A-PRIORI TIME REGULARITY ESTIMATES AND A SIMPLIFIED PROOF OF EXISTENCE IN PERFECT PLASTICITY

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Abstract. We revisit the time-incremental method for proving existence of a quasistatic evolution in perfect plasticity. We show how the time regularity estimates on the stress and the plastic strain can be a-priori recovered from the incremental minimum problems. This allows for a quicker proof of existence: furthermore, this proof bypasses the usual variational reformulation of the problem and directly tackles its original mechanical formulation in terms of an equilibrium condition, a stress constraint, and the principle of maximum plastic work.

1. Introduction

The scope of this note is to revisit and simplify the proof of existence of a quasistatic evolution in small strain linearized perfect elasto-plasticity. This is done through the recovering of some a-priori time regularity estimates. In the usual proof strategy (see [2]) such estimates are instead only available a-posteriori, after that existence for a variational reformulation of the problem has been established.

For a better explaining, we recall both the classical and the variational formulation of the problem. We put ourselves for simplicity in the case of no applied volume and surface forces, so that the evolution is only driven by a prescribed boundary displacement $w(t,x)$, usually taken H^1 regular both in time and space. Given an open set $\Omega \subset \mathbb{R}^n$ and an open subset Γ_0 of $\partial\Omega$, a quasistatic evolution is then a triple $(u(t,x), e(t,x), p(t,x))$ satisfying the following conditions:

- *Kinematic admissibility:* denoting with $Eu(t,x)$ the symmetrised gradient of u, one has
$$\begin{cases} Eu(t,x) = e(t,x) + p(t,x) \text{ in } \Omega \\ u(t,x) = w(t,x) \text{ on } \Gamma_0\,; \end{cases}$$
- *Equilibrium condition and stress constraint:* setting $\sigma(t,x) := \mathbb{C}e(t,x)$, with $\mathbb{C}$ the elasticity tensor, it holds
$$\begin{cases} \operatorname{div}\sigma(t,x) = 0 \text{ in } \Omega\,, \quad \sigma(t,x)\nu(x) = 0 \text{ on } \partial\Omega \setminus \overline{\Gamma}_0\,, \\ \sigma_D(t,x) \in K(x) \text{ for every } x \in \Omega\,. \end{cases}$$
Here $\nu(x)$ is the outward unit normal to $\partial\Omega$, σ_D the orthogonal projection of σ on the space of trace-free $n \times n$ symmetric matrices $\mathbb{M}^{n\times n}_D$, and $K(x)$ is a convex compact neighborhood of 0 in $\mathbb{M}^{n\times n}_D$;
- *flow rule:* for a.e. $x \in \Omega$,
$$\dot p(t,x) \in N_{K(x)}(\sigma_D(t,x))\,,$$
where at the right-hand side we have the normal cone in the sense of Convex Analysis.

By convex duality the flow rule can be equivalently replaced by *Hill's principle of maximum plastic work*:
$$H(x, \dot p(t,x)) = \sigma_D(t,x) : \dot p(t,x)$$
where $H(x,\cdot)$ is the support function of the convex set $K(x)$, accounting for the rate of plastic dissipation, and the colon denotes the scalar product between matrices. Actually, in a rigorous setting, the issue of correctly defining the duality product between stress and plastic strain fields is absolutely nontrivial. It is indeed well-known since the seminal paper of Suquet [9] that p takes in general its values in some space of vector measures, while the stress is typically not continuous. Anyway the issue can be overcome (see [6] and [3]). According to this weak spatial regularity of p

(and consequently of $\dot{p}$), also the definition of plastic dissipation has to be conveniently modified. In the case of a homogeneous material, that is $K(x) = K$ for every x, it can be defined according to the theory of convex function of measures as

$$H(\mu) = H\Big(\frac{\mu}{|\mu|}\Big)\,|\mu|$$

for every bounded Radon measure μ, where $\frac{\mu}{|\mu|}$ is the Radon-Nikodym derivative of μ with respect to its total variation μ. In the case of multiphase materials the issue is more involved and the definition of a correct dissipation potential requires precise kinematic considerations on the behavior of admissible stress fields at interfaces [3] or an abstract point of view [8].

Taking the quadratic form Q associated to $\mathbb{C}$ as the elastic energy, and defining the dissipation functional $\mathcal{H}(p)$ as the total variation of the measure $H(p)$ on $\Omega \cup \Gamma_0$ the variational formulation reinterprets the classical mechanical one within the framework of the variational theory for rate-independent processes (see [5]). A quasistatic evolution is then regarded as the coupling of two conditions, namely

- *global stability:* at each time t the triple $(u(t), e(t), p(t))$ minimizes $Q(\eta) + \mathcal{H}(q - p(t))$ among all (v, η, q) admissible for $w(t)$;
- *energy-dissipation balance:* for every t

$$Q(e(t)) + \mathcal{D}(0, t; p) = Q(e_0) + \int_0^t \int_\Omega \sigma(s, x) : E\dot{w}(s, x)\, dtdx$$

  where the total plastic dissipation $\mathcal{D}$ is defined as the $\mathcal{H}$-total variation in time of $p(t)$, seen as a map from a time interval into the space of bounded radon measures.

We stress that this is a derivative free formulation: indeed, the usual a-priori estimates on the energy only entail that p has bounded variation as a function of the time, while no time regularity can at a first sight be proved for the elastic strain e. As a consequence of this, for instance, compactness of the approximating displacement fields in the existence proof is quite a delicate point, since converging subsequences can be in principle time dependent. Anyway, since globally stable states are unique up to the plastic strain p, this difficulty can be overcome.

It is quite easy to prove that a classical evolution is a variational one: basically, it suffices to integrate Hill's principle to recover the energy balance, while the equilibrium condition and the stress constraint are the Euler conditions for globally stable states (see for instance [2, Theorem 3.6 and Theorem 6.1]). The converse is not that easy, since we must first prove that time derivatives exist. This is however possible, once the energy-dissipation balance holds ([2, Theorem 5.2]).

The purpose of this note is to considerably simplify the path leading to the existence of a classical evolution, by revisiting the time-incremental minimization scheme used in the existence proofs of [2] and [3]. As usual, indeed, the time interval $[0, T]$ is divided into k subintervals (each with vanishing size as $k \to \infty$) by means of points

$$0 = t_k^0 < t_k^1 < \cdots < t_k^{k-1} < t_k^k = T\,,$$

and the approximate solution u_k^i, e_k^i, p_k^i at time t_k^i is defined, inductively, as a minimizer of the functional $Q(e) + \mathcal{H}(p - p_k^{i-1})$ among all triples (u, e, p) admissible for $w(t_k^i)$. We are then able to show (Lemma 3.2) the following key estimate:

$$\|e_k^i - e_k^{i-1}\|_2 \leq C\|Ew(t_k^i) - Ew(t_k^{i-1})\|_2\,.$$

By the H^1-time regularity of $Ew(t)$ this easily implies that the piecewise affine interpolant $e_k(t)$ of the e_k^i's is uniformly bounded in H^1 of the time. Equi-absolute continuity of the interpolants $p_k(t)$ and $u_k(t)$ follows now easily (Lemma 3.4). This implies compactness, as well as that the limit triple $(u(t), e(t), p(t))$ is made of absolutely continuous functions from $[0, T]$ into the respective target spaces. But there is more: already at the level of the interpolants, as a consequence of (3.2) we recover the inequality

$$\mathcal{H}(\dot{p}_k(t)) \leq \langle \sigma_k(t), \dot{p}_k(t)\rangle + \delta_k \tag{1.1}$$

with a small remainder δ_k. This is an approximate version of the $\leq$ inequality in Hill's principle, which is the only nontrivial one because of the stress constraint. With some care, (1.1) passes to the limit (Theorem 3.6) and existence of a quasistatic evolution is now established. This bypasses completely the variational reformulation, that can be anyway easily deduced from the classical one. Besides simplifying the existence proof, we also hope that the introduced technique can be useful for dealing with related models.

We end up this introduction by a short comparison with another method that allows for an existence proof, the viscoplastic approximation used by Suquet in [9]. This is the first existence result of the field, and a very general one, since multiphase behavior is allowed. However, there a weak definition of a solution only in terms of the stress and the displacement is used (see [9, Formula (34)]). The plastic strain p has indeed been eliminated via some formal integration by parts. This avoids the issue of defining both the stress-strain duality and the plastic dissipation, at the price of losing the information on the plastic strain path along an evolution. It has nevertheless been shown in [8] that viscoplastic approximations converge to a variational evolution as the viscosity parameter ε goes to 0. Furthermore, already in Suquet's proof a priori time regularity estimates very close in spirit to the ones in Lemma 3.4 are obtained, with a different technique and in a different context (see also [8, Theorem 4.14]). It is yet worth mentioning that time incremental minimization carries a small but significant advantage with respect to viscoplastic approximation. There we are forced to take a more regular initial displacement $u_0 \in H^1(\Omega)$, in order to obtain existence. In the incremental formulation, instead, the initial displacement u_0 has only the natural $BD(\Omega)$ regularity, and in particular, we are not forced to exclude that a plastic deformation is already present also at the boundary.

2. Preliminaries

For basic notation and preliminary results we refer to [8, Section 2]. For the reader's convenience, we recall only the main assumptions on the data and the constraints appearing in the definition of a quasistatic evolution.

The *reference configuration* Ω is a *bounded connected open set* in $\mathbb{R}^n$, $n \geq 2$, with *Lipschitz boundary* $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup N$. We assume that Γ_0 and Γ_1 are relatively open, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\mathcal{H}^{n-1}(N) = 0$, and

$$\Gamma_0 \neq \emptyset\,. \tag{2.1}$$

The common boundary $\partial\Gamma_0 = \partial\Gamma_1$ (topological notions refer here to the relative topology of $\partial\Omega$) will be assumed to satisfy the Kohn-Temam condition

$$\begin{gathered} \partial\Gamma_0 = \partial\Gamma_1 \quad \text{is a } (N-2)\text{-dimensional } C^2 \text{ manifold}\,, \\ \partial\Omega \quad \text{is } C^2 \text{ in a neighborhood of } \partial\Gamma_0 = \partial\Gamma_1\,. \end{gathered} \tag{2.2}$$

This condition has actually only the role of assuring that (2.9) holds; it could be replaced by any other sufficient condition for (2.9), like for instance the one considered in [3, Theorem 6.6]. We will prescribe a Dirichlet boundary condition on Γ_0 and a traction condition on Γ_1.

For $\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $\operatorname{div}\sigma \in L^2(\Omega;\mathbb{R}^n)$, $[\sigma\nu]$ denotes the normal trace on $\partial\Omega$, in general defined as a distribution. When $\sigma \in C^0(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ we have $[\sigma\nu] = \sigma\nu$ where the right-hand side is the pointwise product between the matrix $\sigma(x)$ and the normal vector $\nu(x)$ at each $x \in \partial\Omega$. Denoting with σ_D the orthogonal projection of σ on the space of trace-free $n\times n$ symmetric matrices $\mathbb{M}^{n\times n}_D$, under our assumptions on Ω if $\sigma_D \in L^\infty(\Omega; \mathbb{M}^{n\times n}_D)$ a tangential component of the trace $[\sigma\nu]^\perp_\nu$ can be defined ([3, Section 1.2]). It satisfies

$$[\sigma\nu]^\perp_\nu \in L^\infty(\partial\Omega;\mathbb{R}^n) \quad \text{and} \quad \|[\sigma\nu]^\perp_\nu\|_\infty \leq \|\sigma_D\|_\infty\,.$$

The *elasticity tensor* is a symmetric positive definite linear operator $\mathbb{C}\colon L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \to L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ satisfying

$$\mathbb{C}e = \mathbb{C}_D e_D + k(\operatorname{tr} e) I$$

where $\mathbb{C}_D$ is an isomorphism of $L^2(\Omega;\mathbb{M}^{n\times n}_D)$ and k is the multiplication by a function $k\in L^\infty(\Omega)$. The stored elastic energy $\mathcal{Q}:L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\to\mathbb{R}$ is given by the quadratic form $\mathcal{Q}$ associated to $\mathbb{C}$

$$\mathcal{Q}(e):=\frac{1}{2}\langle\mathbb{C}e,e\rangle=\frac{1}{2}\int_\Omega(\mathbb{C}e)(x):e(x)\,dx\,.$$

We will use the following simple algebraic identity, following from the simmetry of $\mathbb{C}$: if η and $\hat\eta\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ then

$$\mathcal{Q}(\eta)-\mathcal{Q}(\hat\eta)=\frac{1}{2}\langle\mathbb{C}(\eta+\hat\eta),\eta-\hat\eta\rangle\,. \tag{2.3}$$

The stress constraint will be abstractly modelled by a closed convex subset $\mathcal{K}\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. The constraint will act on the stress σ only through its deviatoric part, namely

$$\sigma\in\mathcal{K}\text{ if and only if }\sigma_D\in\mathcal{K}_D\,, \tag{2.4}$$

where $\mathcal{K}_D\subset L^2(\Omega;\mathbb{M}^{n\times n}_D)$ is closed convex and satisfy the following property: there exist $0<r<R<+\infty$ such that

$$\{\xi\in L^\infty(\Omega;\mathbb{M}^{n\times n}_D):\|\xi\|_\infty\le r\}\subseteq\mathcal{K}_D\subseteq\{\xi\in L^\infty(\Omega;\mathbb{M}^{n\times n}_D):\|\xi\|_\infty\le R\}\,. \tag{2.5}$$

In particular, this implies that, if $\sigma\in\mathcal{K}$, then $\sigma_D\in L^\infty(\Omega;\mathbb{M}^{n\times n}_D)$.

A bounded Radon measure $p\in M_b(\Omega\cup\Gamma_0)$ is said to be an element of the space of plastic strains $\Pi_{\Gamma_0}(\Omega)$ if there exist $u\in BD(\Omega)$, $e\in L^2(\Omega)$, and $w\in H^1(\Omega)$ such that

$$Eu=e+p\quad\text{ in }\Omega\,,$$
$$p=(w-u)\odot\nu\mathcal{H}^{n-1}\quad\text{ in }\Gamma_0$$

where ν is the outer unit normal to $\partial\Omega$ and the right-hand side in the second equality is the absolutley continuous measure with respect to $\mathcal{H}^{n-1}$ having $(w-u)\odot\nu$ as a density. In such a case we say that the triple (u,e,p) belongs to the set of admissible plastic strains for the boundary datum w, denoted by $A(w)$. We will extensively use throughout the paper a notion of generalised duality between the stress and the plastic strain, introduced in [6]. We collect some of its most important properties in the next proposition.

Proposition 2.1. *Let $\sigma\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, with $\operatorname{div}\sigma\in L^n(\Omega;\mathbb{R}^n)$, and $\sigma_D\in L^\infty(\Omega;\mathbb{M}^{n\times n}_D)$, and let $p\in\Pi_{\Gamma_0}(\Omega)$. Define for every $\varphi\in C^\infty_c(\Omega)$ the distribution*

$$\langle[\sigma_D:p]_\Omega,\varphi\rangle=-\langle\varphi\,\sigma,e\rangle-\langle\varphi\operatorname{div}\sigma,u\rangle-\langle\sigma,(u\odot\nabla\varphi)\rangle \tag{2.6}$$

where u and e are such that $(u,\,e,\,p)\in A(w)$. Then $[\sigma_D:p]_\Omega\in M_b(\Omega)$. Furthermore, setting

$$[\sigma_D:p]:=[\sigma_D:p]_\Omega\quad\text{on }\Omega$$
$$[\sigma_D:p]:=[\sigma\nu]^\perp_\nu\cdot(w-u)\mathcal{H}^{n-1}\quad\text{on }\Gamma_0\,,$$

then $[\sigma_D:p]\in M_b(\Omega\cup\Gamma_0)$, does not depend on the choice of (u,e,w) such that $(u,e,p)\in A(w)$, and satisfies the following properties:

(i) *if $\sigma\in C^0_0(\Omega\cup\Gamma_0)$, for any $\varphi\in C^0_0(\Omega\cup\Gamma_0)$*

$$\langle[\sigma_D:p],\varphi\rangle=\langle\varphi\,\sigma_D,p\rangle\,,$$

where $\langle\cdot,\cdot\rangle$ is the standard duality between continuous functions and measures;

(ii) *if p^a and $[\sigma_D:p]^a$ are the Radon-Nikodym derivatives of p and $[\sigma_D:p]$, respectively, with respect to $\mathcal{L}^n$, then*

$$[\sigma_D:p]^a=\sigma_D:p^a$$

Moreover, if $p\in L^1(\Omega)$, then $[\sigma_D:p]\ll\mathcal{L}^n$;

(iii) *for every $\varphi\in C^0(\overline\Omega)$ we have*

$$|\langle[(\sigma_k)_D:p],\varphi\rangle|\le\|\sigma_D\|_\infty\|p\|_1\|\varphi\|_\infty\,; \tag{2.7}$$

(iv) *if* $\sigma_k \rightharpoonup \sigma$ *weakly in* $L^2(\Omega)$, $\operatorname{div}\sigma_k \rightharpoonup \operatorname{div}\sigma$ *weakly in* $L^n(\Omega;\mathbb{R}^n)$, *and* $(\sigma_k)_D$ *is uniformly bounded in* $L^\infty(\Omega;\mathbb{M}^{n\times n}_D)$, *then*

$$\langle[(\sigma_k)_D : p],\varphi\rangle \to \langle[\sigma_D : p],\varphi\rangle$$

(v) *assuming that* $-\operatorname{div}\sigma = f$ *in* Ω, *then*

$$\langle[\sigma_D : p],\varphi\rangle + \langle\varphi\sigma, e - Ew\rangle + \langle\sigma,(u-w)\odot\nabla\varphi\rangle = \langle f,\varphi(u-w)\rangle_\Omega \tag{2.8}$$

for every $\varphi \in C^1(\overline{\Omega})$ *such that* $\varphi = 0$ *in a neighborhood of* $\overline{\Gamma}_1$.

Defining the stress-stain duality $\langle\sigma_D, p\rangle$ *by*

$$\langle\sigma_D, p\rangle := [\sigma_D : p](\Omega\cup\Gamma_0)\,,$$

if additionally $[\sigma\nu] = g \in L^\infty(\Gamma_1;\mathbb{R}^n)$, *and* (2.2) *holds, then*

$$\langle\sigma_D, p\rangle + \langle\sigma, e - Ew\rangle = \langle f, u-w\rangle_\Omega + \langle g, u-w\rangle_{\Gamma_1}. \tag{2.9}$$

Proof. We first observe that (2.6) is well defined also in the case of a Lipschitz boundary $\partial\Omega$ since $u \in L^{\frac{n}{n-1}}(\Omega)$ by the Sobolev embedding, and $\sigma \in L^r(\Omega)$ for any $1 \le r < \infty$ by [3, Proposition 6.1]. Then, the first part of the statement can be proved arguing for instance as in [8, Section 2]. The integration by parts formula (2.9) follows from [3, Theorem 6.5]. □

In defining the plastic dissipation, we follow the general point of view of [8] including all the particular cases considered in [2], [7], and [3]. The *dissipation measure* $\mathbb{H}(p)$ associated to a plastic strain $p \in \Pi_{\Gamma_0}(\Omega)$ is deifined as follows:

$$\mathbb{H}(p) = \sup\{[\sigma_D : p] : \sigma \in \mathcal{K}\,,\ \operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^n)\}\,, \tag{2.10}$$

where the supremum is taken in the sense of measures. The plastic dissipation functional $\mathcal{H}(p)$ is then defined as

$$\mathcal{H}(p) := \mathbb{H}(p)(\Omega\cup\Gamma_0)\,. \tag{2.11}$$

The basic properties of $\mathbb{H}$ and $\mathcal{H}$ are collected in the next proposition.

Proposition 2.2. *Assume* (2.1) *and* (2.2), *and let* $\mathcal{K}$ *be a* C^1*-stable closed convex set in* $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ *satisfying* (2.4) *and* (2.5). *Then*

$$\mathcal{H}(p) = \sup\{\langle\sigma_D, p\rangle : \sigma \in \mathcal{K}\,,\ \operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^n)\}\,. \tag{2.12}$$

The measure $\mathbb{H}(p)$ *and the functional* $\mathcal{H}(p)$ *are nonnegative and satisfy*

$$r\,|p| \le \mathbb{H}(p) \le R\,|p| \quad \textit{and} \quad r\|p\|_1 \le \mathcal{H}(p) \le R\|p\|_1\,, \tag{2.13}$$

with $0 < r < R$ *given by* (2.5). *The functional* $\mathcal{H}$ *is positively* 1*-homogeneous and satifies the triangle inequality*

$$\mathcal{H}(p_1 + p_2) \le \mathcal{H}(p_1) + \mathcal{H}(p_2)\,. \tag{2.14}$$

Finally, if p_k *converges weakly* to* p_∞ *in* $M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ *as* $k\to+\infty$ *and there exist bounded sequences* $u_k \in BD(\Omega)$, $w_k \in H^1(\Omega;\mathbb{R}^n)$ *and* $e_k \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ *such* $(u_k, e_k, p_k) \in A(w_k)$, *then*

$$\mathcal{H}(p_\infty) \le \liminf_{k\to+\infty}\mathcal{H}(p_k)\,. \tag{2.15}$$

Remark 2.3. In the case of an homogeneous material, $\mathcal{K}$ is of the form

$$\mathcal{K} := \{\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) : \sigma_D(x) \in K_D \quad \text{a.e.}\}$$

with K_D a convex compact neighborhood of 0 in $\mathbb{M}^{n\times n}_D$. Then it follows from [2, Proposition 2.4] that $\mathbb{H}(p)$ reduces to the usual convex function of measures $H(p) := H\big(\frac{p}{|p|}\big)|p|$, with H the support function of K. In this case, all the properties in the previous proposition can be derived from the related theory in [4] and [10]. In particular, (2.15) simply follows from Reshetnyak's Theorem ([1, Theorem 2.38]).

About the initial and boundary data, we make the following assumptions. For simplicity of exposition, we consider the case of no applied forces, so that the evolution is simply driven by a prescribed boundary displacement

$$\boldsymbol{w} \in H^1_{loc}([0,+\infty); H^1(\Omega;\mathbb{R}^n))\,. \tag{2.16}$$

This is indeed a minor restriction: one can deal with the case of applied forces with similar methods, provided a uniform safe-load condition is introduced. The only point where adapting the existence proof could be not straightforward is outlined in Remark 3.7. We put $w_0 := \boldsymbol{w}(0)$. The initial datum will be a triple $(u_0, e_0, p_0) \in A(w_0)$ such that, setting $\sigma_0 := \mathbb{C}e_0$, one has

$$\begin{cases} \sigma_0 \in \mathcal{K}\,, \\ -\mathrm{div}\,\sigma_0 = 0 \quad \text{in } \Omega\,; \qquad [\sigma_0\nu] = 0 \quad \text{on } \Gamma_1\,. \end{cases} \tag{2.17}$$

We remark that, differently from [8], here we allow for $u_0 \in BD(\Omega)$ instead of $u_0 \in H^1(\Omega)$.

We finally recall the definition of quasistatic evolution. There (and everywhere in what follows), the time derivative $\dot{\boldsymbol{p}}(t)$ of an absolutely continuous map $\boldsymbol{p}\colon [0,T] \to M_b(\Omega)$ has to be understood in the weak* sense of [2, Theorem 7.1], since the target space is not reflexive, but is the dual of a separable Banach space. The same will apply to the time derivative $\dot{\boldsymbol{u}}(t)$ of an absolutely continuous map $\boldsymbol{u}\colon [0,T] \to BD(\Omega)$.

Definition 2.4. Assume (2.1) and (2.2). Consider a C^1-stable closed convex set $\mathcal{K} \subset L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ satisfying (2.4) and (2.5). take $\boldsymbol{w}$ as in (2.16), (u_0, e_0, p_0) as in (2.17), and fix $T > 0$. We say that $(\boldsymbol{u}, \boldsymbol{e}, \boldsymbol{p})$ is a *quasistatic evolution* with prescribed boundary displacement $\boldsymbol{w}$ and initial condition (u_0, e_0, p_0) in the interval $[0,T]$ if

$$\begin{gathered} \boldsymbol{u} \in AC([0,T]; BD(\Omega))\,, \\ \boldsymbol{e} \in H^1([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))\,, \\ \boldsymbol{p} \in AC([0,T]; M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D))\,, \end{gathered} \tag{2.18}$$

and, setting $\boldsymbol{\sigma}(t) := \mathbb{C}\boldsymbol{e}(t)$ for every $t \in [0,+\infty)$, the following conditions are satisfied:

(ev0) *Initial condition*: $(\boldsymbol{u}(0), \boldsymbol{e}(0), \boldsymbol{p}(0)) = (u_0, e_0, p_0)$.

(ev1) *Weak kinematic admissibility*: for every $t \in [0,+\infty)$, we have $(\boldsymbol{u}(t), \boldsymbol{e}(t), \boldsymbol{p}(t)) \in A(\boldsymbol{w}(t))$.

(ev2) *Equilibrium condition and stress constraint*: for every $t \in [0,+\infty)$

$$\begin{cases} \boldsymbol{\sigma}(t) \in \mathcal{K}\,, \\ -\mathrm{div}\,\boldsymbol{\sigma}(t) = 0 \quad \text{in } \Omega\,, \qquad [\boldsymbol{\sigma}(t)\nu] = 0 \quad \text{on } \Gamma_1. \end{cases} \tag{2.19}$$

(ev3) *Maximum plastic work*: for a.e. $t \in [0,+\infty)$

$$\mathcal{H}(\dot{\boldsymbol{p}}(t)) = \langle \boldsymbol{\sigma}_D(t), \dot{\boldsymbol{p}}(t)\rangle\,. \tag{2.20}$$

3. The existence proof

We start by recalling the following Lemma, proved in [8, Theorem 4.9].

Lemma 3.1. *Assume* (2.1) *and* (2.2). *Let* $w \in H^1(\Omega;\mathbb{R}^n)$. *Consider a triple* $(\hat{u}, \hat{e}, \hat{p}) \in A(w)$ *and a* C^1*-stable closed convex set* $\mathcal{K} \subset L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ *satisfying* (2.4) *and* (2.5). *Then, the following two are equivalent:*

(i) $(\hat{u}, \hat{e}, \hat{p})$ *minimizes* $\mathcal{Q}(e) + \mathcal{H}(p - \hat{p})$ *among all* $(u, e, p) \in A(w)$;

(ii) *setting* $\hat{\sigma} := \mathbb{C}\hat{e}$, *it holds*

$$\begin{cases} \hat{\sigma} \in \mathcal{K}\,, \\ -\mathrm{div}\,\hat{\sigma} = 0 \quad \textit{in } \Omega\,, \qquad [\hat{\sigma}\nu] = 0 \quad \textit{on } \Gamma_1. \end{cases}$$

The existence proof we are going to give rests upon the following key lemma.

Lemma 3.2. *Let w_1, $w_2 \in H^1(\Omega;\mathbb{R}^n)$. Consider two triples (u_1, e_1, p_1) and (u_2, e_2, p_2) in $A(w_1)$ and $A(w_2)$, respectively, such that (u_2, e_2, p_2) minimizes $\mathcal{Q}(e)+\mathcal{H}(p-p_1)$ among all $(u, e, p) \in A(w_2)$. Set $\sigma_1 := \mathbb{C}e_1$, and $\sigma_2 := \mathbb{C}e_2$, and assume that $\sigma_1 \in \mathcal{K}$, $\operatorname{div}\sigma_1 = 0$ in Ω, and that $[\sigma_1\nu] = 0$ on Γ_1. Then there exist two positive constants C_1 and C_2 such that*

$$\|e_2 - e_1\|_2 \le C_1\|E(w_2 - w_1)\|_2 \tag{3.1}$$

and

$$\mathcal{H}(p_2 - p_1) \le \frac{1}{2}\langle(\sigma_1 + \sigma_2)_D, p_2 - p_1\rangle + C_2\|E(w_2 - w_1)\|_2^2 \,. \tag{3.2}$$

Proof. By (2.14) it is not difficult to see that (u_2, e_2, p_2) also minimizes $\mathcal{Q}(e) + \mathcal{H}(p - p_2)$ among all $(u, e, p) \in A(w_2)$. Therefore, Lemma 3.1 assures that $\sigma_2 \in \mathcal{K}$, $\operatorname{div}\sigma_2 = 0$ in Ω, and that $[\sigma_2\nu] = 0$ on Γ_1. On the other hand, it is easy to see that the triple $(u_1 + w_2 - w_1, e_1 + E(w_2 - w_1), p_1)$ belongs to $A(w_2)$ so that by minimality

$$\mathcal{Q}(e_2) + \mathcal{H}(p_2 - p_1) \le \mathcal{Q}(e_1 + E(w_2 - w_1)) \,.$$

By (2.3) with $\eta = e_1 + E(w_2 - w_1)$ and $\hat\eta = e_2$, with simple algebraic manipulations and using the simmetry of the tensor $\mathbb{C}$, we have

$$\begin{aligned}\mathcal{Q}(e_1 + E(w_2 - w_1)) - \mathcal{Q}(e_2) &= \frac{1}{2}\langle\sigma_1 + \sigma_2 + \mathbb{C}E(w_2 - w_1), E(w_2 - w_1) + e_1 - e_2\rangle = \\ \frac{1}{2}\langle\sigma_1 + \sigma_2, E(w_2 - w_1) + e_1 - e_2\rangle &+ \frac{1}{2}\langle\mathbb{C}E(w_2 - w_1), E(w_2 - w_1)\rangle + \frac{1}{2}\langle\mathbb{C}E(w_2 - w_1), e_1 - e_2\rangle = \\ \frac{1}{2}\langle\sigma_1 + \sigma_2, E(w_2 - w_1) - (e_2 - e_1)\rangle &+ \mathcal{Q}(E(w_2 - w_1)) + \frac{1}{2}\langle\sigma_1 - \sigma_2, E(w_2 - w_1)\rangle \,.\end{aligned}$$

We therefore get

$$\mathcal{H}(p_2 - p_1) \le \frac{1}{2}\langle\sigma_1 + \sigma_2, E(w_2 - w_1) - (e_2 - e_1)\rangle + \mathcal{Q}(E(w_2 - w_1)) + \frac{1}{2}\langle\sigma_1 - \sigma_2, E(w_2 - w_1)\rangle \,.$$

Observing that $\operatorname{div}(\frac{\sigma_1+\sigma_2}{2}) = 0$ in Ω, and that $[(\frac{\sigma_1+\sigma_2}{2})\nu] = 0$ on Γ_1, by (2.9) and using the continuity of $\mathcal{Q}$ we arrive at

$$\mathcal{H}(p_2 - p_1) \le \frac{1}{2}\langle(\sigma_1 + \sigma_2)_D, p_2 - p_1\rangle + \beta\|E(w_2 - w_1)\|^2 + \frac{1}{2}\langle\sigma_1 - \sigma_2, E(w_2 - w_1)\rangle \tag{3.3}$$

with β the continuity constant of $\mathcal{Q}$. Equivalently,

$$\frac{1}{2}\langle\sigma_2 - \sigma_1, E(w_2 - w_1)\rangle \le \frac{1}{2}\langle(\sigma_1 + \sigma_2)_D, p_2 - p_1\rangle + \beta\|E(w_2 - w_1)\|^2 - \mathcal{H}(p_2 - p_1) \,. \tag{3.4}$$

Now, since $\operatorname{div}(\sigma_2 - \sigma_1) = 0$ in Ω and $[(\sigma_2 - \sigma_1)\nu] = 0$ on Γ_1, by (2.9) and using the coercity of $\mathcal{Q}$ we have

$$\frac{1}{2}\Big(\langle\sigma_2 - \sigma_1, E(w_2 - w_1)\rangle - \langle(\sigma_2 - \sigma_1)_D, p_2 - p_1\rangle\Big) = \frac{1}{2}\langle\sigma_2 - \sigma_1, e_2 - e_1\rangle \ge \alpha\|e_2 - e_1\|_2^2$$

with α the coercivity constant of $\mathcal{Q}$. Inserting (3.4) into this inequality, we arrive at

$$\begin{aligned}\alpha\|e_2 - e_1\|_2^2 &\le \beta\|E(w_2 - w_1)\|^2 + \tfrac{1}{2}\langle(\sigma_1 + \sigma_2)_D, p_2 - p_1\rangle - \tfrac{1}{2}\langle(\sigma_2 - \sigma_1)_D, p_2 - p_1\rangle - \mathcal{H}(p_2 - p_1) \\ &= \beta\|E(w_2 - w_1)\|^2 + \langle(\sigma_1)_D, p_2 - p_1\rangle - \mathcal{H}(p_2 - p_1) \le \beta\|E(w_2 - w_1)\|^2 \,;\end{aligned}$$

indeed, since $\sigma_1 \in \mathcal{K}$, $\langle(\sigma_1)_D, p_2 - p_1\rangle - \mathcal{H}(p_2 - p_1) \le 0$ by (2.12). This gives (3.1). With this, (3.2) follows immediately from (3.3). □

Remark 3.3. Consider the plastic dissipation measures $\mathbb{H}(p_2 - p_1)$ and the stress-strain duality measure $\frac{1}{2}[(\sigma_1 + \sigma_2)_D : (p_2 - p_1)]$. Since by convexity $\frac{1}{2}(\sigma_1 + \sigma_2) \in \mathcal{K}$, by (2.10) we have that

$$\mathbb{H}(p_2 - p_1) - \frac{1}{2}[(\sigma_1 + \sigma_2)_D : (p_2 - p_1)] \ge 0$$

as a measure. From this and (3.2) we get that for every $\varphi \in C_0^0(\Omega \cup \Gamma_0)$ with $0 \le \varphi \le 1$, it holds

$$\langle\mathbb{H}(p_2 - p_1), \varphi\rangle \le \langle\frac{1}{2}[(\sigma_1 + \sigma_2)_D : (p_2 - p_1)], \varphi\rangle + C_2\|E(w_2 - w_1)\|_2^2 \,. \tag{3.5}$$

In particular, $\frac{1}{2}[(\sigma_1+\sigma_2)_D:(p_2-p_1)]$ is a positive measure up to a higher-order remainder.

Let us now fix a sequence of subdivisions $(t_k^i)_{0\le i\le k}$ of the interval $[0,T]$, with

$$0=t_k^0<t_k^1<\dots<t_k^{k-1}<t_k^k=T\,, \tag{3.6}$$

$$\lim_{k\to\infty}\max_{1\le i\le k}(t_k^{i+1}-t_k^i)=0\,. \tag{3.7}$$

For $i=0,\dots,k$ we set $w_k^i:=\boldsymbol{w}(t_k^i)$ and we define u_k^i, e_k^i, and p_k^i by induction. We set $(u_k^0,e_k^0,p_k^0):=(u_0,e_0,p_0)$, which, by assumption, belongs to $A(w_0)$, and for $i=1,\dots,k$ we define (u_k^i,e_k^i,p_k^i) as a solution to the incremental problem

$$\min_{(u,e,p)\in A(w_k^i)}\{\mathcal{Q}(e)+\mathcal{H}(p-p_k^{i-1})\}\,. \tag{3.8}$$

The existence of such a minimizer is immediately obtained thanks to the coercivity and lower semicontinuity in (e,p) of the functional. Moreover, by the triangle inequality (2.14) the triple (u_k^i,e_k^i,p_k^i) is also a solution of the problem

$$\min_{(u,e,p)\in A(w_k^i)}\{\mathcal{Q}(e)+\mathcal{H}(p-p_k^{i})\}\,. \tag{3.9}$$

For $i=0,\dots,k$ we set $\sigma_k^i:=\mathbb{C}e_k^i$. For every $t\in[0,T]$ we define the **piecewise affine** interpolations

$$\begin{gathered}\boldsymbol{u}_k^\Delta(t):=u_k^i+\tfrac{(t-t_k^i)}{t_k^{i+1}-t_k^i}(u_k^{i+1}-u_k^i)\,,\quad \boldsymbol{e}_k^\Delta(t):=e_k^i+\tfrac{(t-t_k^i)}{t_k^{i}-t_k^{i+1}}(e_k^{i+1}-e_k^i)\,,\\ \boldsymbol{p}_k^\Delta(t):=p_k^i+\tfrac{(t-t_k^i)}{t_k^{i+1}-t_k^i}(p_k^{i+1}-p_k^i)\,,\quad \boldsymbol{\sigma}_k^\Delta(t):=\sigma_k^i+\tfrac{(t-t_k^i)}{t_k^{i+1}-t_k^i}(\sigma_k^{i+1}-\sigma_k^i)\,,\\ \boldsymbol{w}_k^\Delta(t):=w_k^i+\tfrac{(t-t_k^i)}{t_k^{i+1}-t_k^i}(w_k^{i+1}-w_k^i)\,,\end{gathered} \tag{3.10}$$

where i is the largest integer such that $t_k^i\le t$. By construction $(\boldsymbol{u}_k^\Delta(t),\boldsymbol{e}_k^\Delta(t),\boldsymbol{p}_k^\Delta(t))\in A(\boldsymbol{w}_k^\Delta(t))$ (resp. $(\dot{\boldsymbol{u}}_k^\Delta(t),\dot{\boldsymbol{e}}_k^\Delta(t),\dot{\boldsymbol{p}}_k^\Delta(t))\in A(\dot{\boldsymbol{w}}_k^\Delta(t))$) for every (resp. a.e.) $t\in[0,T]$. By Lemma 3.1 and (3.10) we have

$$\boldsymbol{\sigma}_k^\Delta(t)\in\mathcal{K},\quad \operatorname{div}\boldsymbol{\sigma}_k^\Delta(t)=0,\quad [\boldsymbol{\sigma}_k^\Delta(t)\nu]=0 \text{ on } \Gamma_1 \tag{3.11}$$

for every t. We will also later consider a particular piecewise constant interpolation of the stress, namely

$$\tilde{\boldsymbol{\sigma}}_k(t):=\frac{1}{2}(\sigma_k^i+\sigma_k^{i+1}) \tag{3.12}$$

where i is the largest integer such that $t_k^i\le t$. As an almost immediate consequence of (3.1) we have the following a priori estimates.

Lemma 3.4. *There exists $C>0$ such that*

$$\int_0^t\left(\|\dot{\boldsymbol{e}}_k^\Delta(s)\|_2^2+\|\dot{\boldsymbol{p}}_k^\Delta(s)\|_1^2+\|\dot{\boldsymbol{u}}_k^\Delta(s)\|_{BD(\Omega)}^2\right)ds\le C\int_0^t\left(\|E\dot{\boldsymbol{w}}_k^\Delta(s)\|_2^2+\|\dot{\boldsymbol{w}}_k^\Delta(s)\|_{L^1(\Gamma_0;\mathbb{R}^n)}^2\right)ds$$

for every $t\in[0,T]$.

Proof. By (3.10) and (3.1) we immediately get

$$\|\dot{\boldsymbol{e}}_k^\Delta(t)\|_2\le C\|E\dot{\boldsymbol{w}}_k^\Delta(t)\|_2 \tag{3.13}$$

for a.e. $t\in[0,T]$. Since $\boldsymbol{w}_k^\Delta$ is equibounded in $H^1([0,T];H^1(\mathbb{R}^n))$, this implies in particular that $\sup_{t\in[0,T]}\|\boldsymbol{e}_k^\Delta(t)\|_2\le C$ for some constant C independent of k.

Since the triple $(u_k^i+w_k^{i+1}-w_k^i,e_k^i+E(w_k^{i+1}-w_k^i),p_k^i)$ belongs to $A(w_k^{i+1})$, we have by minimality and (2.3)

$$\begin{gathered}\mathcal{H}(p_k^{i+1}-p_k^i)\le\mathcal{Q}(e_k^i+E(w_k^{i+1}-w_k^i))-\mathcal{Q}(e_k^{i+1})=\\ \frac{1}{2}\langle\mathbb{C}(e_k^i+E(w_k^{i+1}-w_k^i)+e_k^{i+1}),e_k^i-e_k^{i+1}+E(w_k^{i+1}-w_k^i)\rangle\,;\end{gathered}$$

dividing by $t_k^{i+1} - t_k^i$, using (3.10) and (2.13) we have

$$r\|\dot{\boldsymbol{p}}_k^\Delta(t)\|_1 \le 2\beta\big(\sup_{t\in[0,T]}\|\boldsymbol{e}_k^\Delta(t)\|_2 + \sup_{t\in[0,T]}\|E\boldsymbol{w}_k^\Delta(t)\|_2\big)\big(\|E\dot{\boldsymbol{w}}_k^\Delta(t)\|_2 + \|\dot{\boldsymbol{e}}_k^\Delta(t)\|_2\big)$$

for a.e. $t \in [0,T]$, with β the continuity constant of $\mathcal{Q}$. From this and (3.13) we get

$$\|\dot{\boldsymbol{p}}_k^\Delta(t)\|_1 \le C\|E\dot{\boldsymbol{w}}_k^\Delta(t)\|_2 \tag{3.14}$$

for a.e. $t \in [0,T]$.

Finally, by [10, Proposition 2.4 and Remark 2.5], for every $u \in BD(\Omega)$ there exists a constant C only depending on Ω and Γ_0 such that

$$\|u\|_{L^1(\Omega;\mathbb{R}^n)} \le C\|u\|_{L^1(\Gamma_0;\mathbb{R}^n)} + C\|Eu\|_{M_b(\Omega;\mathbb{M}^{n\times n}_{sym})}\,. \tag{3.15}$$

Since $(\dot{\boldsymbol{u}}_k^\Delta(t), \dot{\boldsymbol{e}}_k^\Delta(t), \dot{\boldsymbol{p}}_k^\Delta(t)) \in A(\dot{\boldsymbol{w}}_k^\Delta(t))$, by the previous inequality, (3.13), and (3.14) we get the existence of a positive constant still denoted by C such that

$$\|\dot{\boldsymbol{u}}_k^\Delta(t)\|_{BD(\Omega)} \le C\big(\|E\dot{\boldsymbol{w}}_k^\Delta(t)\|_2 + \|\dot{\boldsymbol{w}}_k^\Delta(t)\|_{L^1(\Gamma_0;\mathbb{R}^n)}\big)$$

for a.e. $t \in [0,T]$. Together with (3.13) and (3.14), this implies the conclusion. □

In order to prove that the triple $(\boldsymbol{u}_k^\Delta(t), \boldsymbol{e}_k^\Delta(t), \boldsymbol{p}_k^\Delta(t))$ is converging to a quasistatic evolution in perfect plasticity we need the following Lemma, essentially proved in [8].

Lemma 3.5. *Let $\boldsymbol{u}\colon [0,T] \to BD(\Omega)$, $\boldsymbol{e}\colon [0,T] \to L^2(\Omega)$, $\boldsymbol{p}\colon [0,T] \to M_b(\Omega\cup\Gamma_0)$, and $\boldsymbol{w}\colon [0,T] \to H^1(\Omega;\mathbb{R}^n)$ be absolutely continuous mappings such that $(\boldsymbol{u}(t), \boldsymbol{e}(t), \boldsymbol{p}(t)) \in A(\boldsymbol{w}(t))$ for every t. Consider four sequences $\boldsymbol{u}_k\colon [0,T] \to BD(\Omega)$, $\boldsymbol{e}_k\colon [0,T] \to L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $\boldsymbol{p}_k\colon [0,T] \to M_b(\Omega\cup\Gamma_0)$, and $\boldsymbol{w}_k\colon [0,T] \to H^1(\Omega;\mathbb{R}^n)$ of equi-absolutely continuous mappings such that $(\boldsymbol{u}_k(t), \boldsymbol{e}_k(t), \boldsymbol{p}_k(t)) \in A(\boldsymbol{w}_k(t))$ for every t and every $k \in \mathbb{N}$. Assume that*

$$\boldsymbol{u}_k(t) \rightharpoonup \boldsymbol{u}(t) \text{ weakly}^* - BD(\Omega), \quad \boldsymbol{e}_k(t) \rightharpoonup \boldsymbol{e}(t) \text{ weakly} - L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$$
$$\boldsymbol{p}_k(t) \rightharpoonup \boldsymbol{p}(t) \text{ weakly}^* - M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D), \quad \boldsymbol{w}_k(t) \rightharpoonup \boldsymbol{w}(t) \text{ weakly} - H^1(\Omega;\mathbb{R}^n)$$

for every $t \in [0,T]$. Let $\varphi \in C^1(\overline{\Omega})$ such that $\varphi = 0$ in a neighborhood of $\overline{\Gamma}_1$. Then the functions $t \to \langle \mathbb{H}(\dot{\boldsymbol{p}}_k(t)), \varphi\rangle$, $t \to \mathcal{H}(\dot{\boldsymbol{p}}_k(t))$, $t \to \langle \mathbb{H}(\dot{\boldsymbol{p}}(t)), \varphi\rangle$ and $t \to \mathcal{H}(\dot{\boldsymbol{p}}(t))$ all belong to $L^1([0,T))$ and

$$\int_0^T \langle \mathbb{H}(\dot{\boldsymbol{p}}(t)), \varphi\rangle\,dt \le \liminf_{k\to\infty}\int_0^T \langle \mathbb{H}(\dot{\boldsymbol{p}}_k(t)), \varphi\rangle\,dt \quad \text{and} \quad \int_0^T \mathcal{H}(\dot{\boldsymbol{p}}(t))\,dt \le \liminf_{k\to\infty}\int_0^T \mathcal{H}(\dot{\boldsymbol{p}}_k(t))\,dt\,. \tag{3.16}$$

Proof. By [2, Lemma 5.5] we have $(\dot{\boldsymbol{u}}(t), \dot{\boldsymbol{e}}(t), \dot{\boldsymbol{p}}(t)) \in A(\dot{\boldsymbol{w}}(t))$ and $(\dot{\boldsymbol{u}}_k(t), \dot{\boldsymbol{e}}_k(t), \dot{\boldsymbol{p}}_k(t)) \in A(\dot{\boldsymbol{w}}_k(t))$ for a.e. t and every k. Therefore all the involved integrands are well-defined. They also belong to $L^1([0,T])$ by [8, Lemma 4.6]. The first inequality in (3.16) can be deduced arguing exactly as in the proof of [8, Lemma 4.15]. The second one is an easy consequence of the first one, since $\mathbb{H}(\dot{\boldsymbol{p}}(t))$ and $\mathbb{H}(\dot{\boldsymbol{p}}_k(t))$ are positive measures on $\Omega\cup\Gamma_0$ whose total mass is given by $\mathcal{H}(\dot{\boldsymbol{p}}(t))$, and $\mathcal{H}(\dot{\boldsymbol{p}}_k(t))$, respectively. □

We can finally state and prove the announced result.

Theorem 3.6. *Assume* (2.1) *and* (2.2). *Let $\boldsymbol{w}$ be as in* (2.16), *and assume that u_0, e_0, p_0, and σ_0 satisfy* (2.17). *Consider a C^1-stable closed convex set $\mathcal{K} \subset L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ satisfying* (2.4) *and* (2.5), *and the functional $\mathcal{H}$ defined in* (2.11). *Define $(\boldsymbol{u}_k^\Delta(t), \boldsymbol{e}_k^\Delta(t), \boldsymbol{p}_k^\Delta(t))$ as in* (3.10). *Then, up to a subsequence independent of t, $\boldsymbol{u}_k(t) \rightharpoonup \boldsymbol{u}(t)$ weakly* in $BD(\Omega)$, $\boldsymbol{e}_k(t) \rightharpoonup \boldsymbol{e}(t)$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and $\boldsymbol{p}_k(t) \rightharpoonup \boldsymbol{p}(t)$ weakly* in $M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ for every $t \in [0,T]$. Furthermore $(\boldsymbol{u}(t), \boldsymbol{e}(t), \boldsymbol{p}(t))$ is a quasistatic evolution with datum $\boldsymbol{w}$ and initial condition (u_0, e_0, p_0).*

Proof. By Lemma 3.4 the sequence $\boldsymbol{e}_k^\Delta(t)$ is uniformly bounded in $H^1([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$ so that, by standard theory of the Bochner-Sobolev spaces on the real line, possibly taking a (not relabelled) subsequence we get the existence of a function $\boldsymbol{e}(t) \in H^1([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$ such that $\boldsymbol{e}_k^\Delta(t) \rightharpoonup$

$\boldsymbol{e}(t)$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ for every $t\in[0,T]$. Therefore, setting $\boldsymbol{\sigma}(t):=\mathbb{C}\boldsymbol{e}(t)$, we obviously have

$$\boldsymbol{\sigma}^{\Delta}_k(t)\rightharpoonup\boldsymbol{\sigma}(t)\quad\text{weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \tag{3.17}$$

for every $t\in[0,T]$. It follows from (3.11) and the convexity of $\mathcal{K}$ that

$$\boldsymbol{\sigma}(t)\in\mathcal{K},\quad \operatorname{div}\boldsymbol{\sigma}(t)=0,\quad [\boldsymbol{\sigma}(t)\nu]=0 \text{ on } \Gamma_1 \tag{3.18}$$

for every $t\in[0,T]$.

Now, let M_T be the supremum of the integrals $\int_0^T\|\dot{\boldsymbol{p}}^{\Delta}_k(t)\|_1^2\,dt$, which is finite by Lemma 3.4, and set

$$\mathcal{B}_T:=\{p\in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D):\|p\|_1\le\|p_0\|_1+M_T\sqrt{T}\}\,.$$

There exists a distance d_T on $\mathcal{B}_T$ inducing the weak* convergence such that

$$d_T(p,q)\le\|p-q\|_1\quad\text{for every } p,q\in\mathcal{B}_T. \tag{3.19}$$

By Lemma 3.4 we have that $\boldsymbol{p}^{\Delta}_k(t)\in\mathcal{B}_T$ for every $t\in[0,T]$ and every k and the sequence $\boldsymbol{p}^{\Delta}_k(t)$ is equicontinuous on $[0,T]$ in the L^1-norm. A fortiori, equicontinuity holds also with respect to the distance d_T. By the Arzelà-Ascoli Theorem, up to a subsequence

$$\boldsymbol{p}^{\Delta}_k(t)\rightharpoonup\boldsymbol{p}(t)\text{ weakly}^*\text{ in } M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D) \tag{3.20}$$

for every $t\in[0,T]$. Moreover for every $0\le t_1<t_2$

$$\|\boldsymbol{p}(t_2)-\boldsymbol{p}(t_1)\|_1\le\liminf_{k\to\infty}\|\boldsymbol{p}^{\Delta}_k(t_2)-\boldsymbol{p}^{\Delta}_k(t_1)\|_1\,. \tag{3.21}$$

Being $\boldsymbol{p}^{\Delta}_k(t)$ equi-absolutely continuous by Lemma 3.4, this gives that $\boldsymbol{p}(t)\in AC([0,T];\mathbb{M}^{n\times n}_D)$.

The existence of a function $\boldsymbol{u}\in AC([0,T];BD(\Omega)$ such that $\boldsymbol{u}^{\Delta}_k(t)\rightharpoonup\boldsymbol{u}(t)$ weakly* in $BD(\Omega)$ for every t follows now from (3.15) with similar arguments as before. Now, the initial condition (ev0) of Definition 2.4 is trivially satisfied by the triple $(\boldsymbol{u}(t),\boldsymbol{e}(t),\boldsymbol{p}(t))$. Since, for every t, $(\boldsymbol{u}^{\Delta}_k(t),\boldsymbol{e}^{\Delta}_k(t),\boldsymbol{p}^{\Delta}_k(t))\in A(\boldsymbol{w}^{\Delta}_k(t))$, by [2, Lemma 2.1] we infer that $(\boldsymbol{u}(t),\boldsymbol{e}(t),\boldsymbol{p}(t))\in A(\boldsymbol{w}(t))$, so also condition (ev1) is satisfied. Taking into account (3.18), it only remains to show that $\mathcal{H}(\dot{\boldsymbol{p}}(t))=\langle\boldsymbol{\sigma}(t),\dot{\boldsymbol{p}}(t)\rangle$ for a.e. $t\in[0,T]$. Since the inequality

$$\mathcal{H}(\dot{\boldsymbol{p}}(t))\ge\langle\boldsymbol{\sigma}_D(t),\dot{\boldsymbol{p}}(t)\rangle$$

simply follows by (3.18) and the definition of $\mathcal{H}$, it suffices to prove that

$$\int_0^T\mathcal{H}(\dot{\boldsymbol{p}}(t))\,dt\le\int_0^T\langle\boldsymbol{\sigma}_D(t),\dot{\boldsymbol{p}}(t)\rangle\,dt\,. \tag{3.22}$$

To this end, we introduce the piecewise constant interpolation $\tilde{\boldsymbol{\sigma}}_k(t)$ defined in (3.12). By the equiboundedness of $\boldsymbol{\sigma}^{\Delta}_k$ in $H^1([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$ we easily have that

$$\tilde{\boldsymbol{\sigma}}_k-\boldsymbol{\sigma}^{\Delta}_k\to 0 \tag{3.23}$$

strongly in $L^2([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$, so that in particular $\tilde{\boldsymbol{\sigma}}_k(t)\rightharpoonup\boldsymbol{\sigma}(t)$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ for every $t\in[0,T]$. From this, since $E\dot{\boldsymbol{w}}^{\Delta}_k\to E\dot{\boldsymbol{w}}$ strongly in $L^2([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$, by dominated convergence we deduce that

$$\lim_{k\to\infty}\int_0^T\langle\tilde{\boldsymbol{\sigma}}_k(t),E\dot{\boldsymbol{w}}^{\Delta}_k(t)\rangle\,dt=\lim_{k\to\infty}\int_0^T\langle\tilde{\boldsymbol{\sigma}}_k(t),E\dot{\boldsymbol{w}}(t)\rangle\,dt=\int_0^T\langle\boldsymbol{\sigma}(t),E\dot{\boldsymbol{w}}(t)\rangle\,dt\,. \tag{3.24}$$

On the other hand, using (3.23) and the weak lower semicontinuity of $\mathcal{Q}$ one has

$$\begin{gathered}\liminf_{k\to\infty}\int_0^T\langle\tilde{\boldsymbol{\sigma}}_k(t),\dot{\boldsymbol{e}}^{\Delta}_k(t)\rangle\,dt=\liminf_{k\to\infty}\int_0^T\langle\boldsymbol{\sigma}^{\Delta}_k(t),\dot{\boldsymbol{e}}^{\Delta}_k(t)\rangle\,dt=\\ \liminf_{k\to\infty}\mathcal{Q}(\boldsymbol{e}^{\Delta}_k(t))-\mathcal{Q}(e_0)\ge\mathcal{Q}(\boldsymbol{e}(t))-\mathcal{Q}(e_0)=\int_0^T\langle\boldsymbol{\sigma}(t),\dot{\boldsymbol{e}}(t)\rangle\,dt\,.\end{gathered} \tag{3.25}$$

Putting together (3.24) and (3.25), by means of the integration by parts formula (2.9), (3.11) and (3.18) we get to

$$\begin{aligned}\limsup_{k\to\infty}\int_0^T \langle(\tilde{\boldsymbol{\sigma}}_k)_D(t),\dot{\boldsymbol{p}}_k^\Delta(t)\rangle\,dt &= \limsup_{k\to\infty}\int_0^T \langle\tilde{\boldsymbol{\sigma}}_k(t),E\dot{\boldsymbol{w}}_k^\Delta(t)-\dot{\boldsymbol{e}}_k^\Delta(t)\rangle\,dt\\ &\le \int_0^T\langle\boldsymbol{\sigma}(t),E\dot{\boldsymbol{w}}(t)-\dot{\boldsymbol{e}}(t)\rangle\,dt=\int_0^T\langle\boldsymbol{\sigma}_D(t),\dot{\boldsymbol{p}}(t)\rangle\,dt\,.\end{aligned} \tag{3.26}$$

Finally, by positive 1-homogeneity of $\mathcal{H}$, (3.3) and (3.10) we get

$$\mathcal{H}(\dot{\boldsymbol{p}}_k^\Delta(t))\le\langle(\tilde{\boldsymbol{\sigma}}_k)_D(t),\dot{\boldsymbol{p}}_k^\Delta(t)\rangle+C_2(t_k^{i+1}-t_k^i)\|E\dot{\boldsymbol{w}}_k^\Delta(t)\|_2^2$$

for a.e. $t_k^i\le t\le t_k^{i+1}$. Therefore

$$\int_0^T\mathcal{H}(\dot{\boldsymbol{p}}_k^\Delta(t))\,dt\le\int_0^T\langle\tilde{\boldsymbol{\sigma}}_k(t),\dot{\boldsymbol{p}}_k^\Delta(t)\rangle\,dt+C_2\max_{1\le i\le k}(t_k^{i+1}-t_k^i)\int_0^T\|E\dot{\boldsymbol{w}}_k^\Delta(t)\|_2^2\,dt$$

and (3.22) follows now from (3.7), (3.16), and (3.26). □

Remark 3.7. While considering a nonzero volume force $\boldsymbol{f}(t)$, under the usual assumptions, does not really change the proof of the previous theorem, a minor difficulty has to be overcome in the case where a nonzero surface force $\boldsymbol{g}(t)\in H^1([0,T];L^\infty(\Gamma_1;\mathbb{R}^n))$ is given. Actually, in this case, (3.26) may be no longer satisfied. Indeed, considering the piecewise affine interpolations $\boldsymbol{g}_k^\Delta$ of $\boldsymbol{g}$, from the condition $[\boldsymbol{\sigma}_k^\Delta(t)\nu]=\boldsymbol{g}_k^\Delta(t)$ on Γ_1 and integrating by parts according to (2.9), an additional term

$$\int_0^T\langle\boldsymbol{g}_k^\Delta(t),\dot{\boldsymbol{u}}_k^\Delta(t)-\dot{\boldsymbol{w}}_k^\Delta(t)\rangle_{\Gamma_1}\,dt$$

appears. This latter is in general neither continuous nor semicontinuous. Roughly speaking, this is because the trace of $\boldsymbol{u}_k^\Delta(t)$ on Γ_1 may be not compact in $L^1(\Gamma_1;\mathbb{R}^n)$, although $\boldsymbol{u}_k^\Delta(t)$ is weakly* compact in $BD(\Omega)$. The trace operator is indeed not continuous with respect to weak* convergence.

The proof is to be modified as follows: by an integration by parts argument using (2.8), we first prove that

$$\limsup_{k\to\infty}\int_0^T\langle[(\tilde{\boldsymbol{\sigma}}_k)_D(t):\dot{\boldsymbol{p}}_k^\Delta(t)],\varphi\rangle\,dt\le\int_0^T\langle[\boldsymbol{\sigma}_D(t):\dot{\boldsymbol{p}}(t)],\varphi\rangle\,dt$$

for every $\varphi\in C^1(\overline{\Omega})$ such that $\varphi=0$ in a neighborhood of $\overline{\Gamma}_1$. By (3.5) and (3.16) this gives

$$\int_0^T\langle\mathbb{H}(\dot{\boldsymbol{p}}(t)),\varphi\rangle\,dt\le\int_0^T\langle[\boldsymbol{\sigma}_D(t):\dot{\boldsymbol{p}}(t)],\varphi\rangle\,dt\,.$$

Considering now a sequence $\varphi_j\in C^\infty(\overline{\Omega})$, with $0\le\varphi_j\le1$ in $\overline{\Omega}$ and $\varphi_j=0$ in a neighborhood of $\overline{\Gamma}_1$, such that $\varphi_j(x)\to1$ for every $x\in\Omega\cup\Gamma_0$, we eventually get (3.22) by (2.7) and the dominated convergence theorem.

Acknowledgments

This note originates from some interesting discussions with Prof. G. Dal Maso and Prof. G. Francfort during a minicourse on "Variational Evolution in Plasticity" held by the second one at SISSA, Trieste, April 2013. The two of them, as well as the hospitality of the institute, are therefore gratefully acknowledged. This material is based on work supported by the European Research Council under Grant No. 290888 "Quasistatic and Dynamic Evolution Problems in Plasticity and Fracture".

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