# P 14 Multilevel approaches to optimal control problems with uncertain PDE constraints (K. Kunisch, E. Ullmann) $\rightarrow$ AO, NS

Optimization problems constrained by partial differential equations (PDEs) arise from many applications in engineering and science, e.g. the optimal control of fluid flows, or shape optimization. Since critical input parameters (boundary conditions, material parameters) are often not precisely known we consider optimal control problems (OCPs) where the input data to the PDE constraint is a random function.

**State of the art.** We wish to minimize $J = J(y, u)$ s.t. $c(y, u) = 0$, where $J$ is a real-valued, smooth cost functional, and $c(y, u) = 0$ denotes the PDE constraint with state $y$ and control $u$. Suppose now that the input data $a$ of the PDE is a random function $a\colon D \times \Omega \to \mathbb{R}$, where $D \subset \mathbb{R}^d$ $(d = 2, 3)$, and $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space. We write the PDE in the form $c(y, u, \omega) = 0$. The control may be a random function, i.e. $u = u(x, \omega)$. Let $\mathbb{E}[\cdot]$ denote the expected value w.r.t. the probability measure $\mathbb{P}$. We consider two problems:

$$\bar{u} := \mathbb{E}[\ \operatorname*{arg\,min}_{u=u(x,\omega)} \{\ J(y, u) \mid c(y, u, \omega) = 0\ \}\ ] \tag{1}$$

$$\hat{u}_0 := \operatorname*{arg\,min}_{u=u(x)}\ \{\ \mathbb{E}[J(y, u)] \mid c(y, u, \omega) = 0\ \} \tag{2}$$

Formulation (1) is studied in [4, 6]. Note that $\bar{u}$ does not solve an OCP and is in general not a robust control. In (2) we minimize the expected value of the cost functional subject to an infinite number of PDE constraints. The optimal control $\hat{u}_0$ is assumed deterministic and is robust by construction. This problem is studied in [5, 7, 13, 14, 17, 18, 19, 22, 24]. Alternatively, we could look for a *stochastic* optimal control in (2) (see [8, 9, 10, 21, 25]).

**Thesis project to be supervised by Elisabeth Ullmann.** We go beyond the current state of the art by considering *control constraints*, and by using *Multilevel Monte Carlo* quadrature for the stochastic discretization. With control constraints we do not have analytic regularity of the control w.r.t. the random parameters and sparse grid approaches (see [8, 9, 10, 17, 18, 19, 21]) are of limited use. We employ Monte Carlo based quadrature which also allows us to handle high-dimensional sample spaces. To decrease the computational cost of Monte Carlo we employ Multilevel Monte Carlo (MLMC) [1, 11]. The basic idea is to use a hierarchy of spatial discretizations for the problem under consideration. MLMC has been successfully used for many problems, e.g. elliptic PDEs, or random obstacles [3, 15].

Our starting point is the deterministic OCP with an elliptic PDE constraint

$$\min_{y,u}\ \frac{1}{2}\|y - y_0\|^2_{L^2(D)} + \frac{\alpha}{2}\|u\|^2_{L^2(D)} \quad \text{s.t.} \tag{3}$$

$$-\nabla \cdot (a\nabla y) = u \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D, \quad u \in U_{ad}, \tag{4}$$

where $y_0 \in L^2(D)$ is the desired state, and $U_{ad}$ is the admissible set of controls. We assume that $a$ is a random field. The overall goal of the project is to develop, analyze, and compare numerical methods for the problems (1) and (2). This will be done in three project phases.

(a) We consider formulation (1) and use the standard, additive MLMC estimator (e.g. [11]) to approximate $\bar{u}$. We carry out a convergence analysis and verify the results with numerical experiments. Standard MLMC can easily be combined with (1) and existing solvers for deterministic OCPs can be used as "black box" solver. Moreover, a "pathwise" convergence analysis for a fixed realization $a(\cdot, \omega)$ of the coefficient is possible. However, $a(\cdot, \omega)$ is often only Hölder continuous with coefficient $t < 1/2$ and the analysis does not follow textbook arguments. We plan to use a standard discretization of the control together with a primal-dual active set strategy [2, 20]. Additive MLMC gives a nonconforming approximation $\bar{u}^{MCMC} \notin U_{ad}$ since $U_{ad}$ is not closed w.r.t. arbitrary linear combinations. This has already been observed in [15] for random obstacle problems. We plan to explore alternative ways of constructing MLMC estimators, possibly based on the multilevel correction scheme in [12], or the MG/OPT method in [23], to obtain an approximation in $U_{ad}$.

(b) We consider formulation (2) and approximate the cost functional by quadrature. This has been done with sparse grids in [17, 18, 19]. However, in sparse grids some quadrature weights can be negative, and thus the discretized optimal control problem might not be well-posed [16]. We will use Monte Carlo quadrature where all weights are positive (Quasi Monte Carlo quadrature would also be possible). Coming back to (a) we will again use a primal-dual active set optimization algorithm to realize the constraints. We establish the well-posedness of the discretized OCP, carry out a convergence analysis, and verify the results with numerical experiments.

(c) Finally, we systematically compare formulations (1) and (2) w.r.t. computational costs and robustness of the controls. To this end we estimate the statistics of the tracking error $\|y - y_0\|_{L^2(D)}$ (see the experiments in [4, 6]) using standard, additive MLMC.

**Further topics.** We can consider other types of PDE constraints including the Stokes equation, advection-diffusion problems, or time-dependent, parabolic PDEs. It is possible to include the variance or higher moments of the state in the cost functional (see e.g. [24, 25]), or to include state constraints in (3)–(4).

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