REGULARITY OF RANDOM ATTRACTORS FOR STOCHASTIC REACTION-DIFFUSION EQUATIONS ON UNBOUNDED DOMAINS

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Abstract. The existence of a unique random attractors in $H^1(\mathbb{R}^n)$ for a stochastic reaction-diffusion equation with time-dependent external forces is proved. Because of the presence of both random and non-autonomous deterministic terms, we use a new theory of random attractors which is introduced in [B. Wang, Journal of Differential Equations, 253 (2012), 1544-1583] instead of the usual one. The asymptotic compactness of solutions in $H^1(\mathbb{R}^n)$ is established by combining "tail estimate" technique and some new estimates on solutions. This work improves some recent results about the regularity of random attractors for stochastic reaction diffusion equations.

1. Introduction

Stochastic differential equations arise from many physical systems when random spatio-temporal forcing is taken into account. To study long time behavior of solutions of stochastic differential equations, one use the concept of so-called random attractor, which is an extension of the theory of attractors for deterministic equations. The concept of random attractor for random dynamical systems was introduced in [7, 8, 9] and has been studied extensively in [5, 6, 10, 17] and references therein.

In this paper, we study the following stochastic reaction diffusion equation on $\mathbb{R}^n$,

$$\begin{cases} du + [-\Delta u + f(x,u) + \lambda u]dt = g(t,x)dt + hd\omega, \\ u|_{t=\tau} = u_\tau, \end{cases} \tag{1.1}$$

where $\lambda > 0$, $g \in L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^n))$ and ω is a two-sided real-valued Wiener process on a probability space which will be specified later.

The long time behavior of stochastic reaction diffusion equations with additive noise is studied by many mathematicians in both cases of bounded and unbounded domains. For example, in the case of bounded domains, the authors in [3, 11] proved the existence of random attractors in $L^2(\Omega)$ and $L^p(\Omega)(p > 2)$ respectively. In the case of unbounded domains, the authors in [6] obtained a random attractor for stochastic reaction diffusion equation in $L^2(\mathbb{R}^n)$; $L^p(\mathbb{R}^n)$-random attractor was shown in [24]. Similar results for reaction diffusion equations with multiplicative noise can be found in [12, 20].

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We emphasize that, although in the deterministic case, the higher regularity of attractors for reaction diffusion equations (or its generalized form) is well understood (see e.g. [2, 14, 16, 25]), in the stochastic case, since random pertubations are taken into account, the regularity of random attractors is less known. Up to the best of our knowledge, there are only three results in this direction [1, 22, 23], and all of them dealt with bounded domains and autonomous external forces. The main contribution of this paper is showing the higher regularity of random attractors for (1.1) in unbounded domains without restriction on the growth of nonlinearity. Another interesting feature of the present paper is that we consider stochastic reaction diffusion equation not only with random perturbation but also non-autonomous deterministic terms.

To study equation (1.1), we assume the following hypothesis

(F) For all $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$, the nonlinearity $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfies

$$f(x,s)s \geq \alpha_1 |s|^p + \psi_1(x), \tag{1.2}$$

$$|f(x,s)| \leq \alpha_2 |s|^{p-1} + \psi_2(x), \tag{1.3}$$

$$\frac{\partial f}{\partial s}(x,s) \geq -\ell, \tag{1.4}$$

$$\left| \frac{\partial f}{\partial x}(x,s) \right| \leq \psi_3(x), \tag{1.5}$$

where α_1, α_2, ℓ are positive constants, $p > 2$, $\psi_1 \in L^2(\mathbb{R}^n) \cap L^{p/2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $\psi_2 \in L^2(\mathbb{R}^n) \cap L^{\frac{p}{p-1}}(\mathbb{R}^n)$ and $\psi_3 \in L^2(\mathbb{R}^n)$;

(G) The external force $g \in L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^n))$ satisfies

$$\int_{-\infty}^{\tau} e^{\lambda s} \|g(s)\|^2_{L^2(\mathbb{R}^n)} ds < +\infty \text{ for all } \tau \in \mathbb{R}. \tag{1.6}$$

This implies that, for all $\tau \in \mathbb{R}$,

$$\lim_{k \to +\infty} \int_{-\infty}^{\tau} e^{\lambda s} \int_{|x| \geq k} |g(s,x)|^2 dx ds = 0; \tag{1.7}$$

(H) $h \in L^{2p-2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap H^2(\mathbb{R}^n) \cap W^{2,p}(\mathbb{R}^n)$.

There are some difficulties in studying problem (1.1). First, the equation has not only random but also non-autonomous deterministic terms, thus we have to adapt a new concept of random attractors, which is introduced recently in [19]. Second, the continuity of the random dynamical system with respect to (1.1) in $H^1(\mathbb{R}^n)$, which plays an essential role in obtaining random attractors, is now known. This difficulty can be solved by weakening the concept of usual continuity to norm-to-weak continuity [25] or quasi-continuity [11]. In this work, we use the idea in [24] to overcome the lack of continuity. Roughly speaking, once the attractor in $L^2(\mathbb{R}^n)$ is shown, one can prove the existence of attractors in $H^1(\mathbb{R}^n)$ by checking only the absorption and the asymptotic compactness of the corresponding random dynamical system. The third difficulty, also the main difficulty in this work, is the unboundedness of $\mathbb{R}^n$. This property makes Sobolev embeddings are only continuous but not compact. To get through of it, we use a technique so-called "tail estimates", which is initiated and developed by Wang in both deterministic [15, 16] and stochastic cases [17, 18, 19]. It's worth noticing that, compared to the work [19], the tail-estimates technique we use here is somehow different. Firstly, the tail of solutions which we want to estimate is now in $H^1(\mathbb{R}^n)$ (not in $L^2(\mathbb{R}^n)$ as in [19]), thus computations are more complicated and we have to adapt some new estimates of solutions. The idea originally comes from [4] wherein the authors showed the existence of uniform attractors for reaction diffusion equations on unbounded domains. Secondly, in [19], the authors get the asymptotic compactness of solutions in $L^2(\Omega)$ directly by using $H^1(\Omega) \hookrightarrow L^2(\Omega)$ compactly (Ω is bounded); but, in our case, since we want to establish the asymptotic compactness of solutions in $H^1(\Omega)$, we cannot use such an embedding because the solutions belong only to H^1 and has no higher regularity. This is solved in this paper by using the idea in [4, 17, 18], which uses the eigenfunctions of negative Laplacian to divide solutions into two parts, where one part is bounded in an $m-$dimensional space while the other one tends to zero as $m \to \infty$. Combining the tail estimates and asymptotic compactness in bounded domains, we imply that the random dynamical system is asymptotically compact in $H^1(\mathbb{R}^n)$ and thus obtain the existence of a random attractor in $H^1(\mathbb{R}^n)$.

The rest of the paper is organized as follows: In the next section we give basic concepts related to random attractors for random dynamical systems, and then recall some known results for the random dynamical system which generated by (1.1). The last section is devoted the proof of the main result, the regularity of random attractor for (1.1).

2. Preliminaries

2.1. **Random attractors.** In this section, we recall some basic notions on random attractors for random dynamical systems which are applicable to differential equations with both non-autonomous deterministic and random terms. For further details, readers are referred to [19].

Let Ω_1 be a non-empty set, $(\Omega_2, \mathcal{F}_2, P)$ be a probability space, and $(X, \|\cdot\|)$ be a Banach space with Borel $\sigma-$algebra $\mathcal{B}(X)$.

Suppose that there are two groups $\{\theta_1(t)\}_{t\in\mathbb{R}}$ and $\{\theta_2(t)\}_{t\in\mathbb{R}}$ acting on Ω_1 and Ω_2, respectively. More precisely, $\theta_1 : \mathbb{R} \times \Omega_1 \to \Omega_1$ is a mapping such that $\theta_1(0, \cdot)$ is the identity on Ω_1, $\theta_1(s + t, \cdot) = \theta_1(t, \cdot) \circ \theta_1(s, \cdot)$ for all $t, s \in \mathbb{R}$. Similarly, $\theta_2 : \mathbb{R} \times \Omega_2 \to \Omega_2$ is a $(\mathcal{B}(R) \times \mathcal{F}_2, \mathcal{F}_2)$-measurable mapping such that $\theta_2(0, \cdot)$ is the identity on Ω_2, $\theta_2(s + t, \cdot) = \theta_2(t, \cdot) \circ \theta_2(s, \cdot)$ for all $t, s \in \mathbb{R}$ and $\theta_2(t, \cdot)P = P$ for all $t \in \mathbb{R}$. We will write $\theta_1(t, \cdot)$ and $\theta_2(t, \cdot)$ as $\theta_{1,t}$ and $\theta_{2,t}$ for short. In the sequel, we will call both $(\Omega_1, \{\theta_{1,t}\}_{t\in\mathbb{R}})$ and $(\Omega_2, \mathcal{F}_2, P, \{\theta_{2,t}\}_{t\in\mathbb{R}})$ parametric dynamical systems.

Definition 2.1. *Let $(\Omega_1, \{\theta_{1,t}\}_{t\in\mathbb{R}})$ and $(\Omega_2, \mathcal{F}_2, P, \{\theta_{2,t}\}_{t\in\mathbb{R}})$ be parametric dynamical systems. A mapping $\Phi : \mathbb{R}^+ \times \Omega_1 \times \Omega_2 \times X \to X$ is called a continuous random dynamical system (RDS for short) on X over $(\Omega_1, \{\theta_{1,t}\}_{t\in\mathbb{R}})$ and $(\Omega_2, \mathcal{F}_2, P, \{\theta_{2,t}\}_{t\in\mathbb{R}})$ if for all $\omega_1 \in \Omega_1, \omega_2 \in \Omega_2$ and $t, \tau \in \mathbb{R}^+$, the following conditions are satisfied:*

(i) $\Phi(\cdot, \omega_1, \cdot, \cdot) : \mathbb{R}^+ \times \Omega_2 \times X \to X$ *is* $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F}_2 \times \mathcal{B}(X), \mathcal{B}(X))$*-measurable;*
(ii) $\Phi(0, \omega_1, \omega_2, \cdot)$ *is the identity on* X*;*
(iii) $\Phi(t + \tau, \omega_1, \omega_2, \cdot) = \Phi(t, \theta_{1,\tau}\omega_1, \theta_{2,\tau}\omega_2, \cdot) \circ \Phi(\tau, \omega_1, \omega_2, \cdot)$*;*
(iv) $\Phi(t, \omega_1, \omega_2, \cdot) : X \to X$ *is continuous.*

Hereafter, we always denote by $\mathcal{D}$ a collection of some families of non-empty subsets of X which are parameterized by $\Omega_1 \times \Omega_2$, that is,

$$\mathcal{D} = \{D(\omega_1, \omega_2) \subset X \text{ is bounded}, \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}.$$

Definition 2.2. *Let $K = \{K(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\} \in \mathcal{D}$. Then K is called a $\mathcal{D}$-pullback absorbing set for Φ if for all $\omega_1 \in \Omega_1, \omega_2 \in \Omega_2$ and for every $B \in \mathcal{D}$, there exists $T = T(B, \omega_1, \omega_2) > 0$ such that*

$$\Phi(t, \theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2, B(\theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2)) \subset K(\omega_1, \omega_2) \text{ for all } t \geq T.$$

Definition 2.3. *An RDS Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for all $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, the sequence*

$$\{\Phi(t_n, \theta_{1,-t_n}\omega_1, \theta_{2,-t_n}\omega_2, x_n)\}_{n=1}^{\infty} \text{ has a convergent subsequence in } X$$

provided $t_n \to +\infty$, and $x_n \in B(\theta_{1,-t_n}\omega_1, \theta_{2,-t_n}\omega_2)$ with $\{B(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\} \in \mathcal{D}$.

Definition 2.4. *Let $\mathcal{A} = \{\mathcal{A}(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\} \in \mathcal{D}$. Then $\mathcal{A}$ is called a $\mathcal{D}$-pullback attractor for Φ if the following conditions hold:*

(i) *$\mathcal{A}$ is measurable with respect to the P-completion of $\mathcal{F}_2$ in Ω_2 and $\mathcal{A}(\omega_1, \omega_2)$ is compact for all $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$.*

(ii) *$\mathcal{A}$ is invariant, that is, for every $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$,*

$$\Phi(t, \omega_1, \omega_2, \mathcal{A}(\omega_1, \omega_2)) = \mathcal{A}(\theta_{1,t}\omega_1, \theta_{2,t}\omega_2), \quad \forall t \geq 0.$$

(iii) *$\mathcal{A}$ attracts every member of $\mathcal{D}$, that is, for every $B = \{B(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\} \in \mathcal{D}$ and for every $\omega_1 \in \Omega_1, \omega_2 \in \Omega_2$,*

$$\lim_{t \to +\infty} d(\Phi(t, \theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2, B(\theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2)), \mathcal{A}(\omega_1, \omega_2)) = 0,$$

where d is the Hausdorff semi-distance in X,

$$d(A, B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|_X, \forall A, B \subset X.$$

Theorem 2.1. [19] *Let Φ be a continuous RDS on X over $(\Omega_1, \{\theta_{1,t}\}_{t \in \mathbb{R}})$ and $(\Omega_2, \mathcal{F}_2, P, \{\theta_{2,t}\}_{t \in \mathbb{R}})$. Then Φ has a $\mathcal{D}$-pullback attractor $\mathcal{A}$ in $\mathcal{D}$ if Φ is $\mathcal{D}$-pullback asymptotically compact in X and Φ has $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$. The $\mathcal{D}$-pullback attractor $\mathcal{A}$ is unique and is given by, for each $\omega_1 \in \Omega_2, \omega_2 \in \Omega_2$,*

$$\mathcal{A}(\omega_1, \omega_2) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \Phi(t, \theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2, K(\theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2))}.$$

The continuity of Φ in Theorem 2.1 is a crucial condition to prove the invariance of the attractor. In some cases (as in the case of the present paper), this kind of continuity is unknown. This difficulty can be solved by some weaker kinds of continuity like norm-to-weak continuity (see [25]) or quasi-continuity (see [11]). However, if we know about the existence of a random attractor in another space, which satisfies an "easy" condition, then all we have to check are the existence of an absorbing set and the pullback asymptotic compactness of the RDS. The following is an alternative version of [24, Theorem 2.8].

Theorem 2.2. *Let X, Y be two Banach spaces satisfying that: if $x_n \to x_0$ in X and $x_n \to y_0$ in Y, then $x_0 = y_0$. Let Φ be an RDS in X and an RDS in Y. Then Φ has a unique pullback attractor $\mathcal{A}_Y$ in Y iff*

(i) *Φ has a pullback attractor $\mathcal{A}_X$ in X;*

(ii) *Φ has a random absorbing set $\mathcal{K}_Y = \{K_Y(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$ in $X \cap Y$; and*

(iii) *Φ is pullback asymptotically compact in Y.*

Moreover, $\mathcal{A}_Y$ is defined as

$$A_Y(\omega_1,\omega_2) = \bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau}\Phi(t,\theta_{1,-t}\omega_1,\theta_{2,-t}\omega_2,K_Y(\theta_{1,-t}\omega_1,\theta_{2,-t}\omega_2))}^{Y}.$$

Throughout this paper, we denote by $\|\cdot\|$ and $|\cdot|_p$ the norms in $L^2(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ respectively. For a Banach space X, we will denote by $\|\cdot\|_X$ its norm. We also denote by C an arbitrary constant, which can be different from line to line (even in the same line).

2.2. **Stochastic reaction-diffusion with additive noise on $\mathbb{R}^n$.** In this subsection, we show that problem (1.1) generates a RDS Φ and give some known results for Φ. More details can be seen from [19].

Given $\tau\in\mathbb{R}$ and $t>\tau$, consider the following non-autonomous reaction-diffusion equation defined on $\mathbb{R}^n$,

$$du + (-\Delta u + \lambda u + f(x,u))dt = g(x,t)dt + hd\omega, \tag{2.1}$$

subject to initial data

$$u(x,\tau) = u_\tau(x), \quad x\in\mathbb{R}^n.$$

Let

$$\Omega = \{\omega\in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0\}.$$

Let $\mathcal{F}$ be the Borel σ-algebra induced by the compact-open topology of Ω, and P be the corresponding Wiener measure on $(\Omega,\mathcal{F})$. Define a group $\{\theta_t\}_{t\in\mathbb{R}}$ acting on $(\Omega,\mathcal{F},P)$ by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega\in\Omega, \quad t\in\mathbb{R}.$$

The $(\Omega,\mathcal{F},P,\{\theta_t\}_{t\in\mathbb{R}})$ is a parametric dynamical system.

First, we transfer the stochastic equation into a corresponding non-autonomous deterministic one. Given $\omega\in\Omega$, denote by

$$z(\omega) = -\lambda\int_{-\infty}^{0} e^{\lambda\tau}\omega(\tau)d\tau. \tag{2.2}$$

Then it is easy to check that the random variable z given by (2.2) is a stationary solution of the one-dimensional Ornstein-Uhlenbeck equation:

$$dz(\theta_t\omega) + \lambda z(\theta_t\omega) = d\omega. \tag{2.3}$$

Note that the random variable $|z(\omega)|$ is tempered, that is $\lim_{t\to+\infty} e^{-\lambda t}|z(\theta_{-t}\omega)| = 0$, and $z(\omega)$ is $P-a.e.$ continuous. Therefore, it follows from [3, Proposition 4.3.3] that there exists a tempered function $r(\omega)>0$ such that

$$|z(\omega)|^2 + |z(\omega)|^p + |z(\omega)|^{2p-2} \leq r(\omega), \quad \forall\omega\in\Omega, \tag{2.4}$$

where $r(\omega)$ satisfies, for $P-a.e.$ $\omega\in\Omega$,

$$r(\theta_t\omega) \leq e^{\frac{\lambda}{2}|t|}r(\omega),\ t\in\mathbb{R}. \tag{2.5}$$

This implies that

$$p(\theta_t\omega) = |z(\theta_t\omega)|^2 + |z(\theta_t\omega)|^p + |z(\theta_t\omega)|^{2p-2} \leq e^{\frac{\lambda}{2}|t|}r(\omega). \tag{2.6}$$

We seek a solution to the following equation

$$v_t - \Delta v + \lambda v + f(x, v + hz(\theta_t\omega)) = g(t,x) + z(\theta_t\omega)\Delta h, \tag{2.7}$$

with inital data $v(\tau) = v_\tau = u_\tau - hz(\omega)$, for $t > \tau, x \in \mathbb{R}^n$. Since (2.7) is a deterministic equation, following the arguments of [13], one can show that under assumptions (1.2)-(1.5), for each $\omega \in \Omega, \tau \in \mathbb{R}$ and $v_\tau \in L^2(\mathbb{R}^n)$, equation (2.7) has a unique solution $v(\cdot, \tau, \omega, v_\tau) \in C([\tau, \infty), L^2(\mathbb{R}^n)) \cap L^2(\tau, \tau + T; H^1(\mathbb{R}^n))$ with $v(\tau, \tau, \omega, v_\tau) = v_\tau$ for every $T > 0$. Furthermore, for each $t \geq \tau$, $v(t, \tau, \omega, v_\tau)$ is $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^n)))$-measurable in $\omega \in \Omega$ and continuous in v_τ with respect to the norm of $L^2(\mathbb{R}^n)$. Let $u(t, \tau, \omega, u_\tau) = v(t, \tau, \omega, v_\tau) + hz(\theta_t \omega)$ with $u_\tau = v_\tau + hz(\omega)$. It follows from (2.3) and (2.7) that u is a solution of problem (1.1) which is continuous in both $t \geq \tau$ and $u_\tau \in L^2(\mathbb{R}^n)$ and is $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^n)))$-measurable in $\omega \in \Omega$. We now define an RDS $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ as follows

$$\Phi(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau) = v(t + \tau, \tau, \theta_{-\tau}\omega, v_\tau) + hz(\theta_t \omega), \tag{2.8}$$

where $v_\tau = u_\tau - hz(\omega)$. This implies that

$$\begin{aligned} \Phi(t, \tau - t, \theta_{-t}\omega, u_{\tau-t}) &= u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}) \\ &= v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}) + hz(\omega) \end{aligned} \tag{2.9}$$

where $v_{\tau-t} = u_{\tau-t} - hz(\theta_{-t}\omega)$.

Suppose $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is a family of bounded non-empty subsets of $L^2(\mathbb{R}^n)$ such that, for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{s \to -\infty} e^{\lambda s} \|D(\tau + s, \theta_s \omega)\|^2 = 0, \tag{2.10}$$

where λ is the constant in equation (1.1) and $\|B\| = \sup\{\|x\| : x \in B\}$ for a bounded subset B of $L^2(\mathbb{R}^n)$. Denote by $\mathcal{D}_\lambda$ the collection of all families of bounded empty non-empty subsets of $L^2(\mathbb{R}^2)$ which satisfies (2.10), that is,

$$\mathcal{D}_\lambda = \{D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ satisfies } (2.10)\}.$$

The following result is obtained in [19].

Theorem 2.3. *Under hypothesis* $(\mathbf{F}) - (\mathbf{G}) - (\mathbf{H})$*, the RDS* Φ *generated by* (1.1) *possesses a unique random attractor* $\mathcal{A}_2 = \{A_2(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_\lambda$ *which is compact in* $L^2(\mathbb{R}^n)$ *and attracts all members of* $\mathcal{D}_\lambda$ *in the topology of* $L^2(\mathbb{R}^n)$.

3. Regularity of random attractors

In this section, we prove that the random attractor $\mathcal{A}_2$ in Theorem 2.3 is actually compact in $H^1(\mathbb{R}^n)$ and attracts every member of $\mathcal{D}_\lambda$ in the topology of $H^1(\mathbb{R}^n)$. The strategy is verify three conditions in Theorem 2.2. Condition (i) follows from Theorem 2.3, while the condition (ii) is obtained from Lemma 3.1. We will prove (iii) by adapting the method so-called "tail estimates". Roughly speaking, the idea is to divide

$$\mathbb{R}^n = \{B_K = \{x \in \mathbb{R}^n : |x| \leq K\}\} \cup \{\mathbb{R}^n \backslash B_K\}$$

and then prove that:

- Φ is asymptotically compact in $H^1(B_K)$; and
- Φ can be as small as possible in $H^1(\mathbb{R}^n \backslash B_K)$.

These two points directly imply the asymptotic compactness of Φ in $H^1(\mathbb{R}^n)$.

The following estimates are borrowed from [19]

Lemma 3.1. *Suppose* (1.2)-(1.7) *hold. Then for every* $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_\lambda$, *there exists* $T = T(\tau,\omega,D) > 0$ *such that for all* $t \geq T$, *the solution* v *of* (2.7) *satisfies*

$$\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2_{H^1(\mathbb{R}^n)} \leq R(\tau,\omega), \tag{3.1}$$

and

$$\int_{\tau-t}^{\tau} e^{-\lambda(\tau-s)} \left(\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2_{H^1(\mathbb{R}^n)} + |u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|_p^p\right) ds \leq R(\tau,\omega), \tag{3.2}$$

where $u_{\tau-t} \in D(\tau-t,\theta_{-t}\omega), v_{\tau-t} = u_{\tau-t} - hz(\theta_{-t}\omega)$ *and*

$$R(\tau,\omega) = C\left(1 + r(\omega) + e^{-\lambda\tau}\int_{-\infty}^{\tau} e^{\lambda s}\|g(s)\|^2 ds\right),$$

with constant C *is independent of* t, τ, ω *and* D.

Proof. By [19, Lemma 4.1], we have

$$\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2_{H^1(\mathbb{R}^n)} \leq C\left(1 + e^{-\lambda\tau}\int_{-\infty}^{\tau} e^{\lambda s}\|g(s)\|^2 ds + \int_{-\infty}^{0} e^{\lambda s}|z(\theta_s\omega)|^p ds\right) \tag{3.3}$$

Using (2.6), we get

$$\int_{-\infty}^{0} e^{\lambda s}|z(\theta_s\omega)|^p ds \leq \int_{-\infty}^{0} e^{\lambda s}e^{-\frac{\lambda}{2}s} r(\omega) ds = r(\omega)\int_{-\infty}^{0} e^{\frac{\lambda}{2}s} ds \leq \frac{2}{\lambda} r(\omega). \tag{3.4}$$

Combining (3.3) and (3.4) we obtain (3.1). The proof of (3.2) is very similar, so we omit it. □

Lemma 3.2. *For any* $\tau \in \mathbb{R}$, *any* $D = \{D(\tau,\omega)\} \in D_\lambda$, *we can choose* $T > 0$ *such that*

$$\int_{\tau-1}^{\tau} |v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|_p^p ds \leq CR(\tau,\omega)$$

for all $t \geq T$ *and all* $v_{\tau-t} + hz(\theta_{-t}\omega) = u_{\tau-t} \in D(\tau-t,\theta_{-t}\omega)$, *where* $R(\tau,\omega)$ *is in Lemma 3.1.*

Proof. Using Lemma 3.1, we have

$$\begin{aligned}
& e^{-\lambda}\int_{\tau-1}^{\tau} |u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|_p^p ds \\
\leq & \int_{\tau-1}^{\tau} e^{-\lambda(\tau-s)}|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|_p^p ds \\
\leq & \int_{\tau-t}^{\tau} e^{-\lambda(\tau-s)}|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|_p^p ds \\
& \leq R(\tau,\omega),
\end{aligned} \tag{3.5}$$

thus

$$\int_{\tau-1}^{\tau} |u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|_p^p ds \leq e^{\lambda} R(\tau,\omega). \tag{3.6}$$

Since $u(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}) = v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}) + hz(\theta_{s-\tau}\omega)$, it follows from (3.6) that

$$\begin{aligned}
&\int_{\tau-1}^{\tau} |v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|_p^p \\
&\le \int_{\tau-1}^{\tau} 2^p(|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|_p^p + |z(\theta_{s-\tau}\omega)|^p|h|_p^p)ds \\
&\le 2^p e^{\lambda} R(\tau,\omega) + 2^p|h|_p^p \int_{-1}^{0} |z(\theta_s\omega)|^p ds \\
&\le 2^p e^{\lambda} R(\tau,\omega) + 2^p|h|_p^p e^{\lambda} \int_{-\infty}^{0} e^{\lambda s}|z(\theta_s\omega)|^p ds \\
&\le CR(\tau,\omega).
\end{aligned} \tag{3.7}$$

□

Lemma 3.3. *For any $\tau\in\mathbb{R}$, any $D=\{D(\tau,\omega)\}\in\mathcal{D}_\lambda$, there exists $T\ge 1$ such that*

$$|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|_p^p \le CR(\tau,\omega), \tag{3.8}$$

for all $t\ge T$, $\omega\in\Omega$ and all $v_{\tau-t}+hz(\theta_{-t}\omega)=u_{\tau-t}\in D(\tau-t,\theta_{-t}\omega)$.

Proof. We multiply (2.7) by $v|v|^{p-2}$ then integrate over $\mathbb{R}^n$ to obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}|v|_p^p - \int_{\mathbb{R}^n}\Delta v|v|^{p-2}vdx + \lambda|v|_p^p + \int_{\mathbb{R}^n} f(x,v+hz(\theta_t\omega))|v|^{p-2}vdx \\
&\quad = \int_{\mathbb{R}^n}(g(t)+z(\theta_t\omega)\Delta h)|v|^{p-2}vdx.
\end{aligned} \tag{3.9}$$

Integrating by parts the second term on the left hand side of (2.3) yields

$$-\int_{\mathbb{R}^n}\Delta v|v|^{p-2}vdx = (p-1)\int_{\mathbb{R}^n}|v|^{p-2}|\nabla v|^2dx \ge 0. \tag{3.10}$$

Using (1.3) and Young's inequality, we have

$$f(x,v+hz(\theta_t\omega))v \ge \frac{\alpha_1}{2}|u|^p - |\psi_1| - \frac{1}{2}|\psi_2|^2 - C(|hz(\theta_t\omega)|^2 + |hz(\theta_t\omega)|^p). \tag{3.11}$$

Hence, by Holder's and Young's inequalities

$$\begin{aligned}
&\int_{\mathbb{R}^n} f(x,u)v|v|^{p-2}dx \\
&\ge \int_{\mathbb{R}^n}\left(\frac{\alpha_1}{2^p}|v|^p - |\psi_1| - \frac{1}{2}|\psi_2|^2 - C(|hz(\theta_t\omega)|^2 + |hz(\theta_t\omega)|^p)\right)|v|^{p-2}dx \\
&\ge \frac{\alpha_1}{2^p}|v|_{2p-2}^{2p-2} - \frac{\lambda}{4p}|v|_p^p - C|\psi_1|_{p/2}^{p/2} - \frac{\lambda}{4p}|v|_p^p - C|\psi_2|_p^p \\
&\quad - \frac{\lambda}{2p}|v|_p^p - C|z(\theta_t\omega)|^p|h|_p^p - \frac{\alpha_2}{2^{p+1}}|v|_{2p-2}^{2p-2} - C|z(\theta_t\omega)|^{2p-2}|h|_{2p-2}^{2p-2} \\
&\ge \frac{\alpha_1}{2^{p+1}}|v|_{2p-2}^{2p-2} - \frac{\lambda}{p}|v|_p^p - C(1+|z(\theta_t\omega)|^p + |z(\theta_t\omega)|^{2p-2}).
\end{aligned} \tag{3.12}$$

On the other hand, the right hand side of (3.9) is bounded by

$$\|g(t)\||v|_{2p-2}^{p-1} + |z(\theta_t\omega)|\|\Delta h\||v|_{2p-2}^{p-1} \le \frac{\alpha_1}{2^{p+2}}|v|_{2p-2}^{2p-2} + C(\|g(t)\|^2 + |z(\theta_t\omega)|^2). \tag{3.13}$$

Combining (3.9)-(3.13) gives us

$$\begin{aligned}&\frac{d}{dt}|v|_p^p+\lambda|v|_p^p+\frac{p\alpha_1}{2^{p+2}}|v|_{2p-2}^{2p-2}\\ \le{}& C(1+\|g(t)\|^2+|z(\theta_t\omega)|^2+|z(\theta_t\omega)|^p+|z(\theta_t\omega)|^{2p-2})\\ \le{}& C(1+\|g(t)\|^2+|z(\theta_t\omega)|^{2p-2}).\end{aligned}\tag{3.14}$$

We integrate (3.14) from s to τ, where $s\in(\tau-1,\tau)$, then replace ω by $\theta_{-\tau}\omega$ to get, in particular

$$\begin{aligned}&|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|_p^p\\ \le{}& |v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|_p^p+C(\tau-s+\int_s^\tau\|g(r)\|^2dr+\int_s^\tau|z(\theta_{r-\tau}\omega)|^{2p-2}dr)\\ \le{}& |v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|_p^p+C(1+\int_{\tau-1}^\tau\|g(r)\|^2dr+\int_{\tau-1}^\tau|z(\theta_{r-\tau}\omega)^{2p-2}dr)\end{aligned}\tag{3.15}$$

Integrating (3.15) on $(\tau-1,\tau)$ with respect to s and using Lemma 3.2, we get

$$\begin{aligned}&|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|_p^p\\ \le{}& \int_{\tau-1}^\tau|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|_p^pds+C(1+\int_{\tau-1}^\tau\|g(r)\|^2dr+\int_{-1}^0|z(\theta_r\omega)|^{2p-2}dr)\\ \le{}& CR(\tau,\omega)+C(1+e^\lambda e^{-\lambda\tau}\int_{-\infty}^\tau e^{\lambda s}\|g(s)\|^2ds+e^\lambda\int_{-\infty}^0 e^{\lambda s}|z(\theta_s\omega)|^{2p-2}ds)\\ \le{}& CR(\tau,\omega)+C\int_{-\infty}^0 e^{\lambda s}e^{-\frac{\lambda}{2}s}r(\omega)ds\\ \le{}& CR(\tau,\omega)+\frac{2C}{\lambda}r(\omega)\\ \le{}& CR(\tau,\omega).\end{aligned}\tag{3.16}$$

This completes the proof. □

Proposition 3.4. *Assume that hypothesis* $(\mathbf{F})-(\mathbf{G})-(\mathbf{H})$ *hold. Then the random dynamical system* Φ *has a random absorbing set* $\{B_0(\tau,\omega)\}$ *in* $L^p(\mathbb{R}^n)\cap H^1(\mathbb{R}^n)$.

Proof. By Lemmas 3.1 and 3.3, with the help of (2.4) and (2.9), we obtain

$$\begin{aligned}&\|\Phi(t,\tau-t,\theta_{-t}\omega,u_{\tau-t})\|_{H^1(\mathbb{R}^n)}^2\\ ={}& \|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})+hz(\omega)\|_{H^1(\mathbb{R}^n)}^2\\ \le{}& 2\left(\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|_{H^1(\mathbb{R}^n)}^2+|z(\omega)|^2\|h\|_{H^1(\mathbb{R}^n)}^2\right)\\ \le{}& C\left(1+r(\omega)+e^{-\lambda\tau}\int_{-\infty}^\tau e^{\lambda s}\|g(s)\|^2ds\right).\end{aligned}\tag{3.17}$$

Similarly,

$$\begin{aligned}&|\Phi(t,\tau-t,\theta_{-t}\omega,u_{\tau-t})|_p^p\\ \le{}& 2^p\left(|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|_p^p+|z(\omega)|^p|h|_p^p\right)\\ \le{}& C\left(1+r(\omega)+e^{-\lambda\tau}\int_{-\infty}^\tau e^{\lambda s}\|g(s)\|^2ds\right).\end{aligned}\tag{3.18}$$

Denote by, for $\tau \in \mathbb{R}, \omega \in \Omega$,

$$B_0(\tau,\omega) = \left\{ u \in L^p(\mathbb{R}^n) \cap H^1(\mathbb{R}^n) : |u|_p^p + \|u\|^2_{H^1(\mathbb{R}^n)} \le CR(\tau,\omega) \right\}. \tag{3.19}$$

From (3.17) and (3.18) we have, for any $D = \{D(\tau,\omega)\} \in \mathcal{D}_\lambda$, there exists $T > 0$ such that

$$\Phi(t, \tau - t, \theta_{-t}\omega, u_{\tau-t}) \in B_0(\tau,\omega), \forall t \ge T, \tag{3.20}$$

for all $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$. The proof is complete. □

Lemma 3.5. *Let $\tau \in \mathbb{R}, \omega \in \Omega$ and $D = \{D(\tau,\omega)\} \in \mathcal{D}_\lambda$. Then, for any $\epsilon > 0$, there exist $T = T(\tau,\omega,D,\epsilon) > 0$ and $K = K(\tau,\omega,D,\epsilon) > 0$ such that*

$$\int_{|x|\ge K} |v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})|^2 dx \le \epsilon, \tag{3.21}$$

and

$$\int_{\tau-t}^{\tau} e^{-\lambda(\tau-s)} \int_{|x|\ge K} (|\nabla v(s,\tau-t,\theta_{-\tau}\omega, v_{\tau-t})|^2 + |u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^p) dx ds \le \epsilon, \tag{3.22}$$

for all $t \ge T$, all $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ and $v_{\tau-t} = u_{\tau-t} - hz(\theta_{-t}\omega)$.

Proof. The proof is very similar to that of [19, Lemma 4.4]. Here, we prove an addition estimate (3.22). We define a smooth function $\rho : \mathbb{R}^+ \to [0,1]$ such that $\rho|_{[0,1]} = 0$ and $\rho|_{[2,+\infty)} = 1$. Notice that ρ' is bounded in $\mathbb{R}^+$ and $\rho'(s) = 0$ for all $s \in [0,1] \cup [2,+\infty)$. In the sequel, for the sake of brevity, we will write $\rho(\cdot)$ instead of $\rho\left(\frac{|x|^2}{k^2}\right)$. Multiplying (2.7) by $\rho(\cdot)v$ then integrating on $\mathbb{R}^n$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \rho(\cdot)|v|^2 dx - \int_{\mathbb{R}^n} \rho(\cdot) v\Delta v dx \\ &+ \lambda \int_{\mathbb{R}^n} \rho(\cdot)|v|^2 dx + \int_{\mathbb{R}^n} \rho(\cdot) v f(x, v + z(\theta_t\omega)) dx \\ &= \int_{\mathbb{R}^n} \rho(\cdot) v(g(t,x) + z(\theta_t\omega)\Delta h) v dx. \end{aligned} \tag{3.23}$$

Integrating by parts, we have

$$\begin{aligned} -\int_{\mathbb{R}^n} \rho(\cdot) v\Delta v dx &= \int_{\mathbb{R}^n} \nabla u \cdot \nabla(\rho(\cdot)v) dx \\ &= \int_{\mathbb{R}^n} \rho(\cdot)|\nabla v|^2 dx + \int_{\mathbb{R}^n} v\rho'\left(\frac{|x|^2}{k^2}\right)\frac{2x}{k^2}\cdot\nabla v dx. \end{aligned} \tag{3.24}$$

Since $\rho'(s) = 0$ for $s \in [0,1] \cup [2,+\infty)$,

$$\begin{aligned} \left|\int_{\mathbb{R}^n} v\rho'\left(\frac{|x|^2}{k^2}\right)\frac{2x}{k^2}\cdot\nabla v dx\right| &\le \int_{k\le|x|\le\sqrt{2}k} \left|v\rho'\left(\frac{|x|^2}{k^2}\right)\right| \frac{2|x|}{k^2}|\nabla v| dx \\ &\le \frac{C}{k}\int_{k\le|x|\le\sqrt{2}k} |v|\,|\nabla v| dx \\ &\le \frac{C}{k}\int_{\mathbb{R}^n} |v|\,|\nabla v| dx \\ &\le \frac{C}{k}\|v\|^2_{H^1(\mathbb{R}^n)}. \end{aligned} \tag{3.25}$$

For the nonlinear term, by (1.2)-(1.5), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \rho(\cdot) v f(x,u)dx \\
&= \int_{\mathbb{R}^n} \rho(\cdot) f(x,u)udx - \int_{\mathbb{R}^n} \rho(\cdot) f(x,u) z(\theta_t\omega)hdx \\
&\geq \alpha_1 \int_{\mathbb{R}^n} \rho(\cdot)|u|^p dx - \int_{\mathbb{R}^n} \rho(\cdot)\psi_1(x)dx \\
&-\alpha_2 \int_{\mathbb{R}^n} \rho(\cdot)|u|^{p-1}|z(\theta_t\omega)h|dx - \int_{\mathbb{R}^n} \rho(\cdot)|\psi_2||z(\theta_t\omega)h|dx \\
&\geq \frac{\alpha_1}{2} \int_{\mathbb{R}^n} \rho(\cdot)|u|^p dx - \int_{\mathbb{R}^n} \rho(\cdot)\psi_1(x)dx \\
&- \int_{\mathbb{R}^n} \rho(\cdot)|\psi_2|^2 dx - c_2 \int_{\mathbb{R}^n} \rho(\cdot)(|z(\theta_t\omega)h|^p + |z(\theta_t\omega)h|^2)dx,
\end{aligned} \tag{3.26}
$$

where we have used Young's inequality at the last step. Using Cauchy's inequality,

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \rho(\cdot) v(g(t,x) + z(\theta_t\omega)h)dx \\
&\leq \frac{1}{\lambda} \int_{\mathbb{R}^n} \rho(\cdot)|g(t,x)|^2 dx + \frac{1}{\lambda}|z(\theta_t\omega)|^2 \int_{\mathbb{R}^n} \rho(\cdot)|\Delta h|^2 dx + \frac{\lambda}{2} \int_{\mathbb{R}^n} \rho(\cdot)|v|^2 dx.
\end{aligned} \tag{3.27}
$$

From (3.23)-(3.27), we obtain

$$
\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^n} \rho(\cdot)|v|^2 dx + \lambda \int_{\mathbb{R}^n} \rho(\cdot)|v|^2 + \alpha_1 \int_{\mathbb{R}^n} \rho(\cdot)|u|^p dx + 2\int_{\mathbb{R}^n} \rho(\cdot)|\nabla v|^2 dx \\
&\leq \frac{2c_1}{k}\|v\|^2_{H^1(\mathbb{R}^n)} + 2\int_{\mathbb{R}^n} \rho(\cdot)(|\psi_1| + |\psi_2|^2)dx + \frac{2}{\lambda}\int_{\mathbb{R}^n} \rho(\cdot)|g(t,x)|^2 dx \\
&\quad + c_3 \int_{\mathbb{R}^n} \rho(\cdot)(|hz(\theta_t\omega)|^2 + |hz(\theta_t\omega)|^p + |z(\theta_t\omega)\Delta h|^2)dx.
\end{aligned} \tag{3.28}
$$

Since $\psi_1 \in L^1(\mathbb{R}^n), \psi_2 \in L^2(\mathbb{R}^n), h \in H^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, we can choose k large enough such that

$$
2\int_{\mathbb{R}^n} \rho(\cdot)(|\psi_1| + |\psi_2|^2)dx \leq 2\int_{|x|\geq k} (|\psi_1| + |\psi_2|^2)dx \leq \epsilon_0,
$$

and

$$
\begin{aligned}
&c_3 \int_{\mathbb{R}^n} \rho(\cdot)(|hz(\theta_t\omega)|^2 + |hz(\theta_t\omega)|^p + |z(\theta_t\omega)\Delta h|^2)dx \\
&\leq c_3 \left(|z(\theta_t\omega)|^2 \int_{\mathbb{R}^n} \rho(\cdot)(|h|^2 + |\Delta h|^2)dx + |z(\theta_t\omega)|^p \int_{\mathbb{R}^n} \rho(\cdot)|h|^p dx \right) \\
&\leq c_3 \left(|z(\theta_t\omega)|^2 \int_{|x|\geq k} (|h|^2 + |\Delta h|^2)dx + |z(\theta_t\omega)|^p \int_{|x|\geq k} |h|^p dx \right) \\
&\leq c_3\epsilon_0(|z(\theta_t\omega)|^2 + |z(\theta_t\omega)|^p) \\
&\leq c_4\epsilon_0(1 + |z(\theta_t\omega)|^p).
\end{aligned}
$$

Using these estimates in (3.28) and noticing that $\frac{c_1}{k} \le \epsilon_0$ for large enough k, we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\cdot)|v|^2dx+\lambda\int_{\mathbb{R}^n}\rho(\cdot)|v|^2dx+c_5\int_{\mathbb{R}^n}\rho(\cdot)(|u|^p+|\nabla v|^2)dx\\&\le \epsilon_0\|v\|^2_{H^1(\mathbb{R}^n)}+\frac{2}{\lambda}\int_{\mathbb{R}^n}\rho(\cdot)|g(t,x)|^2dx+c_6\epsilon_0(1+|z(\theta_t\omega)|^p).\end{aligned}\tag{3.29}$$

Multiplying (3.29) by $e^{\lambda t}$ then integrating from $\tau - t$ to τ, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}\rho(\cdot)|v(\tau,\tau-t,\omega,v_{\tau-t})|^2dx\\&\quad+c_5\int_{\tau-t}^{\tau}e^{-\lambda(\tau-s)}\int_{\mathbb{R}^n}\rho(\cdot)(|u(s,\tau-t,\omega,u_{\tau-t})|^p+|\nabla v(s,\tau-t,\omega,v_{\tau-t})|^2)dxds\\&\le e^{-\lambda t}\|v_{\tau-t}\|^2+\epsilon_0\int_{\tau-t}^{\tau}e^{-\lambda(\tau-s)}\|v(s,\tau-t,\omega,v_{\tau-t})\|^2_{H^1(\mathbb{R}^n)}ds\\&\quad+\frac{2}{\lambda}\int_{\tau-t}^{\tau}e^{-\lambda(\tau-s)}\int_{\mathbb{R}^n}\rho(\cdot)|g(s,x)|^2dxds\\&\quad+c_6\epsilon_0\int_{\tau-t}^{\tau}e^{-\lambda(\tau-s)}(1+|z(\theta_s\omega)|^p)ds.\end{aligned}\tag{3.30}$$

In (3.30), we replace ω by $\theta_{-\tau}\omega$ and use the fact $\rho|_{[0,1]} = 0$ and $\rho \ge 0$ to obtain

$$\begin{aligned}&\int_{|x|\ge\sqrt{2}k}|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2dx\\&+c_5\int_{\tau-t}^{\tau}e^{-\lambda(\tau-s)}\int_{|x|\ge\sqrt{2}k}(|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^p+|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2)dxds\\&\le e^{-\lambda t}\|v_{\tau-t}\|^2+\epsilon_0\int_{\tau-t}^{\tau}e^{-\lambda(\tau-s)}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2_{H^1(\mathbb{R}^n)}ds\\&+\frac{2}{\lambda}\int_{\tau-t}^{\tau}e^{-\lambda(\tau-s)}\int_{|x|\ge k}|g(s,x)|^2dxds\\&+c_6\epsilon_0\int_{\tau-t}^{\tau}e^{-\lambda(\tau-s)}(1+|z(\theta_{s-\tau}\omega)|^p)ds.\end{aligned}\tag{3.31}$$

We claim that all terms on the right hand side of (3.31) can be as small as possible. Firstly, since $v_{\tau-t} = u_{\tau-t} - hz(\theta_{-t}\omega)$, we have

$$e^{-\lambda t}\|v_{\tau-t}\|^2 \le 2e^{-\lambda t}\|u_{\tau-t}\|^2+2\|h\|^2e^{-\lambda t}|z(\theta_{-t}\omega)|^2\to 0\tag{3.32}$$

as $t \to +\infty$ because $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ and $|z(\omega)|$ is tempered. Secondly, by Proposition 3.1, we find that

$$\epsilon_0\int_{\tau-t}^{\tau}e^{-\lambda(\tau-s)}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2_{H^1(\mathbb{R}^n)}ds\le\epsilon_0R(\tau,\omega).$$

Thirdly, from (1.7),

$$\frac{\lambda}{2}\int_{\tau-t}^{\tau}e^{-\lambda(\tau-s)}\int_{|x|\ge k}|g(s,x)|^2dxds\le e^{-\lambda\tau}\int_{-\infty}^{\tau}e^{\lambda s}\int_{|x|\ge k}|g(s,x)|^2dxds\to 0\tag{3.33}$$

as $k \to +\infty$. Finally,

$$\begin{aligned}
& c_6\epsilon_0 \int_{\tau-t}^{\tau} e^{-\lambda(\tau-s)}(1+|z(\theta_{s-\tau}\omega)|^p)ds \\
& \le c_6\epsilon_0 e^{-\lambda\tau}\int_{-\infty}^{\tau} e^{\lambda s}ds + c_6\epsilon_0\int_{-\infty}^{0} e^{\lambda s}|z(\theta_s\omega)|^p ds \\
& \le \epsilon_0\frac{c_6}{\lambda} + c_6\epsilon_0\int_{-\infty}^{0} e^{\frac{\lambda}{2}s}r(\omega)ds \\
& \le \epsilon_0\left(\frac{c_6}{\lambda}+\frac{2c_6 r(\omega)}{\lambda}\right)
\end{aligned} \tag{3.34}$$

From (3.31)-(3.34), we can get our desired estimates (3.21) and (3.22) by choosing an appropriate small ϵ_0. □

We are now going to show that it's enough to consider solutions which start from an absorbing set to prove the asymptotic compactness of the RDS.

Lemma 3.6. *Assume that $\{B_0(\tau,\omega)\} \in \mathcal{D}_\lambda$ is an absorbing set for Φ. Assume also that for any $\tau \in \mathbb{R}, \omega \in \Omega$, $t_n \to +\infty$ and $x_n \in B_0(\tau - t_n, \theta_{-t_n}\omega)$, the sequence $\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\}$ is precompact. Then Φ is asymptotically compact.*

Proof. Take an arbitrary random set $\{D(\tau,\omega)\} \in \mathcal{D}_\lambda$, a sequence $t_n \to +\infty$ and $y_n \in D(\tau - t_n, \theta_{-t_n}\omega)$. We have to prove that $\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, y_n)\}$ is precompact.

Since $\{B_0(\tau,\omega)\}$ is a random absorbing of Φ, then there exists $T > 0$ such that, for all $\omega \in \Omega$,

$$\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subset B_0(\tau,\omega) \text{ for all } t \ge T. \tag{3.35}$$

Because $t_n \to +\infty$, we can choose $n_1 \ge 1$ such that $t_{n_1} - 1 \ge T$. In (3.35), replace (t,τ,ω) by $(t_{n_1} - 1, \tau - 1, \theta_{-1}\omega)$, we find that

$$\begin{aligned}
x_1 &:= \Phi(t_{n_1} - 1, \tau - t_{n_1}, \theta_{-t_{n_1}}\omega, y_{n_1}) \\
&\in \Phi(t_{n_1} - 1, \tau - t_{n_1}, \theta_{-t_{n_1}}\omega, D(\tau - t_{n_1}, \theta_{-t_{n_1}}\omega)) \subset B_0(\tau - 1, \theta_{-1}\omega).
\end{aligned} \tag{3.36}$$

Similarly, we can choose a subsequence $\{n_k\}$ of $\{n\}$ such that $n_1 < n_2 < \ldots < n_k \to +\infty$ such that

$$x_k := \Phi(t_{n_k} - k, \tau - t_{n_k}, \theta_{-t_{n_k}}\omega, y_{n_k}) \in B_0(\tau - k, \theta_{-k}\omega). \tag{3.37}$$

Hence, by assumption of Φ, we conclude that

$$\text{the sequence } \{\Phi(k, \tau - k, \theta_{-k}\omega, x_k)\} \text{ is precomact.} \tag{3.38}$$

On the other hand, by (3.37)

$$\begin{aligned}
\Phi(k, \tau - k, \theta_{-k}\omega, x_k) &= \Phi(k, \tau - k, \theta_{-k}\omega, \Phi(t_{n_k} - k, \tau - t_{n_k}, \theta_{-t_{n_k}}\omega, y_{n_k})) \\
&= \Phi(t_{n_k}, \tau - t_{n_k}, \theta_{-t_{n_k}}\omega, y_{n_k}), \forall k \ge 1.
\end{aligned} \tag{3.39}$$

Combining (3.38), (3.39) we obtain that the sequence $\{\Phi(t_{n_k}, \tau - t_{n_k}, \theta_{-t_{n_k}}\omega, y_{n_k})\}$ is precompact, thus $\{\Phi(t_n, \tau - t_n, \theta_{t_n}\omega, y_n)\}$ is precompact. This completes the proof. □

Making use of Lemma 3.6, from now on, we consider initial $u_{\tau-t} \in B_0(\tau - t, \theta_{-t}\omega)$ instead of $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ for some $\{D(\tau,\omega)\} \in \mathcal{D}_\lambda$. The following estimates of $f(u)$ and v_t very useful to obtain the "tail estimate" of solutions in H^1.

Lemma 3.7. *Let $\tau \in \mathbb{R}$. Then, there exists $T > 0$ such that, for any $t \geq T$*

$$\int_{\tau-t}^{\tau} e^{-\lambda(\tau-s)}\|f(x, u(s, \tau-t, \theta_{-\tau}\omega, u_{\tau-t}))\|^2 ds \leq CR(\tau, \omega), \tag{3.40}$$

for all $u_{\tau-t} \in B_0(\tau-t, \theta_{-t}\omega)$.

Proof. Multiply (3.14) by $e^{\lambda t}$ then integrate from $\tau - t$ to τ, we get

$$\begin{aligned} &e^{\lambda\tau}|v(\tau, \tau-t, \omega, v_{\tau-t})|_p^p + C\int_{\tau-t}^{\tau} e^{\lambda s}|v(s, \tau-t, \omega, v_{\tau-t})|_{2p-2}^{2p-2} ds \\ &\leq e^{\lambda(\tau-t)}|v_{\tau-t}|_p^p + C\int_{\tau-t}^{\tau} e^{\lambda s}\left(1 + \|g(s)\|^2 + |z(\theta_s\omega)|^{2p-2}\right) ds \\ &\leq e^{\lambda(\tau-t)}|v_{\tau-t}|_p^p + C\left(e^{\lambda\tau} + \int_{-\infty}^{\tau} e^{\lambda s}\|g(s)\|^2 ds + \int_{\tau-t}^{\tau} e^{\lambda s}|z(\theta_s\omega)|^{2p-2} ds\right). \end{aligned} \tag{3.41}$$

Replacing ω by $\theta_{-\tau}\omega$, we find that

$$\begin{aligned} &C\int_{\tau-t}^{\tau} e^{-\lambda(\tau-s)}|v(s, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})|_{2p-2}^{2p-2} ds \\ &\leq e^{-\lambda t}|v_{\tau-t}|_p^p + C\left(1 + e^{-\lambda\tau}\int_{-\infty}^{\tau} e^{\lambda s}\|g(s)\|^2 ds + \int_{\tau-t}^{\tau} e^{\lambda(s-\tau)}|z(\theta_{s-\tau}\omega)|^{2p-2} ds\right) \\ &\leq e^{-\lambda t}|v_{\tau-t}|_p^p + C\left(1 + e^{-\lambda\tau}\int_{-\infty}^{\tau} e^{\lambda s}\|g(s)\|^2 ds + \int_{-\infty}^{0} e^{\lambda s}|z(\theta_s\omega)|^{2p-2} ds\right) \\ &\leq Ce^{-\lambda t}\left(|u_{\tau-t}|_p^p + |h|_p^p|z(\theta_{-t}\omega)|^p\right) + C\left(1 + r(\omega) + e^{-\lambda\tau}\int_{-\infty}^{\tau} e^{\lambda s}\|g(s)\|^2 ds\right). \end{aligned} \tag{3.42}$$

Since $u_{\tau-t} \in B_0(\tau-t, \theta_{-t}\omega)$ and $|z(\omega)|$ is tempered, when $t \to +\infty$,

$$Ce^{-\lambda t}\left(|u_{\tau-t}|_p^p + |h|_p^p|z(\omega)|^p\right) = Ce^{-\lambda t}|u_{\tau-t}|_p^p + Ce^{-\lambda t}|z(\omega)|^p \to 0. \tag{3.43}$$

Hence, we can get from (3.42) a sufficient large $T > 0$ such that, for all $t \geq T$,

$$\int_{\tau-t}^{\tau} e^{-\lambda(\tau-s)}|v(s, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})|_{2p-2}^{2p-2} ds \leq CR(\tau, \omega). \tag{3.44}$$

We use (1.3) and (3.44) to deduce that

$$\begin{aligned} &\int_{\tau-t}^{\tau} e^{-\lambda(\tau-s)}\|f(x, u(s, \tau-t, \theta_{-\tau}\omega, u_{\tau-t}))\|^2 dx ds \\ &\leq C\int_{\tau-t}^{\tau} e^{-\lambda(\tau-s)}\left(|u(s, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})|_{2p-2}^{2p-2} + \|\psi_2\|^2\right) ds \\ &\leq C\int_{\tau-t}^{\tau} e^{-\lambda(\tau-s)}\left(|v(s, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})|_{2p-2}^{2p-2} + |h|_{2p-2}^{2p-2}|z(\theta_{s-\tau}\omega)|^{2p-2} + \|\psi_2\|^2\right) ds \\ &\leq C\int_{\tau-t}^{\tau} e^{-\lambda(\tau-s)}|v(s, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})|_{2p-2}^{2p-2} ds \\ &\quad + C|h|_{2p-2}^{2p-2}\int_{-t}^{0} e^{\lambda s}|z(\theta_s\omega)|^{2p-2} ds + \frac{C\|\psi_2\|^2}{\lambda} \\ &\leq CR(\tau, \omega). \end{aligned} \tag{3.45}$$

The proof is complete. □

Lemma 3.8. *For any fixed $\tau \in \mathbb{R}$, there exists $T > 0$ satisfying*

$$\int_{\tau-t}^{\tau} e^{-\lambda(\tau-s)} \|v_t(s, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 ds \leq CR(\tau, \omega), \tag{3.46}$$

for all $t \geq T, v_{\tau-t} + hz(\theta_{-t}\omega) = u_{\tau-t} \in B_0(\tau-t, \theta_{-t}\omega)$.

Proof. We multiply equation (2.7) by v_t then integrate over $\mathbb{R}^n$ to get

$$\|v_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + \int_{\mathbb{R}} v_t f(x, v + hz(\theta_t\omega))dx = (g(t,x) + z(\theta_t\omega)\Delta h, v_t). \tag{3.47}$$

The Cauchy inequality gives us

$$\left|\int_{\mathbb{R}^n} v_t f(x,u)dx\right| \leq \frac{1}{4}\|v_t\|^2 + \|f(x,u)\|^2, \tag{3.48}$$

$$(g(t) + z(\theta_t\omega)\Delta h, v_t) \leq \frac{1}{4}\|v_t\|^2 + 2\|g(t)\|^2 + 2|z(\theta_t\omega)|^2\|\Delta h\|^2. \tag{3.49}$$

From these above estimates, we obtain

$$\frac{d}{dt}\|\nabla v\|^2 + \|v_t\|^2 \leq 2\|f(x,u)\|^2 + 4\|g(t)\|^2 + 4|z(\theta_t\omega)|^2\|\Delta h\|^2. \tag{3.50}$$

Multiplying (3.50) by $e^{\lambda t}$ then integrating from $\tau - t$ to τ and replacing ω by $\theta_{-\tau}\omega$, we find that

$$\begin{aligned}
&\|\nabla v(\tau, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 + \int_{\tau-t}^{\tau} e^{-\lambda(\tau-s)}\|v_t(s, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 ds \\
&\quad \leq e^{-\lambda t}\|\nabla v_{\tau-t}\|^2 \\
&\quad +2\int_{\tau-t}^{\tau} e^{-\lambda(\tau-s)}\|f(x, u(s, \tau-t, \theta_{-\tau}\omega, u_{\tau-t}))\|^2 ds \\
&\quad +4e^{-\lambda\tau}\int_{\tau-t}^{\tau} e^{\lambda s}\|g(s)\|^2 ds + 4\|\Delta h\|^2 \int_{\tau-t}^{\tau} e^{\lambda(s-\tau)}|z(\theta_{s-\tau}\omega)|^2 ds \\
&\quad \leq 2e^{-\lambda t}\left(\|\nabla u_{\tau-t}\|^2 + \|\nabla h\|^2|z(\theta_{-t}\omega)|^2\right) \\
&\quad +2\int_{\tau-t}^{\tau} e^{-\lambda(\tau-s)}\|f(x, u(s, \tau-t, \theta_{-\tau}\omega, u_{\tau-t}))\|^2 ds \\
&\quad +4e^{-\lambda\tau}\int_{-\infty}^{\tau} e^{\lambda s}\|g(s)\|^2 ds + 4\|\Delta h\|^2 \int_{-\infty}^{0} e^{\lambda s}|z(\theta_s\omega)|^2 ds \\
&\quad \leq CR(\tau, \omega)
\end{aligned} \tag{3.51}$$

where we have employed Lemma 3.7, $u_{\tau-t} \in B_0(\tau - t\theta_{-t}\omega)$ and the fact that $|z(\omega)|$ is tempered. Thus, we can obtain (3.46) and complete the proof. □

We are now ready to prove the tail estimates of solutions in $H^1(\mathbb{R}^n)$.

Proposition 3.9. *For any fixed $\tau \in \mathbb{R}$ and any $\epsilon > 0$. There exists $T > 0$ and $K > 0$ such that*

$$\int_{|x|\geq K} |\nabla v(\tau, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})|^2 dx \leq \epsilon, \tag{3.52}$$

for all $t \geq T$ and all $v_{\tau-t} + hz(\theta_{-t}\omega) = u_{\tau-t} \in B_0(\tau-t, \theta_{-t}\omega)$.

Proof. Multiplying (2.7) by $-\rho(\cdot)\Delta v$, where ρ is the same as Lemma 3.5, then integrating by parts, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\cdot)|\nabla v|^2dx+\lambda\int_{\mathbb{R}^n}\rho(\cdot)|\nabla v|^2dx+\int_{\mathbb{R}^n}\rho(\cdot)|\Delta v|^2dx\\
&=\frac{-2}{k^2}\int_{\mathbb{R}^n}x\nabla v v_t\rho'(\cdot)dx-\frac{2\lambda}{k^2}\int_{\mathbb{R}^n}x\nabla v v\rho'(\cdot)dx\\
&-\int_{\mathbb{R}^n}\nabla u\frac{\partial f}{\partial x}(x,u)\rho(\cdot)dx-\int_{\mathbb{R}^n}|\nabla u|^2\frac{\partial f}{\partial u}(x,u)\rho(\cdot)dx\\
&-\frac{2}{k^2}\int_{\mathbb{R}^n}x\nabla u f(x,u)\rho'(\cdot)dx+\int_{\mathbb{R}^n}f(x,u)z(\theta_t\omega)\Delta h\rho(\cdot)dx\\
&-\int_{\mathbb{R}^n}g(t)\Delta v\rho(\cdot)dx-\int_{\mathbb{R}^n}z(\theta_t\omega)\Delta h\Delta v\rho(\cdot)dx.
\end{aligned}\tag{3.53}$$

Notice that $\rho'(\cdot)\le C$ for $k\le|x|\le k\sqrt{2}$ and $\rho'(\cdot)=0$ for $|x|<k$ or $x>k\sqrt{2}$. Thus, we have

$$\begin{aligned}
\left|\frac{2}{k^2}\int_{\mathbb{R}^n}x\nabla v v_t\rho'(\cdot)dx\right|&\le\frac{C}{k^2}\int_{k\le|x|\le k\sqrt{2}}|x||\nabla v||v_t|dx\\
&\le\frac{C}{k}\int_{\mathbb{R}^n}|\nabla v||v_t|dx\\
&\le\frac{C}{k}\left(\|\nabla v\|^2+\|v_t\|^2\right).
\end{aligned}\tag{3.54}$$

Similarly,

$$\left|\frac{2\lambda}{k^2}\int_{\mathbb{R}^n}x\nabla v v\rho'(\cdot)dx\right|\le\frac{C}{k}\left(\|\nabla v\|^2+\|v\|^2\right)\tag{3.55}$$

and

$$\begin{aligned}
\left|\frac{2}{k^2}\int_{\mathbb{R}^n}x\nabla u f(x,u)\rho'(\cdot)dx\right|&\le\frac{C}{k}\left(\|\nabla u\|^2+\int_{\mathbb{R}^n}|f(x,u)|^2dx\right)\\
&\le\frac{C}{k}\left(\|\nabla v\|^2+|z(\theta_t\omega)|^2\|\nabla h\|^2+\|f(x,u)\|^2\right)\\
&\le\frac{C}{k}\left(\|\nabla v\|^2+|z(\theta_t\omega)|^2+\|f(x,u)\|^2\right).
\end{aligned}\tag{3.56}$$

Using assumption (1.5) and Cauchy's inequality,

$$\begin{aligned}
\left|\int_{\mathbb{R}^n}\nabla u\frac{\partial f}{\partial x}(x,u)\rho(\cdot)dx\right|&\le\int_{\mathbb{R}^n}|\nabla u||\psi_3|\rho(\cdot)dx\\
&\le\int_{\mathbb{R}^n}\rho(\cdot)|\nabla u|^2dx+\int_{\mathbb{R}^n}\rho(\cdot)|\psi_3|^2dx\\
&\le2\int_{\mathbb{R}^n}\rho(\cdot)(|\nabla v|^2+|z(\theta_t\omega)|^2|\nabla h|^2)dx+\int_{\mathbb{R}^n}\rho(\cdot)|\psi_3|^2dx\\
&\le2\int_{\mathbb{R}^n}\rho(\cdot)|\nabla v|^2+2|z(\theta_t\omega)|^2\int_{\mathbb{R}^n}\rho(\cdot)|\nabla h|^2dx+\int_{\mathbb{R}^n}\rho(\cdot)|\psi_3|^2dx.
\end{aligned}\tag{3.57}$$

By (1.4), we find that

$$
\begin{aligned}
-\int_{\mathbb{R}^n} |\nabla u|^2 \frac{\partial f}{\partial u}(x,u)\rho(\cdot)dx &\le \ell \int_{\mathbb{R}^n} \rho(\cdot)|\nabla u|^2 dx \\
&\le 2\ell \int_{\mathbb{R}^n} \rho(\cdot)|\nabla v|^2 + 2\ell |z(\theta_t\omega)|^2 \int_{\mathbb{R}^n} \rho(\cdot)|\nabla h|^2 dx.
\end{aligned} \tag{3.58}
$$

Applying condition (1.3) and Young's inequality, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^n} f(x,u) z(\theta_t\omega)\Delta h \rho(\cdot)dx \\
&\quad \le \int_{\mathbb{R}^n} |f(x,u)||z(\theta_t\omega)\Delta h|\rho(\cdot)dx \\
&\quad \le \alpha_2 \int_{\mathbb{R}^n} \rho(\cdot)|u|^{p-1}|z(\theta_t\omega)\Delta h|dx + \int_{\mathbb{R}^n} \rho(\cdot)|\psi_2||z(\theta_t\omega)\Delta h|dx \\
&\quad \le C\int_{\mathbb{R}^n} \rho(\cdot)|u|^p dx + C\int_{\mathbb{R}^n} \rho(\cdot)|z(\theta_t\omega)\Delta h|^p dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^n} \rho(\cdot)|\psi_2|^2 dx + \frac{1}{2}\int_{\mathbb{R}^n} \rho(\cdot)|z(\theta_t\omega)\Delta h|^2 dx.
\end{aligned} \tag{3.59}
$$

By Cauchy's inequality, we get

$$
-\int_{\mathbb{R}^n} g\Delta v \rho(\cdot)dx \le \int_{\mathbb{R}^n} \rho(\cdot)|g|^2 dx + \frac{1}{4}\int_{\mathbb{R}^n} \rho(\cdot)|\Delta v|^2 dx \tag{3.60}
$$

and

$$
-\int_{\mathbb{R}^n} z(\theta_t\omega)\Delta h \Delta v \rho(\cdot)dx \le \int_{\mathbb{R}^n} \rho(\cdot)|z(\theta_t\omega)\Delta h|^2 dx + \frac{1}{4}\int_{\mathbb{R}^n} \rho(\cdot)|\Delta v|^2 dx. \tag{3.61}
$$

From (3.53)-(3.61), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n} \rho(\cdot)|\nabla v|^2 dx + \lambda \int_{\mathbb{R}^n} \rho(\cdot)|\nabla v|^2 dx \\
&\quad \le \frac{C}{k}\left(\|v\|^2_{H^1(\mathbb{R}^n)} + \|v_t\|^2 + |z(\theta_t\omega)|^2 + \|f(x,u)\|^2\right) \\
&\quad + C\int_{\mathbb{R}^n} \rho(\cdot)\left(|u|^p + |\nabla v|^2\right)dx + \int_{\mathbb{R}^n} \rho(\cdot)|g(t)|^2 dx \\
&\quad + C|z(\theta_t\omega)|^2 \int_{\mathbb{R}^n} \rho(\cdot)|\nabla h|^2 dx + C|z(\theta_t\omega)|^p \int_{\mathbb{R}^n} \rho(\cdot)|\Delta h|^p dx \\
&\quad + C|z(\theta_t\omega)|^2 \int_{\mathbb{R}^n} \rho(\cdot)|\Delta h|^2 dx + \int_{\mathbb{R}^n} \rho(\cdot)\left(|\psi_2|^2 + |\psi_3|^2\right)dx.
\end{aligned} \tag{3.62}
$$

Since $h \in H^2(\mathbb{R}^n) \cap W^{2,p}(\mathbb{R}^n)$ and $\psi_2, \psi_3 \in L^2(\mathbb{R}^n)$, we can choose K_1 is large enough to satisfy

$$
\int_{\mathbb{R}^n} \rho(\cdot)\left(|\nabla h|^2 + |\Delta h|^2 + |\Delta h|^p\right)dx \le \int_{|x|\ge k} \left(|\nabla h|^2 + |\Delta h|^2 + |\Delta h|^p\right)dx \le \epsilon_0, \tag{3.63}
$$

$$
\int_{\mathbb{R}^n} \rho(\cdot)(|\psi_2|^2 + |\psi_3|^2)dx \le \int_{|x|\ge k} (|\psi_2|^2 + |\psi_3|^2)ds \le \epsilon_0 \text{ and } \frac{C}{k} \le \epsilon_0 \tag{3.64}
$$

where $\epsilon_0 = \dfrac{\epsilon}{R(\tau,\omega)}$, for all $k \ge K_1$.

Inserting (3.63)-(3.64) into (3.62), we obtain, for k is large enough,

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\cdot)|\nabla v|^2dx+\lambda\int_{\mathbb{R}^n}\rho(\cdot)|\nabla v|^2dx\\&\le\epsilon_0\left(\|v\|^2_{H^1(\mathbb{R}^n)}+\|v_t\|^2+\|f(x,u)\|^2\right)\\&+C\int_{|x|\ge k}\left(|u|^p+|\nabla v|^2\right)dx+\int_{|x|\ge k}|g(t)|^2dx\\&+C\epsilon_0(1+|z(\theta_t\omega)|^p).\end{aligned}\tag{3.65}$$

Multiplying (3.65) by $e^{\lambda t}$, integrating from $\tau-t$ to τ then replacing ω by $\theta_{-\tau}\omega$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}\rho(\cdot)|\nabla v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2dx&\le e^{-\lambda t}\int_{\mathbb{R}^n}\rho(\cdot)|\nabla v_{\tau-t}|^2dx\\+\epsilon_0\int_{\tau-t}^{\tau}e^{-\lambda(\tau-s)}\left(\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2_{H^1(\mathbb{R}^n)}+\|v_t(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2\right)ds\\+\epsilon_0\int_{\tau-t}^{\tau}e^{-\lambda(\tau-s)}\|f(x,u(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}))\|^2ds\\+C\int_{\tau-t}^{\tau}e^{-\lambda(\tau-s)}\int_{|x|\ge k}\left(|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^p+|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\right)dxds\\+\int_{\tau-t}^{\tau}e^{-\lambda(\tau-s)}\int_{|x|\ge k}|g(s,x)|^2dxds\\+C\epsilon_0\int_{\tau-t}^{\tau}e^{-\lambda(\tau-s)}(1+|z(\theta_{s-\tau}\omega)|^p)ds.\end{aligned}\tag{3.66}$$

We will estimate all terms on the right hand side of (3.66). Firstly,

$$\begin{aligned}e^{-\lambda t}\int_{\mathbb{R}^n}\rho(\cdot)|\nabla v_{\tau-t}|^2dx&\le e^{-\lambda t}\|v_{\tau-t}\|^2_{H^1(\mathbb{R}^n)}\\&\le e^{-\lambda t}\|u_{\tau-t}-hz(\theta_{-t}\omega)\|^2_{H^1(\mathbb{R}^n)}\\&\le 2e^{-\lambda t}(\|u_{\tau-t}\|^2_{H^1(\mathbb{R}^n)}+\|h\|^2_{H^1(\mathbb{R}^n)}|z(\theta_{-t}\omega)|^2)\\&\le\epsilon\text{ for all }t\ge T_1\end{aligned}\tag{3.67}$$

for some $T_1>0$, since $u_{\tau-t}\in B_0(\tau-t,\theta_{-t}\omega)$ and $|z(\omega)|$ is tempered. Secondly, by Lemmas 3.1, 3.7 and 3.8, we can choose a large enough $T_2>0$ such that, for all $t\ge T_2$,

$$\begin{aligned}\epsilon_0\int_{\tau-t}^{\tau}e^{-\lambda(\tau-s)}\left(\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2_{H^1(\mathbb{R}^n)}+\|v_t(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2\right)ds\\\le\epsilon_0R(\tau,\omega)\le\epsilon\end{aligned}\tag{3.68}$$

and

$$\epsilon_0\int_{\tau-t}^{t}e^{-\lambda(\tau-s)}\|f(x,u(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}))\|^2ds\le\epsilon_0R(\tau,\omega)\le\epsilon.\tag{3.69}$$

Thirdly, from Lemma 3.5, there are $T_3 > 0$ and $K_2 > 0$ satisfying

$$C\int_{\tau-t}^{\tau} e^{-\lambda(\tau-s)} \int_{|x|\geq k} \left(|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^p + |\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\right) dxds \\ \leq C\epsilon \tag{3.70}$$

whenever $t \geq T_3$ and $k \geq K_2$. Fourthly, by assumption (**G**), we get $K_3 > 0$ such that, for all $k \geq K_3$,

$$\int_{\tau-t}^{\tau} e^{-\lambda(\tau-s)} \int_{|x|\geq k} |g(s,x)|^2 dxds \leq \int_{-\infty}^{\tau} e^{-\lambda(\tau-s)} \int_{|x|\geq k} |g(s,x)|^2 dxds \leq \epsilon. \tag{3.71}$$

Finally,

$$\begin{aligned} C\epsilon_0 &\int_{\tau-t}^{\tau} e^{-\lambda(\tau-s)} \left(1+|z(\theta_{s-\tau}\omega)|^p\right) ds \\ &\leq C\epsilon_0 \int_{-t}^{0} e^{\lambda s} \left(1+|z(\theta_s\omega)|^p\right) ds \\ &\leq C\epsilon_0 \int_{-t}^{0} \left(e^{\lambda s} + e^{\lambda s} e^{-\frac{\lambda}{2}s} r(\omega)\right) ds \\ &\leq C\epsilon_0 (1+r(\omega)) \\ &\leq C\epsilon_0 R(\tau,\omega) \leq C\epsilon. \end{aligned} \tag{3.72}$$

Now we set $K = \max\{K_1, K_2, K_3\}$ and $T = \max\{T_1, T_2, T_3\}$, we obtain from (3.66)-(3.72) that

$$\int_{|x|\geq k\sqrt{2}} |\nabla v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2 dx \leq \int_{\mathbb{R}^n} \rho(\cdot)|\nabla v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2 dx \leq C\epsilon \tag{3.73}$$

for all $t \geq T$, $k \geq K$ and $v_{\tau-t} + hz(\theta_{-t}\omega) = u_{\tau-t} \in B_0(\tau-t,\theta_{-t}\omega)$. □

Now, we define a smooth function $\psi = 1-\rho$, where ρ is the cut-off function in Lemma 3.5, and for a given positive number k, define $y(t,x) = \psi\left(\frac{|x|^2}{k^2}\right) v(t,x)$. Then, y is a unique solution to the following initial Cauchy problem

$$\begin{cases} y_t - \Delta y + \lambda y + \psi(\cdot) f(x, v+z(\theta_t\omega)) = v\Delta\psi + 2\nabla\psi\nabla v + \psi(\cdot)\left(g + z(\theta_t\omega)\Delta h\right) \\ y|_{\partial B_{k\sqrt{2}}} = 0, \\ y(\tau) = y_\tau := \psi\left(\frac{|x|^2}{k^2}\right) v_\tau, \end{cases} \tag{3.74}$$

where $\psi(\cdot) = \psi\left(\frac{|x|^2}{k^2}\right)$, $B_{k\sqrt{2}}$ is a ball centered at origin with radius $k\sqrt{2}$. Consider the eigenvalue problem

$$-\Delta w = \lambda w \text{ in } B_{k\sqrt{2}}, \quad \text{with } w|_{\partial B_{k\sqrt{2}}} = 0.$$

It's a classical result that the problem has a family of eigenfunctions $\{e_j\}_{j\geq 1}$ with corresponding eigenvalues $\{\lambda_j\}_{j\geq 1}$ such that $\{e_j\}_{j\geq 1}$ form a orthogonal basis of $H_0^1(B_{k\sqrt{2}})$and $L^2\left(B_{k\sqrt{2}}\right)$ and $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \to \infty$. For a given integer m, we denote by P_m the canonical projector from $H_0^1(B_{k\sqrt{2}})$ onto the subspace *span*$\{e_1, e_2, \ldots, e_m\}$. Let $y \in H_0^1(B_{k\sqrt{2}})$, then y has a unique decomposition $y = y_1 + y_2 = P_m y + (Id - P_m)y$ where Id is the identity of $H_0^1(B_{k\sqrt{2}})$.

To obtain the asymptotic compactness of Φ in bounded domains, we need an Lemma which has a straightforward proof, so we omit it here. One can find in [21, Lemma 3.6] for a similar result.

Lemma 3.10. *Assume that there is $\lambda > 0$ such that, for all $\tau \in \mathbb{R}$,*

$$\sup_{t\geq 0} \int_{\tau-t}^{\tau} e^{-\lambda(\tau-s)} h(s,t) ds < +\infty,$$

where $h : \mathbb{R}^2 \to \mathbb{R}$ is a non-negative function. Then,

$$\lim_{\gamma\to+\infty} \sup_{t\geq 0} \int_{\tau-t}^{\tau} e^{-\gamma(\tau-s)} h(s,t) ds = 0.$$

The following lemma shows the asymptotic compactness of solutions to (3.74), or equivalently, the asymptotic compactness of solutions to (2.7) in bounded domains.

Lemma 3.11. *Let $k > 0$ is fixed. Then, for $\tau \in \mathbb{R}, \omega \in \Omega$ and any $\epsilon > 0$, there exist $m_0 \in \mathbb{N}$ and $T_0 > 0$ satisfying*

$$\|(Id - P_m)y(\tau, \tau - t, \theta_{-\tau}\omega, y_{\tau-t})\|^2_{H_0^1(B_{k\sqrt{2}})} \leq \epsilon \tag{3.75}$$

for all $m \geq m_0$ and for all $t \geq T_0$, where $y_{\tau-t} = \psi(\cdot)v_{\tau-t}$ with $v_{\tau-t} + hz(\theta_{-t}\omega) = u_{\tau-t} \in B(\tau - t, \theta_{-t}\omega)$.

Proof. We rewrite (3.74) in short form

$$y_t - \Delta y + \lambda y + \psi f(x,u) = (g + z(\theta_t\omega)\Delta h)\psi + 2\nabla v \nabla \psi + v\Delta\psi \tag{3.76}$$

and write $y = P_m y + (Id - P_m)y = y_1 + y_2$. Multiplying (3.76) by $-\Delta y_2$ then integrating over $B_{k\sqrt{2}}$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|y_2\|^2_{H_0^1(B_{k\sqrt{2}})} + \|\Delta y_2\|^2_{L^2(B_{k\sqrt{2}})} + \lambda\|y_2\|^2_{H_0^1(B_{k\sqrt{2}})} + \int_{B_{k\sqrt{2}}} \psi f(x,u)(-\Delta y_2)dx \\ &= \int_{B_{k\sqrt{2}}} ((g + z(\theta_t\omega)\Delta h)\psi + 2\nabla v\nabla\psi + v\Delta\psi)(-\Delta y_2)dx. \end{aligned} \tag{3.77}$$

By Cauchy's inequality, we have

$$\begin{aligned} \int_{B_{k\sqrt{2}}} \psi f(x,u)(-\Delta y_2)dx &\leq \int_{B_{k\sqrt{2}}} |\psi||f(x,u)||\Delta y_2|dx \\ &\leq \frac{1}{10}\|\Delta y_2\|^2_{L^2(B_{k\sqrt{2}})} + \frac{5}{2}\int_{B_{k\sqrt{2}}} |\psi|^2|f(x,u)|^2dx \\ &\leq \frac{1}{10}\|\Delta y_2\|^2_{L^2(B_{k\sqrt{2}})} + \frac{5}{2}\|f(x,u)\|^2dx. \end{aligned} \tag{3.78}$$

Similarly,

$$\int_{B_{k\sqrt{2}}} g\psi(-\Delta y_2)dx \leq \frac{1}{10}\|\Delta y_2\|^2_{L^2(B_{k\sqrt{2}})} + \frac{5}{2}\|g\|^2 \tag{3.79}$$

and

$$\int_{B_{k\sqrt{2}}} z(\theta_t\omega)\Delta h\psi(-\Delta y_2)dx \leq \frac{1}{10}\|\Delta y_2\|^2_{L^2(B_{k\sqrt{2}})} + \frac{5}{2}|z(\theta_t\omega)|^2. \tag{3.80}$$

By the definition of ψ, we have $|\psi'| \leq C$, thus

$$
\begin{aligned}
2\int_{B_{k\sqrt{2}}} \nabla v \nabla \psi(-\Delta y_2)dx &= \frac{2}{k^2}\int_{B_{k\sqrt{2}}} x\cdot\nabla v\psi'\left(\frac{|x|^2}{k^2}\right)(-\Delta y_2)dx \\
&\leq \frac{2}{k^2}\int_{B_{k\sqrt{2}}} |x||\nabla v||\psi'\left(\frac{|x|^2}{k^2}\right)||\Delta y_2|dx \\
&\leq \frac{C}{k}\int_{B_{k\sqrt{2}}} |\nabla v||\Delta y_2|dx \\
&\leq \frac{1}{10}\|\Delta y_2\|^2_{L^2(B_{k\sqrt{2}})} + C\|\nabla v\|^2.
\end{aligned}
\tag{3.81}
$$

Similarly,

$$
\int_{B_{k\sqrt{2}}} v\Delta\psi(-\Delta y_2)dx \leq \frac{1}{10}\|\Delta y_2\|^2_{L^2(B_{k\sqrt{2}})} + C\|v\|^2. \tag{3.82}
$$

From (3.77) - (3.82), and using Poincare inequality

$$
\|\Delta y_2\|^2_{L^2(B_{k\sqrt{2}})} \geq \lambda_{m+1}\|y_2\|^2_{H_0^1(B_{k\sqrt{2}})},
$$

we conclude that

$$
\begin{aligned}
&\frac{d}{dt}\|y_2\|^2_{H_0^1(B_{k\sqrt{2}})} + \lambda_{m+1}\|y_2\|^2_{H_0^1(B_{k\sqrt{2}})} \\
&\qquad \leq C\left(\|f(x,u)\|^2 + \|g(t)\|^2 + |z(\theta_t\omega)|^2 + \|\nabla v\|^2 + \|v\|^2\right).
\end{aligned}
\tag{3.83}
$$

Multiplying (3.83) by $e^{\lambda_{m+1}t}$, integrating on $(\tau - t, \tau)$ then replacing ω by $\theta_{-\tau}\omega$, we deduce that

$$
\begin{aligned}
&\|y_2(\tau, \tau - t, \theta_{-\tau}\omega, y_{\tau-t})\|^2_{H_0^1(B_{k\sqrt{2}})} \\
&\leq e^{-\lambda_{m+1}t}\|y_{\tau-t}\|^2_{H_0^1(B_{k\sqrt{2}})} \\
&+ C\int_{\tau-t}^{\tau} e^{-\lambda_{m+1}(\tau-s)}\Big(\|f(x,u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}))\|^2 \\
&\qquad + |z(\theta_{s-\tau}\omega)|^2 + \|g(s)\|^2 + \|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2_{H^1(\mathbb{R}^n)}\Big)ds.
\end{aligned}
\tag{3.84}
$$

On the one hand,

$$
\begin{aligned}
e^{-\lambda_{m+1}t}\|y_{\tau-t}\|^2_{H_0^1(B_{k\sqrt{2}})} &\leq e^{-\lambda_{m+1}t}\|v_{\tau-t}\|^2_{H^1(\mathbb{R}^n)} \\
&\leq 2e^{-\lambda_{m+1}t}\left(\|u_{\tau-t}\|^2_{H^1(\mathbb{R}^n)} + |z(\theta_{-t}\omega)|^2\|h\|^2_{H^1(\mathbb{R}^n)}\right) \\
&\longrightarrow 0 \text{ as } m, t \to +\infty,
\end{aligned}
\tag{3.85}
$$

since $u_{\tau-t} \in B_0(\tau - t, \theta_{-t}\omega)$, $|z(\omega)|$ is tempered and $\lambda_m \to +\infty$. Thus, there exist $T_1 > 0$ and $m_1 \geq 1$ such that

$$
e^{-\lambda_{m+1}t}\|y_{\tau-t}\|^2_{H_0^1(B_{k\sqrt{2}})} \leq \frac{\epsilon}{3} \text{ provided } t \geq T_1 \text{ and } m \geq m_1. \tag{3.86}
$$

On the other hand, by (**G**), Lemmas 3.1, 3.10 and the fact that $\lambda_m \to +\infty$, we can choose $T_2 > 0$ and $m_2 \geq 1$ satisfying

$$\begin{aligned} C\int_{\tau-t}^{\tau} e^{-\lambda_{m+1}(\tau-s)}\|f(x,u(s,\tau-t,\theta_{-t}\omega,u_{\tau-t}))\|^2 ds \\ +C\int_{\tau-t}^{\tau} e^{-\lambda_{m+1}(\tau-s)}\|v(s,\tau-t,\theta_{-t}\omega,v_{\tau-t})\|^2_{H^1(\mathbb{R}^n)} ds \\ +C\int_{\tau-t}^{\tau} e^{-\lambda_{m+1}(\tau-s)}\|g(s)\|^2 ds \\ \leq \frac{\epsilon}{3} \end{aligned} \tag{3.87}$$

whenever $t \geq T_2$ and $m \geq m_2$. Finally

$$\begin{aligned} C\int_{\tau-t}^{\tau} e^{-\lambda_{m+1}(\tau-s)}|z(\theta_{s-\tau}\omega)|^2 ds &= C\int_{-t}^{0} e^{\lambda_{m+1}s}|z(\theta_s\omega)|^2 ds \\ \leq C\int_{-\infty}^{0} e^{\lambda_{m+1}s}e^{-\frac{\lambda}{2}s}r(\omega)ds &\leq \frac{Cr(\omega)}{\lambda_{m+1}-\frac{\lambda}{2}} \leq \frac{\epsilon}{3} \end{aligned} \tag{3.88}$$

for any $m \geq m_3$ with some large enough $m_3 \geq 1$.

Set $T = \max\{T_1, T_2\}$ and $m_0 = \max\{m_1, m_2, m_3\}$. We get from (3.84) that, for all $t \geq T$ and $m \geq m_0$,

$$\|y_2(\tau,\tau-t,\theta_{-\tau}\omega,y_{\tau-t})\|^2_{H_0^1(B_{k\sqrt{2}})} \leq \epsilon. \tag{3.89}$$

This completes the proof. □

Lemma 3.12. *For fixed $\tau \in \mathbb{R}$ and $\omega \in \Omega$. For any $t_n \to +\infty$ and $v_{\tau-t_n} = x_n - hz(\theta_{-t_n}\omega)$, where $x_n \in B_0(\tau-t_n,\theta_{-t_n}\omega)$, we have $\{v(\tau,\tau-t_n,\theta_{-\tau}\omega,v_{\tau-t_n})\}$ is precompact in $H^1(B_K)$ for any $K > 0$.*

Proof. Define

$$y(\tau,\tau-t_n,\theta_{-\tau}\omega,y_{\tau-t_n}) = \psi\left(\frac{|x|^2}{K^2}\right)v(\tau,\tau-t_n,\theta_{-\tau}\omega,v_{\tau-t_n}) \tag{3.90}$$

with $y_{\tau-t_n} = \psi\left(\frac{|x|^2}{K^2}\right)v_{\tau-t_n}$. For the sake of brevity, we denote by $y_n = y(\tau,\tau-t_n,\theta_{-\tau}\omega,y_{\tau-t_n})$. From Lemma 3.1, since $t_n \to +\infty$, we have

$$\{v(\tau,\tau-t_n,\theta_{-\tau}\omega,v_{\tau-t_n})\} \text{ is bounded in } H^1(\mathbb{R}^n). \tag{3.91}$$

Thus

$$\{y_n\}_{n\geq 1} \text{ is bounded in } H_0^1(B_{K\sqrt{2}}). \tag{3.92}$$

Let $\epsilon > 0$ be given. Using Lemma 3.11, there exists $m > 0$ and $N_1 > 0$ such that

$$\|\,(Id_K - P_m)\,y_n\|_{H_0^1(B_{K\sqrt{2}})} \leq \epsilon \tag{3.93}$$

for all $n \geq N_1$, where $Id_K := Id_{H_0^1(B_{K\sqrt{2}})}$.

On the other hand, it follows from (3.92) that $\{P_m(y_n)\}_{n\geq 1}$ is bounded in $P_m(H_0^1(B_{K\sqrt{2}}))$. Hence $\{P_m(y_n)\}_{n\geq 1}$ is precompact since $P_m(H_0^1(B_{K\sqrt{2}}))$ has a finite dimension. Thus, we can take a subsequence $\{n'\}$ of $\{n\}$ and N_2 satisfying

$$\|P_m(y_{n'}) - P_m(y_{k'})\|_{H_0^1(B_{K\sqrt{2}})} \leq \epsilon \text{ for all } n', k' \geq N_2. \tag{3.94}$$

Now, set $N = \max\{N_1, N_2\}$. We have

$$\begin{aligned}
&\|y_{n'} - y_{k'}\|_{H_0^1(B_{K\sqrt{2}})} \\
&\le \|P_m(y_{n'}) - P_m(y_{k'})\|_{H_0^1(B_{K\sqrt{2}})} + \|(Id_K - P_m)y_{n'}\|_{H_0^1(B_{K\sqrt{2}})} \\
&\quad + \|(Id_K - P_m)y_{k'}\|_{H_0^1(B_{K\sqrt{2}})} \\
&\le 3\epsilon
\end{aligned} \tag{3.95}$$

due to (3.93) and (3.94). It follows from (3.90) and (3.95) that

$$\begin{aligned}
&\|v(\tau, \tau - t_{n'}, \theta_{-\tau}\omega, v_{\tau - t_{n'}}) - v(\tau, \tau - t_{k'}, \theta_{-\tau}\omega, v_{\tau - t_{k'}})\|_{H^1(B_K)} \\
&\le \|y_{n'} - y_{k'}\|_{H_0^1(B_{K\sqrt{2}})} \\
&\le 3\epsilon \qquad \text{for all } n', k' \ge N.
\end{aligned} \tag{3.96}$$

This show that $\{v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{\tau - t_n})\}_{n\ge 1}$ is precompact in $H^1(B_K)$ and thus completes the proof. ☐

The main result of this work is now ready to be shown.

Theorem 3.13. *Assume that hypothesis* $(\mathbf{F}) - (\mathbf{G}) - (\mathbf{H})$ *hold. Then the RDS* Φ *corresponding to equation* (1.1) *possesses a pullback attractor* $\mathcal{A}_1 = \{A_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ *in* $H^1(\mathbb{R}^n)$. *Moreover,* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *concide, i.e.,*

$$A_1(\tau, \omega) = A_2(\tau, \omega) \text{ for all } \tau \in \mathbb{R}, \omega \in \Omega.$$

Proof. Theorem 2.3 tells us that Φ has a random attractor in $L^2(\mathbb{R}^n)$. Then, from Theorem 2.2 and Lemma 3.4, it remains to prove the pullback asymptotic compactness of Φ.

Fix $\tau \in \mathbb{R}$ and $\omega \in \Omega$. Let $t_n \to +\infty$ and $x_n \in B_0(\tau - t_n, \theta_{-t_n}\omega)$, thanks to Lemma 3.6, we have to prove that

$$\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\}_{n\ge 1} \text{ is relative compact in } H^1(\mathbb{R}^n). \tag{3.97}$$

By (2.9),

$$\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n) = v(\tau, \tau - t_n, \theta_{-\tau}\omega, x_n - hz(\theta_{-t_n}\omega)) + hz(\omega). \tag{3.98}$$

Set $v_{\tau - t_n} = x_n - hz(\theta_{-t_n}\omega)$. Then it is sufficient to show that

$$\{v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{\tau - t_n})\}_{n\ge 1} \text{ is relative compact in } H^1(\mathbb{R}^n). \tag{3.99}$$

For the sake of brevity, we denote by $v_n = v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{\tau - t_n})$.

Let $\epsilon > 0$ be given. Using Lemmas 3.5 and 3.9, we can choose $N_1 \ge 1$ and $K > 0$ such that

$$\int_{|x|\ge K} (|v_n|^2 + |\nabla v_n|^2)dx \le \epsilon \tag{3.100}$$

for all $n \ge N_1$. On the other hand, from Lemma 3.12, $\{v_n\}_{n\ge 1}$ is precompact in $H^1(B_K)$, i.e., there exists a subsequence of $\{v_n\}$ (not relabeled) and $N_2 \ge 1$ satisfying

$$\|v_n - v_m\|_{H^1(B_K)} \le \epsilon \text{ for all } n, m \ge N_2. \tag{3.101}$$

Set $N = \max\{N_1, N_2\}$. It follows from (3.100) and (3.101) that, for all $n, m \geq N$,

$$\begin{aligned}\|v_n - v_m\|_{H^1(\mathbb{R}^n)} &\leq \|v_n - v_m\|_{H^1(B_K)} + 2\int_{|x|\geq K} (|v_n|^2 + |\nabla v_n|^2)dx \\ &\quad + 2\int_{|x|\geq K} (|v_m|^2 + |\nabla v_m|^2)dx \\ &\leq 5\epsilon.\end{aligned} \tag{3.102}$$

Hence (3.99) holds and the proof is complete. □

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