FINITE ELEMENT ERROR ANALYSIS FOR MEASURE-VALUED OPTIMAL CONTROL PROBLEMS GOVERNED BY THE 1D GENERALIZED WAVE EQUATION

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Abstract. This work is concerned with the optimal control problems governed by the 1D wave equation with variable coefficients and the control spaces $\mathcal{M}_T$ of either measure-valued functions $L^2(I, \mathcal{M}(\Omega))$ or vector measures $\mathcal{M}(\Omega, L^2(I))$. The cost functional involves the standard quadratic terms and the regularization term $\alpha\|u\|_{\mathcal{M}_T}$, $\alpha > 0$. We construct and study three-level in time bilinear finite element discretizations for the problems. The main focus lies on the derivation of error estimates for the optimal state variable and the error measured in the cost functional. The analysis is mainly based on some previous results of the authors. The numerical results are included.

Keywords. Wave equation, optimal control, measure-valued control, vector measure control, finite element method, stability, error estimates.

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1. Introduction

This work is concerned with the discretization and numerical analysis of optimal control problems involving the 1D linear generalized wave equation (with variable coefficients) and controls taking values in certain measure spaces. The discretization of the state equation is based on a space-time finite element method (FEM) introduced in [46]. Related methods are also discussed and analyzed in [1, 20]. See also [2]. The measure-valued control is not directly discretized, cf. the variational control discretization from [24]. However, there exists optimal controls consisting of Dirac measures in the spatial grid points which can be computed, see also [11, 30]. The numerical analysis of the control problem is based on FEM error estimates for the second order hyperbolic equations from [46] and techniques developed in [11, 30]. It requires to overcome significant technical difficulties caused by non-smoothness of controls and states. To the best of our knowledge, this is the first paper providing such numerical analysis for the studied control problems.

Motivated by industrial applications as well as applications in the natural sciences, in which one is interested to place actuators in form of point sources in an optimal way, see, e.g., [4, 8] or in the reconstruction of point sources from given measurements, see, e.g., [31, 41], measure valued optimal control problems involving PDEs gained attention in the last years. These problems can be translated into optimization problems in terms of the coordinates and coefficients of the point sources. However, these optimization problem are non-convex since the solution of the state equation (PDE) depends in a non-linear way on the coordinates of the point sources. Thus one has to deal with multiple local minima. Several authors suggested to cast the control problem resp. inverse problem in form of an optimization problem over a suitable measure space $\mathcal{M}_T$ involving a convex regularization functional R which favors point sources as solutions. In our case we introduce the following problem formulation

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involving the 1D wave equation

$$\begin{aligned} J(u) = F(y) + R(u) \to \min_{u \in \mathcal{M}_T} \\ \text{subject to} \quad \rho \partial_{tt} y - \partial_x(\kappa \partial_x y) = u \quad \text{for } (t,x) \in I \times \Omega = (0,T) \times (0,L) \end{aligned} \tag{1.1}$$

with additional initial and boundary conditions. The functional F is given by a quadratic tracking functional involving $y|_{I\times\Omega}$, $y(T,\cdot)|_\Omega$ and $\partial_t y(T,\cdot)|_\Omega$. The regularization functional R and the control space $\mathcal{M}_T$ are chosen in a way such that $\mathcal{M}_T$ contains point sources of the desired form and R promotes controls of such a form, i.e. linear combinations of point sources with time-dependent intensities or more general controls with a small spatial support. Since problem (1.1) is convex, one need not to deal with several local minima. However, it is not longer guaranteed that the solution consists of a sum of point sources. We enforce such controls via the regularization functional R. Problems of the form (1.1) (also involving other PDEs) have been analysed from theoretical, numerical and algorithmic points of view, see [6, 10–17, 30, 31, 41, 42]. Optimal control problems governed by the linear wave equation were discussed in several different aspects, see [21, 22, 26–29, 32, 33, 37, 38, 47]. In our particular case we consider the control spaces $\mathcal{M}_T$ of measure-valued functions $L^2(I, \mathcal{M}(\Omega))$ and vector measures $\mathcal{M}(\Omega, L^2(I))$ with $R(u) = \alpha\|u\|_{\mathcal{M}_T}$. These two different choices imply different structural properties of the optimal controls. A typical non-regular element from the space $\mathcal{M}(\Omega, L^2(I))$ is given by

$$u = \sum_{i=1}^{n} u_i(t)\delta_{x_i}, \quad u_i \in L^2(I),\ x_i \in \Omega, \tag{1.2}$$

where δ_{x_i} are the Dirac delta functions. Point sources of such type with fixed positions and time-dependent intensities are of interest in acoustics or geology, see [31, 41]. If one is interested in controls involving moving point sources of the form

$$u = \sum_{i=1}^{n} u_i(t)\delta_{x_i(t)}, \quad u_i \in L^2(I), \quad x_i : I \to \Omega \text{ is measurable,} \tag{1.3}$$

then the control space $L^2(I, \mathcal{M}(\Omega))$ rather than $\mathcal{M}(\Omega, L^2(I))$ is more appropriate. The space $\mathcal{M}(\Omega, L^2(I))$ and the functional $\|\cdot\|_{\mathcal{M}(\Omega, L^2(I))}$ are also related to the term directional sparsity resp. joint sparsity, see [19, 23].

The problem like (1.1) for a parabolic/heat state equation is analyzed with $\mathcal{M}_T = \mathcal{M}(\Omega, L^2(I))$ in [30] and $\mathcal{M}_T = L^2(I, \mathcal{M}(\Omega))$ in [11]. In particular, the authors prove existence of optimal controls and derive optimality conditions and FEM error estimates. Our analysis is partly based on these results of [30]. In [31] a similar problem involving the linear wave equation with constant coefficients as state equation is analyzed. In particular, existing regularity results for a Dirac right-hand side are extended to sources from $\mathcal{M}(\Omega, L^2(I))$. Based on these regularity results existence of optimal controls is proved as well as optimal conditions are derived in the 3D case.

Now we briefly sum up the contents of this work. First of all we collect and partially prove required existence and regularity results for the linear wave equation in the 1D setting. In particular, we check that the notions of a weaker solution defined in [46] and more commonly used very weak solution, e.g. [35], are equivalent. Most importantly we prove that the solution of the linear wave equation with variable coefficients from $H^1(\Omega)$ for any source term $u \in \mathcal{M}(\Omega, L^2(I))$ is an element of $\mathcal{C}(\bar{I}, H_0^1(\Omega)) \cap \mathcal{C}^1(\bar{I}, L^2(\Omega))$ provided that the initial data have relevant regularity. The proof is based on a non-standard energy type bound in space, not only in time, cf. [18, 34]. In [31] the same result is proved based on duality techniques which do not extend to the case of variable coefficients. Then, existence of optimal controls and the derivation of optimality conditions are discussed on the basis of results from [30, 31]. In the case $\mathcal{M}_T = \mathcal{M}(\Omega, L^2(I))$ we prove that the optimal control $\bar{u}$ belongs to $\mathcal{C}^1(\bar{I}, \mathcal{M}(\Omega))$.

Further, the FEM discretization of the state equation is introduced. The state variable $y_{h,\tau}$ belongs to the space of bilinear finite elements and is defined by the regularized Galerkin method. The resulting numerical scheme is a three-level method in time. Moreover, we pose and prove the FEM error estimates in $\mathcal{C}(\bar{I}, L^2(\Omega))$ for the discrete state equation which we need for the numerical analysis of the control problem. We base this study mainly on the results from [46] concerning error analysis of FEMs for the second order hyperbolic equations in the classes of the data having integer Sobolev or fractional Nikolskii order of smoothness. Note that their sharpness in a strong sense was stated in [45].

Then we consider a semi-discrete optimal control problem in which the continuous state equation is replaced by its discretized version whereas the controls are not discretized. We prove convergence of the discrete optimal controls to the continuous one and derive optimality conditions based on the Lagrange techniques. Most importantly we derive the discrete adjoint state equation. We can conclude that the first-discretize-then-optimize and first-optimize-then-discretize approaches commute. Therefore an analysis of the discrete adjoint state equation including the error estimates in $\mathcal{C}(\bar{I} \times \bar{\Omega})$ and $L^2(I, \mathcal{C}_0(\Omega))$ can also be based on techniques from [46]. Then we use results from [30] to represent the numerical error of state variable and of the cost functional in terms of FEM errors of the state equation and the adjoint state equation. Let $\bar{u}$ and $\bar{y}$ be the optimal control and the corresponding optimal state, and the variables $\bar{u}_{\tau,h}$ and $\bar{y}_{\tau,h}$ be their discrete counterparts. As the main result of this paper we prove the error estimates

$$\|\bar{y} - \bar{y}_{\tau,h}\|_{L^2(I\times\Omega)} = \mathcal{O}\big((\tau + h)^{\alpha}\big), \quad |J(\bar{u}) - J(\bar{u}_{\tau,h})| = \mathcal{O}\big((\tau + h)^{2/3}\big)$$

where τ is the step in time, h is the maximal step in space and $\alpha = 1/3$ for $\mathcal{M}_T = L^2(I, \mathcal{M}(\Omega))$ or $\alpha = 2/3$ for $\mathcal{M}_T = \mathcal{M}(\Omega, L^2(I))$. The latter higher order is due to the above mentioned improved regularity results for the state and optimal control. Such estimates are proved for the measure-valued controls in the hyperbolic case for the first time. Similar estimates are impossible in multidimensional settings due to much less fractional Sobolev regularity of optimal states and controls.

Finally we discuss the numerical computation of the discrete control $\bar{u}_{h,\tau}$. Based on a control discretization $u_{h,\tau}$ that given by the sum like (1.2) with x_i at the spatial grid points and u_i in the space of linear finite elements, a solution of the semi-discrete control problem can be calculated similarly to [30]. For the actual numerical computation of the optimal control we add the term $(\gamma/2)\|u\|^2_{L^2(I\times\Omega)}$, $\gamma > 0$, to (1.1). This regularized problem is solved by a semi-smooth Newton method, see [40]. In a continuation strategy the regularization parameter γ is made sufficiently small. We complete this work with a numerical example for $\mathcal{M}_T = \mathcal{M}(\Omega, L^2(I))$.

The paper is organized in the following way. In Section 2 we introduce the problem setting and the control spaces resp. the regularization functionals. Section 3 is concerned with regularity properties of the linear wave equation with variable coefficients in the 1D setting. In Section 4 the control problem is analyzed from a theoretical point of view. Section 5 deals with discretization of the state equation. Then we obtain stability bounds and error estimates for the discrete state equation in Section 6. Section 7 is concerned with the analysis of the semi-discrete optimal control problem. The next section discusses stability bounds and error estimates for the discrete adjoint sate equation. In Sections 9 resp. 10 error estimates for the optimal state and cost functional are derived being the main theoretical results of the study. Section 11 deals with the time stepping formulation of the discrete state equation. In Section 12 we discuss the control discretization with Dirac measures at the grid points. Then we introduce the $L^2(I \times \Omega)$ regularized problem and describe its solutions by a semi-smooth Newton method. Finally Section 13 provides a numerical example.

2. Problem setting

We consider optimal control problems of the following form

$$J(y, u) = F(y) + \alpha\|u\|_{\mathcal{M}_T} \to \min_{u,y} \qquad (\mathcal{P})$$

with the parameter $\alpha > 0$ and the tracking functional

$$F(y) := \tfrac{1}{2}\big(\|y - z_1\|^2_{L^2(I,H_\rho)} + \|y(T) - z_2\|^2_{H_\rho} + \|\rho\partial_t y(T) - z_3\|^2_{\mathcal{V}^*_\kappa} \big)$$

using $\mathbf{z} := (z_1, z_2, z_3) \in \mathcal{Y} := L^2(I \times \Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$, subject to *the state equation* which is an initial-boundary value problem for the 1D generalized wave equation

$$\begin{cases} \rho\partial_{tt}y - \partial_x(\kappa\partial_x y) = u & \text{in } I \times \Omega := (0,T) \times (0,L) \\ y = 0 & \text{on } I \times \partial\Omega \\ y = y^0,\ \partial_t y = y^1 & \text{in } \{0\} \times \Omega. \end{cases} \tag{2.1}$$

Here, in particular, the initial data $\mathbf{y} := (y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$, and $L > 0$ and $T > 0$. The coefficients $\rho, \kappa \in H^1(\Omega)$ satisfy $\rho(x) \geq \nu > 0$ and $\kappa(x) \geq \nu$ on Ω.

For brevity we denote $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, $V^2 = H^2(\Omega) \cap V$ and $V^3 = \{v \in V | \partial_x(\kappa\partial_x v) \in V\}$ equipped with the norms

$$\|\cdot\|_V = \|\partial_x \cdot\|_H, \quad \|\cdot\|_{V^2} = \|\partial_{xx} \cdot\|_H, \quad \|\cdot\|_{V^3} = \|\partial_x(\kappa\partial_x \cdot)\|_V.$$

Moreover, we utilize the equivalent coefficient-dependent Hilbert norms on H, V, V^* and $\mathcal{Y}$

$$\|w\|_{H_\rho} = \|\sqrt{\rho}w\|_H, \quad \|w\|_{\mathcal{V}_\kappa} = \|\sqrt{\kappa}\partial_x w\|_H, \quad \|w\|_{\mathcal{V}_\kappa^*} = \sup_{\|v\|_{\mathcal{V}_\kappa} \leq 1} \langle w, v\rangle_\Omega,$$

$$\|\mathbf{z}\|_{\mathcal{Y}} = \left(\|z_1\|^2_{L^2(I,H_\rho)} + \|z_2\|^2_{H_\rho} + \|z_3\|^2_{\mathcal{V}_\kappa^*}\right)^{1/2},$$

where $\langle\cdot,\cdot\rangle_\Omega$ is the duality relation on $V^* \times V$.

For the control space $\mathcal{M}_T$ we consider two choices, either the space of vector measures $\mathcal{M}(\Omega, L^2(I))$ or the space of weak measurable, $\mathcal{M}(\Omega)$-valued functions $L^2(I, \mathcal{M}(\Omega)) := L^2_w(I, \mathcal{M}(\Omega))$. Let correspondingly $\mathcal{C}_T$ be chosen as $\mathcal{C}_0(\Omega, L^2(I))$ or $L^2(I, \mathcal{C}_0(\Omega))$ where $\mathcal{C}_0(\Omega) = \{v \in \mathcal{C}(\bar{\Omega}) |\, v|_{x=0,L} = 0\}$. The following identifications of dual spaces hold

$$\mathcal{C}_0(\Omega, L^2(I))^* \cong \mathcal{M}(\Omega, L^2(I)), \quad L^2(I, \mathcal{C}_0(\Omega))^* \cong L^2(I, \mathcal{M}(\Omega)),$$

i.e. $\mathcal{M}_T = \mathcal{C}_T^*$, see [11] resp. [30], where more details on the properties of these spaces and the norm $\|\cdot\|_{\mathcal{M}_T}$ can be found. In particular, the following embeddings hold

$$\mathcal{M}(\Omega, L^2(I)) \hookrightarrow L^2(I, \mathcal{M}(\Omega)) \hookrightarrow L^2(I, V^*). \tag{2.2}$$

3. Existence and regularity of the state

3.1. **Weak formulations and preliminary existence, uniqueness and regularity results.** In this section we introduce our solution concepts for the state equation (2.1). We begin with defining a weak formulation of (2.1).

Definition 3.1. Let $(u, y^0, y^1) \in X \times V \times H$ with $X = L^2(I, H)$ or $H^1(I, V^*)$ or $\mathcal{M}(\Omega, L^2(I))$. Then $y \in \mathcal{C}(\bar{I}, V) \cap \mathcal{C}^1(\bar{I}, H)$ is called a *weak solution* of (2.1) if it satisfies the integral identity

$$B(y, v) + \big(\rho\partial_t y(T), v(T)\big)_H = \int_I \langle u, v\rangle_\Omega \ \mathrm{d}t + \big(\rho y^1, v(0)\big)_H \quad \forall v \in L^2(I, V) \cap H^1(I, H) \tag{3.1}$$

with the indefinite symmetric bilinear form

$$B(y, v) := -(\rho\partial_t y, \partial_t v)_{L^2(I\times\Omega)} + (\kappa\partial_x y, \partial_x v)_{L^2(I\times\Omega)}, \tag{3.2}$$

and the initial condition $y(0) = y^0$.

The right-hand side in (3.1) is well defined for $X = \mathcal{M}(\Omega, L^2(I))$ too due to embeddings (2.2).

Remark 3.2. It is possible (and more common) to suppose that $v(T) = 0$ in (3.1) when the last term on the left disappears (for example, see [46]). This leads to an equavalent formulation. To check this, it is enough to replace there v by $v\beta_\delta$, where $\beta_\delta(t) = \min\big(1, (T-t)/\delta\big)$, $0 < \delta < T$. Then $\partial_t(v\beta_\delta) = (\partial_t v)\beta_\delta - (1/\delta)v\chi_{(T-\delta,T)}$, where $\chi_{(T-\delta,T)}$ is the characteristic function of $(T-\delta, T)$. Passing to the limit as $\delta \to 0$ with the help of the dominated convergence theorem and the properties of y and v leads to the result.

Another definition of the weak solution is possible.

Definition 3.3. Let $(u, y^0, y^1) \in X \times V \times H$ with $X = L^2(I \times \Omega)$ or $H^1(I, V^*)$ or $\mathcal{M}(\Omega, L^2(I))$. A function $y \in \mathcal{C}(\bar{I}, V) \cap H^2(I, V^*) \hookrightarrow H^1(I, H)$ is called a weak solution of (2.1) if it satisfies

$$\int_I \langle \rho \partial_{tt} y, v\rangle_\Omega + (\kappa \partial_x y, \partial_x v)_H \,\mathrm{d}t = \int_I \langle u, v\rangle_\Omega \,\mathrm{d}t \quad \forall v \in L^2(I, V) \tag{3.3}$$

and $y(0) = y^0$ as well as $\partial_t y(0) = y^1$.

Proposition 3.4. *Definitions 3.1 and 3.3 are equivalent.*

Proof. The equivalence of (3.3) and (3.1) can be proved using integration by parts in time and the density of $\mathcal{C}^\infty(\bar{I}, V)$ in $L^2(I, V) \cap H^1(I, H)$, cf. [35, Chapter 1, Theorem 2.1]. □

Proposition 3.5. *(1) Let $(u, y^0, y^1) \in X \times V \times H$ with $X = L^2(I \times \Omega)$ or $H^1(I, V^*)$. Then* (2.1) *has a unique weak solution satisfying $y \in \mathcal{C}(\bar{I}, V) \cap \mathcal{C}^1(\bar{I}, H) \cap H^2(I, V^*)$ and*

$$\|y\|_{\mathcal{C}(\bar{I},V)} + \|\partial_t y\|_{\mathcal{C}(\bar{I},H)} + \|\partial_{tt} y\|_{L^2(I,V^*)} \le c\,(\|u\|_X + \|\mathbf{y}\|_{V\times H}). \tag{3.4}$$

Hereafter $c > 0$, $c_1 > 0$, etc., are independent of y and the data.
In the case $X = H^1(I, V^*)$ there even holds $y \in \mathcal{C}^2(\bar{I}, V^*)$ as well as

$$\|\partial_{tt} y\|_{\mathcal{C}(\bar{I},V^*)} \le c\,(\|u\|_{H^1(I,V^*)} + \|\mathbf{y}\|_{V\times H}).$$

(2) Let $(u, y^0, y^1) \in X \times V^2 \times V$ with $X = L^2(I, V)$ or $H^1(I, H)$. Then the weak solution y satisfies $y \in \mathcal{C}(\bar{I}, V^2) \cap \mathcal{C}^1(\bar{I}, V) \cap H^2(I, H)$ and

$$\|y\|_{\mathcal{C}(\bar{I},V^2)} + \|\partial_t y\|_{\mathcal{C}(\bar{I},V)} + \|\partial_{tt} y\|_{L^2(I,H)} \le c\,(\|u\|_X + \|\mathbf{y}\|_{V^2\times V}). \tag{3.5}$$

In the case $X = H^1(I, H)$ there even holds $y \in \mathcal{C}^2(\bar{I}, H)$ as well as

$$\|\partial_{tt} y\|_{\mathcal{C}(\bar{I},H)} \le c\,(\|u\|_{H^1(I,H)} + \|\mathbf{y}\|_{V^2\times V}).$$

Moreover, y satisfies the equation $\rho\partial_{tt} y - \partial_x(\kappa \partial_x y) = u$ in $L^2(I \times \Omega)$, i.e. it is the strong solution.

Proof. For example, see [46, Propositions 1.1 and 1.3]. □

Item 2 ensures the regularity of weak solution for more regular data.

For less regular data $(u, y^0, y^1) \in L^2(I, V^*) \times H \times V^*$ one can use other weak formulations. To state the first of them, we define the integration operator $(\mathcal{I}_t v)(t) := \int_0^t v(s)\,\mathrm{d}s$ and its adjoint $(\mathcal{I}_t^* v)(t) := \int_t^T v(s)\,\mathrm{d}s$ on $\bar{I}$.

Definition 3.6. Let $(u, y^0, y^1) \in L^2(I, V^*) \times H \times V^*$. A function $y \in \mathcal{C}(\bar{I}, H)$ with $\mathcal{I}_t y \in \mathcal{C}(\bar{I}, V)$ is called a *weaker solution* of (2.1) if it satisfies

$$\int_I -(\rho y, \partial_t v)_H + (\kappa \partial_x \mathcal{I}_t y, \partial_x v)_H \,\mathrm{d}t + \big(\rho y(T), v(T)\big)_H$$
$$= \int_I \langle u, \mathcal{I}_t^* v\rangle_\Omega \,\mathrm{d}t + \big(\rho y^0, v(0)\big)_H + \langle \rho y^1, (\mathcal{I}_t^* v)(0)\rangle_\Omega \quad \forall v \in L^2(I, V) \cap H^1(I, H). \tag{3.6}$$

As in the case of Definition 3.1, it is sufficient to take $v(T) = 0$ in (3.6), cf. Remark 3.2.

Proposition 3.7. *Let $(u, y^0, y^1) \in L^2(I, V^*) \times H \times V^*$. Then there exists a unique weaker solution $y \in \mathcal{C}(\bar{I}, H) \cap \mathcal{C}^1(\bar{I}, V^*)$ and it satisfies the bound*

$$\|y\|_{\mathcal{C}(\bar{I},H)} + \|\mathcal{I}_t y\|_{\mathcal{C}(\bar{I},V)} + \|\partial_t y\|_{\mathcal{C}(\bar{I},V^*)} \le c\,(\|u\|_{L^2(I,V^*)} + \|\mathbf{y}\|_{H\times V^*}).$$

Proof. See [46, Proposition 1.2]. □

We infer that there are other weak formulations of (2.1) for solutions $y \in \mathcal{C}(\bar{I}, H) \cap \mathcal{C}^1(\bar{I}, V^*)$. One can use the concept of very weak solutions.

Definition 3.8. Let $(u, y^0, y^1) \in L^2(I, V^*) \times H \times V^*$. A function $y \in \mathcal{C}(\bar{I}, H) \cap \mathcal{C}^1(\bar{I}, V^*)$ satisfying

$$\int_I \big(y, \rho\partial_{tt}v - \partial_x(\kappa\partial_x v)\big)_H \,\mathrm{d}t - \big(\rho y(T), \partial_t v(T)\big)_H + \langle \rho\partial_t y(T), v(T)\rangle_\Omega = \int_I \langle u, v\rangle_\Omega \,\mathrm{d}t - \big(\rho y^0, \partial_t v(0)\big)_H + \langle \rho y^1, v(0)\rangle_\Omega \quad (3.7)$$

for any $v \in L^2(I, V^2) \cap H^2(I, H) \hookrightarrow H^1(I, V)$ is called a *very weak solution* of (2.1).

Actually, these two last solution concepts are equivalent for the considered data spaces.

Theorem 3.9. *Definitions 3.6 and 3.8 are equivalent.*

Proof. First of all, we consider the auxiliary integrated in t problem (2.1):

$$\begin{cases} \rho\partial_{tt}\tilde{y} - \partial_x(\kappa\partial_x\tilde{y}) = \mathcal{I}_t u + \rho y^1 & \text{in } I \times \Omega \\ \tilde{y} = 0 & \text{on } I \times \partial\Omega \\ \tilde{y} = 0,\ \partial_t\tilde{y} = y^0 & \text{in } \{0\} \times \Omega \end{cases} \quad (3.8)$$

for $(u, y^0, y^1) \in L^2(I, V^*) \times H \times V^*$. Thus, we have $\mathcal{I}_t u \in H^1(I, V^*)$. According to Proposition 3.5 problem (3.8) has a unique weak solution $\tilde{y} \in \mathcal{C}(\bar{I}, V) \cap \mathcal{C}^1(\bar{I}, H)$. Moreover, we set $y = \partial_t\tilde{y}$. Thus the weak formulation of (3.8) involving $\tilde{y}$ coincides with the weaker formulation of (2.1) involving y. Furthermore there holds $y = \partial_t\tilde{y} \in \mathcal{C}(\bar{I}, H)$ and

$$\partial_t y = \partial_{tt}\tilde{y} = (1/\rho)\big(\mathcal{I}_t u + \partial_x(\kappa\partial_x\tilde{y}) + \rho y^1\big) \in \mathcal{C}(\bar{I}, V^*). \quad (3.9)$$

Now we take any $v \in \mathcal{C}^\infty(\bar{I}, V^2)$ and test (3.6) with $-\partial_t v$ in the role of v:

$$\int_I (\rho y, \partial_{tt}v)_H - \big(\kappa\partial_x\mathcal{I}_t y, \partial_x\partial_t v\big)_H \,\mathrm{d}t - \big(\rho y(T), \partial_t v(T)\big)_H = \int_I \big\langle u, -\mathcal{I}_t^*(\partial_t v)\big\rangle_\Omega \,\mathrm{d}t - \big(\rho y^0, \partial_t v(0)\big)_H + \langle \rho y^1, -(\mathcal{I}_t^*\partial_t v)(0)\rangle_\Omega.$$

Next we rearrange a term on the left integrating by parts in x and t:

$$-\int_I \big(\kappa\partial_x\mathcal{I}_t y, \partial_x\partial_t v\big)_H \,\mathrm{d}t = \int_I \big(\mathcal{I}_t y, L\partial_t v\big)_H \,\mathrm{d}t = -\int_I (y, Lv)_H \,\mathrm{d}t + \big((\mathcal{I}_t y)(T), Lv(T)\big)_H \quad (3.10)$$

with $Lv := \partial_x(\kappa\partial_x v)$. Since $\mathcal{I}_t y \in \mathcal{C}(\bar{I}, V)$, we get

$$\int_I \big(y, \rho\partial_{tt}v - \partial_x(\kappa\partial_x v)\big)_H \,\mathrm{d}t - \big(\rho y(T), \partial_t v(T)\big)_H + \langle \partial_x(\kappa\partial_x\mathcal{I}_t y)(T), v(T)\rangle_\Omega = \int_I \langle u, v\rangle_\Omega \,\mathrm{d}t - \langle (\mathcal{I}_t u)(T), v(T)\rangle_\Omega - \big(\rho y^0, \partial_t v(0)\big)_H + \langle \rho y^1, v(0)\rangle_\Omega - \langle \rho y^1, v(T)\rangle_\Omega.$$

Then (3.9) and the density of $\mathcal{C}^\infty(\bar{I}, V^2)$ in $L^2(I, V^2) \cap H^2(I, H)$ imply that y is a very weak solution of (2.1).

Now let $y \in \mathcal{C}(\bar{I}, H) \cap \mathcal{C}^1(\bar{I}, V^*)$ be a very weak solution of (2.1). Then we take any $v \in \mathcal{C}^\infty(\bar{I}, V^2)$ and test (3.7) with $\mathcal{I}_t^* v$. Thus, we get

$$\int_I (y, -\rho\partial_t v)_H - \big(y, \partial_x(\kappa\partial_x\mathcal{I}_t^* v)\big)_H \,\mathrm{d}t + \big(\rho y(T), v(T)\big)_H = \int_I \big\langle u, \mathcal{I}_t^* v\big\rangle_\Omega \,\mathrm{d}t + \big(\rho y^0, v(0)\big)_H + \langle \rho y^1, (\mathcal{I}_t^* v)(0)\rangle_\Omega \quad (3.11)$$

and then

$$\int_I (y, -\rho\partial_t v)_H + (\mathcal{I}_t y, -\partial_x(\kappa\partial_x v))_H \,\mathrm{d}t + \big(\rho y(T), v(T)\big)_H = \int_I \big\langle \mathcal{I}_t u + \rho y^1, v\big\rangle_\Omega \,\mathrm{d}t + \big(\rho y^0, v(0)\big)_H.$$

The last equation yields that $L\mathcal{I}_t y = -\rho\partial_t y + \mathcal{I}_t u + \rho y^1 \in \mathcal{C}(\bar{I}, V^*)$. Thus $\mathcal{I}_t y \in \mathcal{C}(\bar{I}, V)$ and we can transform a term on the left in (3.11) by replacing v by $\mathcal{I}_t^* v$ in (3.10):

$$\int_I \left(y, -\partial_x(\kappa\partial_x \mathcal{I}_t^* v)\right)_H \,\mathrm{d}t = \int_I \left(\kappa\partial_x \mathcal{I}_t y, \partial_x v\right)_H \,\mathrm{d}t.$$

Then the density of $\mathcal{C}^\infty(\bar{I}, V^2)$ in $L^2(I, V) \cap H^1(I, H)$ shows that y is a weaker solution of (2.1). □

Moreover, there is the concept of solutions by transposition.

Definition 3.10. Let $(u, y^0, y^1) \in L^2(I, V^*) \times H \times V^*$. A *solution by transposition* $y \in \mathcal{C}(\bar{I}, H) \cap \mathcal{C}^1(\bar{I}, V^*)$ of (2.1) is defined by

$$\int_I (\rho y, \phi)_H \,\mathrm{d}t - \left(\rho y(T), p^1\right)_H + \langle \rho\partial_t y(T), p^0\rangle_\Omega = \int_I \langle u, p\rangle_\Omega \,\mathrm{d}t + \langle \rho y^1, p(0)\rangle_\Omega - \left(\rho y^0, \partial_t p(0)\right)_H \quad (3.12)$$

for all $(\phi, p^0, p^1) \in L^2(I \times \Omega) \times V \times H$ where $p \in \mathcal{C}(\bar{I}, V) \cap \mathcal{C}^1(\bar{I}, H)$ is the weak solution of *the adjoint problem*

$$\begin{cases} \rho\partial_{tt} p - \partial_x(\kappa\partial_x p) = \rho\phi & \text{in } I \times \Omega \\ p = 0 & \text{on } I \times \partial\Omega \\ p = p^0, \ \partial_t p = p^1 & \text{in } \{T\} \times \Omega \end{cases} \quad (3.13)$$

Proposition 3.11. *Definitions 3.8 and 3.10 are equivalent too.*

Proof. For $\phi \in H^1(I, H)$ or $L^2(I, V)$, $p^0 \in V^2$ and $p^1 \in V$ there holds $p \in \mathcal{C}(\bar{I}, V^2) \cap \mathcal{C}^1(\bar{I}, V) \cap H^2(I, H)$, see Proposition 3.5. Due to the density of $H^1(I, H)$ resp. $L^2(I, V)$ in $L^2(I \times \Omega)$ as well as V^2 in V and V in H a very weak solution is a solution by transposition. Now let $p \in \mathcal{C}^\infty(\bar{I}, V^2)$ and set $\phi = \partial_{tt} p - (1/\rho)\partial_x(\kappa\partial_x p) \in \mathcal{C}^\infty(\bar{I}, H)$, $p^0 = p(T) \in V^2$ and $p^1 = \partial_t p(T) \in V^2$. Thus p is the solution of (3.13). Then the density of $\mathcal{C}^\infty(\bar{I}, V^2)$ in $L^2(I, V^2) \cap H^2(I, H)$ implies that a solution by transposition is a very weak solution. □

Remark 3.12. For $(u, y^0, y^1) \in L^2(I, H) \times V \times H$, the weaker solution coincides with the weak one.

3.2. **Existence and regularity of the state.** In this section we study the existence, uniqueness and regularity of solution of the state equation for measure valued source terms. We will carry out the analysis for both control spaces. We use the distinct properties of each space in order to show improved regularity of the state.

3.2.1. *The control space $\mathcal{M}(\Omega, L^2(I))$.* The space $\mathcal{M}(\Omega, L^2(I))$ is not so broad as $L^2(I, \mathcal{M}(\Omega))$ and contains no moving point sources but contains the standing δ-sources (1.2). Therefore, we expect that the state has better regularity properties in this case and prove that $y \in \mathcal{C}(\bar{I}, V) \cap \mathcal{C}^1(\bar{I}, H)$. The proof will be based on a priori bound and a density argument. First we state the following density result.

Lemma 3.13. *Let $u \in \mathcal{M}(\Omega, L^2(I))$. Then there exists a sequence $\{u_n\} \subset C_c^\infty(\Omega, L^2(I))$ such that*

$$u_n \rightharpoonup^* u \quad \text{in} \quad \mathcal{M}(\Omega, L^2(I)) \quad \text{as} \quad n \to \infty, \quad \|u_n\|_{\mathcal{M}(\Omega, L^2(I))} \le \|u\|_{\mathcal{M}(\Omega, L^2(I))} \ \forall n \ge 1. \quad (3.14)$$

Proof. We denote by X the locally convex space $\mathcal{M}(\Omega, L^2(I))$ endowed with its weak-star topology and define the absolutely convex set

$$E = \{u \in \mathcal{C}_c^\infty(\Omega, L^2(I)) | \|u\|_{L^1(\Omega, L^2(I))} \le 1\} \subset X.$$

Assume that (3.14) is wrong. Then there exists $u_0 \in \mathcal{M}(\Omega, L^2(I))$, $\|u_0\|_{\mathcal{M}(\Omega, L^2(I))} = 1$ such that $u_0 \notin \bar{E}$ where $\bar{E}$ is the closure of E in X. Owing to the corollary of a theorem on the separation of convex sets [25, Ch. III, Theorem 6] there exists $v \in \mathcal{C}_0(\Omega, L^2(I))$ such that

$$|\langle u, v\rangle_{\mathcal{M}(\Omega, L^2(I)), \mathcal{C}_0(\Omega, L^2(I))}| \le 1 \quad \forall u \in E, \quad 1 < \langle u_0, v\rangle_{\mathcal{M}(\Omega, L^2(I)), \mathcal{C}_0(\Omega, L^2(I))} \le \|v\|_{\mathcal{C}_0(\Omega, L^2(I))}. \quad (3.15)$$

On the other hand, $\mathcal{C}_c^\infty(\Omega, L^2(I))$ is dense in $L^1(\Omega, L^2(I))$ thus

$$\sup_{u \in E} |\langle u, v\rangle_{\mathcal{M}(\Omega, L^2(I)), \mathcal{C}_0(\Omega, L^2(I))}| = \|v\|_{\mathcal{C}_0(\Omega, L^2(I))}$$

that contradicts (3.15). □

Note that clearly $C_c^\infty(\Omega, L^2(I)) \subset L^2(I, V)$.

Preliminarily we prove the following crucial a priori bound.

Lemma 3.14. *Let $(u, y^0, y^1) \in L^1(\Omega, L^2(I)) \times V \times H$ and y be the corresponding strong solution of problem* (2.1)*. Then y satisfies the following a priori bound*

$$\|y\|_{\mathcal{C}(\bar{I},V)} + \|\partial_t y\|_{\mathcal{C}(\bar{I},H)} + \|\kappa\partial_x y\|_{\mathcal{C}(\bar{\Omega},L^2(I))} + \|\partial_t y\|_{\mathcal{C}_0(\Omega,L^2(I))} \le c(\|u\|_{L^1(\Omega,L^2(I))} + \|\mathbf{y}\|_{V\times H}). \quad (3.16)$$

Proof. We first remind the energy equality for problem (2.1)

$$\|\sqrt{\rho}\partial_t y(t)\|_H^2 + \|\sqrt{\kappa}\partial_x y(t)\|_H^2 = \|\sqrt{\rho}y^1\|_H^2 + \|\sqrt{\kappa}\partial_x y^0\|_H^2 + 2\mathcal{I}_t(u(t), \partial_t y(t))_H \quad \text{on } I.$$

After setting

$$E(t) := \|\partial_t y(t)\|_H^2 + \|\partial_x y(t)\|_H^2, \quad E^0 := \|y^1\|_H^2 + \|\partial_x y^0\|_H^2, \quad c_0 := \max\big(\|\rho\|_{L^\infty(\Omega)}, \|\kappa\|_{L^\infty(\Omega)}\big),$$

the energy equality implies

$$\nu\|E\|_{\mathcal{C}(\bar{I})} \le c_0E^0 + 2\max_{\theta\in\bar{I}}\Big|\int_\Omega\int_0^\theta u\partial_t y \,\mathrm{d}t\mathrm{d}x\Big| \le c_0E^0 + 2\nu^{-1}\|u\|_{L^1(\Omega,\,L^2(I))}\|\sqrt{\rho\kappa}\partial_t y\|_{\mathcal{C}_0(\Omega,\,L^2(I))}. \quad (3.17)$$

We also multiply the equation in (2.1) by $-2\kappa\partial_x y$ and integrate over I. Integration by parts in t yields the equality

$$\rho\kappa\partial_x\Big(\|\partial_t y\|_{L^2(I)}^2\Big) + \partial_x\Big(\|\kappa\partial_x y\|_{L^2(I)}^2\Big) = 2\rho\kappa\Big(\partial_t y(T)\partial_x y(T) - y^1\partial_x y^0\Big) - 2(u, \kappa\partial_x y)_{L^2(I)} \text{ on } \Omega. \quad (3.18)$$

We define a function $P := \rho\kappa\|\partial_t y\|_{L^2(I)}^2 + \|\kappa\partial_x y\|_{L^2(I)}^2$ on Ω. Since the left-hand side of (3.18) equals $\partial_x P - (\partial_x(\rho\kappa))\|\partial_t y\|_{L^2(I)}^2$, taking the modulus and integrating over any $(a,b) \subset \Omega$ we derive

$$\begin{aligned}\|\partial_x P\|_{L^1(a,b)} &\le c_0^2(E(T) + E^0) + 2\|u\|_{L^1(\Omega,\,L^2(I))}\|\kappa\partial_x y\|_{\mathcal{C}(\bar{\Omega},\,L^2(I))} + \|(\partial_x(\rho\kappa))\|\partial_t y\|_{L^2(I)}^2\|_{L^1(a,b)} \\ &\le c_0^2(\|E\|_{\mathcal{C}(\bar{I})} + E^0) + 2\|u\|_{L^1(\Omega,\,L^2(I))}\|P\|_{\mathcal{C}(\bar{\Omega})}^{1/2} + \nu^{-2}\|\partial_x(\rho\kappa)\|_{L^1(a,b)}\|P\|_{\mathcal{C}(\bar{\Omega})}. \quad (3.19)\end{aligned}$$

Let $x_0 \in \bar{\Omega}$ be such that $\|P\|_{\mathcal{C}(\bar{\Omega})} = P(x_0)$ hold and let now $[a,b] \ni x_0$. Then the mean value theorem for integrals implies

$$\|P\|_{\mathcal{C}(\bar{\Omega})} \le (b-a)^{-1}\|P\|_{L^1(a,b)} + \|\partial_x P\|_{L^1(a,b)}. \quad (3.20)$$

By the above definitions we clearly have

$$\|P\|_{L^1(\Omega)} \le c_0^2\|E\|_{L^1(I)} \le c_0^2 T\|E\|_{\mathcal{C}(\bar{I})}. \quad (3.21)$$

Inserting (3.19) into (3.20) and using (3.21), we obtain

$$\begin{aligned}\|P\|_{\mathcal{C}(\bar{\Omega})} \le c_0^2(1 + T(b-a)^{-1})\big(\|E\|_{\mathcal{C}(\bar{I})} + E^0\big) + 2\|u\|_{L^1(\Omega,\,L^2(I))}\|P\|_{\mathcal{C}(\bar{\Omega})}^{1/2} \\ + \nu^{-2}(b-a)^{1/2}\|\rho\kappa\|_{H^1(\Omega)}\|P\|_{\mathcal{C}(\bar{\Omega})}. \quad (3.22)\end{aligned}$$

Owing to (3.17) we can write

$$\nu\|E\|_{\mathcal{C}(\bar{I})} \le c_0^2E^0 + 2\nu^{-1}\|u\|_{L^1(\Omega,\,L^2(I))}\|P\|_{\mathcal{C}(\bar{\Omega})}^{1/2}. \quad (3.23)$$

Using this in (3.22) and choosing a small enough (a,b) such that $\nu^{-2}(b-a)^{1/2}\|\rho\kappa\|_{H^1(\Omega)} \le 1/2$, we derive

$$\|P\|_{\mathcal{C}(\bar{\Omega})} \le c_1\big(E^0 + \|u\|_{L^1(\Omega,\,L^2(I))}^2\big). \quad (3.24)$$

Inserting the last bound in (3.23), we also get

$$\|E\|_{\mathcal{C}(\bar{I})} \le c_2\big(E^0 + \|u\|_{L^1(\Omega,\,L^2(I))}^2\big).$$

Finally, this yields bound (3.16). □

Remark 3.15. Lemma 3.14 remains valid for $\rho, \kappa \in W^{1,1}(\Omega)$. Owing to the absolute continuity of the Lebesgue integral we have $\|\partial_x(\rho\kappa)\|_{L^1(a,b)} \le \mu(b-a)$, where $\lim_{\theta\to+0}\mu(\theta) = 0$, thus one can replace $(b-a)^{1/2}\|\rho\kappa\|_{H^1(\Omega)}$ by $\mu(b-a)$ in (3.22) and below in the proof.

Theorem 3.16. *Let $(u, y^0, y^1) \in \mathcal{M}(\Omega, L^2(I)) \times V \times H$. Then there exists a unique weak solution y and it satisfies the bound*

$$\|y\|_{\mathcal{C}(\bar I,V)} + \|\partial_t y\|_{\mathcal{C}(\bar I,H)} \le c\,(\|u\|_{\mathcal{M}(\Omega,L^2(I))} + \|\mathbf{y}\|_{V\times H}). \tag{3.25}$$

Proof. 1. Let first $u = 0$. According to Proposition 3.5 were exists a unique weak solution y of (2.1) for any $\mathbf{y} \in V \times H$ and it satisfies

$$\|y\|_{\mathcal{C}(\bar I,V)} + \|\partial_t y\|_{\mathcal{C}(\bar I,H)} \le c\|\mathbf{y}\|_{V\times H}.$$

2. Now it suffices to consider the case $y^0 = y^1 = 0$. Let first $u \in \mathcal{M}(\Omega, H^1(I)) \hookrightarrow H^1(I, V^*)$ since $\partial_t u \in \mathcal{M}(\Omega, L^2(I)) \hookrightarrow L^2(I, V^*)$. Then according to Proposition 3.5 there exists a unique weak solution $y \in \mathcal{C}(\bar I, V) \cap \mathcal{C}^1(\bar I, H)$ of (2.1) and it satisfies bound (3.4). Moreover, it is also a weaker solution.

So it remains to prove the bound

$$\|y\|_{\mathcal{C}(\bar I,V)} + \|\partial_t y\|_{\mathcal{C}(\bar I,H)} \le c\,\|u\|_{\mathcal{M}(\Omega,\,L^2(I))} \quad \text{for} \quad u \in \mathcal{M}(\Omega, H^1(I)). \tag{3.26}$$

To this end, according to Lemma 3.13 we approximate u by functions $\{u_n\} \subset L^2(I, V)$ satisfying (3.14). The strong solution y_n of (2.1) corresponding to $u = u_n$ satisfies the bound like (3.16) and in particular

$$\|y_n\|_{\mathcal{C}(\bar I,V)} + \|\partial_t y_n\|_{\mathcal{C}(\bar I,H)} \le c\,\|u_n\|_{\mathcal{M}(\Omega,\,L^2(I))} \le c\|u\|_{\mathcal{M}(\Omega,L^2(I))}.$$

Therefore there exists a subsequence of $\{y_n\}$ (not relabelled) and $\tilde y \in L^\infty(I, V) \cap W^{1,\infty}(I, H)$ such that y_n converges to $\tilde y$ in the weak-star sense of $L^\infty(I, V) \cap W^{1,\infty}(I, H)$. This is sufficient to pass to the limit in the last bound and in (3.6) for $y = y_n$, $u = u_n$ and $v(T) = 0$, see Remark 3.2. Thus $\tilde y$ both satisfies the bound

$$\|\tilde y\|_{L^\infty(\bar I,V)} + \|\partial_t \tilde y\|_{W^{1,\infty}(\bar I,H)} \le c\,\|u\|_{\mathcal{M}(\Omega,\,L^2(I))}$$

and is a weaker solution of (2.1). Due to its uniqueness there holds $\tilde y = y$, and bound (3.26) is proved.

2. Let now $u \in \mathcal{M}(\Omega, L^2(I))$ and y be the corresponding weaker solution of (2.1), see Proposition 3.7. Since $\mathcal{M}(\Omega, H^1(I))$ is dense in $\mathcal{M}(\Omega, L^2(I))$, cf. [31, Proposition 2.1], there exists a sequence $\{u_n\} \subset \mathcal{M}(\Omega, H^1(I))$ such that $u_n \to u$ in $\mathcal{M}(\Omega, L^2(I))$ as $n \to \infty$. Let $y_n \in \mathcal{C}(\bar I, V) \cap \mathcal{C}^1(\bar I, H)$ be the above weak solution of (2.1) corresponding to $u = u_n$. Since $\{u_n\}$ is a Cauchy sequence in $\mathcal{M}(\Omega, L^2(I))$, $\{y_n\}$ is a Cauchy sequence in $\mathcal{C}(\bar I, V) \cap \mathcal{C}^1(\bar I, H)$ too due to bound (3.26) for $u = u_n$. Thus $y_n \to \hat y$ in $\mathcal{C}(\bar I, V) \cap \mathcal{C}^1(\bar I, H)$ and

$$\|\hat y\|_{\mathcal{C}(\bar I,V)} + \|\partial_t \hat y\|_{\mathcal{C}(\bar I,H)} \le c\,\|u\|_{\mathcal{M}(\Omega,\,L^2(I))}.$$

Then we pass to the limit in (3.1) for $y = y_n$, $u = u_n$ and $v(T) = 0$ and see that $\hat y$ is a weak solution of (2.1). Due to uniqueness of the weaker solution we get $\hat y = y$, and the proof is complete. ☐

3.2.2. *The control space $\mathcal{M}_T = L^2(I, \mathcal{M}(\Omega))$.* Recall that the space $\mathcal{M}_T = L^2(I, \mathcal{M}(\Omega))$ contains the moving point sources (1.3). We set

$$H^{(-1)} = V^*, \quad H^{(0)} = H, \quad H^{(1)} = V, \quad H^{(2)} = V^2, \quad H^{(3)} = V^3$$

and introduce the interpolation spaces

$$H^{(\lambda)} := \left(H^{(\ell)}, H^{(\ell+1)}\right)_{\lambda-\ell,\infty}, \quad \ell := \lfloor\lambda\rfloor,$$

for non-integer $\lambda \in (-1, 3)$ using the real $K_{\lambda,q}$-interpolation method of Banach spaces for $q = \infty$, see [3]. Recall that the value $q = \infty$ leads to the broadest intermediate spaces. Their explicit description in terms of the subspaces in the Nikolskii spaces is known, see [39, 44, 46]. In particular,

$$H^{(\lambda)} = H^{\lambda,2}(\Omega) \ \text{ for } \ 0 < \lambda < \tfrac12, \quad H^{(\lambda)} = \tilde H^{1/2,2}(\Omega) \ \text{ for } \ \lambda = \tfrac12, \quad H^{(\lambda)} = H_0^{\lambda,2}(\Omega) \ \text{ for } \ \tfrac12 < \lambda < 1,$$

where $\tilde{H}^{1/2,2}(\Omega) := \{w \in H^{1/2,2}(\Omega) |\, \mathsf{o}w \in H^{1/2,2}(-L, 2L)\}$ and $\mathsf{o}w$ is the odd extension of w with respect to $x = 0, L$ oustide Ω. It is well known that the last space contains discontinuous but piecewise continuously differentiable functions.

Lemma 3.17. *The following embeddings hold*

$$L^2(I, \mathcal{M}(\Omega)) \hookrightarrow L^2(I, H^{(-1/2)}), \tag{3.27}$$

$$\mathcal{C}^1(\bar{I}, \mathcal{M}(\Omega)) \hookrightarrow H^1(I, H^{(-1/2)}). \tag{3.28}$$

Proof. Both the embeddings follow from $\mathcal{M}(\Omega) \hookrightarrow H^{(-1/2)}$. Due to [3] the last embedding means that

$$\sup_{0<h<X} h^{-1/2} \inf_{v \in H} (\|w - v\|_{V^*} + h\|v\|_H) \le c\|w\|_{\mathcal{M}(\Omega)} \quad \forall w \in \mathcal{M}(\Omega).$$

Equivalently, for any $0 < h < L$, there exists $w_h \in H$ such that

$$\|w - w_h\|_{V^*} \le c_1 h^{1/2} \|w\|_{\mathcal{M}(\Omega)}, \quad \|w_h\|_H \le c_1 h^{-1/2} \|w\|_{\mathcal{M}(\Omega)} \quad \forall w \in \mathcal{M}(\Omega). \tag{3.29}$$

Any $w \in \mathcal{M}(\Omega)$ can be represented as $w = D_x W$ with $\|w\|_{\mathcal{M}(\Omega)} = \operatorname{var}_{\bar{\Omega}} W$, where D_x is the distributional derivative of a function $W \in NBV(\Omega)$ [9, Ch. 2]. Here $NBV(\Omega)$ is the space of normalized functions of bounded variation on $\bar{\Omega}$ that are continuous from the right at $x = 0$ and continuous from the left at any $x \in (0, L]$. Notice that the following inequalities hold

$$\|W - W(0)\|_{L^\infty(\Omega)} \le \operatorname{var}_{\bar{\Omega}} W, \quad \sup_{0<h<L} \|\delta_{-h} W\|_{L^1(h,L)} \le \operatorname{var}_{\bar{\Omega}} W \quad \forall W \in NBV(\Omega), \tag{3.30}$$

where $\delta_{-h} W(x) := h^{-1}\big(W(x) - W(x - h)\big)$ is the backward difference quotient (the latter inequality follows from the definition of the Riemann integral).

We can choose $W(0) = 0$ and extend $W(x) = 0$ for $x < 0$. Then for $0 < h < L$ we define the backward average $W_h(x) := h^{-1} \int_{-h}^{0} W(x + \xi)\, \mathbf{d}\xi$ and set $w_h := \partial_x W_h = \delta_{-h} W$. Owing to the inequalities

$$\|\varphi\|_H \le \|\varphi\|_{L^\infty(\Omega)}^{1/2} \|\varphi\|_{L^1(\Omega)}^{1/2} \quad \forall \varphi \in L^\infty(\Omega)$$

and (3.30) we can prove estimates (3.29):

$$\|w - w_h\|_{V^*} \le \|W - W_h\|_H \le \big(2\|W\|_{L^\infty(\Omega)}\big)^{1/2} \big(h \sup_{0<h<L} \|\delta_{-h} W\|_{L^1(\Omega)}\big)^{1/2} \le c h^{1/2} \operatorname{var}_{\bar{\Omega}} W,$$

$$\|w_h\|_H = \|\delta_{-h} W\|_H \le \big(2h^{-1} \|W\|_{L^\infty(\Omega)}\big)^{1/2} \|\delta_{-h} W\|_{L^1(\Omega)}^{1/2} \le c h^{-1/2} \operatorname{var}_{\bar{\Omega}} W.$$

□

4. Analysis of the control problem

According to Theorem 3.16 and Proposition 3.7 the state equation (2.1) is uniquely solvable for any u in either $\mathcal{M}(\Omega, L^2(I))$ or $L^2(I, \mathcal{M}(\Omega))$ and the solution y depends continuously on the data. Therefore, we can introduce the linear and bounded operator $\hat{S}\colon (u, y^0, y^1) \mapsto (y, y(T), \rho\partial_t y(T))$. The control-to-state mapping

$$S\colon \mathcal{M}_T \to \mathcal{Y}, \quad u \mapsto (y, y(T), \rho\partial_t y(T))$$

is given by $Su = \hat{S}(u, 0, 0) + \hat{S}(0, y^0, y^1)$ for fixed y^0 and y^1 and it is an affine and bounded operator. So we can rewrite the original control problem $(\mathcal{P})$ in its reduced form

$$j(u) = \tfrac{1}{2} \|Su - \mathbf{z}\|_{\mathcal{Y}}^2 + \alpha \|u\|_{\mathcal{M}_T} \to \min_{u \in \mathcal{M}_T}.$$

Proposition 4.1. *Problem $(\mathcal{P})$ has a unique solution $\bar{u} \in \mathcal{M}_T$.*

Proof. The control-to-state operator S is weak-star-to-strong sequential continuous, i.e., if $\{u_n\} \subset \mathcal{M}_T$ and $u_n \rightharpoonup^* u$ in $\mathcal{M}_T$, then $Su_n \to Su$ in $\mathcal{Y}$. The proof of this continuity property is similar to [31, Lemma 6.1] in the case of solutions by transposition resp. very weak solutions. The strong continuity follows from the compact embeddings and well known Aubin-Lions-Lemma. Then the direct method of calculus of variations combined with the sequential Banach-Alaoglu theorem ($\mathcal{C}_T$ is separable) can be applied to show existence of an optimal control. Additionally the control is unique since the control-to-state operator S is injective and the data tracking functional is strictly convex. □

Owing to Proposition 3.7 the optimal control $\bar{u} \in \mathcal{M}_T$ satisfies the inequalities

$$\alpha\|\bar{u}\|_{\mathcal{M}_T} \le j(\bar{u}) \le j(0) = \tfrac{1}{2}\|S(0) - \mathbf{z}\|_{\mathcal{Y}}^2 \le c\big(\|y^0\|_H + \|y^1\|_{V^*} + \|\mathbf{z}\|_{\mathcal{Y}}\big)^2 \tag{4.1}$$

and thus

$$\|\bar{u}\|_{\mathcal{M}_T} \le c\big(\|\mathbf{y}\|_{H\times V^*} + \|z\|_{\mathcal{Y}}\big)^2 \le C. \tag{4.2}$$

Hereafter $C > 0$ depends on the norms of data.

Next we discuss first order optimality conditions. We introduce the adjoint control-to-solution operator $S^\star \colon \mathcal{Y} \to C(\bar{I}, V) \hookrightarrow \mathcal{C}_T$, $(\phi, p^1, p^0) \mapsto p$ where p is a weak solution of (3.13). This operator is well defined and bounded according to Proposition 3.5.

We also need the operator $A^{-1} \colon V^* \to V$, $f \mapsto w$ where $w \in V$ is the unique solution of

$$(\kappa\partial_x w, \partial_x v)_H = \langle f, v\rangle_\Omega \quad \forall v \in V. \tag{4.3}$$

The next result provides the necessary and sufficient optimality condition for the optimal pair $(\bar{p}, \bar{u})$.

Proposition 4.2. *An element $\bar{u} \in \mathcal{M}_T$ is an optimal control of $(\mathcal{P})$ if and only if*

$$-\bar{p} \in \alpha\partial\|\bar{u}\|_{\mathcal{M}_T}, \tag{4.4}$$

or equivalently

$$\langle -\bar{p}, u - \bar{u}\rangle_{\mathcal{C}_T,\mathcal{M}_T} + \alpha\|\bar{u}\|_{\mathcal{M}_T} \le \alpha\|u\|_{\mathcal{M}_T} \quad \forall u \in \mathcal{M}_T \tag{4.5}$$

where $\bar{p} = S^*(\bar{y} - z_1, -(\bar{y}(T) - z_2), A^{-1}(\rho\partial_t\bar{y} - z_3))$ with $(\bar{y}, \bar{y}(T), \rho\partial_t\bar{y}(T)) = \hat{S}(\bar{u}, y^0, y^1)$.

Proof. For $\mathcal{M}_T = \mathcal{M}(\Omega, L^2(I))$ a proof in [31] remains valid; for $\mathcal{M}_T = L^2(I, \mathcal{M}(\Omega))$ it is similar. □

To discuss further the properties of the optimal control $\bar{u}$, we introduce the Jordan decomposition of a signed measure $\mu \in \mathcal{M}(\Omega)$, see [5]. There exists unique elements $\mu^\pm \in \mathcal{M}(\Omega)^+$ such that $\mu = \mu^+ - \mu^-$. Moreover, we recall the polar decomposition of a vector measure $\mu \in \mathcal{M}(\Omega, L^2(I))$: $\mathrm{d}\mu = \mu'\mathrm{d}|\mu|$, where μ' is the Radon-Nikodym-derivative of μ with respect to $|\mu|$.

The subgradient condition in Proposition 4.2 implies the following conditions.

Proposition 4.3. *Let $\bar{u} \in \mathcal{M}_T$ be the optimal control of $(\mathcal{P})$ and $\bar{p} \in \mathcal{C}_T$ be the corresponding adjoint state. Then there holds $\|\bar{p}\|_{\mathcal{C}_T} \le \alpha$.*

In the cases $\mathcal{M}_T = L^2(I, \mathcal{M}(\Omega))$ and $\mathcal{M}_T = \mathcal{M}(\Omega, L^2(I))$ there respectively hold

$$\operatorname{supp} \bar{u}^\pm(t) \subset \{x \in \Omega \mid \bar{p}(t,x) = \mp\|\bar{p}(t,\cdot)\|_{\mathcal{C}_0(\Omega)}\} \quad \text{for a.a. } t \in I$$

and

$$\operatorname{supp}|\bar{u}| \subset \{x \in \Omega \mid \|\bar{p}(\cdot,x)\|_{L^2(I)} = \alpha\}, \quad \bar{u}' = -\alpha^{-1}\bar{p} \ \text{ in } \ L^1(\Omega, |\bar{u}|, L^2(I)). \tag{4.6}$$

Proof. A detailed discussion of the proof of these results can be found in [11, 30]. □

The regularity of the adjoint state $\bar{p}$ is now applied to show improved regularity of the optimal control $\bar{u}$.

Theorem 4.4. *Let $\mathcal{M}_T = \mathcal{M}(\Omega, L^2(I))$, $\mathbf{z} \in \mathcal{Y}^1 := L^2(I, V) \times V \times H$, $\mathbf{y} \in V \times H$ and $\bar{u}$ be the optimal control of $(\mathcal{P})$. Then $\bar{u} \in \mathcal{C}^1(\bar{I}, \mathcal{M}(\Omega))$ and the following bound holds*

$$\|\bar{u}\|_{\mathcal{C}^1(\bar{I},\mathcal{M}(\Omega))} \le C = C(\|\mathbf{y}\|_{V\times H}, \|\mathbf{z}\|_{\mathcal{Y}^1}).$$

Proof. There holds $\bar{y} \in \mathcal{C}(\bar{I}, V) \cap \mathcal{C}^1(\bar{I}, H)$ according to Theorem 3.16. Thus, the optimal adjoint state has the following regularity $\bar{p} \in \mathcal{C}(\bar{I}, V^2) \cap \mathcal{C}^1(\bar{I}, V)$ by Proposition 3.5. We have $\bar{u} = -\alpha^{-1}\bar{p}\,|\bar{u}|$ according to (4.6). Moreover, we define the function

$$w = -\alpha^{-1}(\partial_t\bar{p})|\bar{u}|$$

and show that it serves the time derivative of $\bar{u}$. For any $t_0, t \in \bar{I}$ and $t_0 \neq t$, we define the difference quotient $\bar{u}(t_0;t) = (\bar{u}(t) - \bar{u}(t_0))/(t - t_0)$. Then we consider

$$\begin{aligned}\|\bar{u}(t_0,t) - w(t_0)\|_{\mathcal{M}(\Omega)} &= \alpha^{-1} \sup_{\|\phi\|_{\mathcal{C}_0(\Omega)}\le 1} \int_\Omega (\bar{p}(t_0,t) - \partial_t p(t_0))\phi \, \mathrm{d}|\bar{u}| \\ &\le \alpha^{-1}\|\bar{p}(t_0,t) - \partial_t p(t_0)\|_{\mathcal{C}_0(\Omega)}\|\bar{u}\|_{\mathcal{M}(\Omega,L^2(I))} \\ &\le c\alpha^{-1}\|\bar{p}(t_0,t) - \partial_t p(t_0)\|_V\|\bar{u}\|_{\mathcal{M}(\Omega,L^2(I))} \to 0\end{aligned}$$

as $t \to t_0$ since $\bar{p} \in \mathcal{C}^1(\bar{I}, V)$. Next, quite similarly we get

$$\|w(t) - w(t_0)\|_{\mathcal{M}(\Omega)} \le c\alpha^{-1}\|\partial_t\bar{p}(t) - \partial_t\bar{p}(t_0)\|_V\|\bar{u}\|_{\mathcal{M}(\Omega,L^2(I))} \to 0$$

as $t \to t_0$. Consequently $\partial_t\bar{u} = w \in \mathcal{C}(\bar{I}, \mathcal{M}(\Omega))$. Finally, we bound $\partial_t\bar{u}$ as follows

$$\begin{aligned}\|\partial_t\bar{u}\|_{\mathcal{C}(\bar{I},\mathcal{M}(\Omega))} &\le c\alpha^{-1}\|\partial_t\bar{p}\|_{\mathcal{C}(\bar{I},V)}\|\bar{u}\|_{\mathcal{M}(\Omega,L^2(I))} \\ &\le c_1\alpha^{-1}(\|\bar{y} - z_1\|_{L^2(I,V)} + \|\bar{y}(T) - z_2\|_V + \|\rho\partial_t\bar{y}(T) - z_3\|_H)\|\bar{u}\|_{\mathcal{M}(\Omega,L^2(I))} \\ &\le c_2\alpha^{-1}(\|\bar{u}\|_{\mathcal{M}(\Omega,L^2(I))} + \|\mathbf{y}\|_{V\times H} + \|\mathbf{z}\|_{\mathcal{Y}^1})\|\bar{u}\|_{\mathcal{M}(\Omega,L^2(I))}\end{aligned}$$

owing to Proposition 3.5 and Theorem 3.16. Utilizing bound (4.2) for $\bar{u}$, we complete the proof. □

5. Discretization of the state equation

We introduce the uniform grid $t_m = m\tau$ in time with the step $\tau = T/M$ and a non-uniform grid $0 = x_0 < x_1 < \ldots < x_N = L$ in space with the steps $h_j = x_j - x_{j-1}$, where $M \ge 2$ and $N \ge 2$. Let also $h = \max_{j=1,\ldots,N} h_j$, $h_{\min} = \min_{j=1,\ldots,N} h_j$ and $\vartheta = (\tau, h)$. We assume that the space grid is *quasi-uniform*, i.e., $h \le c_1 h_{\min}$. Hereafter c, c_1, C, etc., are grid-independent.

Let $V_\tau \subset H^1(I)$ and $V_h \subset V$ be the spaces of piecewise linear finite elements with respect to the introduced grids on $\bar{I}$ and $\bar{\Omega}$.

We approximate the state variable y by $y_\vartheta \in V_\vartheta := V_\tau \otimes V_h \subset H^1(I, V)$ and additionally $\partial_t y(T)$ by $y^1_{Th} \in V_h$. For $(u, y^0, y^1) \in \mathcal{M}_T \times H \times V^*$ *the discrete state equation* has the following form

$$B_\sigma(y_\vartheta, v) + (\rho y^1_{Th}, v(T))_H = \langle u, v\rangle_{\mathcal{M}_T,\mathcal{C}_T} + \langle \rho y^1, v(0)\rangle_\Omega \quad \forall v \in V_\vartheta, \tag{5.1}$$

$$(\rho y_\vartheta(0), \varphi)_H = (\rho y^0, \varphi)_H \quad \forall\varphi \in V_h, \tag{5.2}$$

involving the indefinite symmetric bilinear form

$$B_\sigma(y, v) := -(\rho\partial_t y, \partial_t v)_{L^2(I\times\Omega)} - (\sigma - \tfrac{1}{6})\tau^2(\kappa\partial_x\partial_t y, \partial_x\partial_t v)_{L^2(I\times\Omega)} + (\kappa\partial_x y, \partial_x v)_{L^2(I\times\Omega)}, \tag{5.3}$$

with the grid independent parameter σ, cf. (3.1). This definition follows [46] but notice carefully that normally y_ϑ is uniquely defined by (5.1) with $v(T) = 0$ and (5.2). To treat general v, we need y^1_{Th}.

Remark 5.1. The second term in (5.3) regularizes the Galerkin (i.e. projection) method with respect to bilinear form (3.2). It is included to ensure unconditional stability for suitable values of σ. Moreover, the term $-(1/6)\,\tau^2(\kappa\partial_x\partial_t y, \partial_x\partial_t v)_{L^2(I\times\Omega)}$ is the error term of the compound trapezoidal rule applied for the calculation of the temporal integral in $(\kappa\partial_x y, \partial_x v)_{L^2(I\times\Omega)}$. So that, in particular, for $\sigma = 0$ in (5.3) this temporal integral is calculated using this rule whereas for $\sigma = 1/6$ it is not approximated.

Next we recall the inverse inequality

$$\|\varphi\|_{\mathcal{V}_\kappa} \le \alpha_h\|\varphi\|_{H_\rho} \quad \forall\varphi \in V_h \tag{5.4}$$

where the least constant satisfies $c_1 h^{-1} \le \alpha_h \le c_2 h^{-1}$ for the quasi-uniform grid. For $\sigma \le 1/4$ we need to state conditions linking the temporal and spatial grids to ensure stability of the numerical method.

Assumption 5.2. In what follows, let

$$\text{if } \ \sigma < \tfrac{1}{4}, \ \text{ then } \ \tau^2\alpha_h^2(\tfrac{1}{4} - \sigma) \le 1 - \varepsilon_0^2 \ \text{ for some } \ 0 < \varepsilon_0 < 1, \tag{5.5}$$

$$\text{if } \ \sigma \le \tfrac{1}{4}, \ \text{ then } \ \tau^2\alpha_h^2(\tfrac{1+\varepsilon_1^2}{4} - \sigma) \le 1 \ \text{ for some } \ 0 < \varepsilon_1 \le 1. \tag{5.6}$$

Remark 5.3. The parameters ε_0 and ε_1 can be chosen arbitrarily small but then constants in the stability and error estimates for our FEM can tend to infinity.

Remark 5.4. As we see below in Section 11, the method is related to well known time-stepping methods, in particular, to the explicit Leap-Frog-method for $\sigma = 0$. Then conditions (5.5) and (5.6) reduce to a CFL-type one $\tau\alpha_h \le 2\sqrt{1-\varepsilon_0^2}$. For $\sigma = 1/4$ the method is related to the Crank-Nicolson scheme and is unconditionally stable but in a weaker norm than we need to derive our error estimates so that we impose a very weak CFL-type condition $\tau\alpha_h \le 2/\varepsilon_1$.

Below in proofs we utilize the auxiliary squared norms

$$\|\varphi\|^2_{H^0_\tau} := \|\varphi\|^2_{H_\rho} + (\sigma - \tfrac{1}{4})\tau^2\|\varphi\|^2_{\mathcal{V}_\kappa}, \quad \|y\|^2_{\mathcal{C}_\tau(H_E)} = \max_{1\le m\le M}\left(\tfrac{1}{\tau}\|y_m - y_{m-1}\|^2_{H^0_\tau} + \tfrac{1}{2}\|y_m + y_{m-1}\|^2_{\mathcal{V}_\kappa}\right)$$

for $\varphi \in V_h$ and $y \in V_\tau \otimes V_h$. We need to bound them by standard norms.

Lemma 5.5. *Under conditions* (5.5) *and* (5.6) *the following inequalities hold*

$$\varepsilon_0\|\varphi\|_{H_\rho} \le \|\varphi\|_{H^0_\tau} \quad \forall\varphi \in V_h, \tag{5.7}$$

$$\|y\|_{\mathcal{C}_\tau(\mathcal{V}_\kappa)} := \max_{0\le m\le M}\|y(t_m)\|_{\mathcal{V}_\kappa} \le \tfrac{\sqrt{2}}{\varepsilon_1}\|y\|_{\mathcal{C}_\tau(H_E)} \quad \forall y \in V_\vartheta$$

with $\varepsilon_0 := 1$ *for* $\sigma \ge 1/4$ *and* $\varepsilon_1 := \sqrt{4\sigma - 1}$ *for* $\sigma > 1/4$.

Proof. For $\sigma \ge 1/4$, the first inequality is obvious; for $\sigma < 1/4$ it can be checked by a direct calculation using (5.4). The proof of the second inequality is covered in [46, Corollary 2.1]. □

Now we discuss some properties of y^1_{Th} and $\partial_t y(T)$ that are essential below.

Proposition 5.6. *Let* $(y_\vartheta, y^1_{Th}) \in V_\vartheta \times V_h$ *be the solution of* (5.1)-(5.2). *Then there holds*

$$(\rho y^1_{Th}, \varphi)_H = -\Big(\kappa\partial_x \int_I y_\vartheta \,\mathrm{d}t, \partial_x\varphi\Big)_H + \int_I \langle u, \varphi\rangle_\Omega \,\mathrm{d}t + (\rho y^1, \varphi)_H \quad \forall\varphi \in V_h. \tag{5.8}$$

Proof. This is proved by testing (5.1) with time constant functions $v = \varphi \in V_h$. □

The non-local in time identity (5.8) is convenient for our error analysis but not for the implementation; for the latter issue see Section 11. Identities similar to (5.8) also hold on the continuous level.

Proposition 5.7. *(1) Let* $y \in \mathcal{C}(\bar{I}, V) \cap \mathcal{C}^1(\bar{I}, H)$ *be the weak solution of* (2.1) *for* $\mathcal{M}_T = \mathcal{M}(\Omega, L^2(I))$. *Then there holds*

$$(\rho\partial_t y(T), \varphi)_H = -\Big(\kappa\partial_x \int_I y \,\mathrm{d}t, \partial_x\varphi\Big)_H + \Big\langle \int_I u \,\mathrm{d}t, \varphi\Big\rangle_\Omega + (\rho y^1, \varphi)_H \quad \forall\varphi \in V. \tag{5.9}$$

(2) Let $y \in \mathcal{C}(\bar{I}, H) \cap \mathcal{C}^1(\bar{I}, V^*)$ *be the weaker (very weak) solution of* (2.1) *for* $\mathcal{M}_T = L^2(I, \mathcal{M}(\Omega))$. *Then there holds*

$$\langle \rho\partial_t y(T), \varphi\rangle_\Omega = -\Big(\kappa\partial_x \int_I y \,\mathrm{d}t, \partial_x\varphi\Big)_H + \int_I \langle u, \varphi\rangle_\Omega \,\mathrm{d}t + \langle \rho y^1, \varphi\rangle_\Omega \quad \forall\varphi \in V. \tag{5.10}$$

Proof. For $\mathcal{M}_T = \mathcal{M}(\Omega, L^2(I))$ identity (5.9) is proved by testing (3.1) with time constant function $v = \varphi \in V$. For $\mathcal{M}_T = L^2(I, \mathcal{M}(\Omega))$ we test (3.7) with any $\varphi \in V^2$ and get

$$\langle \rho\partial_t y(T), \varphi\rangle_\Omega = ((\mathcal{I}_t y)(T), \partial_x(\kappa\partial_x\varphi))_H + \langle u, \varphi\rangle_{\mathcal{M}_T, \mathcal{C}_T}$$

According to Proposition 3.7 we have $\mathcal{I}_t y \in \mathcal{C}(\bar{I}, V)$. Thus there holds

$$\langle \rho\partial_t y(T), \varphi\rangle_\Omega = -(\kappa\partial_x(\mathcal{I}_t y)(T), \partial_x\varphi)_H + \langle u, \varphi\rangle_{\mathcal{M}_T, \mathcal{C}_T} + \langle \rho y^1, \varphi\rangle_\Omega.$$

The density of V^2 in V implies (5.10). □

For our analysis, we need some projection and interpolation operators. We introduce the standard projectors $\pi_h^0\colon H_\rho \to V_h$ and $\pi_h^1\colon \mathcal{V}_\kappa \to V_h$ defined by

$$(\rho\pi_h^0 w, \varphi)_H = (\rho w, \varphi)_H \quad \forall\varphi \in V_h, \tag{5.11}$$

$$(\kappa\partial_x\pi_h^1 w, \partial_x\varphi)_H = (\kappa\partial_x w, \partial_x\varphi)_H \quad \forall\varphi \in V_h. \tag{5.12}$$

Clearly $\|\pi_h^0 w\|_{H_\rho} \le \|w\|_{H_\rho}$ and $\|\pi_h^1 w\|_{\mathcal{V}_\kappa} \le \|w\|_{\mathcal{V}_\kappa}$. Identity (5.2) means that $y_\vartheta(0) = \pi_h^0 y^0$.

Moreover the following property holds

$$(w, \tilde{w})_{\mathcal{V}_\kappa} - (\pi_h^1 w, \pi_h^1\tilde{w})_{\mathcal{V}_\kappa} = (w - \pi_h^1 w, \tilde{w} - \pi_h^1\tilde{w})_{\mathcal{V}_\kappa} \quad \forall w, \tilde{w} \in V. \tag{5.13}$$

Following [46], we also introduce *the regularized H_ρ projector* $\pi_{h,\sigma_0}^0\colon V \to V_h$ defined by

$$(\rho\pi_{h,\sigma_0} w, \varphi)_H + \sigma_0\tau^2(\kappa\partial_x\pi_{h,\sigma_0} w, \partial_x\varphi)_H = (\rho w, \varphi)_H \quad \forall\varphi \in V_h. \tag{5.14}$$

with the grid independent parameter $\sigma_0 \ge \sigma - 1/4$. Clearly $\pi_{h,\sigma_0} = \pi_h^0$ for $\sigma_0 = 0$.

Let $i_\tau\colon \mathcal{C}(\bar{I}) \to V_\tau$ be the interpolation operator such that $i_\tau w(t_m) = w(t_m)$ for all $m = 0, \ldots, M$.

Next we define the operator $A_h^{-1}\colon V^* \to V_h$, $f \mapsto w_h$ where $w_h \in V_h$ is the unique solution of

$$(\kappa\partial_x w_h, \partial_x\varphi)_H = \langle f, \varphi\rangle_\Omega \quad \forall\varphi \in V_h. \tag{5.15}$$

Clearly $A_h^{-1} = \pi_h^1 A^{-1}$, see (4.3) with $w = A^{-1}f$, and the norm in $\mathcal{V}_\kappa^*$ and its discrete counterpart can be written as

$$\|f\|_{\mathcal{V}_\kappa^*} = \|A^{-1}f\|_{\mathcal{V}_\kappa} = \|w\|_{\mathcal{V}_\kappa}, \quad \|f\|_{H_h^{-1}} := \|A_h^{-1}f\|_{\mathcal{V}_\kappa} = \|w_h\|_{\mathcal{V}_\kappa} \le \|w\|_{\mathcal{V}_\kappa}.$$

Moreover, we set $r_h A^{-1} := A^{-1} - A_h^{-1} = A^{-1} - \pi_h^1 A^{-1}$. First we note that

$$A^{-1}\colon H^{(\lambda)} \to H^{(\lambda+2)}, \quad -1 \le \lambda \le 1. \tag{5.16}$$

Then by the standard FEM error analysis [7] and operator interpolation theory we have

$$\|r_h A^{-1} f\|_V = \|w - \pi_h^1 w\|_V \le ch^{1+\lambda}\|f\|_{H^{(\lambda)}} \quad \forall f \in H^{(\lambda)}, \quad -1 \le \lambda \le 0, \tag{5.17}$$

$$\|r_h A^{-1} f\|_H = \|w - \pi_h^1 w\|_H \le ch^{2+\lambda}\|f\|_{H^{(\lambda)}} \quad \forall f \in H^{(\lambda)}, \quad -1 \le \lambda \le 0. \tag{5.18}$$

6. Stability and error estimates for the discrete state equation

In this section we present error estimates for the state equation. We begin with an auxiliary result.

Lemma 6.1. *For $\sigma_0 \ge \sigma - 1/4 \ge 0$, the following estimate holds*

$$\|\pi_{h,\sigma_0} w - \pi_h^0 w\|_{H_\tau^0} \le c(\tau + h)^\lambda\|w\|_{H^{(\lambda)}} \quad \forall w \in H^{(\lambda)}, \quad \textit{for} \;\; 1 \le \lambda \le 2. \tag{6.1}$$

Proof. We recall the well known estimates

$$\|\pi_h^0 w\|_V \le c\|w\|_V \quad \forall w \in V, \tag{6.2}$$

$$\|w - \pi_h^0 w\|_V \le ch\|w\|_{V^2} \quad \forall w \in V^2, \tag{6.3}$$

which are valid using the inverse inequality (5.4). We also remind inequality (5.7) and notice also that for $\sigma_0 \ge 0$ the following additional inequality holds

$$\sqrt{\sigma_0}\tau\|\varphi\|_{\mathcal{V}_\kappa} \le \|\varphi\|_{H_\tau^0} \quad \forall\varphi \in V_h. \tag{6.4}$$

Let $w \in V$ and $\varphi \in V_h$. We apply identities (5.11) and (5.14) and get

$$\begin{aligned}\big(\rho(\pi_{h,\sigma_0}^0 w - \pi_h^0 w), \varphi\big)_H + \sigma_0\tau^2\big(\kappa\partial_x(\pi_{h,\sigma_0}^0 w - \pi_h^0 w), \partial_x\varphi\big)_H &= -\sigma_0\tau^2\big(\kappa\partial_x\pi_h^0 w, \partial_x\varphi\big)_H \\ &= \sigma_0\tau^2\big(\kappa\partial_x(w - \pi_h^0 w), \partial_x\varphi\big)_H + \sigma_0\tau^2\langle\partial_x(\kappa\partial_x w), \varphi\rangle_\Omega.\end{aligned}$$

Now we set $\varphi = \pi_{h,\sigma_0}^0 w - \pi_h^0 w$ and from the former and latter equalities together with estimates (6.2) and (6.3) we obtain the estimate

$$\|\pi_{h,\sigma_0}^0 w - \pi_h^0 w\|_{H_\tau^0} \le c\tau(\tau + h)^{\lambda-1}\|w\|_{H^{(\lambda)}}$$

for $\lambda = 1, 2$ respectively.

By using the $K_{\lambda,\infty}$-method, we complete the proof. □

Now we get a stability bound and error estimates in $\mathcal{C}(\bar{I}, H) \times H_h^{-1}$ for the discrete state equation.

Proposition 6.2. *Let y and (y_ϑ, y^1_{Th}) be the solutions to the state equation* (2.1) *and the discrete state equation* (5.1)-(5.2).

(1) For $(u, y^0, y^1) \in L^2(I, V^*) \times V \times V^*$, the following stability bound holds:

$$\|y_\vartheta\|_{\mathcal{C}(\bar{I},H)} + \|\rho y^1_{Th}\|_{H_h^{-1}} \le c\,(\|u\|_{L^2(I,V^*)} + \|\mathbf{y}\|_{V\times V^*}). \tag{6.5}$$

(2) For $(u, y^0, y^1) \in L^2(I, H^{(-1/2)}) \times V \times H^{(-1/2)}$, the following error estimate holds:

$$\|y - y_\vartheta\|_{\mathcal{C}(\bar{I},H)} + \|\rho(\partial_t y(T) - y^1_{Th})\|_{H_h^{-1}} \le c\,(\tau + h)^{1/3}(\|u\|_{L^2(I,H^{(-1/2)})} + \|\mathbf{y}\|_{V\times H^{(-1/2)}}). \tag{6.6}$$

(3) For $(u, y^0, y^1) \in H^1(I, H^{(-1/2)}) \times V \times H$, the higher order error estimate holds:

$$\|y - y_\vartheta\|_{\mathcal{C}(\bar{I},H)} + \|\rho(\partial_t y(T) - y^1_{Th})\|_{H_h^{-1}} \le c\,(\tau + h)^{2/3}(\|u\|_{H^1(I,H^{(-1/2)})} + \|\mathbf{y}\|_{V\times H}). \tag{6.7}$$

Proof. 1. According to [46, Theorem 2.1 (1)], the bound

$$\|y_\vartheta\|_{\mathcal{C}(\bar{I},\,H)} + \Big\|\int_I y_\vartheta \ \mathrm{d}t\Big\|_V \le c\,(\|u\|_{L^2(I,V^*)} + \|y_\vartheta(0)\|_{H^0_\tau} + \|y^1\|_{V^*}) \tag{6.8}$$

is valid for any $y_\vartheta(0) \in V_h$. We have $y_\vartheta(0) = \pi^0_h y^0$. In the case $\sigma \le 1/4$, there clearly holds

$$\|\pi^0_h y^0\|_{H^0_\tau} \le \|\pi^0_h y^0\|_{H_\rho} \le \|y^0\|_{H_\rho}.$$

For $\sigma > 1/4$, we alternatively get using (6.1) for $\lambda = 1$

$$\|\pi^0_h y^0\|_{H^0_\tau} \le \|\pi^0_h y^0 - \pi^h_{h,\sigma_0} y^0\|_{H^0_\tau} + \|\pi^0_{h,\sigma_0} y^0\|_{H^0_\tau} \le c(\tau + h)\|y^0\|_V + \|y^0\|_{H_\rho}$$

for any $\sigma_0 \ge \sigma - 1/4$.

We proceed with the bound for y^T_h. Identity (5.8) and bound (6.8) together with the generalized Minkowski inequality imply

$$\|\rho y^T_h\|_{H_h^{-1}} \le c\left(\Big\|\int_I y_\vartheta \ \mathrm{d}t\Big\|_V + \Big\|\int_I u \ \mathrm{d}t\Big\|_{V^*} + \|\rho y^1\|_{V^*}\right) \le c_1\,(\|u\|_{L^2(I,V^*)} + \|\mathbf{y}\|_{V\times H}). \tag{6.9}$$

Finally we derive bound (6.5).

2. Let $\tilde{y}_\vartheta$ be the solution of equation (5.1) for $\tilde{y}_\vartheta(0) = \pi^0_{h,\sigma_0} y^0$. From [46, Theorem 4.1] we get the error estimate

$$\|y - \tilde{y}_\vartheta\|_{\mathcal{C}(\bar{I},H)} + \Big\|\int_I (\pi^1_h y - \tilde{y}_\vartheta) \ \mathrm{d}t\Big\|_V \le c(\tau + h)^{1/3}(\|u\|_{L^2(I,\,H^{(-1/2)})} + \|\mathbf{y}\|_{H^{(1/2)}\times H^{(-1/2)}}).$$

In the case $\sigma \le 1/4$ we can choose $\sigma_0 = 0$, then $y_\vartheta(0) = \pi_{h,\sigma_0} y^0 = \pi^0_h y^0$ and $\tilde{y}_\vartheta = y_\vartheta$. In the case $\sigma \ge 1/4$ we can use the stability bound (6.8) and estimate (6.1) to get

$$\|\tilde{y}_\vartheta - y_\vartheta\|_{\mathcal{C}(\bar{I},H)} + \Big\|\int_I (\tilde{y}_\vartheta - y_\vartheta) \ \mathrm{d}t\Big\|_V \le c\,\|\pi^0_{h,\sigma_0} y^0 - \pi^0_h y^0\|_{H^0_\tau} \le c_1(\tau + h)\|y^0\|_V. \tag{6.10}$$

Then by subtracting (5.8) from (5.10) and applying identity (5.12) we find

$$\langle \rho(\partial_t y(T) - y^1_{Th}), \varphi\rangle_\Omega = -\left(\kappa\partial_x \int_I (y - y_\vartheta) \ \mathrm{d}t, \partial_x\varphi\right)_H = -\left(\kappa\partial_x \int_I (\pi^1_h y - y_\vartheta) \ \mathrm{d}t, \partial_x\varphi\right)_H \quad \forall\varphi \in V_h,$$

consequently

$$\|\rho(\partial_t y(T) - y^1_{Th})\|_{H_h^{-1}} \le c\Big\|\int_I (\pi^1_h y - y_\vartheta) \ \mathrm{d}t\Big\|_V. \tag{6.11}$$

Thus we obtain (6.6).

3. Once again we apply [46, Theorem 4.1] and first get the estimate

$$\|y - \tilde{y}_\vartheta\|_{\mathcal{C}(\bar{I},H)} + \Big\|\int_I (\pi^1_h y - \tilde{y}_\vartheta) \ \mathrm{d}t\Big\|_V \le c\,(\tau + h)^{2/3}(\|u\|_{L^2(I,H)} + \|\mathbf{y}\|_{V\times H}).$$

Combining it together with (6.10), we derive

$$\|y-y_\vartheta\|_{\mathcal{C}(\bar I,H)}+\Big\|\int_I(\pi_h^1y-y_\vartheta)\,\mathrm{d}t\Big\|_V\le c\,(\tau+h)^{2/3}\big(\|u\|_{L^2(I,H)}+\|\mathbf{y}\|_{V\times H}\big). \tag{6.12}$$

In this proof, we apply this estimate in the case $u=0$ only (but in general case below).

In the remaining case $\mathbf{y}=0$, from [46, Theorem 4.1] we also get the higher order error estimate

$$\|y-y_\vartheta\|_{\mathcal{C}(\bar I,H)}+\Big\|\int_I(\pi_h^1y-y_\vartheta)\,\mathrm{d}t\Big\|_V\le c\,(\tau+h)^{4/3}\|u\|_{H^1(I,H)}\quad\text{for}\quad\forall u\in H^1(I,H). \tag{6.13}$$

Moreover owing to Proposition 3.7 and bound (6.5) (both for $\mathbf{y}=0$) we have

$$\begin{aligned}\|y-y_\vartheta\|_{\mathcal{C}(\bar I,H)}+\Big\|\int_I(\pi_h^1y-y_\vartheta)\,\mathrm{d}t\Big\|_V&\le\|y\|_{\mathcal{C}(\bar I,H)}+c\|\mathcal{I}_ty\|_{\mathcal{C}(\bar I,V)}+\|y_\vartheta\|_{\mathcal{C}(\bar I,H)}+\Big\|\int_I y_\vartheta\,\mathrm{d}t\Big\|_V\\&\le c_1\|u\|_{L^2(I,V^*)}\quad\text{for}\quad\forall u\in L^2(I,V^*).\end{aligned}$$

The last bound and estimate (6.13) imply by the $K_{1/2,\infty}$-method:

$$\|y-y_\vartheta\|_{\mathcal{C}(\bar I,H)}+\Big\|\int_I(\pi_h^1y-y_\vartheta)\,\mathrm{d}t\Big\|_V\le c\,(\tau+h)^{2/3}\|u\|_{B_{1/2}}\quad\forall u\in B_{1/2}:=(L^2(I,V^*),H^1(I,H))_{1/2,\infty}.$$

Due to the simple embedding

$$H^1(I,H^{(-1/2)})=(H^1(I;V^*),H^1(I;H))_{1/2,\infty}\hookrightarrow(L^2(I,V^*),H^1(I,H))_{1/2,\infty}$$

and inequality (6.11) we complete the proof. □

Remark 6.3. A priori stability bound (6.5) implies the unique solvability of the discrete state equation (5.1)-(5.2).

Remark 6.4. According to the given proof, for $\tilde y_\vartheta$ in place of y_ϑ the norms of $\mathbf{y}$ in (6.5) and (6.6) can be weakened down to respectively $\|\mathbf{y}\|_{H\times V^*}$ and $\|\mathbf{y}\|_{H^{(1/2)}\times H^{(-1/2)}}$. For $\sigma\le 1/4$, we have $\tilde y_\vartheta=y_\vartheta$. The same can be shown for y_ϑ also for $\sigma>1/4$ provided that $\tau\alpha_h\le c_0$ with any $c_0>0$.

7. Discrete control problem

First we introduce the discrete mapping $\hat S_\vartheta\colon(u,y_0,y_1)\mapsto(y_\vartheta,y_\vartheta(T),\rho y_{Th}^1)$ and the discrete affine linear control-to-state mapping

$$S_\vartheta\colon\mathcal{M}_T\to\mathcal{Y}_\vartheta=V_\vartheta\times V_h\times(\rho\times V_h),\ u\mapsto(y_\vartheta,y_\vartheta(T),\rho y_{Th}^1)$$

defined by $S_\vartheta u=\hat S_\vartheta(u,0,0)+\hat S_\vartheta(0,y_0,y_1)$. The mapping S_ϑ is a composition of

$$u\mapsto\vec u=\{\langle u,e_{m,n}^\vartheta\rangle_{\mathcal{M}_T,\mathcal{C}_T}\}_{m,n=1}^{M,N-1},\quad\mathcal{M}_T\to\mathbb{R}^{M(N-1)},$$

where $\{e_{m,n}^\vartheta\}$ is a basis in V_ϑ, and $\vec u\mapsto(y_\vartheta,y_\vartheta(T),\rho y_{Th}^1)$. The former mapping is bounded due to $e_{m,n}^\vartheta\in\mathcal{C}_T$ and the latter one is finite dimensional. Thus S_ϑ is a bounded operator. Then we consider the following semi-discrete optimal control problem

$$j_\vartheta(u)=\tfrac12\,\|S_\vartheta u-\mathbf{z}\|_{\mathcal{Y}_h}^2+\alpha\|u\|_{\mathcal{M}_T}\to\min_{u\in\mathcal{M}_T} \tag{$\mathcal{P}_\vartheta$}$$

with the squared semi-norm corresponding to the inner product

$$(\mathbf{z},\tilde{\mathbf{z}})_{\mathcal{Y}_h}=(\rho z_1,\tilde z_1)_{L^2(I\times\Omega)}+(\rho z_2,\tilde z_2)_H+\big(A_h^{-1}z_3,A_h^{-1}\tilde z_3\big)_{\mathcal{V}_\kappa}\quad\forall\mathbf{z},\tilde{\mathbf{z}}\in\mathcal{Y}.$$

Using the similar argument as in the continuous case it can be shown that ($\mathcal{P}_\vartheta$) has a solution $\bar u_\vartheta$ which is not unique in general, and due to the optimality, the stability bound (6.5) and property (5.16) (for $\lambda=-1$) one gets

$$\alpha\|\bar u_\vartheta\|_{\mathcal{M}_T}\le j_\vartheta(\bar u_\vartheta)\le j_\vartheta(0)=\tfrac12\|S_\vartheta(0)-z\|_{\mathcal{Y}_h}^2\le c\,\big(\|\mathbf{y}\|_{V\times V^*}+\|z\|_{\mathcal{Y}}\big)^2,$$

cf. (4.1), and consequently

$$\|\bar u_\vartheta\|_{\mathcal{M}_T}\le c\big(\|\mathbf{y}\|_{V\times V^*}+\|z\|_{\mathcal{Y}}\big)^2\le C. \tag{7.1}$$

Theorem 7.1. *Let* $\mathbf{z} \in \mathcal{Y}$, $\mathbf{y} \in V \times H^{(-1/2)}$ *and* $\bar{u}, \bar{u}_\vartheta \in \mathcal{M}_T$ *be the optimal controls of respectively problems* $(\mathcal{P})$ *and* $(\mathcal{P}_\vartheta)$. *Then there holds*

$$\bar{u}_\vartheta \rightharpoonup^* \bar{u} \text{ in } \mathcal{M}_T, \quad \|\bar{u}_\vartheta\|_{\mathcal{M}_T} \to \|\bar{u}\|_{\mathcal{M}_T} \quad \text{as} \quad \vartheta \to 0.$$

Proof. Owing to (7.1) there exists a sequence $\{\vartheta_n\}$, $\vartheta_n \to 0$, and $u \in \mathcal{M}_T$ such that $\bar{u}_{\vartheta_n} \rightharpoonup^* u$ in $\mathcal{M}_T$ as $n \to \infty$. Next we prove that this implies that

$$\|S_{\vartheta_n}\bar{u}_{\vartheta_n} - \mathbf{z}\|_{\mathcal{Y}_{\vartheta_n}} \to \|Su - \mathbf{z}\|_{\mathcal{Y}}. \tag{7.2}$$

To this end, we write the chain of inequalities

$$\begin{aligned} \big|\|S_{\vartheta_n}\bar{u}_{\vartheta_n} - \mathbf{z}\|_{\mathcal{Y}_{\vartheta_n}} - \|Su - \mathbf{z}\|_{\mathcal{Y}}\big| &\le \big|\|S_{\vartheta_n}\bar{u}_{\vartheta_n} - \mathbf{z}\|_{\mathcal{Y}_{\vartheta_n}} - \|Su - \mathbf{z}\|_{\mathcal{Y}_{\vartheta_n}}\big| + \big|\|Su - \mathbf{z}\|_{\mathcal{Y}_{\vartheta_n}} - \|Su - \mathbf{z}\|_{\mathcal{Y}}\big| \\ &\le \|S_{\vartheta_n}\bar{u}_{\vartheta_n} - Su\|_{\mathcal{Y}_{\vartheta_n}} + \big|\|Su - \mathbf{z}\|_{\mathcal{Y}_{\vartheta_n}} - \|Su - \mathbf{z}\|_{\mathcal{Y}}\big| \\ &\le \|S_{\vartheta_n}\bar{u}_{\vartheta_n} - S\bar{u}_{\vartheta_n}\|_{\mathcal{Y}_{\vartheta_n}} + \|S\bar{u}_{\vartheta_n} - Su\|_{\mathcal{Y}_{\vartheta_n}} + \big|\|Su - \mathbf{z}\|_{\mathcal{Y}_{\vartheta_n}} - \|Su - \mathbf{z}\|_{\mathcal{Y}}\big|. \end{aligned}$$

The first term on the right in the last inequality converges to zero according to the error estimate (6.6). The convergence of the second term follows from the weak-star-to-strong continuity of $S\colon \mathcal{M}_T \to \mathcal{Y}$ and the stability of π_h^1 in V. Finally, property (5.13) for $\tilde{w} = w$ implies the convergence of the last term. Then (7.2) and the weak-star lower semicontinuity of $\|\cdot\|_{\mathcal{M}_T}$ in $\mathcal{M}_T$ implies

$$j(u) \le \liminf_{n\to\infty} j_{\vartheta_n}(\bar{u}_{\vartheta_n}) \le \limsup_{n\to\infty} j_{\vartheta_n}(\bar{u}_{\vartheta_n}) \le \limsup_{n\to\infty} j_{\vartheta_n}(\bar{u}) = j(\bar{u}).$$

Thus, the uniqueness of $\bar{u}$ means that $u = \bar{u}$ and in addition implies the convergence of the whole sequence $\bar{u}_\vartheta \rightharpoonup^* \bar{u}$ in $\mathcal{M}_T$ as $\vartheta \to 0$. Moreover, we have $j_\vartheta(\bar{u}_\vartheta) \to j(\bar{u})$. This and (7.2) lead to $\|\bar{u}_\vartheta\|_{\mathcal{M}_T} \to \|\bar{u}\|_{\mathcal{M}_T}$. □

For convenience we set $F_h(\mathbf{z}) = (1/2)\|\mathbf{z}\|^2_{\mathcal{Y}_h}$. In the following the directional derivative of a functional $g\colon \mathcal{M}_T \to \mathbb{R}$ at $u \in \mathcal{M}_T$ in direction $\delta u \in \mathcal{M}_T$ is denoted by $Dg(u)\delta u$. In the case $Dg(u) \in \mathcal{M}_T^*$, g is the Gateaux differentiable in u. Moreover, we make use of the convex subdifferential of $\|\cdot\|_{\mathcal{M}_T}$. Let $\hat{u} \in \mathcal{M}_T$ and $p \in \mathcal{C}_T$. Then there holds $p \in \partial\|\hat{u}\|_{\mathcal{M}_T}$ if and only if

$$\langle p, u - \hat{u}\rangle_{\mathcal{C}_T,\mathcal{M}_T} + \alpha\|\hat{u}\|_{\mathcal{M}_T} \le \alpha\|u\|_{\mathcal{M}_T} \quad \forall u \in \mathcal{M}_T.$$

An element $\bar{u}_\vartheta \in \mathcal{M}_T$ is an optimal solution of $(\mathcal{P}_\vartheta)$ if and only if $-D((F_h \circ S_\vartheta)(\bar{u}_\vartheta)) \in \alpha\partial\|\bar{u}_\vartheta\|_{\mathcal{M}_T}$. To calculate $D((F_h \circ S_\vartheta)(u))$ for $u \in \mathcal{M}_T$, we apply the Lagrange technique and define the Lagrange functional by

$$\begin{aligned} L(u, y_\vartheta, y^1_{Th}, p_\vartheta, p^1_{0h}) = F_h(y_\vartheta, y_\vartheta(T), y^1_{Th}) - B_\sigma(y_\vartheta, p_\vartheta) - (\rho y^1_{Th}, p_\vartheta(T))_H \\ + \langle u, p_\vartheta\rangle_{\mathcal{M}_T,\mathcal{C}_T} + (\rho y^1, p_\vartheta(0))_H + (\rho(y_\vartheta(0) - y^0), p^1_{0h})_H \end{aligned}$$

with $(p_\vartheta, p^1_{0h}) \in V_\vartheta \times V_h$ (where we base on identities (5.1)-(5.2)). We obviously have

$$(F_h \circ S_\vartheta)(u) = L(u, S_\vartheta u, p_\vartheta, p^1_{0h}) \quad \forall (p_\vartheta, p^1_{0h}) \in V_\vartheta \times V_h.$$

Thus there holds

$$D((F_h \circ S_\vartheta)(u))\delta u = D_u L(u, y_\vartheta, y^1_{Th}, p_\vartheta, p^1_{0h})\delta u = \langle p_\vartheta, \delta u\rangle_{\mathcal{C}_T,\mathcal{M}_T} \quad \forall \delta u \in \mathcal{M}_T$$

provided that $(p_\vartheta, p^1_{0h}) \in V_\vartheta \times V_h$ is the solution of the discrete problem

$$\begin{aligned} -D_{y_\vartheta} L(u, y_\vartheta, y^1_{Th}, p_\vartheta, p^1_{0h})v = B_\sigma(v, p_\vartheta) - (\rho(y_\vartheta - z_1), v)_{L^2(I\times\Omega)} \\ - (\rho(y_\vartheta(T) - z_2), v(T))_H - (\rho v(0), p^1_{0h})_H = 0 \quad \forall v \in V_\vartheta \end{aligned}$$

and

$$-D_{y^1_{Th}} L(u, y_\vartheta, y^1_{Th}, p_\vartheta, p^1_{0h})\varphi = (\rho\varphi, p_\vartheta(T))_H - \left(\rho A_h^{-1}(\rho y^1_{Th} - z_3), \varphi\right)_H = 0 \quad \forall \varphi \in V_h.$$

Therefore *the discrete optimality system* consists of *the discrete state equation*

$$\begin{aligned} B_\sigma(\bar{y}_\vartheta, v) + (\rho\bar{y}^1_{Th}, v(T))_H &= \langle \bar{u}_\vartheta, v\rangle_{\mathcal{M}_T,\mathcal{C}_T} + (\rho y^1, v(0))_H \quad &\forall v \in V_\vartheta, \\ (\rho\bar{y}_\vartheta(0), \varphi)_H &= (\rho y^0, \varphi)_H &\forall \varphi \in V_h, \end{aligned} \tag{7.3}$$

the discrete adjoint state equation

$$\begin{aligned} B_\sigma(v,\bar p_\vartheta)-(\rho v(0),\bar p_{0h}^1)_H &= (\rho(\bar y_\vartheta - z_1),v)_{L^2(I\times\Omega)}+(\rho(\bar y_\vartheta(T)-z_2),v(T))_H \quad \forall v\in V_\vartheta,\\ (\rho\varphi,\bar p_\vartheta(T))_H &= \big(\rho A_h^{-1}(\rho\bar y_{Th}^1 - z_3),\varphi\big)_H \quad \forall\varphi\in V_h \end{aligned} \tag{7.4}$$

and *the discrete variational inequality*

$$\langle -\bar p_\vartheta, u-\bar u_\vartheta\rangle_{\mathcal{C}_T,\mathcal{M}_T}+\|\bar u_\vartheta\|_{\mathcal{M}_T}\le\|u\|_{\mathcal{M}_T}\quad \forall u\in\mathcal{M}_T. \tag{7.5}$$

8. Stability and error estimates for the discrete adjoint state equation

We define *the general discrete adjoint state equation*

$$B_\sigma(v,p_\vartheta)-(\rho v(0),p_{0h}^1)_H=(\rho(y-z_1),v)_{L^2(I\times\Omega)}+(\rho(y(T)-z_2),v(T))_H \quad \forall v\in V_\vartheta, \tag{8.1}$$

$$(\rho\varphi,p_\vartheta(T))_H=\big(\rho A_h^{-1}(\rho\partial_t y(T)-z_3),\varphi\big)_H \quad \forall\varphi\in V_h. \tag{8.2}$$

Here y is the solution to the state equation (2.1). Clearly identity (8.2) means simply that $p_\vartheta(T)=A_h^{-1}q_T=\pi_1^h A^{-1}q_T$ with $q_T:=\rho\partial_t y(T)-z_3$.

Now we get a stability bound and error estimates in $\mathcal{C}(\bar I,H)\times H_h^{-1}$ and $\mathcal{C}_T$ for the discrete adjoint state equation.

Proposition 8.1. *Let* $p=S^*\big(y-z_1,-(y(T)-z_2),A^{-1}(\rho\partial_t y(T)-z_3)\big)$ *and* (p_ϑ,p_{0h}^1) *be the solution of the corresponding general discrete adjoint state equation* (8.1)-(8.2)*.*

(1) If $y\in\mathcal{C}(\bar I,H)\cap\mathcal{C}^1(\bar I,V^*)$ *and* $\mathbf{z}\in\mathcal{Y}$*, then the following stability bound holds*

$$\|p_\vartheta\|_{\mathcal{C}(\bar I,V)}+\|\rho p_{0h}^1\|_{H_h^{-1}}\le c\,(\|y-z_1\|_{L^2(I\times\Omega)}+\|y(T)-z_2\|_H+\|\rho\partial_t y(T)-z_3\|_{V^*}). \tag{8.3}$$

(2) If $u\in L^2(I,V^*)$*,* $\mathbf{z}\in\mathcal{Y}$ *and* $\mathbf{y}\in H\times V^*$*, then the following error estimate holds*

$$\|p-p_\vartheta\|_{\mathcal{C}(\bar I,H)}+\|\rho(\partial_t p(0)-p_h^0)\|_{H_h^{-1}}\le c(\tau+h)^{2/3}(\|u\|_{L^2(I,V^*)}+\|\mathbf{z}\|_{\mathcal{Y}}+\|\mathbf{y}\|_{H\times V^*}). \tag{8.4}$$

(3) If $u\in L^2(I,H^{(-1/2)})$*,* $\mathbf{z}\in\mathcal{Y}^{1/2}:=L^2(I,H^{(1/2)})\times H^{(1/2)}\times H^{(-1/2)}$ *and* $\mathbf{y}\in H^{(1/2)}\times H^{(-1/2)}$*, then the following error estimate holds*

$$\|p-p_\vartheta\|_{\mathcal{C}(\bar I\times\bar\Omega)}\le c(\tau+h)^{2/3}(\|u\|_{L^2(I,H^{(-1/2)})}+\|\mathbf{z}\|_{\mathcal{Y}^{1/2}}+\|\mathbf{y}\|_{H^{(1/2)}\times H^{(-1/2)}}). \tag{8.5}$$

(4) If $u\in H^1(I,H^{(-1/2)})$*,* $\mathbf{z}\in\mathcal{Y}^{3/2}:=L^2(I,H^{(3/2)})\times H^{(3/2)}\times H^{(1/2)}$ *and* $\mathbf{y}\in H^{(3/2)}\times H^{(1/2)}$*, then the following higher order error estimate holds*

$$\|p-p_\vartheta\|_{L^2(I,\mathcal{C}_0(\Omega))}\le c(\tau+h)^{4/3}(\|u\|_{H^1(I,H^{(-1/2)})}+\|\mathbf{z}\|_{\mathcal{Y}^{3/2}}+\|\mathbf{y}\|_{H^{(3/2)}\times H^{(1/2)}}). \tag{8.6}$$

Proof. 1. According to [46, Theorem 2.1 (2)] the following energy bound hold

$$\|p_\vartheta\|_{\mathcal{C}(\bar I,V)}+\|\partial_t p_\vartheta\|_{L^\infty(\bar I,H)}\le c\,(\|y-z_1\|_{L^2(I\times\Omega)}+\|p_\vartheta(T)\|_V+\|y(T)-z_2\|_H)$$

for any $p_\vartheta(T)\in V_h$. Using (6.2), $A_h^{-1}=\pi_h^1 A^{-1}$ and (5.16) we get

$$\|p_\vartheta(T)\|_V\le c\|A^{-1}q_T\|_V\le c_1\|q_T\|_{V^*}. \tag{8.7}$$

By applying also the counterpart of inequalities (6.9) we derive bound (8.3).

2. The counterpart of the error estimate (6.12) for the adjoint state equation case and bound (8.7) give

$$\begin{aligned}\|p-p_\vartheta\|_{\mathcal{C}(\bar I,H)}+\Big\|\int_I(\pi_h^1 p-p_\vartheta)\,\mathrm{d}t\Big\|_V &\le c\,(\tau+h)^{2/3}(\|y-z_1\|_{L^2(I,H)}+\|A_h^{-1}q_T\|_V+\|y(T)-z_2\|_H)\\ &\le c_1\,(\tau+h)^{2/3}(\|y\|_{\mathcal{C}(\bar I,H)}+\|\partial_t y\|_{\mathcal{C}^1(\bar I,V^*)}+\|\mathbf{z}\|_{\mathcal{Y}}).\end{aligned}$$

Owing to inequality (6.11) and Proposition 3.7 we obtain estimate (8.4).

3. Below we need the multiplicative inequalities

$$\|w\|_{\mathcal{C}(\bar{I}\times\bar{\Omega})} \le c\|w\|^{1/2}_{\mathcal{C}(\bar{I},H)}\|w\|^{1/2}_{\mathcal{C}(\bar{I},V)} \quad \forall w \in C(\bar{I},V), \tag{8.8}$$

$$\|w\|_{L^2(I,\mathcal{C}_0(\Omega))} \le c\|w\|^{1/2}_{L^2(I,H)}\|w\|^{1/2}_{L^2(I,V)} \quad \forall w \in L^2(I,V). \tag{8.9}$$

Let $\check{p}_\vartheta$ be the auxiliary solution to (8.1) for $\check{p}_\vartheta(T) = \pi_h^0 A^{-1}q_T$. Owing to inequality (8.8) and the stability bounds [46, Theorem 2.1] we get

$$\|p_\vartheta - \check{p}_\vartheta\|_{\mathcal{C}(\bar{I}\times\bar{\Omega})} \le c\|(p_\vartheta - \check{p}_\vartheta)(T)\|^{1/2}_{H^0_\tau}\|(p_\vartheta - \check{p}_\vartheta)(T)\|^{1/2}_V.$$

Consequently, for $q_T \in H^{(\alpha-2)}$, by (6.2), (5.17) and (5.18) the following chain of inequalities hold

$$\|p_\vartheta - \check{p}_\vartheta\|_{\mathcal{C}(\bar{I}\times\bar{\Omega})} \le c\big(\|(p_\vartheta - \check{p}_\vartheta)(T)\|^{1/2}_H\|(p_\vartheta - \check{p}_\vartheta)(T)\|^{1/2}_V + \tau^{1/2}\|(p_\vartheta - \check{p}_\vartheta)(T)\|_V\big)$$
$$\le c_1\big(\|r_hA^{-1}q_T\|^{1/2}_H\|r_hA^{-1}q_T\|^{1/2}_V + \tau^{1/2}\|r_hA^{-1}q_T\|_V\big) \le c_2(\tau+h)^{\alpha-1/2}\|q_T\|_{H^{(\alpha-2)}}$$

for $1 \le \alpha \le 2$. Thus it is enough to prove error estimates (8.5) and (8.6) for $\check{p}_\vartheta$ instead of p_ϑ.

According to [46, Theorem 5.3 and estimate (5.18)] we have the error estimate

$$\|i_\tau p - \check{p}_\vartheta\|_{\mathcal{C}(\bar{I}\times\bar{\Omega})} = \|p - \check{p}_\vartheta\|_{\mathcal{C}_\tau(\bar{I},\mathcal{C}(\bar{\Omega}))} := \max_{0\le m\le M}\|(p - \check{p}_\vartheta)(t_m)\|_{\mathcal{C}(\bar{\Omega})}$$
$$\le c(\tau+h)^{2(\alpha-1/2)/3}\big(\|y - z_1\|_{L^2(I,H^{(\alpha-1)})} + \|y(T) - z_2\|_{H^{(\alpha-1)}} + \|q_T\|_{H^{(\alpha-2)}}\big), \quad \alpha = 1,2. \tag{8.10}$$

We emphasize that due to [46, Theorem 4.3 (2) (e)] and (6.1) this estimate holds for $\check{p}_\vartheta(T) = \pi_h^0 A^{-1}q_T$.

Inequality (8.8), Proposition 3.5 (applied to the adjoint state problem) and property (5.16) imply the following error estimate for the time interpolation

$$\|p - i_\tau p\|_{\mathcal{C}(\bar{I}\times\bar{\Omega})} \le c\big(\tau\|\partial_t p\|_{\mathcal{C}(\bar{I},H)}\big)^{1/2}\|(\tau\partial_t)^{\alpha-1}p\|^{1/2}_{\mathcal{C}(\bar{I},V)}$$
$$\le c_1\tau^{\alpha/2}\big(\|y - z_1\|_{L^2(I,H^{(\alpha-1)})} + \|y(T) - z_2\|_{H^{(\alpha-1)}} + \|q_T\|_{H^{(\alpha-2)}}\big), \tag{8.11}$$

for $\alpha = 1,2$. Owing to estimates (8.10) and (8.11) as well as Propositions 3.7 and 3.5 we get

$$\|p - \check{p}_\vartheta\|_{\mathcal{C}(\bar{I}\times\bar{\Omega})} \le c(\tau+h)^{2(\alpha-1/2)/3}\big(\|y\|_{C(\bar{I},H^{(\alpha-1)})} + \|\partial_t y\|_{C(\bar{I},H^{(\alpha-2)})} + \|\mathbf{z}\|_{\mathcal{Y}^{(\alpha-1)}}\big)$$
$$\le c_1(\tau+h)^{2(\alpha-1/2)/3}\big(\|u\|_{L^2(I,H^{(\alpha-2)})} + \|\mathbf{y}\|_{H^{(\alpha-1)}\times H^{(\alpha-2)}} + \|\mathbf{z}\|_{\mathcal{Y}^{(\alpha-1)}}\big), \quad \alpha = 1,2, \tag{8.12}$$

where $\mathcal{Y}^{(\alpha)} := L^2(I,H^{(\alpha)}) \times H^{(\alpha)} \times H^{(\alpha-1)}$.

By applying the $K_{1/2,\infty}$-method, we get (8.5) for $\check{p}_\vartheta$ in the role of p_ϑ.

4. First notice that the multiplicative inequality (8.9), Proposition 3.5 (2) (applied for the adjoint state problem) and property (5.16) imply another error estimate for the time interpolation

$$\|p - i_\tau p\|_{L^2(I,\mathcal{C}_0(\Omega))} \le c\big(\tau^2\|\partial_{tt}p\|_{L^2(I,H)}\big)^{1/2}\big(\tau\|\partial_t p\|_{\mathcal{C}(\bar{I},V)}\big)^{1/2}$$
$$\le c_1\tau^{3/2}\big(\|y - z_1\|_{L^2(I,V)} + \|y(T) - z_2\|_V + \|q_T\|_H\big).$$

Then Proposition 3.5 (1) leads to

$$\|p - i_\tau p\|_{L^2(I,\mathcal{C}_0(\Omega))} \le c\tau^{3/2}\big(\|u\|_{H^1(I,V^*)} + \|z\|_{\mathcal{Y}} + \|\mathbf{y}\|_{V\times H}\big). \tag{8.13}$$

Next we derive the error estimate

$$\|i_\tau p - \check{p}_\vartheta\|_{\mathcal{C}(\bar{I}\times\bar{\Omega})} \le c(\tau+h)^{4/3}\big(\|u\|_{H^1(I,H^{(-1/2)})} + \|\mathbf{z}\|_{\mathcal{Y}^{3/2}} + \|\mathbf{y}\|_{H^{(3/2)}\times H^{(1/2)}}\big). \tag{8.14}$$

According to [46, Theorem 5.3 and estimate (5.18)] and Propositions 3.7 and 3.5 the following three estimates hold

$$\|i_\tau p - \check{p}_\vartheta\|_{\mathcal{C}(\bar{I}\times\bar{\Omega})} \le c(\tau+h)^{4/3}\|\mathbf{z}\|_{\mathcal{Y}^{3/2}} \quad \text{for} \quad u = 0,\ \mathbf{y} = 0, \tag{8.15}$$

$$\|i_\tau p - \check{p}_\vartheta\|_{\mathcal{C}(\bar{I}\times\bar{\Omega})} \le c(\tau+h)\|y\|_{H^1(I,H)} \le c_1(\tau+h)\big(\|u\|_{H^1(I,V^*)} + \|\mathbf{y}\|_{V\times H}\big) \quad \text{for} \quad \mathbf{z} = 0,$$

$$\|i_\tau p - \check{p}_\vartheta\|_{\mathcal{C}(\bar I\times\bar\Omega)} \le c(\tau+h)^{5/3}(\|\partial_{tt}y\|_{L^2(I,H)} + \|\mathbf{y}\|_{V\times H})$$
$$\le c_1(\tau+h)^{5/3}(\|u\|_{H^1(I,H)} + \|\mathbf{y}\|_{V^2\times V}) \text{ for } \mathbf{z}=0$$

for $\check{p}_\vartheta(T) = \pi_h^0 A^{-1} q_T$ (for the same reason as above). Then applying the $K_{1/2,\infty}$-method to the two last estimates we get

$$\|i_\tau p - \check{p}_\vartheta\|_{\mathcal{C}(\bar I\times\bar\Omega)} \le c(\tau+h)^{4/3}(\|u\|_{H^1(I,H^{(-1/2)})} + \|\mathbf{y}\|_{H^{(3/2)}\times H^{(1/2)}}) \text{ for } \mathbf{z}=0.$$

By combining this estimate and (8.15) we obtain (8.14).

Estimates (8.13) and (8.14) imply

$$\|p - \check{p}_\vartheta\|_{L^2(I,\mathcal{C}_0(\Omega))} \le c(\tau+h)^{4/3}(\|u\|_{H^1(I,H^{(-1/2)})} + \|\mathbf{z}\|_{\mathcal{Y}^{3/2}} + \|\mathbf{y}\|_{H^{(3/2)}\times H^{(1/2)}})$$

that completes the proof of (8.6) for $\check{p}_\vartheta$ in the role of p_ϑ. □

Remark 8.2. A priori stability bound (6.5) (taken for $y=0$) implies the unique solvability of the general discrete adjoint state equation (8.1)-(8.2).

9. Error estimates for the state variable

We introduce the discrete adjoint control-to-state operator $S_\vartheta^\star\colon L^2(I\times\Omega)\times V\times H \to V_\vartheta$, $(\phi, p^1, p^0)\mapsto p_\vartheta$ defined by

$$B_\sigma(v, p_\vartheta) = (\rho\phi, v)_{L^2(I\times\Omega)} - (\rho p^1, v(T))_H \quad \forall v\in V_\vartheta,\ v(0)=0$$

with $p_\vartheta(T) = \pi_h^0 p^0$. Similarly to bound (8.3) and Remark 8.2 it is well defined and satisfies

$$\|S_\vartheta^\star(\phi, p_0, p_1)\|_{\mathcal{C}(\bar I, V)} \le c\,(\|\phi\|_{L^2(I\times\Omega)} + \|p^0\|_V + \|p^1\|_H).$$

Let for brevity $W, W_h\colon \mathcal{Y}\to\mathcal{Y}^*$ be the duality mappings defined by

$$W(y_1,y_2,y_3) = (y_1, -y_2, A^{-1}y_3), \quad W_h(y_1,y_2,y_3) = (y_1,-y_2,A_h^{-1}y_3) \ \ \forall (y_1,y_2,y_3)\in\mathcal{Y}.$$

With this notation, the function $p_\vartheta = S_\vartheta^*(y - z_1, -(y(T)-z_2), A_h^{-1}(\rho\partial_t y(T) - z_3)) = S_\vartheta^* W_h(Su - z)$ solves the general discrete adjoint state equation (8.1)-(8.2).

Proposition 9.1. *Let* $\mathbf{z}\in\mathcal{Y}$ *and* $\mathbf{y}\in V\times V^*$. *Then the following estimate holds*

$$\|S\bar u - S_\vartheta \bar u_\vartheta\|_{\mathcal{Y}_h} \le \|S\bar u - S_\vartheta\bar u\|_{\mathcal{Y}_h} + C\|S^\star W(S\bar u - \mathbf{z}) - S_\vartheta^\star W_h(S\bar u - \mathbf{z})\|_{\mathcal{C}_T}^{1/2}. \tag{9.1}$$

Proof. We recall that $\bar p = S^\star W(S\bar u - \mathbf{z})$ and $\bar p_\vartheta = S_\vartheta^\star W_h(S_\vartheta\bar u_\vartheta - \mathbf{z})$ and test the continuous subgradient condition (4.5) with the discrete optimal control $\bar u_\vartheta$ and the discrete subgradient condition (7.5) with the continuous optimal control $\bar u$. Then we subtract the first inequality from the second one and get

$$\langle \bar u - \bar u_\vartheta, \bar p - \bar p_\vartheta\rangle_{\mathcal{M}_T,\mathcal{C}_T} \le 0.$$

We define $\hat p_\vartheta := S_\vartheta^\star W_h(S\bar u - \mathbf{z})$, insert it between $\bar p$ and $\bar p_\vartheta$ and obtain

$$0 \le \langle \bar u_\vartheta - \bar u, \bar p - \hat p_\vartheta\rangle_{\mathcal{M}_T,\mathcal{C}_T} + \langle \bar u_\vartheta - \bar u, \hat p_\vartheta - \bar p_\vartheta\rangle_{\mathcal{M}_T,\mathcal{C}_T}. \tag{9.2}$$

For convenience we introduce the variables $(\hat y_\vartheta, \hat y_\vartheta(T), \rho\hat y_{Th}^1) = S_\vartheta\bar u$ and remark that the state equations for $(\bar y_\vartheta, \bar y_{Th}^1)$ and $(\hat y_\vartheta, \hat y_{Th}^1)$ have the same initial data. With the help of them we rewrite the second term on the right in (9.2) taking first the difference of the discrete state equations (7.3) and (5.1) (taken for $(\hat y_\vartheta, \hat y_{Th}^1)$) for $v = \hat p_\vartheta - \bar p_\vartheta$, next the difference of the discrete adjoint state equations (7.4) and (8.1)-(8.2) (taken for $\hat p_\vartheta$) for $v = \bar y_\vartheta - \hat y_\vartheta$ and $\varphi = \bar y_{Th}^1 - \hat y_{Th}^1$ and finally using (5.15)

$$\langle \bar u_\vartheta - \bar u, \hat p_\vartheta - \bar p_\vartheta\rangle_{\mathcal{M}_T,\mathcal{C}_T} = B_\sigma(\bar y_\vartheta - \hat y_\vartheta, \hat p_\vartheta - \bar p_\vartheta) + (\rho(\bar y_{Th}^1 - \hat y_{Th}^1), (\hat p_\vartheta - \bar p_\vartheta)(T))_H$$
$$= (\rho(\bar y_\vartheta - \hat y_\vartheta), \bar y - \bar y_\vartheta)_{L^2(I\times\Omega)} + (\rho(\bar y_\vartheta - \hat y_\vartheta)(T), (\bar y - \bar y_\vartheta)(T))_H + (\rho(\bar y_{Th}^1 - \hat y_{Th}^1), A_h^{-1}(\rho(\partial_t\bar y(T) - \bar y_{Th}^1)))_H$$
$$= (S_\vartheta\bar u_\vartheta - S_\vartheta\bar u, S\bar u - S_\vartheta\bar u_\vartheta)_{\mathcal{Y}_h}.$$

Further we easily get

$$\langle \bar u_\vartheta - \bar u, \hat p_\vartheta - \bar p_\vartheta\rangle_{\mathcal M_T,\mathcal C_T} = (S\bar u - S_\vartheta \bar u_\vartheta, S_\vartheta \bar u_\vartheta - S_\vartheta \bar u)_{\mathcal Y_h} = (S\bar u - S_\vartheta \bar u_\vartheta, S\bar u - S_\vartheta \bar u)_{\mathcal Y_h} - \|S\bar u - S_\vartheta \bar u_\vartheta\|^2_{\mathcal Y_h}$$
$$\le \tfrac12 \|S\bar u - S_\vartheta \bar u\|^2_{\mathcal Y_h} - \tfrac12\|S\bar u - S_\vartheta \bar u_\vartheta\|^2_{\mathcal Y_h}.$$

Thus (9.2) implies

$$\|S\bar u - S_\vartheta \bar u_\vartheta\|^2_{\mathcal Y_h} \le 2\langle \bar u_\vartheta - \bar u, \bar p - \hat p_\vartheta\rangle_{\mathcal M_T,\mathcal C_T} + \|S\bar u - S_\vartheta \bar u\|^2_{\mathcal Y_h}$$
$$\le 2(\|\bar u_\vartheta\|_{\mathcal M_T} + \|\bar u\|_{\mathcal M_T})\|\bar p - \hat p_\vartheta\|_{\mathcal C_T} + \|S\bar u - S_\vartheta \bar u\|^2_{\mathcal Y_h}.$$

Finally by applying bounds (4.2) and (7.1) we derive (9.1). □

This proposition is important since it allows one to derive estimates for $\bar y - \bar y_\vartheta$ with the help of the above error estimates for the discrete state and adjoint state equations.

Theorem 9.2. *(1) Let $\mathcal M_T = L^2(I,\mathcal M(\Omega))$, $\mathbf z \in \mathcal Y^{1/2}$ and $\mathbf y \in V\times H$. Then the following error estimate holds*

$$\|\bar y - \bar y_\vartheta\|_{L^2(I\times\Omega)} + \|(\bar y - \bar y_\vartheta)(T)\|_H + \|\rho(\partial_t \bar y(T) - \bar y^1_{Th})\|_{H_h^{-1}} \le C(\tau+h)^{1/3}. \tag{9.3}$$

(2) Let $\mathcal M_T = \mathcal M(\Omega, L^2(I))$, $\mathbf z \in \mathcal Y^{3/2}$ and $\mathbf y \in H^{(3/2)}\times H^{(1/2)}$. Then the following higher order error estimate holds

$$\|\bar y - \bar y_\vartheta\|_{L^2(I\times\Omega)} + \|(\bar y - \bar y_\vartheta)(T)\|_H + \|\rho(\partial_t \bar y(T) - \bar y^1_{Th})\|_{H_h^{-1}} \le C(\tau+h)^{2/3}. \tag{9.4}$$

Proof. 1. Let us base on Proposition 9.1. First, Proposition 6.2 (4) implies

$$\|S\bar u - S_\vartheta \bar u\|_{\mathcal Y_h} \le c(\tau+h)^{1/3}(\|\bar u\|_{L^2(I,H^{(-1/2)})} + \|\mathbf y\|_{V\times H}).$$

Second, Proposition 8.1 (3) leads to

$$\|S^\star W(S\bar u - \mathbf z) - S^\star_\vartheta W_h(S\bar u - \mathbf z)\|_{\mathcal C_T} \le c(\tau+h)^{2/3}(\|\bar u\|_{L^2(I,H^{(-1/2)})} + \|\mathbf z\|_{\mathcal Y^{1/2}} + \|\mathbf y\|_{H^{(1/2)}\times H^{(-1/2)}}).$$

Now owing to Proposition 9.1, embedding (3.27) and bound (4.2) for $\bar u$ error estimate (9.3) is proved.

2. First, Proposition 6.2 (3) implies

$$\|S\bar u - S_\vartheta \bar u\|_{\mathcal Y_h} \le c(\tau+h)^{2/3}\left(\|\bar u\|_{H^1(I,H^{(-1/2)})} + \|\mathbf y\|_{H^{(3/2)}\times H^{(1/2)}}\right).$$

Second, Proposition 8.1 (4) leads to

$$\|S^\star W(S\bar u - \mathbf z) - S^\star_\vartheta W_h(S\bar u - \mathbf z)\|_{\mathcal C_T} \le c(\tau+h)^{4/3}(\|\bar u\|_{H^1(I,H^{(-1/2)})} + \|\mathbf z\|_{\mathcal Y^{3/2}} + \|\mathbf y\|_{H^{(3/2)}\times H^{(1/2)}}).$$

Now owing to Proposition 9.1, embedding (3.28) and Theorem 4.4 for $\bar u$ error estimate (9.4) is proved too. □

Remark 9.3. Note that our error bounds could be better provided that one would improve the last term on the right in (9.1) by increasing the power 1/2. But this seems a complicated problem.

10. Error estimate for the cost functional

In this section we derive error estimate for the cost functional. We first observe the inequalities

$$j(\bar u) \le j(\bar u_\vartheta),\ \ j_\vartheta(\bar u_\vartheta) \le j_\vartheta(\bar u)$$

which can be equivalently rewritten in the form

$$j(\bar u) - j_\vartheta(\bar u) \le j(\bar u) - j_\vartheta(\bar u_\vartheta) \le j(\bar u_\vartheta) - j_\vartheta(\bar u_\vartheta). \tag{10.1}$$

Therefore, to bound $|j(\bar u) - j_\vartheta(\bar u_\vartheta)|$ below we apply the following result.

Proposition 10.1. *Let* $\mathbf{y} \in V \times H$*. Then for any* $u \in \mathcal{M}_T$

$$|j(u) - j_\vartheta(u)| \le c\Big(\|Su - S_\vartheta u\|^2_{\mathcal{Y}_h} + \\ + (\|u\|_{\mathcal{M}_T} + \|\mathbf{y}\|_{V\times H})\Big(\|p - p_\vartheta\|_{\mathcal{C}_T} + \|p(0) - p_\vartheta(0)\|_H + h\|\partial_t p(0)\|_H + \|\rho(\partial_t p(0) - p^1_{0h})\|_{H_h^{-1}}\Big) \\ + \|r_h A^{-1}(\rho\partial_t y(T))\|^2_V + \|r_h A^{-1} z_3\|^2_V\Big) \quad (10.2)$$

with $(y, y(T), \rho\partial_t y(T)) = Su$ *and the same* p *and* (p_ϑ, p^1_{0h}) *as in Proposition 8.1.*

Proof. Let $u \in \mathcal{M}_T$. According to the definitions of the continuous and discrete cost functionals and property (5.13) for $\tilde{w} = w$ and $\tilde{w}_h = w_h$ we get

$$\begin{aligned} j(u) - j_\vartheta(u) &= \tfrac{1}{2}\|Su - \mathbf{z}\|^2_{\mathcal{Y}} - \tfrac{1}{2}\|S_\vartheta u - \mathbf{z}\|^2_{\mathcal{Y}_h} \\ &= \tfrac{1}{2}(Su - S_\vartheta u, Su + S_\vartheta u - 2\mathbf{z})_{\mathcal{Y}_h} + \tfrac{1}{2}\|A^{-1}(\rho\partial_t y(T) - z_3)\|^2_{\mathcal{V}_\kappa} - \tfrac{1}{2}\|A_h^{-1}(\rho\partial_t y(T) - z_3)\|^2_{\mathcal{V}_\kappa} \\ &= -\tfrac{1}{2}\|Su - S_\vartheta u\|^2_{\mathcal{Y}_h} + (Su - S_\vartheta u, Su - \mathbf{z})_{\mathcal{Y}_h} + \tfrac{1}{2}\|r_h A^{-1}(\rho\partial_t y(T) - z_3)\|^2_{\mathcal{V}_\kappa}. \quad (10.3) \end{aligned}$$

We set $p_{Th} := A_h^{-1}(\rho\partial_t y(T) - z_3)$. Owing to the adjoint problem (3.12) with $(\phi, p^1, p^0) = W(y - z_1, y(T) - z_2, \rho\partial_t y(T) - z_3)$ we get

$$\begin{aligned} (Su, Su - \mathbf{z})_{\mathcal{Y}_h} - (A_h^{-1}(\rho\partial_t y(T)), p_{Th})_{\mathcal{V}_\kappa} &= (\rho y, y - z_1)_{L^2(I\times\Omega)} + (\rho y(T), y(T) - z_2)_H \\ &= \langle u, p\rangle_{\mathcal{M}_T,\mathcal{C}_T} + (\rho y^1, p(0))_H - \langle \rho\partial_t y(T), p^0\rangle_\Omega - (\rho y^0, \partial_t p(0))_H. \end{aligned}$$

Similarly owing to the general discrete adjoint state equation (8.1)-(8.2) for $v = y_\vartheta$ and the discrete state equation (5.1)-(5.2) for $v = p_\vartheta$ and $\varphi = p^1_{0h}$ we get

$$\begin{aligned} (S_\vartheta u, Su - \mathbf{z})_{\mathcal{Y}_h} - (A_h^{-1}(\rho y^1_{Th}), p_{Th})_{\mathcal{V}_\kappa} &= (\rho y_\vartheta, y - z_1)_{L^2(I\times\Omega)} + (\rho y_\vartheta(T), y(T) - z_2)_H \\ &= B_\sigma(y_\vartheta, p_\vartheta) - (\rho y_\vartheta(0), p^1_{0h})_H \\ &= \langle u, p_\vartheta\rangle_{\mathcal{M}_T,\mathcal{C}_T} + (\rho y^1, p_\vartheta(0))_H - (\rho y^1_{Th}, p_\vartheta(T))_H - (\rho y^0, p^1_{0h})_H. \end{aligned}$$

In addition owing to the definitions (8.2) of $p_\vartheta(T)$ and (5.15) of A_h^{-1}, we can write

$$(\rho y^1_{Th}, p_\vartheta(T))_H = (\rho y^1_{Th}, p_{Th})_H = (A_h^{-1}(\rho y^1_{Th}), p_{Th})_{\mathcal{V}_\kappa}.$$

Consequently we obtain

$$\begin{aligned} (Su - S_\vartheta u, Su - \mathbf{z})_{\mathcal{Y}_h} &= (Su, Su - \mathbf{z})_{\mathcal{Y}_h} - (S_\vartheta u, Su - \mathbf{z})_{\mathcal{Y}_h} \\ &= \langle u, p - p_\vartheta\rangle_{\mathcal{M}_T,\mathcal{C}_T} - (\rho y^0, \partial_t p(0) - p^1_{0h})_H + (\rho y^1, p(0) - p_\vartheta(0))_H \\ &\quad + (A_h^{-1}(\rho\partial_t y(T)), p_{Th})_{\mathcal{V}_\kappa} - \langle \rho\partial_t y(T), p^0\rangle_\Omega. \quad (10.4) \end{aligned}$$

In addition using property (5.13) we derive

$$\begin{aligned} (A_h^{-1}(\rho\partial_t y(T)), p_{Th})_{\mathcal{V}_\kappa} - \langle \rho\partial_t y(T), p^0\rangle_\Omega &= (A_h^{-1}(\rho\partial_t y(T)), p_{Th})_{\mathcal{V}_\kappa} - (A^{-1}(\rho\partial_t y(T)), p^0)_{\mathcal{V}_\kappa} \\ &= -(r_h A^{-1}(\rho\partial_t y(T)), r_h A^{-1}(\rho\partial_t y(T) - z_3))_{\mathcal{V}_\kappa}. \quad (10.5) \end{aligned}$$

Next, for the term $(\rho y^0, \partial_t p(0) - p^1_{0h})_H$ in (10.4) we have

$$\begin{aligned} |(\rho y^0, \partial_t p(0) - p^1_{0h})_H| &= |(\rho(y^0 - \pi_h^0 y^0), \partial_t p(0) - p^1_{0h})_H + (\rho\pi_h^0 y^0, \partial_t p(0) - p^1_{0h})_H| \\ &\le |(\rho(y^0 - \pi_h^0 y^0), \partial_t p(0))_H| + c\|\pi_h^0 y^0\|_V \|\rho(\partial_t p(0) - p^1_{0h})\|_{H_h^{-1}} \\ &\le c_1\, \|y^0\|_V \Big(h\|\partial_t p(0)\|_H + \|\rho(\partial_t p(0) - p^1_{0h})\|_{H_h^{-1}}\Big) \quad (10.6) \end{aligned}$$

due to the bounds $\|y^0 - \pi_h^0 y^0\|_{H_\rho} \le \|y^0 - \pi_h^1 y^0\|_{H_\rho}$, (5.18) and (6.2). Clearly also $|(\rho y^1, p(0) - p_\vartheta(0))_H| \le \|y^1\|_H \|p(0) - p_\vartheta(0)\|_H$. Finally from (10.3)-(10.6) we derive (10.2). $\square$

Now we prove for the cost functional squared error estimate (9.3) for the state variable.

Theorem 10.2. *Let $\mathcal{M}_T = L^2(I, \mathcal{M}(\Omega))$, $\mathbf{z} \in \mathcal{Y}^{1/2}$ and $\mathbf{y} \in V \times H$. Then the following error estimate for the cost functional holds*

$$|j(\bar{u}) - j_\vartheta(\bar{u}_\vartheta)| \leq C(\tau + h)^{2/3}.$$

Proof. Let us base on Proposition 10.1 and take any $u \in L^2(I, \mathcal{M}(\Omega))$. Owing to Proposition 6.2 (2) we have

$$\|Su - S_\vartheta u\|_{\mathcal{Y}_h} \leq c(\tau + h)^{1/3}\big(\|u\|_{L^2(I,H^{(-1/2)})} + \|\mathbf{y}\|_{V\times H^{(-1/2)}}\big).$$

Proposition 8.1 (3) leads to

$$\|p - p_\vartheta\|_{\mathcal{C}(\bar{I}\times\bar{\Omega})} \leq c(\tau + h)^{2/3}\big(\|u\|_{L^2(I,H^{(-1/2)})} + \|\mathbf{z}\|_{\mathcal{Y}^{1/2}} + \|\mathbf{y}\|_{H^{(1/2)}\times H^{(-1/2)}}\big).$$

Owing to Propositions 3.5(1) (applied to the adjoint state problem) and 3.7 we have

$$\|\partial_t p(0)\|_H \leq \|\partial_t p\|_{C(\bar{I},H)} \leq c\big(\|u\|_{L^2(I,V^*)} + \|\mathbf{y}\|_{H\times V^*} + \|\mathbf{z}\|_{\mathcal{Y}}\big)$$

(like in estimates (8.11)-(8.12) for $\alpha = 1$). By using estimate (5.17) for $\lambda = -1/2$ we obtain

$$\|r_h A^{-1}(\rho\partial_t y(T))\|_V + \|r_h A^{-1} z_3\|_V \leq ch^{1/2}\big(\|\partial_t y(T)\|_{H^{(-1/2)}} + \|z_3\|_{H^{(-1/2)}}\big).$$

By collecting all these estimates together with embedding (3.27), Proposition 8.1 (2) to bound $\|\rho(\partial_t p(0) - p^1_{0h})\|_{H_h^{-1}}$ and applying Proposition 10.1, we derive

$$|j(u) - j_\vartheta(u)| \leq c(\tau + h)^{2/3}\big(\|u\|_{L^2(I,\mathcal{M}(\Omega))} + \|\mathbf{z}\|_{\mathcal{Y}^{1/2}} + \|\mathbf{y}\|_{V\times H}\big)^2.$$

Owing to inequalities (10.1) together with bounds (4.2) for $\bar{u}$ and (7.1) for $\bar{u}_\vartheta$ the proof is complete. □

Remark 10.3. In the case $\mathcal{M}_T = \mathcal{M}(\Omega, L^2(I))$ the lack of the bound $\|\bar{u}_\vartheta\|_{H^1(I,M(\Omega))} \leq C$, cf. Theorem 4.4 for $\bar{u}$, does not allow one to prove the error estimate $|j(\bar{u}) - j_\vartheta(\bar{u}_\vartheta)| \leq C(\tau + h)^{4/3}$.

11. Time-stepping formulation

In this section we discuss the time-stepping formulation of the discrete state equation (5.1)-(5.2) and the discrete adjoint state equation (7.4). We introduce the piecewise-linear "hat" functions such that $e^\tau_m(t_k) = \delta_{m,k}$ for any $k, m = 0, \ldots, M$, where $\delta_{m,k}$ is the Kroneker delta. We recall that e^τ_m are "half" hat functions for $m = 0, M$. There holds $V_\tau = \operatorname{span}\{e^\tau_0, \ldots, e^\tau_M\}$. Similarly, we introduce the spatial hat functions such that $e^h_j(x_k) = \delta_{j,k}$ for any $j = 1, \ldots, N-1$ and $k = 0, \ldots, N$; then $V_h = \operatorname{span}\{e^h_1, \ldots, e^h_{N-1}\}$.

Then the approximate state variable $y_\vartheta \in V_\vartheta$ can be represented in the following forms

$$y_\vartheta(t,x) = \sum_{m=0}^{M}\sum_{j=1}^{N-1} y_{m,j} e^h_j(x) e^\tau_m(t) = \sum_{m=0}^{M} y^h_m(x) e^\tau_m(t) = \sum_{j=1}^{N-1} y^\tau_j(t) e^h_j(x), \quad (t,x) \in \bar{I}\times\bar{\Omega} \tag{11.1}$$

with $y_{m,j} \in \mathbb{R}$, $y^h_m \in V_h$ and $y^\tau_j \in V_\tau$.

We also define the forward and backward difference and the average in time operators

$$\delta_t v_m = \frac{v_{m+1} - v_m}{\tau}, \quad \bar{\delta}_t v_m = \frac{v_m - v_{m-1}}{\tau},$$

$$B^\tau v_0 = \tfrac{1}{3}v_0 + \tfrac{1}{6}v_1, \quad B^\tau v_m = \tfrac{1}{6}v_{m-1} + \tfrac{2}{3}v_m + \tfrac{1}{6}v_{m+1}, \ 1 \leq m \leq M-1, \quad B^\tau v_M = \tfrac{1}{6}v_{M-1} + \tfrac{1}{3}v_M.$$

We define the self-adjoint positive-definite operators B_h and L_h acting in V_h (in other words, the mass and stiffness matrices) such that

$$(B_h\varphi_h, \psi_h)_{V_h} = (\rho\varphi_h, \psi_h)_H, \quad (L_h\varphi_h, \psi_h)_{V_h} = (\kappa\partial_x\varphi_h, \partial_x\psi_h)_H \quad \forall\varphi, \psi \in V_h.$$

For $w \in V^*$ and $u \in L^2(I, V^*)$ we define the vectors $w^h = \{\langle w, e^h_j\rangle_\Omega\}_{j=1}^{N-1}$ and

$$u^\vartheta_m = \frac{1}{\tau}\Big\{\big(\langle u, e^h_j\rangle_\Omega, e^\tau_m\big)_{L^2(I)}\Big\}_{j=1}^{N-1}, \ 1 \leq m \leq M-1, \quad u^\vartheta_m = \frac{2}{\tau}\Big\{\big(\langle u, e^h_j\rangle_\Omega, e^\tau_m\big)_{L^2(I)}\Big\}_{j=1}^{N-1}, \ m = 0, M.$$

We recall the form of the discrete state (11.1).

The forward time-stepping is implemented as follows. The integral identities (5.1)-(5.2) are equivalent to the operator equations

$$(B_h+\sigma\tau^2L_h)\delta_t\bar\delta_t y_{\vartheta,m}+L_hy_{\vartheta,m}=u_m^\vartheta,\quad m=2,\dots,M-1, \tag{11.2}$$

$$(B_h+\sigma\tau^2L_h)\delta_t y_{\vartheta,1}+\tfrac{\tau}{2}L_hy_{\vartheta,0}=(\rho y^1)^h+\tfrac{\tau}{2}u_0^\vartheta, \tag{11.3}$$

$$B_hy_{\vartheta,0}=(\rho y^0)^h \tag{11.4}$$

followed by the counterpart of (11.3) at time T for y_{Th}^1:

$$B_hy_{Th}^1=(B_h+\sigma\tau^2L_h)\bar\delta_t y_{\vartheta,M}-\tfrac{\tau}{2}L_hy_{\vartheta,M}+\tfrac{\tau}{2}u_M^\vartheta. \tag{11.5}$$

Next *the adjoint (backward) time-stepping* is implemented in a similar manner. Namely, the integral identities (7.4) are equivalent to the operator equations

$$(B_h+\sigma\tau^2L_h)\delta_t\bar\delta_t p_{\vartheta,m}+L_hp_{\vartheta,m}=B_hB^\tau y_{\vartheta,m}-(\rho z_1)_m^\vartheta,\quad m=M-1,\dots,1, \tag{11.6}$$

$$-(B_h+\sigma\tau^2L_h)\bar\delta_t p_{\vartheta,M}+\tfrac{\tau}{2}L_hp_{\vartheta,M}=B_hy_{\vartheta,M}-(\rho z_2)^h+\tfrac{\tau}{2}\big(B_hB^\tau y_{\vartheta,M}-(\rho z_1)_M^\vartheta\big), \tag{11.7}$$

$$L_hp_{\vartheta,M}=B_hy_{Th}^1-z_3^h, \tag{11.8}$$

followed by the counterpart of (11.5) for p_{0h}^1:

$$B_hp_{0h}^1=(B_h+\sigma\tau^2L_h)\delta_t p_{\vartheta,0}+\tfrac{\tau}{2}L_hp_{\vartheta,0}-\tfrac{\tau}{2}\big(B_hB^\tau y_{\vartheta,0}-(\rho z_1)_0^\vartheta\big). \tag{11.9}$$

Remark 11.1. For $\sigma=1/4$ the three-level time stepping scheme (11.2)-(11.5) is closely related to the well-known two-level Crank-Nicolson method applied to the first order in time system

$$\begin{cases}\partial_t y=v,\ \rho\partial_t v-\partial_x(\kappa\partial_x y)=u & \text{in } I\times\Omega\\ y=0 & \text{on } I\times\partial\Omega\\ y=y^0,\ v=y^1 & \text{in } \{0\}\times\Omega,\end{cases}$$

see [46, Section 8] for details, as well as to the Petrov-Galerkin method described in [29]. After the mass lumping, for $\sigma=0$ our method becomes explicit and is related to the Leap-Frog method; moreover, for any σ it becomes close to three-level finite-difference schemes with such weight in time, eg. see [43].

12. Control discretization. Solution process and $L^2(I\times\Omega)$-regularization

Now we discuss in more detail solving of the semi-discrete optimization problem $(\mathcal{P}_\vartheta)$ in the case $\mathcal{M}_T=\mathcal{M}(\Omega,L^2(I))$.

An important point is that we can seek its solution in the form

$$\bar u_\vartheta\in\mathcal{M}_\vartheta:=V_\tau\otimes\mathcal{M}_h,\quad \mathcal{M}_h:=\operatorname{span}\{\delta_{x_1},\dots,\delta_{x_{N-1}}\}\subset\mathcal{M}(\Omega).$$

To show that, let π_τ^0 be the projector in $L^2(I)$ on V_τ. Note that, for $\eta\in L^2(I)$, it satisfies

$$(B^\tau\pi_\tau^0\eta)_m=\tfrac{1}{\tau}(\eta,e_m^\tau)_{L^2(I)}\ \text{ for }\ 1\le m\le M-1,\quad (B^\tau\pi_\tau^0\eta)_m=\tfrac{2}{\tau}(\eta,e_m^\tau)_{L^2(I)}\ \text{ for }\ m=0,M.$$

Then we define $\Pi_h\colon\mathcal{M}(\Omega)\to\mathcal{M}_h$ by $\Pi_hw:=\sum_{j=1}^{N-1}\langle w,e_j^h\rangle_\Omega\delta_{x_j}$ and $\Pi_\vartheta=\pi_\tau^0\Pi_h$. The following identity holds

$$\langle\Pi_\vartheta u,v\rangle_{\mathcal{M}_T,\mathcal{C}_T}=\langle u,\pi_\tau^0 i_hv\rangle_{\mathcal{M}_T,\mathcal{C}_T}\quad\forall u\in\mathcal{M}_T,v\in\mathcal{C}_T$$

with the interpolation operator $i_h\colon\mathcal{C}_0(\Omega)\to V_h$ such that $i_hw(x_j)=w(x_j)$ for all $j=0,\dots,N$. In particular, if $v\in V_\vartheta$, then

$$\langle\Pi_\vartheta u,v\rangle_{\mathcal{M}_T,\mathcal{C}_T}=\langle u,v\rangle_{\mathcal{M}_T,\mathcal{C}_T},$$

and consequently (like in [30, Lemma 3.11]) we have $S_\vartheta=S_\vartheta\circ\Pi_\vartheta$ as well as $\|\Pi_\vartheta u\|_{\mathcal{M}_T}\le\|u\|_{\mathcal{M}_T}$. Thus for each solution $\tilde u_\vartheta$ of problem $(\mathcal{P}_\vartheta)$, *the discrete control* $\Pi_\vartheta\tilde u_\vartheta$ satisfies

$$j_\vartheta(\tilde u_\vartheta)=j_\vartheta(\Pi_\vartheta\tilde u_\vartheta).$$

Therefore $\Pi_\vartheta \tilde{u}_\vartheta$ is also a solution of $(\mathcal{P}_\vartheta)$. This is a justification for solving the fully discrete problem

$$j_\vartheta(u_\vartheta) = \tfrac{1}{2}\,\|S_\vartheta u_\vartheta - \mathbf{z}\|^2_{\mathcal{Y}_h} + \alpha\|u_\vartheta\|_{\mathcal{M}(\Omega,L^2(I))} \to \min_{u_\vartheta \in \mathcal{M}_\vartheta} \tag{12.1}$$

in order to get a solution of $(\mathcal{P}_\vartheta)$.

The direct solution of (12.1) by means of a generalized Newton type method is a challenging problem since a proper globalization strategy is needed, see [36]. Thus we propose a solution strategy based on an additional $L^2(I \times \Omega)$-regularization of (12.1) with a parameter $\gamma > 0$ and a continuation method. For high values of γ the corresponding Newton type method converges independently of the initial guess in numerical practice. Thus the continuation strategy can be seen as simple globalization strategy.

On the continuous level we consider the following regularized problem

$$j_\gamma(u) = \tfrac{1}{2}\,\|Su - \mathbf{z}\|^2_{\mathcal{Y}} + \alpha\|u\|_{\mathcal{M}(\Omega,L^2(I))} + \tfrac{\gamma}{2}\|u\|^2_{L^2(I\times\Omega)} \to \min\nolimits_{u \in L^2(I\times\Omega)}\,. \tag{12.2}$$

It is possible to formulate a semi-smooth Newton method for this problem on the continuous level which is based on the following necessary and sufficient optimality condition

$$\bar{u}_\gamma(t,x) = -\frac{1}{\gamma}\max\left(0, 1 - \frac{\alpha}{\|\bar{p}(\cdot,x)\|_{L^2(I)}}\right)\bar{p}(t,x), \quad (t,x) \in I \times \Omega, \tag{12.3}$$

with $\bar{p} = S^\star W_h(S\bar{u}_\gamma - \mathbf{z})$. Moreover, this semi-smooth Newton method is superlinear convergent. Let $\bar{u}_\gamma$ and $\bar{u}$ be the unique solutions of (12.2) and $(\mathcal{P})$. Then we have $\bar{u}_\gamma \rightharpoonup^* \bar{u}$ in $\mathcal{M}(\Omega, L^2(I))$, see [23, 30, 40]. This justifies the use of a continuation strategy in γ. The control discretization described above can not be used for (12.2). Instead we propose to use discrete controls from V_ϑ, i.e.,

$$u_\vartheta(t,x) = \sum_{m=0}^{M}\sum_{j=1}^{N-1} u_{m,j} e^\tau_m(t) e^h_j(x) = \sum_{j=1}^{N-1} u_j(t) e^h_j(x) = \sum_{m=0}^{M} u_m(x) e^\tau_m(t),$$

cf. (11.1). In particular, we solve the following fully discrete regularized problem

$$j^\gamma_\vartheta(u_\vartheta) = \tfrac{1}{2}\,\|S_\vartheta(l_\vartheta u_\vartheta) - \mathbf{z}\|^2_{\mathcal{Y}_h} + \alpha\|u_\vartheta\|_{\mathcal{M}(\Omega,L^2(I)),h} + \tfrac{\gamma}{2}\|u_\vartheta\|^2_{L^2(I\times\Omega),h} \to \min\nolimits_{u_\vartheta \in V_\vartheta} \tag{12.4}$$

with

$$\|u_\vartheta\|_{\mathcal{M}(\Omega,L^2(I)),h} = \sum_{j=1}^{N-1} d_j \|u_j\|_{L^2(I)}, \quad \|u_\vartheta\|^2_{L^2(I\times\Omega),h} = \sum_{m=0}^{M} (B^\tau u_m)^t D (B^\tau u_m)$$

where $D = \operatorname{diag}(d_1, \dots, d_{N-1})$ is the lumped mass matrix. Moreover, the operator l_ϑ defined by

$$(l_\vartheta u_\vartheta, v_\vartheta)_{L^2(I\times\Omega),h} = \sum_{m=0}^{M} (B^\tau u_m)^t D (B^\tau v_m) \quad \forall u_\vartheta, v_\vartheta \in V_\vartheta.$$

The use of D allows us to derive the following optimality conditions for (12.4)

$$\bar{u}^\gamma_{m,j} = -\frac{1}{\gamma}\max\left(0, 1 - \frac{\alpha}{\|\bar{p}_{\vartheta\,\cdot,j}\|_{L^2(I)}}\right)\bar{p}_{\vartheta\,m,j}, \tag{12.5}$$

for all m and j, with $\bar{p}_\vartheta = S^\star_\vartheta W_h(S_\vartheta \bar{u}_\vartheta - \mathbf{z})$, cf. (12.3). Based on (12.5) we can set up a semi-smooth Newton method. Since problem (12.4) is a discretization of (12.2), we can expect that this method behaves mesh independently. Let $\bar{u}^\gamma_\vartheta = \sum_{j=1}^{N-1} u_j(t) e^h_j$ be the solution of (12.4) and we define

$$\tilde{u}^\gamma_\vartheta = \sum_{j=1}^{N-1} \frac{u_j(t)}{d_j}\delta_{x_j}.$$

As $\gamma \to 0$ the control $\tilde{u}^\gamma_\vartheta$ tends to a solution of (12.1) justifying the use of this control discretization and the continuation strategy. For more details see [40].

13. Numerical results

In this section, we present results of numerical experiments. We consider an example involving the wave equation $\partial_{tt}y - \partial_{xx}y = 0$ in $I \times \Omega = (0,1) \times (0,1)$, zero initial data $y^0 = y^1 = 0$, the control space $\mathcal{M}_T = \mathcal{M}(\Omega, L^2(I))$ and the tracking functional

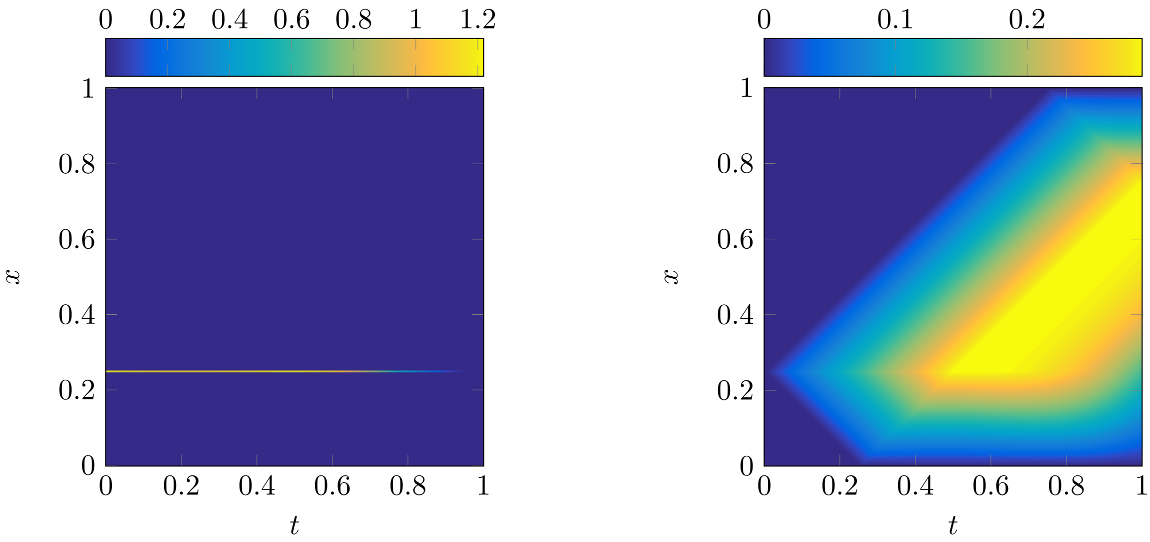
$$F(y) = \tfrac{1}{2}\|y - z\|^2_{L^2(I\times\Omega)}$$

with the particular desired state

$$z(x) = \tfrac{1}{\sqrt{2\pi}\rho} e^{-\frac{(x-\lambda)^2}{2\rho^2}}$$

where $\rho = 0.1$ and $\lambda = \pi/20$. The time independent function z is a gaussian centered in an irrational point λ. For sufficiently large α ($\alpha = 0.1$), we expect that the optimal control $\bar{u}$ consists of one point source with a position close to λ. If the gaussian would move through the domain, a point source shaped $\bar{u}$ is not able to follow the center of the gaussian since $\mathcal{M}(\Omega, L^2(I))$ contains no moving point sources. The optimal control would rather consist of some additional fixed point sources. This would not lower the regularity of the state whereas a moving point source can cause it.

The domain Ω and the time interval I are discretized by uniform grids with $M = 2^{r_\tau}$ and $N = 2^{r_h}$ where $r_\tau, r_h = 2, 3, \ldots$. The stability parameter is fixed to its lowest value $\sigma = 1/4$ ensuring unconditional stability of the time-stepping method. The discrete control problem is solved for $r_h = 2, 3, \ldots, r^{\max}$ and a fixed r_τ and vice versa. The solution process has been described above in Section 12. Numerically the desired state z is replaced by $i_h z$ for simplicity, moreover the corresponding error $\mathcal{O}(h^2)$ is negligible. Since the optimal pair $(\bar{u}, \bar{y})$ is not known in our example, we replace it by a reference solution $(\hat{u}, \hat{y})$ which is chosen as the approximate solution on the finest grid level. We depict $(\hat{u}, \hat{y})$ in Figure 1. As expected, the optimal control $\hat{u}$ consists only of one point source positioned in



(A) $\hat{u}$ (on a coarser grid) (B) $\hat{y}$

Figure 1. Reference solution

the vicinity of λ. Thus, the state $\hat{y}$ has a kink at this position. Due to reflections at the boundary, $\hat{y}$ has also kinks at other positions.

Next, we discuss the convergence results. In Figure 2, we see the convergence rate of $\|\bar{y}_\sigma - \hat{y}\|_{L^2(I\times\Omega)}$ and of the functional for a sequence of h refinements. The state error behaves in a linear way and the rate for the functional is close to two; as usual the latter is approximately the doubled rate of the former. Both are better than above theoretical rates. In Figure 3, we see the similar results for a sequence of τ refinements. The error of the functional stagnates at the last τ refinement that is caused

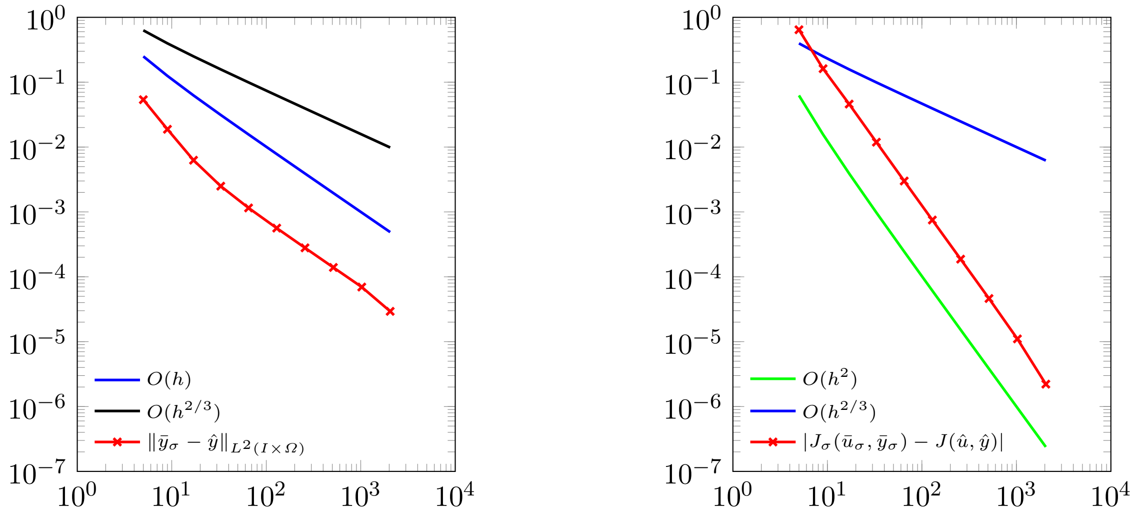


FIGURE 2. Errors as h refines and $M = 2^{10}$

by a too coarse space grid. Nevertheless, we observe reduced rates for $\hat{y}$ much less than two caused by its reduced regularity (kinks).

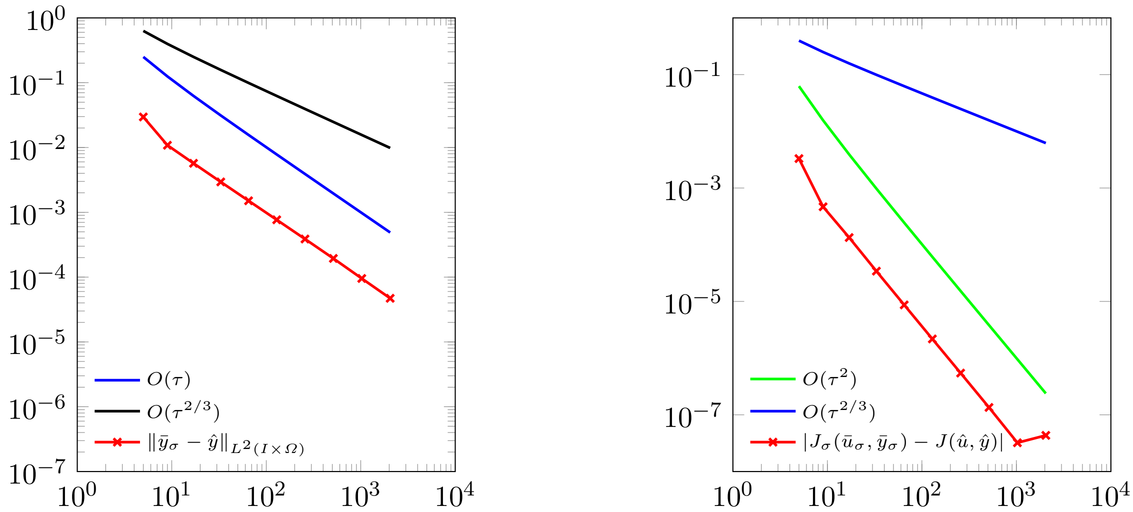


FIGURE 3. Errors as τ refines and $N = 2^{10}$

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