STABILIZATION OF VISCOELASTIC WAVE EQUATIONS WITH DISTRIBUTED OR BOUNDARY DELAY

GILBERT PERALTA

Abstract. The wave equation with viscoelastic boundary damping and internal or boundary delay is considered. The memory kernel is assumed to be integrable and completely monotonic. Under certain conditions on the damping factor, delay factor and the memory kernel it is shown that the energy of the solutions decay to zero either asymptotically or exponentially. In the case of internal delay, the result is obtained through spectral analysis and the Gearhart-Prüss Theorem, whereas in the case of boundary delay, it is obtained using the energy method.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with C^2-boundary. Consider the wave equation with interior delay and viscoelastic boundary damping

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) + a_0 u_t(t,x) + a_1 u_t(t-\tau,x) = 0, \quad \text{in } (0,\infty)\times\Omega, \\ \dfrac{\partial u}{\partial \nu}(t,x) + a \star u_t(t,x) = 0, \quad \text{on } (0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \quad \text{in } \Omega, \\ u_t(t,x) = f(t,x), \quad \text{on } (-\tau,0)\times\Omega, \end{cases} \tag{1.1}$$

where $\tau > 0$ is a constant delay parameter, a_0 is the damping factor and a_1 is the delay factor. Here, ν is the unit outward vector normal to the boundary $\partial\Omega$ of Ω, and the convolution $a \star v$ is defined by

$$a \star v(t,\cdot) = \int_0^t a(t-s)v(s,\cdot)\,\mathrm{d}s, \qquad t > 0.$$

The system (1.1) models the evolution of sound in a compressible fluid within a viscoelastic surface without accounting for viscoelasticity and the variable u represents the acoustic pressure, see [**17**] for example. The energy of a solution of (1.1), without viscoelasticity and delay, is defined by

$$E_w(t) = \int_\Omega |u_t(t,x)|^2 + |\nabla u(t,x)|^2\,\mathrm{d}x. \tag{1.2}$$

Our goal is to prove that $E_w(t)$ decays to zero as t tends to infinity.

Date: February 19, 2015.

Key words and phrases. Wave equations, viscoelasticity, feedback delays, stabilization, completely monotonic kernels

Mathematics Subject Classification: 35L20, 47D06, 93D15

Address. Department of Mathematics and Computer Science, University of the Philippines Baguio, Governor Pack Road, Baguio City, Philippines 2600.

E-mail. grperalta@upb.edu.ph .

It is well known that delay can have a destabilizing effect to systems that are asymptotically stable in the absence of delay [**1, 3, 4, 8, 14, 16**]. However, if the damping factor is larger than the delay factor then one can show exponential stability for the wave equation. In particular, consider the wave equation with homogeneous Dirichlet boundary condition on a part of the boundary

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) + a_0 u_t(t,x) + a_1 u_t(t-\tau,x) = 0, \quad \text{in } (0,\infty)\times\Omega, \\ \dfrac{\partial u}{\partial \nu}(t,x) = 0, \quad \text{on } (0,\infty)\times\Gamma_D, \\ \dfrac{\partial u}{\partial \nu}(t,x) + k u_t(t,x) = 0, \quad \text{in } (0,\infty)\times\Gamma_N, \end{cases} \tag{1.3}$$

where $\Gamma_D \neq \emptyset$, $\Gamma_D \cup \Gamma_N = \partial\Omega$, $\overline{\Gamma}_D \cap \overline{\Gamma}_N = \emptyset$ and the domain Ω satisfies some geometric conditions. If $k = 0$ and $a_0 > a_1 \geq 0$ then the exponential decay of the energy of the solutions has been shown by Nicaise and Pignotti [**14**] using observability estimates for the wave equation with mixed Dirichlet-Neumann boundary conditions. For $k > 0$, $a_0 = 0$ and sufficiently small $a_1 > 0$, it has been shown in [**1**] that (1.3) is uniformly exponentially stable. This is achieved by rewriting the initial-boundary value problem into a pure initial value problem in an extended state space and using multipliers to derive the necessary decay property. However, in the case $k = 0$ and $a_0 = a$, there are solutions with constant energies. In other words, the delay component $a_1 u(\cdot - \tau)$ cancels the dissipative effect of the damping term $a_0 u_t$ in (1.3).

In this paper, we consider completely monotonic and integrable kernels for (1.1) as in [**5**]. A function $a \in C^\infty((0,\infty);\mathbb{R})$ is called *completely monotonic* if

$$(-1)^j a^{(j)}(t) \geq 0, \text{ for all } t > 0,\ j = 0, 1, \dots .$$

According to Bernstein Theorem [**9, Theorem 2.5**], a is completely monotonic if and if only there exists a locally finite positive measure $\mu \in \mathcal{M}_{\mathrm{loc}}((0,\infty);\mathbb{R})$ such that

$$a(t) = \int_0^\infty e^{-st}\,\mathrm{d}\mu(s), \qquad t > 0.$$

Furthermore, for a completely monotonic function a, we have $a \in L^1((0,\infty);\mathbb{R})$ if and only if

$$\mu(\{0\}) = 0 \qquad \text{and} \qquad \hat{a}(0) = \int_0^\infty \frac{1}{s}\,\mathrm{d}\mu(s) < \infty.$$

Let $a \in L^1((0,\infty);\mathbb{R})$ be completely monotonic with corresponding measure $\mu \neq 0$. Then the Laplace transform of a is given by

$$\hat{a}(\lambda) = \int_0^\infty \frac{1}{\lambda + s}\,\mathrm{d}\mu(s), \qquad \Re\lambda > 0, \tag{1.4}$$

and admits a holomorphic extension to $\mathbb{C} \setminus (-\infty, 0]$.

In the absence of delay and damping, that is, $a_0 = a_1 = 0$, the asymptotic stability of (1.1) has been shown in [**5**] using the well-known Arendt-Batty-Lyubic-Vu Theorem. This is the best we can obtain since it is possible to have eigenvalues arbitrarily close to the imaginary axis, see for instance [**6**]. We will show that if $0 < a_1 = a_0$, that is, the damping factor and the delay factor are equal, then the dissipative effect of the viscoelastic damping is strong enough to preserve the asymptotic stability of the wave equation (1.1). In the case $0 \leq a_1 < a_0$ we further have exponential stability. Because the boundary condition in (1.1) do not have a Dirichlet part, we cannot apply directly the energy method employed in the references mentioned above. Instead, we use the frequency-domain approach. Our proof relies on a generalized Lax-Milgram Lemma and the Gearhart-Prüss Theorem.

We also consider the case where the delay occurs at the boundary

$$
\begin{cases}
u_{tt}(t,x) - \Delta u(t,x) = 0, \quad \text{in } (0,\infty)\times\Omega, \\
u(t,x) = 0, \quad \text{on } (0,\infty)\times\Gamma_D, \\
\dfrac{\partial u}{\partial \nu}(t,x) + a\star u_t(t-\tau,x) + cu_t(t,x) = 0, \quad \text{on } (0,\infty)\times\Gamma_N, \\
u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \quad \text{in } \Omega, \\
u_t(t,x) = f(t,x), \quad \text{on } (-\tau,0)\times\Gamma_N,
\end{cases}
\tag{1.5}
$$

and show that if $\hat{a}(0) < c$ and Ω satisfies a suitable geometric condition, then the energy of the solution decays to zero exponentially. This assumption is natural, since if $\hat{a}(\lambda) = k$ for some constant k then formally the convolution becomes

$$a\star u_t = \mathscr{L}^{-1}(\mathscr{L}(a)\mathscr{L}(u_t)) = ku_t$$

where $\mathscr{L}$ denotes the Laplace transform. Then the condition $\hat{a}(0) < c$ coincides with the one given in [**14**].

The difficult task is to modify the energy functional E_w suitable to prove the decay property. For the delay variable this is standard. In fact, the energy associated with it is given by

$$E_d(t) = \frac{c}{2}\int_{-\tau}^{0}\int_{\Gamma_N} |u_t(t+\theta,x)|^2 \,\mathrm{d}x\,\mathrm{d}\theta.$$

Aside from this, we also need to add the contribution of viscoelasticity to the energy. For this, we define the following energy corresponding to the memory term

$$E_m(t) = \frac{1}{2}\int_0^\infty\int_{\Gamma_N}\left|\int_0^t e^{-s(t-r)}u_t(r-\tau,x)\,\mathrm{d}r\right|^2 \mathrm{d}x\,\mathrm{d}\mu(s).$$

The total energy for (1.5) is then defined as

$$E(t) = E_w(t) + E_d(t) + E_m(t), \qquad t\ge 0.$$

We would like to point out that our stability result for (1.1) is only possible for a factor space of the state space whereas the stability result for (1.5) is valid for the whole state space. Other works related to wave equations with memory and delay can be found in [**2, 11, 15**] to name a few.

2. Semigroup Well-Posedness

In this section, we will reformulate (1.1) and (1.5) as first order Cauchy problems on suitable state spaces and prove the well-posedness using semigroup theory. First let us consider the problem (1.1) with internal delay. Let $v(t,x) = u_t(t,x)$, $w(t,x) = \nabla u(t,x)$ and $z(t,\theta,x) = u_t(t+\theta,x) = v(t+\theta,x)$ for $t>0$, $x\in\Omega$ and $\theta\in(-\tau,0)$. In order to keep track of the memory, we introduce another variable $\psi : (0,\infty)\times(0,\infty)\times\partial\Omega\to\mathbb{C}^n$ defined by

$$\psi(t,s,x) = \int_0^t e^{-s(t-r)}v(r,x)\,\mathrm{d}r, \qquad t,s>0,\ x\in\partial\Omega.$$

The convolution in (1.1) can be written in terms of ψ as

$$\begin{aligned} a \star v(t,x) &= \int_0^t \left(\int_0^\infty e^{-s(t-r)}\,\mathrm{d}\mu(s) \right) v(r,x)\,\mathrm{d}r \\ &= \int_0^\infty \psi(t,s,x)\,\mathrm{d}\mu(s). \end{aligned}$$

Then (1.1) is equivalent to the linear system

$$\begin{cases} v_t(t,x) - \operatorname{div} w(t,x) + a_0 v(t,x) + a_1 z(t,-\tau,x) = 0, \quad \text{in } (0,\infty)\times\Omega, \\ w_t(t,x) - \nabla v(t,x) = 0, \quad \text{in } (0,\infty)\times\Omega, \\ z_t(t,\theta,x) = z_\theta(t,\theta,x), \quad \text{in } (0,\infty)\times(-\tau,0)\times\Omega, \\ \psi_t(t,s,x) = -s\psi(t,s,x) + v(t,x), \quad \text{on } (0,\infty)\times(0,\infty)\times\partial\Omega, \\ (w\cdot\nu)(t,x) + \displaystyle\int_0^\infty \psi(t,s,x)\,\mathrm{d}\mu(s) = 0, \quad \text{on } (0,\infty)\times\partial\Omega, \\ v(0,x) = u_1(x), \quad w(0,x) = \nabla u_0(x), \quad \text{in } \Omega, \\ z(0,\theta,x) = f(\theta,x), \quad \text{in } (-\tau,0)\times\Omega, \\ \psi(0,s,x) = 0, \quad \text{on } (0,\infty)\times\partial\Omega. \end{cases}$$

We consider the state space to be complex-valued because we will use some information about the spectrum of the generator.

For simplicity, we introduce the abbreviations $L^p_\mu := L^p((0,\infty); L^2(\partial\Omega;\mathbb{C}^n), \mathrm{d}\mu)$ for $p \geq 1$ and $L^2_\tau := L^2((-\tau,0); L^2(\Omega;\mathbb{C}^n))$. These are the state spaces for the memory and delay variables, respectively. Let

$$X = L^2(\Omega;\mathbb{C}^n) \times L^2(\Omega;\mathbb{C}^{n\times n}) \times L^2_\tau \times L^2_\mu$$

be the Hilbert space equipped with the inner product

$$\begin{aligned} \langle (v_1,w_1,z_1,\psi_1),(v_2,w_2,z_2,\psi_2)\rangle_X &= \int_\Omega (v_1(x)\cdot v_2(x) + w_1(x)\cdot w_2(x))\,\mathrm{d}x \\ + \kappa \int_{-\tau}^0 \int_\Omega z_1(\theta,x)\cdot z_2(\theta,x)\,\mathrm{d}x\,\mathrm{d}\theta &+ \int_0^\infty \int_{\partial\Omega} \psi_1(s,x)\cdot\psi_2(s,x)\,\mathrm{d}x\,\mathrm{d}s \end{aligned}$$

where $\kappa = a_0$ if $a_0 > 0$ and $\kappa = 1$ if $a_0 = 0$. The dot represents either the inner product in $\mathbb{C}^n$ or $\mathbb{C}^{n\times n}$ where it is applicable. Let $L^2_{\mathrm{div}}(\Omega) = \{w \in L^2(\Omega;\mathbb{C}^{n\times n}) : \operatorname{div} w \in L^2(\Omega;\mathbb{C}^n)\}$, where div is the distributional divergence. Recall that there exists a generalized normal trace operator $w \mapsto w\cdot\nu \in \mathcal{L}(L^2_{\mathrm{div}}(\Omega), H^{-\frac12}(\partial\Omega;\mathbb{C}^n))$ such that the following generalized Green's identity

$$\int_\Omega \operatorname{div} w(x)\cdot u(x)\,\mathrm{d}x = \langle w\cdot\nu, \Gamma u\rangle_{H^{-\frac12}(\partial\Omega)\times H^{\frac12}(\partial\Omega)} - \int_\Omega w(x)\cdot\nabla u(x)\,\mathrm{d}x$$

holds for all $w \in L^2_{\mathrm{div}}(\Omega)$ and $u \in H^1(\Omega;\mathbb{C}^n)$, see **[18]** for example. Here $\Gamma : H^1(\Omega;\mathbb{C}^n) \to H^{\frac12}(\partial\Omega;\mathbb{C}^n)$ is the usual trace operator.

Define the operator $A : D(A) \subset X \to X$ by

$$A\begin{pmatrix} v \\ w \\ z \\ \psi \end{pmatrix} = \begin{pmatrix} \operatorname{div} w - a_0 v - a_1 z(-\tau) \\ \nabla v \\ z_\theta \\ -s\psi + \Gamma v \end{pmatrix}$$

where its domain is given by

$$D(A) = \{(v, w, z, \psi) \in X : v \in H^1(\Omega; \mathbb{C}^n),\ z \in H^1((-\tau, 0); L^2(\Omega; \mathbb{C}^n)),$$
$$w \in L^2_{\mathrm{div}}(\Omega),\ -s\psi(s) + \Gamma v \in L^2_\mu,\ z(0) = v,\ w \cdot \nu + \int_0^\infty \psi(s)\,\mathrm{d}\mu(s) = 0\}.$$

Note that $-s\psi(s) + \Gamma v \in L^2_\mu$ implies $\psi \in L^1_\mu$. Indeed, this follows from the equality

$$\psi(s) = \frac{1}{1+s}\psi(s) + \frac{1}{1+s}\Gamma v - \frac{\Gamma v - s\psi(s)}{1+s}$$

and the fact that $s \mapsto \frac{1}{1+s} \in L^1_\mu \cap L^2_\mu$. The problem (1.1) can now be written as a first order evolution equation in X

$$\begin{cases} \dot{U}(t) = AU(t), & t > 0, \\ U(0) = U_0, \end{cases} \tag{2.1}$$

where $U_0 = (u_1, \nabla u_0, f, 0)$.

Theorem 2.1. *The operator A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X. If $0 \leq a_1 \leq a_0$ then the semigroup consists of contractions. In particular, for every $U_0 \in X$ (resp. $U_0 \in D(A)$) the Cauchy problem (2.1) has a unique solution $U \in C([0,\infty); X)$ (resp. $U \in C^1([0,\infty); X) \cap C([0,\infty); D(A))$).*

Proof. Let $(v, w, z, \psi) \in D(A)$. Applying the generalized Green's identity and the boundary conditions $z(0) = v$ and $w \cdot \nu = -\int_0^\infty \psi(s)\,\mathrm{d}\mu(s)$ we have

$$\begin{aligned}
\langle A(v,w,z,\psi), (v,w,z,\psi)\rangle_X &= \int_\Omega \operatorname{div} w(x) \cdot v(x)\,\mathrm{d}x - a_0 \int_\Omega |v(x)|^2\,\mathrm{d}x \\
&- a_1 \int_\Omega z(-\tau, x) \cdot v(x)\,\mathrm{d}x + \int_\Omega \nabla v(x) \cdot w(x)\,\mathrm{d}x \\
&+ \kappa \int_{-\tau}^0 \int_\Omega z_\theta(\theta, x) \cdot z(\theta, x)\,\mathrm{d}x\,\mathrm{d}\theta - \int_0^\infty \int_{\partial\Omega} s|\psi(s,x)|^2\,\mathrm{d}x\,\mathrm{d}\mu(s) \\
&+ \int_0^\infty \int_{\partial\Omega} \Gamma v(x) \cdot \psi(s,x)\,\mathrm{d}x\,\mathrm{d}\mu(s) \\
&= -\left(a_0 - \frac{\kappa}{2}\right) \int_\Omega |v(x)|^2\,\mathrm{d}x - a_1 \int_\Omega z(-\tau,x) \cdot v(x)\,\mathrm{d}x - \frac{\kappa}{2} \int_\Omega |z(-\tau,x)|^2\,\mathrm{d}x \\
&- \int_0^\infty \int_{\partial\Omega} s|\psi(s,x)|^2\,\mathrm{d}x\,\mathrm{d}\mu(s) + i\kappa \Im \int_{-\tau}^0 \int_\Omega z_\theta(\theta,x) \cdot z(\theta,x)\,\mathrm{d}x\,\mathrm{d}\theta \\
&+ 2i\Im \left(\int_\Omega \nabla v(x) \cdot w(x)\,\mathrm{d}x + \int_0^\infty \int_{\partial\Omega} \Gamma v(x) \cdot \psi(s,x)\,\mathrm{d}x\,\mathrm{d}\mu(s) \right).
\end{aligned}$$

Taking the real part and using the Cauchy-Schwarz inequality we obtain

$$\Re(A(v,w,z,\psi), (v,w,z,\psi))_X \leq -\int_0^\infty s\|\psi(s)\|^2_{L^2}\,\mathrm{d}\mu(s) + k \int_\Omega |v(x)|^2\,\mathrm{d}x \tag{2.2}$$

where $k = \frac{1}{2}(a_1^2 + 1)$ if $a_0 = 0$ and $k = \frac{1}{2}\left(a_1^2/a_0 - a_0\right)$ if $a_0 > 0$. The first integral is finite since $s|\psi(s)|^2 = \Gamma v \cdot \psi(s) - (-s\psi(s) + \Gamma v) \cdot \psi(s) \in L^1_\nu$. In particular, if $a_0 \geq a_1 > 0$ then $k \leq 0$ and therefore A is dissipative. The inequality (2.2) also implies that $A - kI$ is dissipative. The case where $a_0 = a_1 = 0$ was already established in [**5**].

The next step is to show the range condition $R(\lambda I - A) = X$ for all $\lambda > 0$. Let $(f, g, h, \phi) \in X$. The equation $(\lambda I - A)(v, w, z, \psi) = (f, g, h, \phi)$ for $(v, w, z, \psi) \in D(A)$ is equivalent to the system

$$\begin{aligned}
\lambda v - \operatorname{div} w + a_0 v + a_1 z(-\tau) &= f && (2.3)\\
\lambda w - \nabla v &= g && (2.4)\\
\lambda z(\theta) - z_\theta(\theta) &= h(\theta) && (2.5)\\
z(0) &= v && (2.6)\\
(\lambda + s)\psi(s) - \Gamma v &= \phi(s) && (2.7)\\
w \cdot \nu + \int_0^\infty \psi(s)\,\mathrm{d}\mu(s) &= 0. && (2.8)
\end{aligned}$$

The variation of parameters formula applied to (2.5) and (2.6) gives

$$z(\theta) = e^{\lambda\theta} v + \int_\theta^0 e^{\lambda(\theta - \vartheta)} h(\vartheta)\,\mathrm{d}\vartheta, \qquad \theta \in (-\tau, 0). \tag{2.9}$$

Solving for w and ψ in (2.4) and (2.7), respectively, we get

$$w = \frac{1}{\lambda}(g + \nabla v) \tag{2.10}$$

and

$$\psi(s) = \frac{1}{\lambda + s}(\phi(s) + \Gamma v), \qquad s > 0. \tag{2.11}$$

Taking the inner product in $L^2(\Omega; \mathbb{C}^n)$ of (2.3) with λu for $u \in H^1(\Omega, \mathbb{C}^n)$ and using (2.9) yield

$$\lambda(\lambda + a_0 + a_1 e^{-\lambda\tau}) \int_\Omega v \cdot u\,\mathrm{d}x - \int_\Omega \operatorname{div}(\lambda w) \cdot u\,\mathrm{d}x = \lambda \int_\Omega f_\lambda \cdot u\,\mathrm{d}x \tag{2.12}$$

where

$$f_\lambda := f - a_1 \int_{-\tau}^0 e^{-\lambda(\tau + \vartheta)} h(\vartheta)\,\mathrm{d}\vartheta.$$

Green's identity together with (2.8), (2.10) and (2.11) yields

$$\begin{aligned}
\int_\Omega \operatorname{div}(\lambda w) \cdot u\,\mathrm{d}x \quad = \quad & -\lambda \int_{\partial\Omega} \left(\int_0^\infty \frac{\phi(s)}{\lambda + s}\,\mathrm{d}\mu(s) \right) \cdot \Gamma u\,\mathrm{d}x - \lambda \hat{a}(\lambda) \int_{\partial\Omega} \Gamma v \cdot \Gamma u\,\mathrm{d}x\\
& - \int_\Omega (\nabla v + g) \cdot \nabla u\,\mathrm{d}x.
\end{aligned}$$

Plugging the latter equality in (2.12) and rearranging the terms, we obtain the variational equation

$$a(v, u) = F(u), \qquad \text{for all } u \in H^1(\Omega; \mathbb{C}^n) \tag{2.13}$$

where $a : H^1(\Omega; \mathbb{C}^n) \times H^1(\Omega; \mathbb{C}^n) \to \mathbb{C}$ and $F : H^1(\Omega; \mathbb{C}^n) \to \mathbb{C}$ are the sesquilinear and antilinear forms defined by

$$a(v, u) = \lambda(\lambda + a_0 + a_1 e^{-\lambda\tau}) \int_\Omega v \cdot u\,\mathrm{d}x + \int_\Omega \nabla v \cdot \nabla u\,\mathrm{d}x + \lambda \hat{a}(\lambda) \int_{\partial\Omega} \Gamma v \cdot \Gamma u\,\mathrm{d}x$$

and

$$F(u) = \lambda \int_\Omega f_\lambda \cdot u\,\mathrm{d}x - \int_\Omega g \cdot \nabla u\,\mathrm{d}x - \lambda \int_{\partial\Omega} \left(\int_0^\infty \frac{\phi(s)}{\lambda + s}\,\mathrm{d}\mu(s) \right) \cdot \Gamma u\,\mathrm{d}x.$$

Since a is H^1-coercive and a and F are both continuous, it follows from Lax-Milgram Lemma that there exists a unique $v \in H^1(\Omega;\mathbb{C}^n)$ such that (2.13) is satisfied. Defining z, w and ψ by (2.9), (2.10) and (2.11), respectively, and integrating by parts we can see that $(v, w, z, \psi) \in D(A)$ where v is the solution of (2.13). Thus $R(\lambda I - A) = X$ for all $\lambda > 0$.

Suppose that $a_0 = 0 < a_1$. In this case, we have $k > 0$ and so $R(\lambda I - (A - kI)) = R((\lambda + k)I - A) = X$ for all $\lambda > 0$. Thus by the Lumer-Phillips Theorem, the operator $A - kI$ generates a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ and therefore $A = (A - kI) + kI$ generates the strongly continuous semigroup $(e^{kt}S(t))_{t\geq 0}$ on X by the perturbation theorem. If $a_0 \geq a_1 \geq 0$ then A is dissipative and hence A generates a strongly continuous semigroup of contractions on X. □

Now let us turn to the problem (1.5) with boundary delay. In this case we assume that the states are real-valued. Suppose that $\Gamma_D \neq \emptyset$, $\Gamma_D \cup \Gamma_N = \partial\Omega$, $\overline{\Gamma}_D \cap \overline{\Gamma}_N = \emptyset$ and there exists a strictly convex $m \in C^2(\overline{\Omega})$, that is, there is $\alpha > 0$ such that $\nabla^2 m(x)\xi \cdot \xi \geq \alpha|\xi|^2$ for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n$, and $\nabla m(x) \cdot \nu(x) \leq 0$ for all $x \in \Gamma_D$. Here, $\nabla^2 m$ denotes the Hessian of m. The existence of m allows us to apply a classical observability estimate for the wave equation.

Let $v(t,x) = u_t(t,x)$ for $(t,x) \in (0,\infty) \times \Omega$, $z(t,\theta,x) = u_t(t+\theta,x) = v(t+\theta,x)$ for $(t,\theta,x) \in (0,\infty) \times (-\tau,0) \times \Gamma_N$ and

$$\psi(t,s,x) = \int_0^t e^{-s(t-r)} u_t(r-\tau,x)\,\mathrm{d}r = \int_0^t e^{-s(t-r)} z(r,-\tau,x)\,\mathrm{d}r \tag{2.14}$$

for $(t,s,x) \in (0,\infty) \times (0,\infty) \times \Gamma_N$. Then (1.5) is equivalent to the system

$$\begin{cases} u_t(t,x) - v(t,x) = 0, \quad \text{in } (0,\infty) \times \Omega, \\ v_t(t,x) - \Delta u(t,x) = 0, \quad \text{in } (0,\infty) \times \Omega, \\ z_t(t,\theta,x) = z_\theta(t,\theta,x), \quad \text{on } (0,\infty) \times \Gamma_N, \\ \psi_t(t,s,x) = -s\psi(t,s,x) + z(t,-\tau,x), \quad \text{on } (0,\infty) \times (0,\infty) \times \Gamma_N, \\ u(t,x) = 0, \quad \text{on } (0,\infty) \times \Gamma_D, \\ \dfrac{\partial u}{\partial \nu}(t,x) + \displaystyle\int_0^\infty \psi(t,s,x)\,\mathrm{d}\mu(s) + cv(t,x) = 0, \quad \text{on } (0,\infty) \times \Gamma_N, \\ u(0,x) = u_0(x), \quad v(0,x) = u_1(x), \quad \text{in } \Omega, \\ z(0,\theta,x) = f(\theta,x), \quad \text{on } (-\tau,0) \times \Gamma_N, \\ \psi(0,s,x) = 0, \quad \text{on } (0,\infty) \times \Gamma_N. \end{cases}$$

Due to the homogeneous Dirichlet boundary conditions on Γ_D, we will pose the problem on the state space

$$\tilde{X} = H^1_{\Gamma_D}(\Omega) \times L^2(\Omega) \times L^2((-\tau,0);L^2(\Gamma_N)) \times \tilde{L}^2_\mu$$

where $H^1_{\Gamma_D}(\Omega) = \{u \in H^1(\Omega) : u_{|\Gamma_D} = 0\}$ and $\tilde{L}^2_\mu = L^2_\mu((0,\infty);L^2(\Gamma_N),\mathrm{d}\mu)$. Equipped with the inner product

$$\langle (u_1,v_1,z_1,\psi_1),(u_2,v_2,z_2,\psi_2)\rangle_{\tilde{X}} = \int_\Omega (\nabla v_1(x) \cdot \nabla v_2(x) + w_1(x)w_2(x))\,\mathrm{d}x$$
$$+ \hat{a}(0) \int_{-\tau}^0 \int_{\Gamma_N} z_1(\theta,x)z_2(\theta,x)\,\mathrm{d}x\,\mathrm{d}\theta + \int_0^\infty \int_{\Gamma_N} \psi_1(s,x)\psi_2(s,x)\,\mathrm{d}s\,\mathrm{d}x,$$

$\tilde{X}$ is a Hilbert space. Let $E(\Delta) = \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}$ be equipped with the graph norm $\|u\|_{E(\Delta)} = (\|u\|^2_{H^1(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)})^{\frac{1}{2}}$ where Δ denotes the distributional Laplacian. Recall that there exists a generalized first order trace operator $u \mapsto \frac{\partial u}{\partial \nu} \in \mathcal{L}(E(\Delta); H^{-\frac{1}{2}}(\Gamma_N))$ such that the following generalized Green's identity holds

$$\int_\Omega (\Delta u) w \,\mathrm{d}x = \left\langle \frac{\partial u}{\partial \nu}, w \right\rangle_{H^{-\frac{1}{2}}(\Gamma_N)\times H^{\frac{1}{2}}(\Gamma_N)} - \int_\Omega \nabla u \cdot \nabla w \,\mathrm{d}x$$

for every $u \in E(\Delta)$ and $w \in H^1_{\Gamma_D}(\Omega)$, see [**10**].

Define the operator $\tilde{A} : D(\tilde{A}) \subset \tilde{X} \to \tilde{X}$ by

$$\tilde{A}\begin{pmatrix} u \\ v \\ z \\ \psi \end{pmatrix} = \begin{pmatrix} v \\ \Delta u \\ z_\theta \\ -s\psi + z(-\tau) \end{pmatrix}$$

with domain

$$\begin{aligned} D(\tilde{A}) = \{&(u, v, z, \psi) \in \tilde{X} : u \in E(\Delta),\ z \in H^1((-\tau, 0); L^2(\partial\Omega)),\ v \in H^1(\Omega), \\ &- s\psi(s) + z(-\tau) \in \tilde{L}^2_\mu,\ z(0) = \Gamma v,\ \frac{\partial u}{\partial \nu} + \int_0^\infty \psi(s)\,\mathrm{d}\mu(s) + cv = 0\}. \end{aligned}$$

Then (1.5) can be written as a first order evolution equation in $\tilde{X}$

$$\begin{cases} \dot{U}(t) = \tilde{A}U(t), & t > 0, \\ U(0) = U_0 = (u_0, u_1, f, 0). \end{cases} \tag{2.15}$$

Using similar methods as in the proof of the previous theorem, the following well-posedness theorem can be proved. The details are omitted.

Theorem 2.2. *If $\hat{a}(0) \le c$ then $\tilde{A}$ generates a strongly continuous semigroup of contractions in $\tilde{X}$. For every $U_0 \in D(\tilde{A})$ there exists a unique solution $U \in C^1([0,\infty); \tilde{X}) \cap C([0,\infty); D(\tilde{A}))$ of (2.15).*

3. Internal Delay: Spectral Analysis and Stability

The first step is to prove that the spectrum of A not lying on the negative real axis consists only of eigenvalues.

Lemma 3.1. *It holds that $\sigma(A) \cap (\mathbb{C} \setminus (-\infty, 0]) = \sigma_p(A)$ where $\sigma(A)$ and $\sigma_p(A)$ denote the spectrum and point spectrum of A.*

To prove this, we need the following generalization of the Lax-Milgram Lemma. The proof of this lemma is contained in the proof of [**5, Theorem 3**].

Lemma 3.2 (Lax-Milgram-Fredholm)**.** *Let V and H be Hilbert spaces such that the embedding $V \subset H$ is compact and dense. Suppose that $a_V : V \times V \to \mathbb{C}$ and $a_H : H \times H \to \mathbb{C}$ are two bounded sesquilinear forms such that a_V is V-coercive and $F : V \to \mathbb{C}$ is a continuous conjugate linear form. The equation*

$$a_H(v, u) + a_V(v, u) = F(u), \qquad \forall u \in V \tag{3.1}$$

has either a unique solution $v \in V$ for all $F \in V'$ or has a nontrivial solution for $F = 0$.

Proof of Lemma 3.1. Using (1.4) it can be seen that $\lambda\hat{a}(\lambda) \in \mathbb{C} \setminus (-\infty, 0]$ and $\inf_{q\geq 0} |1 + q\lambda\hat{a}(\lambda)| > 0$ whenever $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, see [**5**] for details. We split the sesquilinear form a as $a = a_H + a_V$ where $a_V : H^1(\Omega;\mathbb{C}^n) \times H^1(\Omega;\mathbb{C}^n) \to \mathbb{C}$ and $a_H : L^2(\Omega;\mathbb{C}^n) \times L^2(\Omega;\mathbb{C}^n) \to \mathbb{C}$ are the two bounded sesquilinear forms defined by

$$a_V(v, u) = \int_\Omega (v \cdot u + \nabla v \cdot \nabla u)\,\mathrm{d}x + \lambda\hat{a}(\lambda) \int_{\partial\Omega} \Gamma v \cdot \Gamma u\,\mathrm{d}x$$

and

$$a_H(v, u) = (\lambda(\lambda + a_0 + a_1 e^{-\lambda\tau}) - 1) \int_\Omega v \cdot u\,\mathrm{d}x$$

respectively. According to the Lax-Milgram-Fredholm Lemma, the variational equality

$$a_H(v, u) + a_V(v, u) = G(u) \qquad \text{for all } u \in H^1(\Omega;\mathbb{C}^n) \tag{3.2}$$

has either a unique solution $v \in H^1(\Omega;\mathbb{C}^n)$ for all $G \in [H^1(\Omega;\mathbb{C}^n)]'$ or has a nontrivial solution for $G = 0$. As in the proof of the range condition in Theorem 2.1, it can be shown that the equation $(\lambda I - A)(v, w, z, \psi) = (f, g, h, \phi)$, for $(v, w, z, \psi) \in D(A)$ and for a given $(f, g, h, \phi) \in X$ and $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, is equivalent to (3.2). Therefore $\lambda \in \mathbb{C}\setminus(-\infty, 0]$ is either in the resolvent set of A or in the point spectrum of A. □

The next step is to prove that under the condition $0 \leq a_1 \leq a_0$, the generator A has no purely imaginary eigenvalues except for the origin.

Lemma 3.3. *The kernel of the operator A is given by*

$$\ker A = \{0\} \times Y \times \{0\} \times \{0\}$$

where

$$Y = \{w \in L^2_{\mathrm{div}}(\Omega) : \operatorname{div} w = 0, w \cdot \nu = 0\}. \tag{3.3}$$

If $0 \leq a_1 \leq a_0$ then the operator A has no purely imaginary eigenvalues, in other words, $\sigma_p(A) \cap i\mathbb{R} = \{0\}$.

Proof. Suppose that $A(v, w, z, \psi) = 0$. Then it follows that $z(\theta) = v$ in $H^1(\Omega;\mathbb{C}^n)$ for all $\theta \in (-\tau, 0)$, $\nabla v = 0$ and $\psi(s) = \frac{\Gamma v}{s}$. Thus, v is constant. Applying the generalized Green's identity and the boundary conditions

$$\begin{aligned}
(a_0 + a_1) \int_\Omega |v|^2\,\mathrm{d}x \;&=\; \int_\Omega \operatorname{div} w \cdot v\,\mathrm{d}x \\
=&- \langle \int_0^\infty \psi(s)\,\mathrm{d}\mu(s), \Gamma v\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} - \int_\Omega w \cdot \nabla v\,\mathrm{d}x \\
=&- \hat{a}(0) \int_{\partial\Omega} |\Gamma v|^2\,\mathrm{d}x.
\end{aligned}$$

Since the measure μ is positive this implies that $\Gamma v = 0$ and therefore $v = 0$. Consequently, $z = 0$, $\psi = 0$ and $w \in Y$. This proves that $\ker A \subset \{0\}\times Y\times\{0\}\times\{0\}$. The other inclusion is trivial.

Now let us show the second statement. We prove it by contradiction. Suppose that $ir \in \sigma_p(A)$ for some $r \in \mathbb{R} \setminus \{0\}$. Hence there exists a nonzero $(v, w, z, \psi) \in$

$D(A)$ such that

$$\begin{aligned} irv - \operatorname{div} w + a_0 v + a_1 z(-\tau) &= 0 && (3.4)\\ irw - \nabla v &= 0 && (3.5)\\ irz(\theta) - z_\theta(\theta) &= 0 && (3.6)\\ (ir+s)\psi(s) - \Gamma v &= 0. && (3.7) \end{aligned}$$

From (3.6) and the initial condition $z(0) = v$ we have $z(\theta) = e^{ir\theta}v$ and plugging this in (3.4) and using (3.5) we obtain

$$\Delta v = ir(ir + a_0 + a_1 e^{-ir\tau})v. \tag{3.8}$$

The boundary conditions and (3.5) imply

$$\frac{\partial v}{\partial \nu} = irw\cdot\nu = -ir\int_0^\infty \psi(s)\,\mathrm{d}\mu(s) = -ir\hat{a}(ir)\Gamma v. \tag{3.9}$$

Thus, $v \in H^2(\Omega;\mathbb{C}^n)$ from the regularity theory of elliptic equations [**10**]. Using Green's formula and (3.8)

$$\begin{aligned} ir(ir + a_0 + a_1 e^{-ir\tau})\int_\Omega |v|^2\,\mathrm{d}x &= \int_\Omega \Delta v\cdot v\,\mathrm{d}x\\ &= -ir\hat{a}(ir)\int_{\partial\Omega}|\Gamma v|^2\,\mathrm{d}x - \int_\Omega |\nabla v|^2\,\mathrm{d}x. \quad (3.10) \end{aligned}$$

Note that $\Im(ir\hat{a}(ir)) \neq 0$. Indeed,

$$\begin{aligned} ir\hat{a}(ir) &= \int_0^\infty \frac{ir}{ir+s}\,\mathrm{d}\mu(s)\\ &= r^2\int_0^\infty \frac{1}{r^2+s^2}\,\mathrm{d}\mu(s) + ir\int_0^\infty \frac{s}{r^2+s^2}\,\mathrm{d}\mu(s). \end{aligned}$$

Taking the imaginary part of (3.10) we have

$$\frac{r(a_0 + a_1\cos(r\tau))}{\Im(ir\hat{a}(ir))}\int_\Omega |v|^2 + \int_{\partial\Omega}|\Gamma v|^2 = 0. \tag{3.11}$$

Since $a_0 \geq a_1 \geq 0$ it holds that

$$\frac{r(a_0 + a_1\cos(r\tau))}{\Im(ir\hat{a}(ir))} = (a_0 + a_1\cos(r\tau))\left(\int_0^\infty \frac{s}{r^2+s^2}\,\mathrm{d}\mu(s)\right)^{-1} \geq 0.$$

Hence (3.11) implies that $\Gamma v = 0$ and consequently $\frac{\partial v}{\partial\nu} = 0$ from (3.9). Thus $v \in H_0^2(\Omega;\mathbb{C}^n)$ and therefore $v \in H^2(\mathbb{R}^n;\mathbb{C}^n)$ by extending v by zero outside Ω. Hence $v \in H^2(\mathbb{R}^n;\mathbb{C}^n)$ satisfies (3.8) which is a contradiction to the fact that the Laplacian Δ in $\mathbb{R}^n$ has an empty point spectrum. Therefore, we must have $ir \notin \sigma_p(A)$ for any nonzero real number r. This completes the proof of the lemma. □

The following lemma states that $(\ker A)^\perp = L^2(\Omega;\mathbb{C}^n)\times Y^\perp\times L^2_\tau\times L^2_\mu$ is invariant under the resolvent $(\lambda I - A)^{-1}$ for all positive λ.

Lemma 3.4. *For every $\lambda > 0$ we have $(\lambda I - A)^{-1}((\ker A)^\perp) \subset (\ker A)^\perp \cap D(A)$.*

Proof. According to the Helmholtz orthogonal decomposition [**18**] we have

$$L^2(\Omega;\mathbb{C}^{n\times n}) = Y \oplus Y^\perp$$

where Y is defined by (3.3) and its orthogonal complement is given by

$$Y^\perp = \{\nabla p \in L^2(\Omega;\mathbb{C}^{n\times n}) : p \in L^2(\Omega;\mathbb{C}^n)\}.$$

Let us show that if $\lambda > 0$, $(f,g,h,\phi) \in (\ker A)^{\perp}$ and $(v,w,z,\psi) \in D(A)$ satisfy $(\lambda I - A)(v,w,z,\psi) = (f,g,h,\phi)$ then $w \in Y^{\perp}$. Indeed, since $g \in Y^{\perp}$ we have $g = \nabla p$ for some $p \in L^2(\Omega;\mathbb{C}^n)$. Thus, according to (2.10) we have $w = \nabla(\lambda^{-1}(p + v)) \in Y^{\perp}$ since $\lambda^{-1}(p+v) \in L^2(\Omega;\mathbb{C}^n)$. □

Our stabilization results are based on the following theorems. For their proofs, we refer to [**7, Corollary V.2.22**] and [**7, Theorem V.1.11**], respectively.

Theorem 3.5 (Arendt-Batty-Lyubich-Vu)**.** *Let A be the generator of a bounded strongly continuous semigroup on a reflexive Banach space X. If*

(1) $\sigma_p(A) \cap i\mathbb{R} = \emptyset$ *and*
(2) $\sigma(A) \cap i\mathbb{R}$ *is countable*

then $(e^{At})_{t\geq 0}$ is strongly stable, that is, $e^{At}U \to 0$ in X for all $U \in X$.

Theorem 3.6 (Gearhart-Prüss)**.** *Let A be the generator of a bounded strongly continuous semigroup $T(t)$, $t \geq 0$, on a Hilbert space X. Then $T(t)$ is uniformly exponentially stable if and only if $\{\lambda \in \mathbb{C} : \Re\lambda > 0\} \subset \rho(A)$ and*

$$\sup_{\Re\lambda > 0} \|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} < \infty$$

where $\mathcal{L}(X)$ denotes the space of bounded linear operators in X into itself.

From Lemma 3.4 and [**19, Proposition 2.4.3**], the closed subspace $(\ker A)^{\perp}$ of X is invariant under the semigroup generated by A. Furthermore, the restricted semigroup $(T_p(t))_{t>0}$ defined by $T_p(t) = T(t)_{|(\ker A)^{\perp}}$ is a strongly continuous semigroup on $(\ker A)^{\perp}$ whose generator is given by the part of A in $(\ker A)^{\perp}$, that is, the operator $A_p : D(A_p) \to (\ker A)^{\perp}$ defined by $A_pU = AU$ for all $U \in D(A_p)$, where $D(A_p) = \{U \in D(A) \cap (\ker A)^{\perp} : AU \in (\ker A)^{\perp}\}$.

In the following theorem, we denote by Z the space consisting of functions $u \in L^2(\Omega;\mathbb{C}^n)$ such that $\nabla u \in Y^{\perp} \cap L^2_{\text{div}}(\Omega)$.

Theorem 3.7. *Let $\Pi : X \to \ker A$ be the orthogonal projection of X onto $\ker A$. If $0 \leq a_1 = a_0$ then for every $U \in X$ we have*

$$\lim_{t\to\infty} \|T(t)U - \Pi U\|_X = 0,$$

and in particular, $E(t) \to 0$ as $t \to \infty$ for every solution of (1.1) *with initial data*

$$(u_0, u_1, f) \in Z \times H^1(\Omega;\mathbb{C}^n) \times H^1((-\tau,0);L^2(\Omega;\mathbb{C}^n)). \tag{3.12}$$

If $0 \leq a_1 < a_0$ then there exist constants $M \geq 1$ and $\alpha > 0$ such that for all $t \geq 0$

$$\|T(t) - \Pi\|_{\mathcal{L}(X)} \leq Me^{-\alpha t},$$

in particular, $E(t) \leq Me^{-\alpha t}E(0)$, $t \geq 0$, for every solution of (1.1) *with initial data satisfying* (3.12)*.*

Proof. Since $T(t) = T(t)\Pi + T(t)(I - \Pi) = \Pi + T_p(t)(I - \Pi)$, it is enough to prove that

$$\lim_{t\to\infty} \|T_p(t)U\|_X = 0, \qquad \text{for all } U \in (\ker A)^{\perp}, \tag{3.13}$$

if $0 \leq a_1 \leq a_0$ and

$$\|T_p(t)U\|_X \leq Me^{-\alpha t}\|U\|_X, \qquad \text{for all } U \in (\ker A)^{\perp},\ t > 0, \tag{3.14}$$

in the case $0 \leq a_1 < a_0$. In both cases we have $\sigma(A_p) \subset \{\lambda \in \mathbb{C} : \Re\lambda \leq 0\}$ since A_p is dissipative. Using Lemma 3.1 and Lemma 3.3 it can be seen that

$\{\lambda \in \mathbb{C} : \Re\lambda \geq 0\} \subset \rho(A_p)$, where $\rho(A_p)$ is the resolvent set of A_p. The asymptotic stability (3.13) now follows immediately from Theorem 3.5.

Now let us prove (3.14). Suppose this is not the case so that according to Theorem 3.6 we have $\sup_{\Re\lambda>0} \|(\lambda I - A_p)^{-1}\|_{\mathcal{L}(X)} = \infty$. Hence, by the uniform boundedness principle, there exists $(v, w, z, \psi) \in X$ such that $\sup_{\Re\lambda>0} \|(\lambda I - A_p)^{-1}(v, w, z, \psi)\|_{\mathcal{L}(X)} = \infty$. Because the resolvent is holomorphic on every compact subset of $\rho(A_p)$, there exists a sequence of normalized vectors

$$Y_m := (v_m, w_m, z_m, \psi_m) \in D(A_p)$$

and a sequence of complex numbers $b_m + ic_m$, where $b_m \geq 0$ and $c_m \in \mathbb{R}$ for all m, such that

$$|b_m + ic_m| \to \infty \quad \text{and} \quad \|U_m\|_X \to 0 \tag{3.15}$$

where $U_m := (f_m, g_m, h_m, \phi_m) = ((b_m + ic_m)I - A_p)Y_m$. The latter equation is equivalent to the system

$$\begin{aligned} f_m &= (b_m + ic_m)v_m - \operatorname{div} w_m + a_0 v_m + a_1 z_m(-\tau) && (3.16)\\ g_m &= (b_m + ic_m)w_m - \nabla v_m && (3.17)\\ h_m(\theta) &= (b_m + ic_m)z_m(\theta) - z_{m\theta}(\theta) && (3.18)\\ \phi_m(s) &= (b_m + ic_m + s)\psi_m(s) - \Gamma v_m && (3.19) \end{aligned}$$

with the boundary conditions $z_m(0) = v_m$ and $w_m \cdot \nu + \int_0^\infty \psi_m(s)\,\mathrm{d}\mu(s) = 0$. According to (3.18) we have

$$z_m(\theta) = e^{(b_m+ic_m)\theta}v_m + \int_\theta^0 e^{(b_m+ic_m)(\theta-\vartheta)}h_m(\vartheta)\,\mathrm{d}\vartheta, \quad \theta \in (-\tau, 0). \tag{3.20}$$

The dissipativity of A_p, see (2.2), implies that

$$\begin{aligned} \Re\langle U_m, Y_m\rangle_X &= \Re((b_m + ic_m) - \langle A_p Y_m, Y_m\rangle_X)\\ &\geq b_m + \int_0^\infty \int_\Omega s|\psi_m|^2\,\mathrm{d}x\,\mathrm{d}\mu(s) - k\int_\Omega |v_m|^2\,\mathrm{d}x \end{aligned} \tag{3.21}$$

where $k = \frac{1}{2}(a_1^2/a_0 - a_0) < 0$. Since $|\langle U_m, Y_m\rangle| \leq \|U_m\|_X \to 0$ and all the terms in (3.21) are nonnegative it follows that $b_m \to 0$ and

$$v_m \to 0 \quad \text{strongly in } L^2(\Omega; \mathbb{C}^n). \tag{3.22}$$

Consequently, $|c_m| \to \infty$ as $m \to \infty$ by (3.15). From (3.20) and the Cauchy-Schwarz inequality we have

$$\begin{aligned} \int_{-\tau}^0 \int_\Omega |z_m|^2\,\mathrm{d}x\,\mathrm{d}\theta \;&\leq\; 2\left(\int_{-\tau}^0 e^{2b_m\theta}\,\mathrm{d}\theta\right)\int_\Omega |v_m|^2\,\mathrm{d}x\\ &\quad + 2\left(\int_{-\tau}^0 \int_\theta^0 e^{2b_m(\theta-\vartheta)}\,\mathrm{d}\vartheta\,\mathrm{d}\theta\right)\int_{-\tau}^0 \int_\Omega |h_m|^2\,\mathrm{d}x\,\mathrm{d}\vartheta. \end{aligned} \tag{3.23}$$

Since b_m is uniformly bounded in m, (3.22), (3.23) and (3.15) imply that

$$z_m \to 0 \quad \text{strongly in } L^2_\tau. \tag{3.24}$$

Taking the inner product of (3.16)−(3.19) with v_m, w_m, z_m and ψ_m in $L^2(\Omega;\mathbb{C}^n)$, $L^2(\Omega;\mathbb{C}^{n\times n})$, L^2_τ and L^2_μ, respectively, we obtain

$$\begin{aligned}\int_\Omega f_m\cdot v_m\,\mathrm{d}x &= (a_0+b_m+ic_m)\int_\Omega |v_m|^2\,\mathrm{d}x+\int_\Omega w_m\cdot\nabla v_m\,\mathrm{d}x \\ &\quad+\int_0^\infty\int_{\partial\Omega}\psi_m\cdot\Gamma v_m\,\mathrm{d}x\,\mathrm{d}\mu(s)+a_1\int_\Omega z_m(-\tau)\cdot v_m\,\mathrm{d}x\end{aligned}\tag{3.25}$$

$$\int_\Omega g_m\cdot w_m\,\mathrm{d}x=(b_m+ic_m)\int_\Omega |w_m|^2\,\mathrm{d}x-\int_\Omega \nabla v_m\cdot w_m\,\mathrm{d}x\tag{3.26}$$

$$\begin{aligned}\int_{-\tau}^0\int_\Omega h_m\cdot z_m\,\mathrm{d}x\,\mathrm{d}\theta &= (b_m+ic_m)\int_{-\tau}^0\int_\Omega |z_m|^2\,\mathrm{d}x\,\mathrm{d}\theta \\ &\quad-\int_{-\tau}^0\int_\Omega z_{m\theta}\cdot z_m\,\mathrm{d}x\,\mathrm{d}\theta\end{aligned}\tag{3.27}$$

$$\begin{aligned}\int_0^\infty\int_{\partial\Omega}\phi_m\cdot\psi_m\,\mathrm{d}x\,\mathrm{d}\mu(s) &= (b_m+ic_m)\int_0^\infty\int_{\partial\Omega}|\psi_m|^2\,\mathrm{d}x\,\mathrm{d}\mu(s) \\ &\quad+\int_0^\infty\int_{\partial\Omega}s|\psi_m|^2\,\mathrm{d}x\,\mathrm{d}\mu(s)-\int_0^\infty\int_{\partial\Omega}\Gamma v_m\cdot\psi_m\,\mathrm{d}x\,\mathrm{d}\mu(s).\end{aligned}\tag{3.28}$$

All of these terms tend to 0 as m tends to infinity according to (3.15). Dividing (3.27) by c_m, taking the imaginary part and then passing to the limit we obtain

$$\int_{-\tau}^0\int_\Omega |z_m|^2\,\mathrm{d}x\,\mathrm{d}\theta-\frac{1}{c_m}\Im\int_{-\tau}^0\int_\Omega z_{m\theta}\cdot z_m\,\mathrm{d}x\,\mathrm{d}\theta\to 0.$$

Invoking (3.24) we have

$$\frac{1}{c_m}\Im\int_{-\tau}^0\int_\Omega z_{m\theta}\cdot z_m\,\mathrm{d}x\,\mathrm{d}\theta\to 0.\tag{3.29}$$

Taking the real part of (3.27) and letting $m\to\infty$ we have

$$b_m\int_{-\tau}^0\int_\Omega |z_m|^2\,\mathrm{d}x\,\mathrm{d}\theta-\int_\Omega |v_m|^2\,\mathrm{d}x+\int_\Omega |z_m(-\tau)|^2\,\mathrm{d}x\to 0.$$

Using (3.22) and (3.24) the latter limit implies that

$$z_m(-\tau)\to 0\quad\text{strongly in } L^2(\Omega;\mathbb{C}^n).\tag{3.30}$$

Adding (3.26), (3.27) and (3.28) and then subtracting (3.25) we obatin

$$\begin{aligned}\varrho_m \;:=\; &(b_m+ic_m)\left(1-2\int_\Omega |v_m|^2\,\mathrm{d}x\right)-a_0\int_\Omega |v_m|^2\,\mathrm{d}x-a_1\int_\Omega z_m(-\tau)\cdot v_m\,\mathrm{d}x \\ &+\int_0^\infty\int_\Omega s|\psi_m|^2\,\mathrm{d}x\,\mathrm{d}\mu(s)-\int_{-\tau}^0\int_\Omega z_{m\theta}\cdot z_m\,\mathrm{d}x\,\mathrm{d}\theta \\ &-2\Re\int_0^\infty\int_{\partial\Omega}\psi_m\cdot\Gamma v_m\,\mathrm{d}x\,\mathrm{d}\mu(s)-2\Re\int_\Omega w_m\cdot\nabla v_m\,\mathrm{d}x\end{aligned}$$

where $\varrho_m\to 0$ as $m\to\infty$. Dividing by c_m, taking the imaginary part and passing to the limit yield

$$1-2\int_\Omega |v_m|^2\,\mathrm{d}x-\frac{a_1}{c_m}\Im\int_\Omega z_m(-\tau)\cdot v_m\,\mathrm{d}x-\frac{1}{c_m}\Im\int_{-\tau}^0\int_\Omega z_{m\theta}\cdot z_m\,\mathrm{d}x\,\mathrm{d}\theta\to 0.$$

This together with (3.21), (3.29) and (3.30) we obtain $\int_\Omega |v_m|^2 \,\mathrm{d}x \to \frac{1}{2}$ which is a contradiction to (3.22). Therefore (3.14) must hold. This completes the proof of the theorem. □

4. Boundary Delay: Stability Through the Energy Method

For this section we use the energy method to prove the exponential stability of the solution of (1.5) under the condition $\hat{a}(0) < c$, see **[12]** for a related problem. For this purpose, we recall the total energy

$$\begin{aligned} E(t) &= E_w(t) + \frac{1}{2}\int_0^\infty \int_{\Gamma_N} \left| \int_0^t e^{-s(t-r)} u_t(r-\tau, x) \,\mathrm{d}r \right|^2 \mathrm{d}x \,\mathrm{d}\mu(s) \\ &\quad + \frac{c}{2}\int_{-\tau}^0 \int_{\Gamma_N} |u_t(t+\theta, x)|^2 \,\mathrm{d}x \,\mathrm{d}\theta. \end{aligned}$$

The first step is to prove the following decay property of the energy.

Theorem 4.1. *Suppose that $\hat{a}(0) < c$. Every solution of* (1.5) *with initial data in $D(\tilde{A})$ has a decreasing energy. More precisely,*

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \le -\frac{1}{2}(c - \hat{a}(0)) D(t), \qquad t > 0, \tag{4.1}$$

where

$$D(t) = \int_{\Gamma_N} |u_t(t,x)|^2 + |u_t(t-\tau, x)|^2 \,\mathrm{d}x.$$

Proof. Taking the derivative of E and defining ψ by (2.14) we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E(t) &= \int_\Omega (u_t u_{tt} + \nabla u \cdot \nabla u_t) \,\mathrm{d}x - \int_0^\infty \int_{\Gamma_N} s|\psi(t,s,x)|^2 \,\mathrm{d}x \,\mathrm{d}\mu(s) \\ &\quad + \int_0^\infty \int_{\Gamma_N} \psi(t,s,x) u_t(t-\tau, x) \,\mathrm{d}x \,\mathrm{d}\mu(s) \\ &\quad + c\int_{-\tau}^0 \int_{\Gamma_N} u_t(t+\theta, x) u_{tt}(t+\theta, x) \,\mathrm{d}x \,\mathrm{d}\theta \end{aligned} \tag{4.2}$$

By Green's identity and Young's inequality

$$\begin{aligned} &\int_\Omega (u_t u_{tt} + \nabla u \cdot \nabla u_t) \,\mathrm{d}x = \int_{\Gamma_N} u_t \frac{\partial u}{\partial \nu} \,\mathrm{d}x \\ &= -\int_{\Gamma_N} u_t(t,x) \left(\int_0^\infty \psi(t,s,x) \,\mathrm{d}\mu(s) + c u_t(t,x) \right) \mathrm{d}x \\ &\le -c\int_{\Gamma_N} |u_t(t,x)|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\Gamma_N} \int_0^\infty \left(\frac{1}{s}|u_t(t,x)|^2 + s|\psi(t,s,x)|^2 \right) \mathrm{d}\mu(s) \,\mathrm{d}x \\ &= -\left(c - \frac{\hat{a}(0)}{2} \right) \int_{\Gamma_N} |u_t(t,x)|^2 \,\mathrm{d}x + \frac{1}{2}\int_0^\infty \int_{\Gamma_N} s|\psi(t,s,x)|^2 \,\mathrm{d}x \,\mathrm{d}\mu(s). \end{aligned} \tag{4.3}$$

On the other hand we also have

$$\begin{aligned} &\int_0^\infty \int_{\Gamma_N} \psi(t,s,x) u_t(t-\tau, x) \,\mathrm{d}x \,\mathrm{d}\mu(s) \\ &\le \frac{\hat{a}(0)}{2} \int_{\Gamma_N} |u_t(t-\tau, x)|^2 \,\mathrm{d}x + \frac{1}{2}\int_0^\infty \int_{\Gamma_N} s|\psi(t,s,x)|^2 \,\mathrm{d}x \,\mathrm{d}\mu(s). \end{aligned} \tag{4.4}$$

Since $u_t(t+\theta,x) = u_\theta(t+\theta,x)$ and $u_{tt}(t+\theta,x) = u_{\theta\theta}(t+\theta,x)$ we have, by Fubini's Theorem,

$$\int_{-\tau}^{0}\int_{\Gamma_N} u_t(t+\theta,x)u_{tt}(t+\theta,x)\,\mathrm{d}x\,\mathrm{d}\theta = \frac{1}{2}\int_{\Gamma_N}(|u_t(t,x)|^2 - |u_t(t-\tau,x)|^2)\,\mathrm{d}x. \quad (4.5)$$

Combining (4.2)–(4.5) proves the decay property (4.1). □

Using Theorem 4.1 and a standard density argument, we have the following a priori trace regularity on u_t and $u_t(\cdot-\tau)$.

Corollary 4.2. *The map $U_0 \mapsto (u_t, u_t(\cdot-\tau)) : D(\tilde{A}) \to L^2(0,T;L^2(\Gamma_N)^2)$ has a unique continuous extension to $\tilde{X}$.*

The next step is the following inverse observability estimate as in [**14**].

Theorem 4.3. *There exists $T^* > 0$ such that for all $T > T^*$ there is a constant $C_T > 0$ satisfying*

$$E(0) \le C_T \int_0^T D(t)\,\mathrm{d}t.$$

Proof. According to the observability estimate in [**13, Proposition 6.3**] there is $\tilde{T} > 0$ such that for all $T > \tilde{T}$ there exists a constant $c_T > 0$ such that

$$E_w(0) \le c_T \int_0^T\int_{\Gamma_N}\left(\left|\frac{\partial u}{\partial\nu}\right|^2 + |u_t|^2\right)\mathrm{d}x\,\mathrm{d}t + c_T\|u\|_{H^{\frac{1}{2}+\epsilon}((0,T)\times\Omega)}$$

for any $\epsilon > 0$. The boundary condition implies that

$$\begin{aligned}\int_0^T\int_{\Gamma_N}\left|\frac{\partial u}{\partial\nu}\right|^2\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_{\Gamma_N}\left|\int_0^\infty\psi(t,s,x)\,\mathrm{d}\mu(s) + cu_t(t,x)\right|^2\mathrm{d}x\,\mathrm{d}t\\ &\le 2\int_0^T\int_{\Gamma_N}\left|\int_0^\infty\psi(s,t,x)\,\mathrm{d}\mu(s)\right|^2\mathrm{d}x\,\mathrm{d}t + 2c^2\int_0^T\int_{\Gamma_N}|u_t(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t. \quad (4.6)\end{aligned}$$

By Hölder's inequality it holds that

$$\begin{aligned}&\int_0^T\int_{\Gamma_N}\left|\int_0^\infty\psi(s,t,x)\,\mathrm{d}\mu(s)\right|^2\mathrm{d}x\,\mathrm{d}t\\ &\le \hat{a}(0)\int_0^T\int_0^\infty\int_{\Gamma_N} s|\psi(s,t,x)|^2\,\mathrm{d}x\,\mathrm{d}\mu(s)\,\mathrm{d}t. \quad (4.7)\end{aligned}$$

Multiplying the equation $\psi_t(t,s,x) = -s\psi(t,s,x) + u_t(t-\tau,x)$ by $\psi(t,s,x)$, integrating over $(0,T)\times(0,\infty)\times\Gamma_N$ and using $\psi(0,s,x) = 0$ we have

$$\begin{aligned}\frac{1}{2}\int_0^\infty\int_{\Gamma_N}|\psi(T,s,x)|^2\,\mathrm{d}x\,\mathrm{d}\mu(s) &= \int_0^T\int_0^\infty\int_{\Gamma_N}\psi_t(t,s,x)\psi(t,s,x)\,\mathrm{d}x\,\mathrm{d}\mu(s)\,\mathrm{d}t\\ &= \int_0^T\int_0^\infty\int_{\Gamma_N}(-s|\psi(s,t,x)|^2 + u_t(t-\tau,x)\psi(t,s,x))\,\mathrm{d}x\,\mathrm{d}\mu(s)\,\mathrm{d}t\\ &\le \int_0^T\int_0^\infty\int_{\Gamma_N}\left(-\frac{s}{2}|\psi(s,t,x)|^2 + \frac{1}{2s}|u_t(t-\tau,x)|^2\right)\mathrm{d}x\,\mathrm{d}\mu(s)\,\mathrm{d}t\\ &= -\frac{1}{2}\int_0^T\int_0^\infty\int_{\Gamma_N} s|\psi(s,t,x)|^2\,\mathrm{d}x\,\mathrm{d}\mu(s)\,\mathrm{d}t + \frac{\hat{a}(0)}{2}\int_0^T\int_{\Gamma_N}|u_t(t-\tau,x)|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Therefore it follows that

$$\int_0^T\int_0^\infty\int_{\Gamma_N} s|\psi(s,t,x)|^2\,\mathrm{d}x\,\mathrm{d}\mu(s)\,\mathrm{d}t \leq \frac{\hat{a}(0)}{2}\int_0^T\int_{\Gamma_N}|u_t(t-\tau,x)|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{4.8}$$

The change of variable $t = \theta + \tau$ implies that

$$E_d(0) = \frac{c}{2}\int_{-\tau}^0\int_{\Gamma_N}|u_t(\theta,x)|^2\,\mathrm{d}x\,\mathrm{d}\theta = \frac{c}{2}\int_0^\tau\int_{\Gamma_N}|u_t(t-\tau,x)|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Taking $T^* = \max(\tilde{T},\tau)$ it follows that for all $T > T^*$

$$E(0) = E_w(0) + E_d(0) \leq C_T\int_0^T D(t)\,\mathrm{d}t + C_T\|u\|_{H^{\frac{1}{2}+\epsilon}((0,T)\times\Omega)}. \tag{4.9}$$

for some constant $C_T > 0$. To finish the proof of the theorem, one can use a standard compactness-uniqueness argument as in [**14, Proposition 4.2**] to show that the second term on the right hand side of (4.9) can be absorbed by the first term, that is,

$$\|u\|_{H^{\frac{1}{2}+\epsilon}((0,T)\times\Omega)} \leq C\int_0^T D(t)\,\mathrm{d}t.$$

This completes the proof the theorem. □

From the proof of the previous theorem, one can obtain the following trace regularity.

Corollary 4.4. *The map $U_0 \mapsto a\star u_t(\cdot-\tau) : D(\tilde{A}) \to L^2(0,T;L^2(\Gamma_N))$ admits a unique continuous extension to $\tilde{X}$. As a consequence, the map $U_0 \mapsto \frac{\partial u}{\partial\nu} : D(\tilde{A}) \to L^2(0,T;L^2(\Gamma_N))$ admits a unique continuous extension to $\tilde{X}$.*

Proof. The first statement follows from (4.7), (4.8) and Corollary 4.2. The second part follows from the first one together with the estimates (4.6)–(4.8). □

Theorem 4.5. *Suppose that $\hat{a}(0) < c$. Then there exist $M \geq 1$ and $\alpha > 0$ such that for every solution of* (1.5) *we have*

$$E(t) \leq Me^{-\alpha t}E(0), \qquad t > 0. \tag{4.10}$$

Proof. Let $U_0 \in D(\tilde{A})$. Using Theorem 4.1 and Theorem 4.3 one obtains

$$E(T) \leq E(0) \leq C_T\int_0^T D(t)\,\mathrm{d}t \leq \frac{2C_T}{c-\hat{a}(0)}(E(0)-E(T))$$

for every $T > T^*$. Therefore, for $T > T^*$ it holds that

$$E(T) \leq \frac{2C_T}{2C_T + c - \hat{a}(0)}E(0). \tag{4.11}$$

Since $2C_T(2C_T + c - \hat{a}(0))^{-1} < 1$, a standard argument shows that (4.11) implies (4.10). By the density of $D(\tilde{A})$ in $\tilde{X}$, the estimate (4.10) holds for every initial data in $\tilde{X}$. □

Acknowledgments

The author would like to thank Georg Propst for his helpful comments and suggestions.

References

[1] K. Ammari, S. Nicaise and C. Pignotti, *Feedback boundary stabilisation of wave equations with interior delay*, System and Control Letters **59**, pp. 623-628, 2010.

[2] P. Cornilleau and S. Nicaise, *Energy decay of solutions of the wave equation with general memory boundary conditions*, Differential and Integral Equations **22**, No. 11/12, pp. 1173–1192, 2009.

[3] R. Datko, *Not all feedback stabilised hyperbolic systems are robust with respect to small time delays in their feedback*, SIAM J. Control Optim. **26**, pp. 697-713, 1988.

[4] R. Datko, J. Lagnese and P. Polis, *An example on the effect of time delays in boundary feedback stabilization of wave equations*, SIAM J. Control Optim **24**, pp. 152-156, 1985.

[5] W. Desch, E. Fašangová, J. Milota and G. Propst, *Stabilization through viscoelastic boundary damping: a semigroup approach*, Semigroup Forum **80**, pp. 405-415, 2010.

[6] W. Desch, E. Fašangová, J. Milota and G. Propst, *Spectrum of a viscoelastic boundary damping*, J. Integral Equations Appl. **23**, pp. 521-539, 2011.

[7] K. J. Engel and R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, 2nd ed., Springer, Berlin, 2000.

[8] S. Gerbi and B. Said-Houari, *Existence and exponential stability for a damped wave equation with dynamic boundary conditions and a delay term*, Applied Mathematics and Computation **218**, No. 24, pp. 11900-11910, 2012.

[9] G. Gripenberg, S.-O. Londen and O. Staffans, Volterra Integral and Functional Equations, Cambridge University Press, Cambridge, 1990.

[10] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Monogr. Stud. Math. 21, Pitman, Boston-London-Melbourne, 1985.

[11] M. Kirane and B. Said-Houari, *Existence and asymptotic stability of a viscoelastic wave equation with a delay*, Z. Angew. Math. Phys. **62**, pp. 1065–1082, 2011.

[12] J. Lagnese, *Decay of solutions of the wave equations in a bounded region with boundary dissipation*, J. Differential Equations **50**, pp. 163-182, 1983.

[13] I. Lasiecka, R. Triggiani and P. F. Yao, *Inverse/observability estimates for second-order hyperbolic equations with variable coefficients*, J. Math. Anal. Appl. **235**, pp. 13-57, 1999.

[14] S. Nicaise and C. Pignotti, *Stability and instability results of the wave equation with a delay term in the boundary of internal feedbacks*, Siam J. Control Optim. **45**, pp. 1561-1585, 2006.

[15] S. Nicaise and C. Pignotti, *Stabilization of the wave equation with boundary or internal distributed delay*, Differential and Integral Equations **21**, No.9-10, pp. 801-100, 2008.

[16] C. Pignotti, *A note on stabilisation of locally damped wave equations with time delay*, Systems and Control Letters **61**, pp. 92-97, 2012.

[17] G. Propst and J. Prüss, *On wave equations with boundary dissipation of memory type*, J. Integral Equations Appl. **8**, pp. 99-123, 1996.

[18] R. Temam, Navier-Stokes Equations, Theory and Numerical Analysis, North-Holland, Amsterdam, 1979.

[19] M. Tucsnak and G. Weiss, Observation and Control for Operator Semigroups, Birkhäuser-Verlag, Basel, 2009.

[20] G. Q. Xu, S. P. Yung and L. K. Li, *Stabilzation of wave systems with input delay in the boundary control*, ESAIM Control Optim Calc. Var. **12**, pp. 770-785, 2006.