P 13 Model reduction in HJB-based feedback design for optimal control of evolution problems (F. Bornemann, K. Kunisch) → AO, NS, IS

This project studies the construction of max-plus finite element methods and corresponding projection-based model reduction techniques for Hamilton–Jacobi–Bellman (HJB) equations and closed-loop optimal control. The construction of max-plus discretizations naturally decomposes into an expensive offline and a curse-of-dimensionality free real-time phase. The systematic construction of such methods will be based on locally exact evaluations of the Lax–Oleinik semigroup together with transforming classical algorithms into their max-plus counterparts by Maslov dequantization.

State of the art. By dynamical programming and Bellman's optimality principle, the optimal feedback map $u(t) = u^*(x(t), t)$ of the finite-horizon optimal control problem

$$\begin{cases} \displaystyle\int_0^T L(x(t), u(t))\, dt + \varphi(x(T) = \max ! \\ \dot{x} = f(x, u), \quad x(0) = x_0, \quad u(t) \in U, \end{cases} \tag{P}$$

is given by the maximizer $u^*(x, t)$ of the expression defining the Hamilton–Jacobi–Bellman (HJB) equation:

$$\frac{\partial}{\partial t} V(x, t) = \max_{u \in U} (\langle \nabla_x V(x, t), f(x, u) \rangle + L(x, u)), \qquad V(x, 0) = \varphi(x), \tag{HJB}$$

where $V(x, t)$ denotes the value function associated to (P). Once $u^*(x, t)$ is available, one arrives at the closed loop system which is given by $\dot{x} = f(x, u^*(x, t))$. Let us note that the spatial dimension of (HJB) is that of the state space of the control system. For large or infinite state space dimensions, the Hamilton-Jacobi-Bellman based approach for computing the value function V becomes computationally intractable (curse of dimensionality). The feasibility of general closed-loop approaches thus depends on techniques for a (rough) approximation or reduced model of the feedback map u^*.

On the one hand, for an infinite horizon problem (where V and u^* are stationary), in [6] such a model reduction based on a proper orthogonal decomposition (POD) — obtained from the solution of the controlled system at certain snapshots in time — was suggested to reduce the state space dimension to just a few essential degrees of freedom. Even though the feasibility was demonstrated in a proof-of-concept study of optimal boundary feedback control for the one-dimensional Burgers equations [6], much remains to be done for this approach to become really efficient, the bottleneck being the cost of solving the HJB equation (even when the dynamical system in (P) is of moderate order, say less than 5–10).

On the other hand, Maslov [8] observed that the Lax–Oleinik semigroup S^t solving (HJB), defined by

$$S^t : \varphi \mapsto V(\cdot, t),$$

is accessible to methods of *idempotent analysis* [5] in *semirings*, namely, it is *max-plus linear*

$$S^t(f \oplus g) = S^t f \oplus S^t g, \qquad S^t(\lambda \otimes f) = \lambda \otimes S^t f,$$

with pointwise application of the operations $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for elements a, b in the tropical semiring $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$. For linear-quadratic optimal control problems, Fleming and McEneaney [4, 9] introduced a max-plus based method by approximating the value function $V(\cdot, t)$ by a max-plus linear ansatz

$$V_h(x,t) = \sup_{j=1:N} (\lambda_j(t) + w_j(x)) = \bigoplus_{j=1}^{n} (\lambda_j(t) \otimes w_j(x)).$$

Here, the basis functions w_j are pre-computed offline, while the coefficients λ_j are obtained by means of a direct, curse-of-dimensionality free real-time computation based on dynamic programming. Akian, Gaubert and Lakhoua [1] turned this method into a Petrov-Galerkin method (max-plus finite element method) based on the max-plus inner product

$$\langle u|v \rangle = \sup_{x \in X} (u(x) + v(x)) = \int_X^{\oplus} u(x) \otimes v(x)\, dx.$$

Core to the method and the underlying choice of basis functions $w_j(x)$ is the (approximate) evaluation of the application of the Lax–Oleinik semigroup $S^t w_j(x)$. Denoting the time step by τ and the resolution of the spatial discretization by h, error estimates of the form $\tau + h/\tau$ have been established.

Thesis project to be supervised by Folkmar Bornemann. In this project the systematic offline construction of Petrov–Galerkin bases for the max-plus finite element method is studied and will be extended to snapshot-based projection-type model reduction techniques. A guide to such constructions is given by the *Maslov dequantization* [7], which gets max-plus analysis as the semiclassic limit $h \to 0$ of the classical analysis under the transform $Q_h : x \mapsto h \log x$, namely

$$a \oplus_h b = Q_h(Q_h^{-1}(a) + Q_h^{-1}(b)), \qquad a \otimes_h b = Q_h(Q_h^{-1}(a) \times Q_h^{-1}(b)) = a + b.$$

By dequantization one can lift algorithms from the classical realm into the max-plus setting. The project is organized along the following steps:

1. Offline construction of basis functions based on a local variational principle (Lax–Hopf formula), that allows the approximate local evaluation of the Lax–Oleinik semigroup. This technique was introduced by Bornemann and Rasch [2] for the analysis of linear finite elements in Hamilton–Jacobi equations. Recently, Mirebeau [10] extended this approach to a direct one sweep fast-marching solver for highly anisotropic problems based on lattice reduction techniques.

2. Transformation of projection-based model reduction techniques to the max-plus setting.

3. Generalizing from linear-quadratic optimal control problems to more general nonlinear problems using idempotent Newton's method as recently introduced by Esparza and coworkers [3].

4. Error analysis of the methods analogously to [1].

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