gamma_0)} + \|u_1^*\|_H). \tag{3.11}$$

Therefore by trace theory we obtain from (3.11)

$$\|\tilde{p}\|_{L^2(\Gamma_0)} + \left\|\frac{\partial u}{\partial \nu}\right\|_{L^2(\Gamma_0)} \le C(\|\Delta\varphi_1^*\|_{L^2(\Gamma_0)} + \|u_1^*\|_H). \tag{3.12}$$

Notice that $p = \tilde{p} + p^*$, where p^* is a constant, is still an admissible pressure for the Stokes problem and satisfies the stability estimate

$$\|u\|_{[H^2(\Omega)]^3} + \|p\|_{H^1(\Omega)} \le C(\|\Delta\varphi_1^*\|_{L^2(\Gamma_0)} + \|u_1^*\|_H), \tag{3.13}$$

and hence (3.12) where $\tilde{p}$ is replaced by p.

In virtue of the Lax-Milgram lemma, the heat equation (3.10) has a unique solution $\theta \in D(A_R)$ and it satisfies

$$\|\theta\|_{V_R} \le C(\|\Delta\varphi_1^*\|_{L^2(\Gamma_0)} + \|\theta^*\|_{L^2(\Gamma_0)}). \tag{3.14}$$

By standard elliptic theory, the biharmonic problem (3.9) admits a unique solution $\varphi_1 \in H^3(\Gamma_0) \cap H_0^2(\Gamma_0)$, see [20, p. 152] for example. However, there is no reason that φ_1 has average zero. This will be done, following the method in [8], by choosing the appropriate constant p^*.

Let $\zeta \in H^4(\Gamma_0) \cap H_0^2(\Gamma_0)$ be the solution of the biharmonic problem

$$\begin{cases} \Delta^2\zeta = 1, & \text{in } \Gamma_0, \\ \zeta = \dfrac{\partial\zeta}{\partial\nu} = 0, & \text{on } \Sigma_0. \end{cases} \tag{3.15}$$

We test equation (3.9) with ζ to obtain

$$\begin{aligned} \langle A\varphi_1, \zeta\rangle_{D(A^{\frac{1}{2}})' \times D(A^{\frac{1}{2}})} &= (-\Delta\theta + \tilde{p} + p^* - \nu \cdot \frac{\partial u}{\partial \nu}, \zeta)_{L^2(\Gamma_0)} \\ &\quad - \langle P_\gamma \varphi_2^*, \zeta\rangle_{H^{-1}(\Gamma_0) \times H_0^1(\Gamma_0)}. \end{aligned} \tag{3.16}$$

Acording to the following equations

$$\begin{aligned}\langle A\varphi_1,\zeta\rangle_{D(A^{\frac12})'\times D(A^{\frac12})} &= \int_{\Gamma_0}\varphi_1\,\mathrm{d}\Gamma_0\\ \langle P_\gamma\varphi_2^*,\zeta\rangle_{D(P_\gamma^{\frac12})'\times D(P_\gamma^{\frac12})} &= \int_{\Gamma_0}\varphi_2^* P_\gamma\zeta\,\mathrm{d}\Gamma_0\\ (p^*,\zeta)_{L^2(\Gamma_0)} &= \int_{\Gamma_0} p^*|\Delta\zeta|^2\,\mathrm{d}\Gamma_0\end{aligned}$$

we obtain from (3.16) that $\varphi_1\in\widehat{L}^2(\Gamma_0)$ if and only if

$$p^* = \left(\int_{\Gamma_0}|\Delta\zeta|^2\,\mathrm{d}\Gamma_0\right)^{-2}\int_{\Gamma_0}\Big[\Big(\Delta\theta-\tilde{p}+\nu\cdot\frac{\partial u}{\partial\nu}\Big)\zeta+\varphi_2^*P_\gamma\zeta\Big]\mathrm{d}\Gamma_0. \tag{3.17}$$

Therefore we choose p^* according to (3.17).

Replacing ζ by φ_1 in the above calculations, we have instead

$$\|\Delta\varphi_1\|^2_{L^2(\Gamma_0)} = -(\theta,\Delta\varphi_1)_{L^2(\Gamma_0)} + (p-\nu\cdot\gamma_1 u,\varphi_1)_{L^2(\Gamma_0)} - (\varphi_2^*,P_\gamma\varphi_1)_{L^2(\Gamma_0)}.$$

Using the Cauchy-Schwarz and Poincaré inequalities together with (3.12) and (3.14) in the latter equality we obtain

$$\begin{aligned}\|\Delta\varphi_1\|_{L^2(\Gamma_0)} &\le C(\|\theta\|_{L^2(\Gamma_0)}+\|p\|_{L^2(\Gamma_0)}+\|\gamma_1 u\|_{L^2(\Gamma_0)}+\|\varphi_2^*\|_{L^2(\Gamma_0)})\\ &\le C(\|\theta^*\|_{L^2(\Gamma_0)}+\|\Delta\varphi_1^*\|_{L^2(\Gamma_0)}+\|u_1^*\|_H+\|\varphi_2^*\|_{L^2(\Gamma_0)}). \qquad (3.18)\end{aligned}$$

for some constant $C>0$. Furthermore, since $\varphi_2=\varphi_1^*$ it holds that

$$\|\varphi_2\|_{L^2(\Gamma_0)}+\|\nabla\varphi_2\|_{L^2(\Gamma_0)}\le C\|\Delta\varphi_1^*\|_{L^2(\Gamma_0)}. \tag{3.19}$$

Combining the estimates (3.13), (3.14), (3.18) and (3.19) proves (3.7).

Finally, it remains to check that p satisfies (3.1). Since both u and u^* are in H

$$\Delta p = \mathrm{div}(\nabla p) = \mathrm{div}(\Delta u - u^*) = 0.$$

Applying P_γ^{-1} to both sides of (3.9) produces

$$P_\gamma^{-1}(\Delta^2\varphi_1+\Delta\theta+\nu\cdot\gamma_1 u) = P_\gamma^{-1}p-\varphi_2^* \text{ in } H_0^1(\Gamma_0).$$

Taking into account the definition of the state space $\mathcal{H}$ we have $\varphi_2^* = u^*\cdot\nu = \Delta u\cdot\nu - \nabla p\cdot\nu$ on Γ_0 and hence

$$\frac{\partial p}{\partial\nu}+P_\gamma^{-1}p = P_\gamma^{-1}(\Delta^2\varphi_1+\Delta\theta+\nu\cdot\gamma_1 u)+\Delta u\cdot\nu \text{ in } H^{-\frac12}(\Gamma_0).$$

On the other hand, $\Delta u\cdot\nu-\nabla p\cdot\nu = u^*\cdot\nu = 0$ on Γ_1 and hence

$$\frac{\partial p}{\partial\nu} = \Delta u\cdot\nu \text{ in } H^{-\frac12}(\Gamma_1).$$

Thus $p=G(u,\varphi_1,\theta)$ and we obtain from the above discussions that $0\in\rho(\mathcal{A})$.

Therefore $\mathcal{A}$ generates a strongly continuous semigroup of contractions on $\mathcal{X}$ by the Lumer-Philipps Theorem. The additional regularity (3.3) for u and θ is a consequence of the identity

$$\int_0^T \|\nabla u\|^2_{[L^2(\Omega)]^3}+\|\theta\|^2_{V_R}\,\mathrm{d}t = \|U^0\|^2_{\mathcal{X}}-\|e^{T\mathcal{A}}U^0\|^2_{\mathcal{X}},\quad \forall\, U_0\in D(\mathcal{A}),\ T>0, \tag{3.20}$$

and the density of $D(\mathcal{A})$ in $\mathcal{X}$. This completes the proof of the theorem. □

Remark 3.2. If $U^0 \in D(\mathcal{A})$ then standard semigroup theory gives us the additional regularity of the strong solutions of (3.2)

$$(u, \varphi, \varphi_t, \theta) \in C([0,\infty), D(\mathcal{A})) \cap C^1([0,\infty), \mathcal{H}).$$

The analysis in the proof of Theorem 3.1 shows that

$$p \in C([0,\infty), H^1(\Omega)), \qquad u \in C([0,\infty), [H^2(\Omega)]^3 \cap V). \tag{3.21}$$

Indeed, for every $t \in [0,\infty)$ we have

$$\begin{cases} -\Delta u(t) + \nabla p(t) = -\, u_t(t), & \text{in } \Omega, \\ \operatorname{div} u(t) = 0, & \text{in } \Omega, \\ u(t) = 0, & \text{on } \Gamma_1, \\ u(t) = \varphi_t(t)\nu, & \text{on } \Gamma_0, \end{cases} \tag{3.22}$$

where the first equation holds in $[L^2(\Omega)]^3$. Hence one has the estimate

$$\|u(t)\|_{[H^2(\Omega)]^3} + \|p(t)\|_{H^1(\Omega)} \le C\Big(\|u_t(t)\|_{L^2[(\Omega)]^3} + \|\varphi_t(t)\|_{H_0^2(\Gamma_0)}\Big), \quad t \ge 0.$$

The continuity of p and u as mentioned in (3.21) follows from this estimate and the fact that $u_t \in C([0,\infty), [L^2(\Omega)]^3)$ and $\varphi_t \in C([0,\infty), H_0^2(\Omega))$.

If the initial data U^0 lies in $D(\mathcal{A})$ then we have the following equations

$$\varphi_{tt}(t) - \gamma\Delta\varphi_{tt}(t) = -\Delta^2\varphi(t) - \Delta\theta(t) - \nu \cdot \frac{\partial u(t)}{\partial \nu} + p(t) \text{ in } H^{-1}(\Gamma_0), \tag{3.23}$$

$$\theta_t(t) = \Delta\theta(t) - \sigma\theta + \Delta\varphi_t(t) \text{ in } L^2(\Gamma_0), \tag{3.24}$$

hold pointwise-in-time. If in addition, $U^0 \in D(\mathcal{A}^2)$ then (3.23) holds in $L^2(\Gamma_0)$ and

$$(\varphi, \varphi_t, \theta) \in C([0,\infty), H^4(\Gamma_0) \times H^3(\Gamma_0) \times H^2(\Gamma_0)).$$

The above regularity properties will justify the calculations that is provided in the succeeding section.

Remark 3.3. By trace theory and Poincaré inequality we obtain from the boundary condition on Γ_0 in (3.22) that

$$\int_0^T \int_{\Gamma_0} |\varphi_t|^2 \, \mathrm{d}\Gamma_0 \, \mathrm{d}t \le C \int_0^T \int_\Omega |\nabla u|^2 \, \mathrm{d}\Omega \, \mathrm{d}t \tag{3.25}$$

for some constant $C > 0$ independent T, φ_t and u. The estimate (3.25) implies that diffusion of the fluid implies the dissipation of the velocity φ_t of the plate. This observation plays a crucial role in the derivation of energy estimates that are required in the proof of Theorem 3.4 below.

3.3. **Main Result.** We now state the main theorem of this paper.

Theorem 3.4. *For every initial data in $\mathcal{H}$, the energy of the solutions of the Cauchy problem* (3.2) *decay exponentially. In other words, there exist constants $M \ge 1$ and $a > 0$ such that*

$$E(t) \le ME(0)e^{-at}, \quad t \ge 0. \tag{3.26}$$

We shall prove this theorem using suitable multipliers. To establish (3.26), it is sufficient to estimate the total energy on sufficiently large time intervals $[0, T]$ in terms of the final energy, the initial energy and the total dissipation on $[0,T]$. In other words, we want to derive the energy estimate

$$\int_0^T E(t)\,\mathrm{d}t \leq C\left(E(T)+E(0)+\int_0^T D(t)\,\mathrm{d}t\right), \quad T>T^*, \tag{3.27}$$

for some constants $C>0$ (independent of T) and $T^*\geq 0$, where D is the *dissipation term* given by

$$D(t)=\frac{1}{2}(\|\nabla u(t)\|^2_{[L^2(\Omega)]^3}+\|\theta(t)\|^2_{V_R}). \tag{3.28}$$

Indeed, from (3.20) and the fact that the energy is decreasing, (3.27) implies

$$TE(T)\leq 2C(E(T)+E(0))$$

Thus, if $T>\max(T^*,4C)$ then

$$E(T)\leq \delta_T E(0)$$

for some constant $0<\delta_T<1$. The exponential decay property (3.26) now follows from the latter inequality together with induction and the evolution property.

We note that as in Avalos and Lasiecka [6], the decay rate in Theorem 3.4 is not uniform in $\gamma>0$. However, in the case of $\gamma=0$ one may proceed using the same techniques presented here.

4. Modification of Multipliers

As stated in the introduction, we will utilize suitable multipliers to derive the energy estimates required to obtain exponential stability of the system (1.1)–(1.7). First, we discuss a simple but efficient way of modifying the multipliers suitable to the Stokes map S in Theorem 2.1. Fix a smooth cut-off function $\rho\in C_0^\infty(\Gamma_0)$ such that $\rho\geq 0$ in Γ_0 and $\int_{\Gamma_0}\rho\,\mathrm{d}\Gamma_0=1$. For a function f defined on $(0,T)\times\Gamma_0$ we set

$$(I_\rho f)(t,x):=\rho(x)\int_{\Gamma_0} f(t,y)\,\mathrm{d}\Gamma_{0y}, \quad (t,x)\in(0,T)\times\Gamma_0,$$

whenever the integral makes sense. Note that supp $I_\rho f(t,\cdot)\subset$ supp ρ and hence $I_\rho f(t,\cdot)$ is also compactly supported in Γ_0. The operator I_ρ is regularizing with respect to space in the sense that

$$I_\rho\in\mathcal{L}(L^p(0,T;L^2(\Gamma_0)),L^p(0,T;H_0^s(\Gamma_0))) \tag{4.1}$$

for every $s\geq 0$ and $1\leq p\leq\infty$, with operator norm independent of T and depending only on (p,s,ρ,Γ_0).

Let us define the map

$$f\mapsto M_\rho f:=f-I_\rho f.$$

In the subsequent discussions, if f is a specific multiplier then $M_\rho f$ is called the *modified multiplier*. According to the definition of $M_\rho f$ we have

$$\int_{\Gamma_0}(M_\rho f)(t)\,\mathrm{d}\Gamma_0=0, \quad t\in(0,T),$$

for suitable f. Therefore $M_\rho f$ is a suitable argument for the Stokes map S whenever f is sufficiently regular. This choice of multiplier $M_\rho f$ is similar to the one given by Haraux [12], see also [15].

If $f \in L^2(0,T;\widehat{L}^2(\Gamma_0))$ then $(M_\rho f)(t) = f(t)$ in $\widehat{L}^2(\Gamma_0)$ for a.e. $t \in (0,T)$, that is, M_ρ is invariant under functions that take values in $\widehat{L}^2(\Gamma_0)$. For each nonnegative integers k and j, if $f \in H^j(0,T;H^k(\Gamma_0))$ then $M_\rho f \in H^j(0,T;\widehat{H}^k(\Gamma_0))$ and

$$\sum_{i=1}^{j} \int_0^T \|\partial_t^i M_\rho f(t)\|^2_{H^k(\Gamma_0)}\,\mathrm{d}t \le C \sum_{i=1}^{j} \int_0^T \|\partial_t^i f(t)\|^2_{\widehat{H}^k(\Gamma_0)}\,\mathrm{d}t,$$

for some constant $C > 0$ independent of f. Therefore

$$M_\rho \in \mathcal{L}(H^j(0,T;H^k(\Gamma_0)), H^j(0,T;\widehat{H}^k(\Gamma_0))). \tag{4.2}$$

Lemma 4.1. *Let $T > 0$, $\varepsilon > 0$ and $f \in L^2(0,T;H_0^1(\Gamma_0)) \cap H^1(0,T;L^2(\Gamma_0))$. Then for every initial data in $D(\mathcal{A})$, the component u of the solution and the pressure p satisfy the estimate*

$$\begin{aligned}
&\left| \int_0^T \int_{\Gamma_0} \left(\nu \cdot \frac{\partial u}{\partial \nu} - p\right) M_\rho f \,\mathrm{d}\Gamma_0\,\mathrm{d}t \right| \\
&\le C_\rho \int_\Omega |u(T)|^2 + |u(0)|^2 \,\mathrm{d}\Omega + C_\rho \int_{\Gamma_0} |f(T)|^2 + |f(0)|^2 \,\mathrm{d}\Gamma_0 \\
&\quad + C_{\varepsilon,\rho} \int_0^T \int_\Omega |\nabla u|^2 \,\mathrm{d}\Omega\,\mathrm{d}t + \varepsilon \int_0^T \int_{\Gamma_0} |f|^2 + |\nabla f|^2 + |f_t|^2 \,\mathrm{d}\Gamma_0\,\mathrm{d}t,
\end{aligned} \tag{4.3}$$

for some constants $C_\rho > 0$ and $C_{\varepsilon,\rho} > 0$ independent of T, u and f.

Proof. We multiply the linearized Navier-Stokes equation (1.1) by $SM_\rho f$, which is admissible since $M_\rho f \in L^2(0,T;\widehat{H}_0^1(\Gamma_0)) \cap H^1(0,T;\widehat{L}^2(\Gamma_0))$ from (4.2), and use the divergence theorem and Green's identity to obtain

$$\begin{aligned}
0 &= \int_0^T \int_\Omega (u_t - \Delta u + \nabla p) \cdot SM_\rho f \,\mathrm{d}\Omega\,\mathrm{d}t \\
&= \left[\int_\Omega u \cdot SM_\rho f \,\mathrm{d}\Omega \right]_0^T - \int_0^T \int_\Omega u \cdot SM_\rho f_t + \nabla u \cdot \nabla SM_\rho f \,\mathrm{d}\Omega\,\mathrm{d}t \\
&\quad - \int_0^T \int_{\Gamma_0} \left(\nu \cdot \frac{\partial u}{\partial \nu} - p\right) M_\rho f \,\mathrm{d}\Gamma_0\,\mathrm{d}t
\end{aligned} \tag{4.4}$$

after using $\operatorname{div} SM_\rho f = 0$ in Ω and the boundary conditions $u = 0$ on Γ_1 and $SM_\rho f = (M_\rho f)\nu$ on Γ_0.

By Theorem 2.1, the properties of the map M_ρ and the embedding $[H^{\frac{1}{2}}(\Omega)]^3 \subset [L^2(\Omega)]^3$ we have

$$\begin{aligned}
\int_\Omega |SM_\rho f(t)|^2 \,\mathrm{d}\Omega &\le C \int_{\Gamma_0} |M_\rho f(t)|^2 \,\mathrm{d}\Gamma_0 \\
&\le C_\rho \int_{\Gamma_0} |f(t)|^2 \,\mathrm{d}\Gamma_0
\end{aligned} \tag{4.5}$$

for every $t \in [0,T]$ and

$$\int_0^T \int_\Omega |SM_\rho f_t|^2 \,\mathrm{d}\Omega\,\mathrm{d}t \le C_\rho \int_0^T \int_{\Gamma_0} |f_t|^2 \,\mathrm{d}\Gamma_0\,\mathrm{d}t. \tag{4.6}$$

Similarly, the embedding $[H^{\frac{3}{2}}(\Omega)]^3 \subset [H^1(\Omega)]^3$ and Theorem 2.1 imply

$$\begin{aligned}\int_0^T\int_\Omega |\nabla S M_\rho f|^2 \,\mathrm{d}\Omega\,\mathrm{d}t &\leq C\int_0^T\int_{\Gamma_0} |M_\rho f|^2 + |\nabla(M_\rho f)|^2 \,\mathrm{d}\Gamma_0\,\mathrm{d}t \\ &\leq C_\rho \int_0^T\int_{\Gamma_0} |f|^2 + |\nabla f|^2 \,\mathrm{d}\Gamma_0\,\mathrm{d}t. \end{aligned} \tag{4.7}$$

The desired estimate (4.3) follows from (4.4)–(4.7) together with Young's and Poincaré inequalities. □

We will utilize this lemma with f being $h\cdot\nabla\varphi$, $A_D^{-1}\theta$ or φ, where h is a smooth extension of ν to Γ_0 (see Remark 5.4 below for a short historical account regarding these multipliers). The additional terms obtained by enforcing the compatibility condition will be estimated with the help of the following lemma.

Lemma 4.2. *Let $T > 0$, $\varepsilon > 0$ and $f \in H^1(0,T;L^2(\Gamma_0))$. For every data in $D(\mathcal{A}^2)$ the components u and θ of the solution satisfy the estimate*

$$\begin{aligned}&\left|\int_0^T\int_{\Gamma_0} (\varphi_{tt} - \gamma\Delta\varphi_{tt} + \Delta^2\varphi + \Delta\theta) I_\rho f \,\mathrm{d}\Gamma_0\,\mathrm{d}t\right| \\ &\leq C\int_{\Gamma_0} |\varphi_t(T)|^2 + |\nabla\varphi_t(T)| + |\varphi_t(0)|^2 + |\nabla\varphi_t(0)|^2 \,\mathrm{d}\Gamma_0 \\ &+ C\int_{\Gamma_0} |f(T)|^2 + |f(0)|^2 \,\mathrm{d}\Gamma_0 + C_\varepsilon \int_0^T\int_{\Gamma_0} |f|^2 + |f_t|^2 \,\mathrm{d}\Gamma_0\,\mathrm{d}t \\ &+ C\int_0^T\int_{\Gamma_0} |\varphi_t|^2 + |\nabla\theta|^2 \,\mathrm{d}\Gamma_0\,\mathrm{d}t + \varepsilon\int_0^T\int_{\Gamma_0} |\Delta\varphi|^2 + |\nabla\varphi_t|^2 \,\mathrm{d}\Gamma_0\,\mathrm{d}t\end{aligned}$$

for some constants $C = C_{\rho,\gamma,\Gamma_0} > 0$ and $C_\varepsilon = C_{\varepsilon,\rho,\gamma,\Gamma_0} > 0$.

Proof. Integrating by parts in time, using Green's identities, the boundary conditions and the fact that $I_\rho f$ vanishes on Σ_0 yield

$$\begin{aligned}&\int_0^T\int_{\Gamma_0} (\varphi_{tt} - \gamma\Delta\varphi_{tt} + \Delta^2\varphi + \Delta\theta) I_\rho f \,\mathrm{d}\Gamma_0\,\mathrm{d}t \\ &= \left[\int_{\Gamma_0} \varphi_t I_\rho f \,\mathrm{d}\Gamma_0\right]_0^T - \int_0^T\int_{\Gamma_0} \varphi_t I_\rho f_t \,\mathrm{d}\Gamma_0\,\mathrm{d}t + \left[\gamma\int_{\Gamma_0} \nabla\varphi_t\cdot\nabla(I_\rho f)\,\mathrm{d}\Gamma_0\right]_0^T \\ &- \gamma\int_0^T\int_{\Gamma_0} \nabla\varphi_t\cdot\nabla(I_\rho f_t)\,\mathrm{d}\Gamma_0\,\mathrm{d}t + \int_0^T\int_{\Gamma_0} \Delta\varphi\,\Delta(I_\rho f)\,\mathrm{d}\Gamma_0\,\mathrm{d}t \\ &- \int_0^T\int_{\Gamma_0} \nabla\theta\cdot\nabla(I_\rho f)\,\mathrm{d}\Gamma_0\,\mathrm{d}t.\end{aligned} \tag{4.8}$$

According to the regularizing property of I_ρ, see (4.1), we have the following estimates

$$\int_{\Gamma_0} |I_\rho f(t)|^2 + |\nabla(I_\rho f)(t)|^2 \,\mathrm{d}\Gamma_0 \leq C\int_{\Gamma_0} |f(t)|^2 \,\mathrm{d}\Gamma_0, \quad t\in[0,T], \tag{4.9}$$

$$\int_0^T\int_{\Gamma_0} |I_\rho f_t|^2 + |\nabla(I_\rho f_t)|^2 \,\mathrm{d}\Gamma_0\,\mathrm{d}t \leq C\int_0^T\int_{\Gamma_0} |f_t|^2 \,\mathrm{d}\Gamma_0\,\mathrm{d}t, \tag{4.10}$$

$$\int_0^T\int_{\Gamma_0} |\nabla(I_\rho f)|^2 + |\Delta(I_\rho f)|^2 \,\mathrm{d}\Gamma_0\,\mathrm{d}t \leq C\int_0^T\int_{\Gamma_0} |f|^2 \,\mathrm{d}\Gamma_0\,\mathrm{d}t, \tag{4.11}$$

for some constant $C = C_{\rho,\Gamma_0} > 0$. Using Young's inequality in (4.8) and then applying the estimates (4.9)–(4.11) we obtain the estimate of the lemma. □

5. Proof of Uniform Stability

First, we give a *hidden trace regularity* for the plate component that is similar to [6, 14, 19].

Theorem 5.1. *For every initial data $U^0 \in \mathcal{H}$ the component φ of the solution of* (3.2) *satisfies $\Delta\varphi \in L^2(0,T;L^2(\Sigma_0))$ and*

$$\int_0^T\int_{\Sigma_0} |\Delta\varphi|^2 \,\mathrm{d}\Sigma_0\,\mathrm{d}t \le C\left(E(T)+E(0)+\int_0^T (E(t)+D(t))\,\mathrm{d}t\right) \tag{5.1}$$

where $C = C_{\rho,\gamma,\Gamma_0} > 0$ is independent of U^0 and D is the function defined in (3.28).

Proof. By a density argument, we may suppose that the initial data lies in $D(\mathcal{A}^2)$. Let $h \in [C^2(\overline{\Gamma}_0)]^2$ be a vector field such that $h = \nu$ on Σ_0. We multiply the plate equation in (1.3) by the multiplier $M_\rho(h\cdot\nabla\varphi)$ and integrate over time and space to obtain

$$\begin{aligned}&\int_0^T\int_{\Gamma_0} (\varphi_{tt} - \gamma\Delta\varphi_{tt} + \Delta^2\varphi + \Delta\theta) M_\rho(h\cdot\nabla\varphi)\,\mathrm{d}\Gamma_0\,\mathrm{d}t\\ &= \int_0^T\int_{\Gamma_0}\left(p - \nu\cdot\frac{\partial u}{\partial\nu}\right) M_\rho(h\cdot\nabla\varphi)\,\mathrm{d}\Gamma_0\,\mathrm{d}t\end{aligned} \tag{5.2}$$

According to Remark 3.2, $h\cdot\nabla\varphi \in C^1([0,T],H_0^1(\Gamma_0))$ for each $T>0$. Thus, we can apply Lemma 4.1 and 4.2. Using Lemma 4.1 with $f = h\cdot\nabla\varphi$ and $\varepsilon = 1$ we have

$$\begin{aligned}&\left|\int_0^T\int_{\Gamma_0}\left(\nu\cdot\frac{\partial u}{\partial\nu} - p\right) M_\rho(h\cdot\nabla\varphi)\,\mathrm{d}\Gamma_0\,\mathrm{d}t\right|\\ &\le C\int_\Omega |u(T)|^2 + |u(0)|^2\,\mathrm{d}\Omega + C\int_{\Gamma_0} |\nabla\varphi(T)|^2 + |\nabla\varphi(0)|^2\,\mathrm{d}\Gamma_0\\ &\quad + C\int_0^T\int_\Omega |\nabla u|^2\,\mathrm{d}\Omega\,\mathrm{d}t + C_\gamma\int_0^T\int_{\Gamma_0} |\Delta\varphi|^2 + \gamma|\nabla\varphi_t|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t,\end{aligned} \tag{5.3}$$

On the other hand, Lemma 4.2 with $f = h\cdot\nabla\varphi$ and $\varepsilon = 1$ produces the estimate

$$\begin{aligned}&\left|\int_0^T\int_{\Gamma_0} (\varphi_{tt} - \gamma\Delta\varphi_{tt} + \Delta^2\varphi + \Delta\theta) I_\rho(h\cdot\nabla\varphi)\,\mathrm{d}\Gamma_0\,\mathrm{d}t\right|\\ &\le C\int_{\Gamma_0} |\varphi_t(T)|^2 + |\nabla\varphi_t(T)| + |\varphi_t(0)|^2 + |\nabla\varphi_t(0)|^2\,\mathrm{d}\Gamma_0\\ &\quad + C_\gamma\int_0^T\int_{\Gamma_0} |\Delta\varphi|^2 + |\varphi_t|^2 + \gamma|\nabla\varphi_t|^2 + |\nabla\theta|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t\end{aligned} \tag{5.4}$$

To estimate the remaining terms in (5.2) we shall proceed as in [6, pp. 168–169]. Indeed, integrating by parts and using the fact that $\varphi = \varphi_t = \frac{\partial\varphi}{\partial\nu} = \frac{\partial\varphi_t}{\partial\nu} = 0$ and

$h = \nu$ on Σ_0 we have

$$
\begin{aligned}
&\int_0^T\int_{\Gamma_0} (\varphi_{tt} - \gamma\Delta\varphi_{tt} + \Delta^2\varphi + \Delta\theta) h\cdot\nabla\varphi \,\mathrm{d}\Gamma_0\,\mathrm{d}t \\
&= \left[\int_{\Gamma_0} \varphi_t h\cdot\nabla\varphi\,\mathrm{d}\Gamma_0\right]_0^T - \int_0^T\int_{\Gamma_0} \varphi_t h\cdot\nabla\varphi_t\,\mathrm{d}\Gamma_0\,\mathrm{d}t \\
&+ \left[\gamma\int_{\Gamma_0} \nabla\varphi_t\cdot\nabla(h\cdot\nabla\varphi)\,\mathrm{d}\Gamma_0\right]_0^T - \gamma\int_0^T\int_{\Gamma_0} \nabla\varphi_t\cdot\nabla(h\cdot\nabla\varphi_t)\,\mathrm{d}\Gamma_0\,\mathrm{d}t \\
&- \int_0^T\int_{\Gamma_0} \nabla(\Delta\varphi)\cdot\nabla(h\cdot\nabla\varphi)\,\mathrm{d}\Gamma_0\,\mathrm{d}t - \int_0^T\int_{\Gamma_0} \nabla\theta\cdot\nabla(h\cdot\nabla\varphi)\,\mathrm{d}\Gamma_0\,\mathrm{d}t. \qquad (5.5)
\end{aligned}
$$

We shall estimate each terms on the right hand side of (5.5).

The first three terms can be simply estimated using the Cauchy-Schwarz and Poincaré inequalities. For the fourth term, we use the divergence theorem to get

$$\int_0^T\int_{\Gamma_0} \nabla\varphi_t\cdot\nabla(h\cdot\nabla\varphi_t)\,\mathrm{d}\Gamma_0\,\mathrm{d}t = \frac{1}{2}\int_0^T\int_{\Gamma_0} \mathrm{div}(|\nabla\varphi_t|^2 h)\,\mathrm{d}\Gamma_0\,\mathrm{d}t + R_1(T) \qquad (5.6)$$

where the remainder term $R_1(T)$ can be estimated by

$$|R_1(T)| \le C\int_0^T\int_{\Gamma_0} |\nabla\varphi_t|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t. \qquad (5.7)$$

Applying the divergence theorem once again together with $|\nabla\varphi_t|^2 = 0$ and $h\cdot\nu = 1$ on Σ_0 we obtain

$$\int_{\Gamma_0} \mathrm{div}(|\nabla\varphi_t|^2 h)\,\mathrm{d}\Gamma_0 = \int_{\Sigma_0} |\nabla\varphi_t|^2\,\mathrm{d}\Sigma_0 = 0. \qquad (5.8)$$

Consequently, from (5.6)–(5.8) one acquires the estimate

$$\left|\int_0^T\int_{\Gamma_0} \nabla\varphi_t\cdot\nabla(h\cdot\nabla\varphi_t)\,\mathrm{d}\Gamma_0\,\mathrm{d}t\right| \le C\int_0^T\int_{\Gamma_0} |\nabla\varphi_t|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t. \qquad (5.9)$$

Let us estimate the fifth term on the right hand side of (5.5). Before we proceed, we note the following standard identities

$$
\begin{aligned}
2\nabla(\Delta\varphi)\cdot((\Delta\varphi)h) &= h\cdot\nabla(|\Delta\varphi|^2) \qquad (5.10)\\
\nabla(h\cdot\nabla\varphi) &= H\nabla\varphi + (\nabla^2\varphi)h \qquad (5.11)\\
(\Delta\varphi)h &= (\nabla^2\varphi)h + P(\varphi,h) \qquad (5.12)
\end{aligned}
$$

where H is the Jacobian of h, $\nabla^2\varphi$ is the Hessian of φ and

$$P(\varphi,h) = \begin{pmatrix} h_1\varphi_{yy} - h_2\varphi_{xy} \\ h_2\varphi_{xx} - h_1\varphi_{xy} \end{pmatrix}.$$

Thus from (5.10)–(5.12) we have

$$
\begin{aligned}
&-\int_0^T\int_{\Gamma_0} \nabla(\Delta\varphi)\cdot\nabla(h\cdot\nabla\varphi)\,\mathrm{d}\Gamma_0\,\mathrm{d}t = -\int_0^T\int_{\Gamma_0} \nabla(\Delta\varphi)\cdot H\nabla\varphi\,\mathrm{d}\Gamma_0\,\mathrm{d}t \\
&- \frac{1}{2}\int_0^T\int_{\Gamma_0} h\cdot\nabla(|\Delta\varphi|^2)\,\mathrm{d}\Gamma_0\,\mathrm{d}t + \int_0^T\int_{\Gamma_0} \nabla(\Delta\varphi)\cdot P(\varphi,h)\,\mathrm{d}\Gamma_0\,\mathrm{d}t. \qquad (5.13)
\end{aligned}
$$

Using the fact that $\frac{\partial^2 u}{\partial \tau^2} = ((\nabla^2\varphi)\tau)\cdot\tau = 0$ on Γ_0, where $\tau = (-\nu_2, \nu_1)$, one can show through (tedious) integration by parts that

$$\left| \int_0^T \int_{\Gamma_0} \nabla(\Delta\varphi)\cdot P(\varphi,h)\,\mathrm{d}\Gamma_0\,\mathrm{d}t \right| \le C \int_0^T \int_{\Gamma_0} |\Delta\varphi|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t. \tag{5.14}$$

On the other hand, according to the divergence theorem and $|\nabla\varphi| = 0$ on Γ_0

$$\left| \int_0^T \int_{\Gamma_0} \nabla(\Delta\varphi)\cdot H\nabla\varphi\,\mathrm{d}\Gamma_0\,\mathrm{d}t \right| \le C \int_0^T \int_{\Gamma_0} |\Delta\varphi|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t. \tag{5.15}$$

The second term on the right hand side of (5.13) can be expressed using the divergence theorem as

$$\begin{aligned}
&\int_0^T \int_{\Gamma_0} h\cdot\nabla(|\Delta\varphi|^2)\,\mathrm{d}\Gamma_0\,\mathrm{d}t \\
&= \int_0^T \int_{\Gamma_0} \operatorname{div}(h|\Delta\varphi|^2) - (\operatorname{div} h)|\Delta\varphi|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t \\
&= \int_0^T \int_{\Sigma_0} |\Delta\varphi|^2\,\mathrm{d}\Sigma_0\,\mathrm{d}t - \int_0^T \int_{\Gamma_0} (\operatorname{div} h)|\Delta\varphi|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t.
\end{aligned} \tag{5.16}$$

Therefore from (5.13)–(5.16) we have

$$\begin{aligned}
&\frac{1}{2}\int_0^T \int_{\Sigma_0} |\Delta\varphi|^2\,\mathrm{d}\Sigma_0\,\mathrm{d}t \\
&\le \int_0^T \int_{\Gamma_0} \nabla(\Delta\varphi)\cdot\nabla(h\cdot\nabla\varphi)\,\mathrm{d}\Gamma_0\,\mathrm{d}t + C\int_0^T \int_{\Gamma_0} |\Delta\varphi|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t.
\end{aligned} \tag{5.17}$$

The last term on the right hand side of (5.5) can be estimated as

$$\begin{aligned}
&\left| \int_0^T \int_{\Gamma_0} \nabla\theta\cdot\nabla(h\cdot\nabla\varphi)\,\mathrm{d}\Gamma_0\,\mathrm{d}t \right| \\
&\le \frac{1}{2}\int_0^T \int_{\Gamma_0} |\nabla\theta|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t + C\int_0^T \int_{\Gamma_0} |\Delta\varphi|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t.
\end{aligned} \tag{5.18}$$

Combining (5.2)–(5.5), (5.9), (5.17) and (5.18) proves the inequality (5.1). □

The next step is to estimate the total kinetic energy of the plate component. Due to (3.25) it is enough to consider the total energy of $\nabla\varphi_t$. This is achieved by the modified multiplier $M_\rho(A_D^{-1}\theta)$. We recall that the pseudodifferential operator A_D^{-1} is smoothing in the sense that

$$\|A_D^{-1}\theta\|_{H^2(\Gamma_0)} \le C\|\theta\|_{L^2(\Gamma_0)}, \tag{5.19}$$

see (2.2). Note that the heat equation in (1.3) can be rewritten in terms of the pseudodifferential operator A_D^{-1} as

$$A_D^{-1}\theta_t + (I - D\gamma_0)\theta + \sigma A_D^{-1}\theta + \varphi_t = 0. \tag{5.20}$$

This follows from the representation (2.3). Therefore from (5.19), (5.20) and $D\gamma_0 \in \mathcal{L}(H^1(\Gamma_0))$

$$\begin{aligned}
\|A_D^{-1}\theta_t\|_{H^1(\Gamma_0)} &\le \|(I - D\gamma_0)\theta\|_{H^1(\Gamma_0)} + \sigma\|A_D^{-1}\theta\|_{H^1(\Gamma_0)} + \|\varphi_t\|_{H^1(\Gamma_0)} \\
&\le C(\|\theta\|_{H^1(\Gamma_0)} + \|\varphi_t\|_{L^2(\Gamma_0)} + \|\nabla\varphi_t\|_{L^2(\Gamma_0)}).
\end{aligned} \tag{5.21}$$

Similarly, from (5.19) it holds that

$$\|A_D^{-1}\theta_t\|_{L^2(\Gamma_0)} \leq C(\|\theta\|_{H^1(\Gamma_0)} + \|\varphi_t\|_{L^2(\Gamma_0)}). \tag{5.22}$$

Lemma 5.2. *Let $T > 0$. For every data in $D(\mathcal{A}^2)$ the component φ of the solution of* (3.2) *satisfy the estimate*

$$\begin{aligned}&\frac{\gamma}{2}\int_0^T\int_{\Gamma_0} |\nabla\varphi_t|^2 \,\mathrm{d}\Gamma_0\,\mathrm{d}t\\ &\leq \varepsilon\int_0^T\int_{\Gamma_0} |\Delta\varphi|^2 \,\mathrm{d}\Gamma_0\,\mathrm{d}t + C_\varepsilon\left(E(T) + E(0) + \int_0^T D(t)\,\mathrm{d}t\right)\end{aligned}$$

for every $0 < \varepsilon < \frac{\gamma}{2}$.

Proof. The proof is similar as in the proof of the previous theorem, but now using the modified multiplier $M_\rho(A_D^{-1}\theta)$. Because $\theta \in C^1([0,T], L^2(\Gamma_0))$ for each $T > 0$ we have $A_D^{-1}\theta \in C^1([0,T], H^2(\Gamma_0))$, and thus Lemma 4.1 and Lemma 4.2 are applicable. The estimate will be derived from the following identity

$$\begin{aligned}&\int_0^T\int_{\Gamma_0} (\varphi_{tt} - \gamma\Delta\varphi_{tt} + \Delta^2\varphi + \Delta\theta)M_\rho(A_D^{-1}\theta)\,\mathrm{d}\Gamma_0\,\mathrm{d}t\\ &= \int_0^T\int_{\Gamma_0}\left(p - \nu\cdot\frac{\partial u}{\partial\nu}\right)M_\rho(A_D^{-1}\theta)\,\mathrm{d}\Gamma_0\,\mathrm{d}t\end{aligned} \tag{5.23}$$

obtained by multiplying the plate equation in (1.3) by $M_\rho(A_D^{-1}\theta)$ and integrating over time and space.

According to Lemma 4.1 with $f = A_D^{-1}\theta$, (5.19) and (3.25) we can estimate the right hand side of (5.23) as follows

$$\begin{aligned}&\left|\int_0^T\int_{\Gamma_0}\left(\nu\cdot\frac{\partial u}{\partial\nu} - p\right)M_\rho(A_D^{-1}\theta)\,\mathrm{d}\Gamma_0\,\mathrm{d}t\right| \leq C(E(T) + E(0))\\ &+ C_\varepsilon\int_0^T\int_\Omega |\nabla u|^2\,\mathrm{d}\Omega\,\mathrm{d}t + \frac{\varepsilon}{3}\int_0^T\int_{\Gamma_0} |\theta|^2 + |\nabla\theta|^2 + |\nabla\varphi_t|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t.\end{aligned} \tag{5.24}$$

On the other hand, using Lemma 4.2 with $f = A_D^{-1}\theta$, (5.19), (5.21) and (5.22) we obtain

$$\begin{aligned}&\left|\int_0^T\int_{\Gamma_0}(\varphi_{tt} - \gamma\Delta\varphi_{tt} + \Delta^2\varphi + \Delta\theta)I_\rho(A_D^{-1}\theta)\,\mathrm{d}\Gamma_0\,\mathrm{d}t\right|\\ &\leq\ C(E(T) + E(0)) + C_\varepsilon\int_0^T\int_\Omega |\nabla u|^2\,\mathrm{d}\Omega\,\mathrm{d}t + C_\varepsilon\int_0^T\int_{\Gamma_0} |\theta|^2 + |\nabla\theta|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t\\ &+ \frac{\varepsilon}{3}\int_0^T\int_{\Gamma_0} |\Delta\varphi|^2 + |\nabla\varphi_t|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t\end{aligned} \tag{5.25}$$

after majorizing every term involving φ_t via (3.25).

Now we estimate the remaining terms in (5.23) separately. First we have from (5.20)

$$\begin{aligned}
&\int_0^T\int_{\Gamma_0}(\varphi_{tt}-\gamma\Delta\varphi_{tt})A_D^{-1}\theta\,\mathrm{d}\Gamma_0\,\mathrm{d}t\\
&=\left[\int_{\Gamma_0}\varphi_t A_D^{-1}\theta+\gamma\nabla\varphi_t\cdot\nabla(A_D^{-1}\theta)\,\mathrm{d}\Gamma_0\right]_0^T\\
&+\int_0^T\int_{\Gamma_0}\varphi_t(I-D\gamma_0)\theta+\sigma\varphi_t A_D^{-1}\theta\Gamma_0\,\mathrm{d}t\\
&+\int_0^T\int_{\Gamma_0}\gamma\nabla\varphi_t\cdot\nabla(I-D\gamma_0)\theta+\gamma\sigma\nabla\varphi_t\cdot\nabla(A_D^{-1}\theta)\,\mathrm{d}\Gamma_0\,\mathrm{d}t\\
&+\int_0^T\int_{\Gamma_0}|\varphi_t|^2+\gamma|\nabla\varphi_t|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t. \qquad (5.26)
\end{aligned}$$

By trace theory and (2.4) there holds

$$\begin{aligned}
&\int_0^T\int_{\Gamma_0}|(I-D\gamma_0)\theta|^2+|\nabla(I-D\gamma_0)\theta|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t\\
&\leq C\int_0^T\int_{\Gamma_0}|\theta|^2+|\nabla\theta|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t. \qquad (5.27)
\end{aligned}$$

If J_1 is the second term on the right hand side of the equation (5.26) then

$$|J_1|\leq\frac{\varepsilon}{3}\int_0^T\int_{\Gamma_0}|\nabla\varphi_t|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t+C_{\gamma,\varepsilon}\int_0^T D(t)\,\mathrm{d}t \qquad (5.28)$$

using Young's inequality, (3.25), (5.19), and (5.27). Since $A_D^{-1}\theta=0$ on Σ_0 we obtain from Green's identity

$$\begin{aligned}
&\int_0^T\int_{\Gamma_0}(\Delta^2\varphi+\Delta\theta)A_D^{-1}\theta\,\mathrm{d}\Gamma_0\,\mathrm{d}t \qquad (5.29)\\
&=-\int_0^T\int_{\Sigma_0}\Delta\varphi\frac{\partial}{\partial\nu}(A_D^{-1}\theta)\,\mathrm{d}\Sigma_0\,\mathrm{d}t+\int_0^T\int_{\Gamma_0}(\Delta\varphi)\theta-\nabla\theta\cdot\nabla(A_D^{-1}\theta)\,\mathrm{d}\Gamma_0\,\mathrm{d}t.
\end{aligned}$$

Let J_2 be the right hand side of the latter equation.

Trace theory and (5.19) imply

$$\int_0^T\int_{\Sigma_0}\Big|\frac{\partial}{\partial\nu}(A_D^{-1}\theta)\Big|^2\mathrm{d}\Sigma_0\,\mathrm{d}t\leq C\int_0^T\int_{\Gamma_0}|\theta|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t. \qquad (5.30)$$

Therefore from Theorem 5.1, (3.25), (5.30) and Young's inequality we deduce

$$\begin{aligned}
|J_2| &\leq \frac{\varepsilon}{3C}\int_0^T\int_{\Sigma_0}|\Delta\varphi|^2\,\mathrm{d}\Sigma_0\,\mathrm{d}t+\frac{\varepsilon}{3}\int_0^T\int_{\Gamma_0}|\Delta\varphi|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t+C_\varepsilon\int_0^T D(t)\,\mathrm{d}t\\
&\leq \frac{\varepsilon}{3}\int_0^T\int_{\Gamma_0}|\nabla\varphi_t|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t+\frac{2\varepsilon}{3}\int_0^T\int_{\Gamma_0}|\Delta\varphi|^2\,\mathrm{d}\Gamma_0\,\mathrm{d}t\\
&\quad+C_\varepsilon\left(E(T)+E(0)+\int_0^T D(t)\,\mathrm{d}t\right), \qquad (5.31)
\end{aligned}$$

where C is the constant appearing in Theorem 5.1. Combining (5.23)–(5.26) and (5.28)–(5.31), using the equivalence of the norms in $H^1(\Gamma_0)$ and V_R, and choosing $\varepsilon > 0$ as stated in the lemma prove the desired estimate. □

Lemma 5.3. *Let $T > 0$. Then for every data in $D(\mathcal{A}^2)$ we have the estimate*

$$\int_0^T \int_{\Gamma_0} |\Delta\varphi|^2 \,\mathrm{d}\Gamma_0 \,\mathrm{d}t \le 2\gamma \int_0^T \int_{\Gamma_0} |\nabla\varphi_t|^2 \,\mathrm{d}\Gamma_0 \,\mathrm{d}t + C\left(E(T) + E(0) + \int_0^T D(t)\,\mathrm{d}t\right).$$

Proof. For the proof we will utilize the multiplier $M_\rho\varphi$ which coincides with φ. Integrating by parts and using the boundary conditions

$$\begin{aligned}
0 &= \int_0^T \int_{\Gamma_0} \left(\varphi_{tt} - \gamma\Delta\varphi_{tt} + \Delta^2\varphi + \Delta\theta + \nu\cdot\frac{\partial u}{\partial \nu} - p\right)\varphi \,\mathrm{d}\Gamma_0\,\mathrm{d}t \\
&= \left[\int_{\Gamma_0} \varphi_t\varphi + \gamma\nabla\varphi_t\cdot\nabla\varphi \,\mathrm{d}\Gamma_0\right]_0^T - \int_0^T\int_{\Gamma_0} |\varphi_t|^2 + \gamma|\nabla\varphi_t|^2 \,\mathrm{d}\Gamma_0\,\mathrm{d}t \\
&+ \int_0^T\int_{\Gamma_0} |\Delta\varphi|^2 \,\mathrm{d}\Gamma_0\,\mathrm{d}t - \int_0^T\int_{\Gamma_0} \nabla\theta\cdot\nabla\varphi \,\mathrm{d}\Gamma_0\,\mathrm{d}t \\
&- \int_0^T\int_{\Gamma_0} \left(p - \nu\cdot\frac{\partial u}{\partial\nu}\right)\varphi \,\mathrm{d}\Gamma_0\,\mathrm{d}t. \qquad (5.32)
\end{aligned}$$

Because $\varphi \in C^1([0,T], \widehat{H}_0^2(\Gamma_0))$, Lemma 4.1 is applicable. According to Lemma 4.1, the Poincaré inequality and (3.25)

$$\begin{aligned}
&\left|\int_0^T\int_{\Gamma_0} \left(p - \nu\cdot\frac{\partial u}{\partial\nu}\right)\varphi \,\mathrm{d}\Gamma_0\,\mathrm{d}t\right| \\
&\le \frac{1}{4}\int_0^T\int_{\Gamma_0} |\Delta\varphi|^2 \,\mathrm{d}\Gamma_0\,\mathrm{d}t + C\left(E(T) + E(0) + \int_0^T D(t)\,\mathrm{d}t\right). \qquad (5.33)
\end{aligned}$$

Using Young's and Poincaré inequalities one obtains

$$\left|\int_0^T\int_{\Gamma_0} \nabla\theta\cdot\nabla\varphi \,\mathrm{d}\Gamma_0\,\mathrm{d}t\right| \le \frac{1}{4}\int_0^T\int_{\Gamma_0} |\Delta\varphi|^2 \,\mathrm{d}\Gamma_0\,\mathrm{d}t + C\int_0^T\int_{\Gamma_0} |\nabla\theta|^2 \,\mathrm{d}\Gamma_0\,\mathrm{d}t. \quad (5.34)$$

Solving for the integral term involving $\Delta\varphi$ in (5.32) and then invoking (5.33) and (5.34) proves the lemma. □

Remark 5.4. We would like to point out that the multipliers φ and $h\cdot\nabla\varphi$ are now standard and they were used in various plate equations and even for wave equations, e.g. [13, 16] and the references therein. The multiplier $A_D^{-1}\theta$ was first introduced by Avalos and Lasiecka [5, 6] for certain thermoelastic plate systems. It was used later for certain thermoelastic von Karman plate models by Perla Menzala and Zuazua [23, 24, 25]. The idea of using the Stokes map S originated from the recent work of Chuesov and Ryzhkova [10]. It was used in deriving suitable estimates for a Lyapunov functional associated with a fluid-plate interaction model.

Now, let us finish the proof of Theorem 3.4. From Lemma 5.2 and Lemma 5.3

$$\begin{aligned}
\frac{\gamma}{2}\int_0^T\int_{\Gamma_0} |\nabla\varphi_t|^2 \,\mathrm{d}\Gamma_0\,\mathrm{d}t \quad &\le \quad 2\gamma\varepsilon\int_0^T\int_{\Gamma_0} |\nabla\varphi_t|^2 \,\mathrm{d}\Gamma_0\,\mathrm{d}t \\
&+ C_\varepsilon\left(E(T) + E(0) + \int_0^T D(t)\,\mathrm{d}t\right)
\end{aligned}$$

for every $0 < \varepsilon < \frac{\gamma}{2}$. Choosing $\varepsilon > 0$ in such a way that $2\varepsilon \leq \min(\frac{1}{4}, \gamma)$ we obtain from the above inequality the estimate

$$\frac{\gamma}{4}\int_0^T\int_{\Gamma_0} |\nabla\varphi_t|^2 \,\mathrm{d}\Gamma_0 \,\mathrm{d}t \leq C_\gamma \left(E(T) + E(0) + \int_0^T D(t)\,\mathrm{d}t \right) \tag{5.35}$$

and consequently from Lemma 5.3

$$\int_0^T\int_{\Gamma_0} |\Delta\varphi|^2 \,\mathrm{d}\Gamma_0 \,\mathrm{d}t \leq C_\gamma \left(E(T) + E(0) + \int_0^T D(t)\,\mathrm{d}t \right). \tag{5.36}$$

From (3.25) and Poincaré inequality we have

$$\int_0^T\int_{\Omega} |u|^2 \,\mathrm{d}\Omega \,\mathrm{d}t + \int_0^T\int_{\Gamma_0} |\varphi_t|^2 + |\theta|^2 \,\mathrm{d}\Gamma_0 \,\mathrm{d}t \leq C \int_0^T D(t)\,\mathrm{d}t \tag{5.37}$$

for some constant $C > 0$. Combining (5.35)–(5.37) yields (3.27) with $T^* = 0$ and in return (3.26). The proof of Theorem 3.4 is now completed.

References

[1] G. Avalos and F. Bucci, *Spectral analysis and rational decay rates of strong solutions to a fluid-structure PDE system*, e-Print arXiv:1312.48.12 [math.AP], December 2013, 19 pp.

[2] G. Avalos and F. Bucci, *Exponential decay properties of a mathematical model for a certain fluid-structure interaction*, e-Print arXiv:1403.6281v1 [math.AP], March 2014, 15 pp.

[3] G. Avalos and T. Clark, *Well-posedness and backward uniqueness for a PDE model of fluid-structure interaction*, to appear in Evol. Equ. Control Theory.

[4] G. Avalos and M. Dvorak, *A new maximality argument for a coupled fluid-structure interaction model with implications for a divergence-free finite element method*, Appl. Math. (Warsaw) **35**, No. 3, pp. 259–280, 2008.

[5] G. Avalos and I. Lasiecka, *Exponential stability of a thermoelastic system without mechanical dissipation*, IMA Preprint Series #1357, November 1995.

[6] G. Avalos and I. Lasiecka, *Exponential stability of a thermoelastic system with free boundary conditions without mechanical dissipation*, SIAM J. Math. Anal. **29**, No. 1, pp. 155–182, 1998.

[7] G. Avalos and R. Triggiani, *The coupled PDE system arising in fluid-structure interaction, Part I: Explicit semigroup generator and its spectral properties*, Contemporary Mathematics **440**, pp. 15–54, 2007.

[8] G. Avalos and R. Triggiani, *Fluid structure interaction with and without internal dissipation of the structure : A contrast study in stability*, Evol. Equ. Control Theory **2**, pp. 563–598, 2013.

[9] I. Chuesov, *A global attractor for a fluid-plate interaction model accounting only for longitudinal deformations of the plate*, Math. Methods Appl. Sci. **34**, pp. 1801-1812, 2011.

[10] I. Chuesov and I. Ryzhkova, *A global attractor for a fluid-plate interaction model*, Comm. Pure Appl. Anal. **12**, No. 4, pp. 1635–1656, 2013.

[11] H. Cohen and S. I. Rubinow, *Some mathematical topics in Biology*, Proc. Symp. on System Theory Polytechnic Press, New York, pp. 321–337, 1965.

[12] A. Haraux, *Decay rate of the range component of solutions to some semilinear evolution equations*, NoDea **13**, pp. 435–45, 2006.

[13] V. Komornik, Exact Controllability and Stabilization: The Multiplier Method, Research in Applied Mathematics, John Wiley & Sons, New York, 1994.

[14] J. Lagnese, Boundary Stabilization of Thin Plates, SIAM Stud. Appl. Math. 10, SIAM, Philadelphia, PA, 1989.

[15] I. Lasiecka and Y. Lu, *Interface feedback control stabilisation of a nonlinear fluid-structure interaction*, Nonlinear Analysis **75**, pp. 1449–1460, 2012.

[16] I. Lasiecka and R. Triggiani, *Exact controllability and uniform stabilisation of Kirchoff plates with boundary control only on $\Delta\omega|_\Sigma$ and homogeneous boundary displacements*, J. Differential Equations **88**, pp. 62–101, 1991.

[17] I. Lasiecka and R. Triggiani, *Analyticity of thermo-elastic semigroups with coupled hinged/Neumann B.C.*, Abstr. Appl. Anal. **3**, No. 1-2, pp. 1–236, 1998.
[18] J. L. Lions, Quelques méthodes de résolution des probelèmes aux limites non linéares, Dunod, Paris, 1969.
[19] J. L. Lions, Contrôlabilité exacte, perturbations et stabilization de systèmes distribués, Vol. 1, Masson, Paris, 1989.
[20] J. L. Lions and E. Magenes, Non-homogeneous Boundary Value Problems and Applications, Vol.1, Springer-Verlag, New York, 1972.
[21] J. L. Lions and E. Zuazua, *Approximate controllability of a hydro-elastic coupled system*, ESAIM: Control, Optimisation and Calculus of Variations **1**, pp. 1–15, 1995.
[22] Z. Liu and M. Renardy, *A note on the equations of a thermoelastic plate*, Appl. Math. Lett. **8**, pp. 1–6, 1995.
[23] G. Perla Menzala and E. Zuazua, *Explicit exponential decay rates for solutions of von Karman's system of thermoelastic plates*, C.R. Acad. Sci. Paris **324**, pp. 49-54, 1997.
[24] G. Perla Menzala and E. Zuazua, *Energy decay rates for the von Karman system of thermoelastic plates*, Differential and Integral Equations **11**, No. 5, pp. 775–770, 1998.
[25] G. Perla Menzala and E. Zuazua, *The energy decay rate for the modified von Karman system of thermoelastic plates: an improvement*, App. Math. Lett. **16**, pp. 531–534, 2003.
[26] A. Quarteroni and A. Valli, Numerical Approximations of Partial Differential Equations, Springer, Heidelberg, 2008.
[27] R. Temam, Navier-Stokes Equations, Theory and Numerical Analysis, North-Holland, Amsterdam, 1979.
[28] M. Tucsnak and G. Weiss, Observation and Control for Operator Semigroups, Birkhäuser-Verlag, Basel, 2009.