Reduced Basis Methods

Parametrized Optimal Control Problems

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Martin Grepl
Institut für Geometrie und Praktische Mathematik, RWTH Aachen

Problem Statement

Given $\mu \in \mathcal{D}$, find $u^*(\mu) \in \mathcal{U}$ such that

$$u^*(\mu) = \arg\min_{u \in \mathcal{U}} J(y(\mu), u(\mu); \mu)$$

where $y(\mu) \in Y$ satisfies[†]

$$a(y, \phi; \mu) = b(\phi; \mu) u, \quad \forall \phi \in Y.$$

Assumptions:

- FE-Space: $Y \subset Y^{\mathrm{e}}, \dim(Y) = \mathcal{N}$, $\qquad H_0^1(\Omega) \subset Y^{\mathrm{e}} \subset H^1(\Omega)$
- Control input: $u \in \mathcal{U} \equiv \mathbb{R}$
- Bilinear form a is continuous and coercive
- Linear form b is continuous
- Affine parameter dep.: $a(w, v; \mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu)\, a^q(w, v)$

† We will often drop the dependence on μ, i.e., $y = y(\mu)$, ...

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where $y(\mu) \in Y$ satisfies[†]

$$a(y, \phi; \mu) = b(\phi; \mu)u, \quad \forall \phi \in Y.$$

Cost Functional

$$J(y, u; \mu) = \frac{1}{2} \|y - y_d\|^2_{L^2(D)} + \frac{\lambda}{2} \|u - u_d\|^2_{\mathcal{U}}$$

where:

- $D \subset \Omega$ (or $D \subset \Gamma$) is a measurable set
- y_d and u_d are the desired state and control input
- $\lambda > 0$ is the regularization parameter

Truth Problem Statement

- Cost functional

$$J(y,u;\mu) = \frac{1}{2}\,\|y(\mu) - y_d\|^2_{L^2(D)} + \frac{\lambda}{2}\,\|u - u_d\|^2_{\mathcal{U}}$$

- Lagrangian

$$\mathcal{L}(y,u,p;\mu) = J(y,u;\mu) + a(y,p;\mu) - b(p;\mu)u$$

First order necessary conditions $\vartheta = (\varphi, \psi, \phi)$

$$\nabla\mathcal{L}(y,u,p;\mu)(\vartheta) = 0, \quad \forall \vartheta \in X \equiv Y \times \mathcal{U} \times Y$$

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$$\begin{aligned} a(y^*,\phi;\mu) &= b(\phi;\mu)u^*, & \forall\phi \in Y, \\ a(\varphi,p^*;\mu) &= (y_d - y^*, \varphi)_{L^2(D)}, & \forall\varphi \in Y, \\ -b(p^*;\mu) + \lambda(u^* - u_d) &= 0. \end{aligned}$$

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$\Rightarrow$ Solution expensive ($\mathcal{N}$-dependent cost)

Motivation I/II

Why do we care?

- Optimization over the parameter (many-query context)
- Model Predictive Control (real-time context)

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One possible solution: Surrogate model approach

- Introduce reduced basis space $\boldsymbol{Y_N} \subset \boldsymbol{Y}$.
- Replace high-dimensional problem, $\boldsymbol{y} \in \boldsymbol{Y}$, by reduced basis approximation, $\boldsymbol{y_N} \in \boldsymbol{Y_N}$.

RB Problem Statement

- Cost functional

$$J_N(y_N, u_N; \mu) = \frac{1}{2} \|y_N(\mu) - y_d\|^2_{L^2(D)} + \frac{\lambda}{2} \|u_N - u_d\|^2_{\mathcal{U}}$$

- Lagrangian

$$\mathcal{L}_N(y_N, u_N, p_N; \mu) = J_N(y_N, u_N; \mu) + a(y_N, p_N; \mu) - b(p_N; \mu) u_N$$

First order necessary conditions $\vartheta = (\varphi, \psi, \phi)$

$$\nabla \mathcal{L}_N(y_N, u_N, p_N; \mu)(\vartheta) = 0, \quad \forall \vartheta \in X_N \equiv Y_N \times \mathcal{U} \times Y_N$$

Given $\mu \in \mathcal{D}$, find $(y_N^*, p_N^*, u_N^*) \in X_N$ such that

$$\begin{aligned} a(y_N^*, \phi; \mu) &= b(\phi; \mu) u_N^*, & \forall \phi \in Y_N, \\ a(\varphi, p_N^*; \mu) &= (y_d - y_N^*, \varphi)_{L^2(D)}, & \forall \varphi \in Y_N, \\ -b(p_N^*; \mu) + \lambda(u_N^* - u_d) &= 0. & \end{aligned}$$

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$\Rightarrow$ **Solution inexpensive (N-dependent cost), but suboptimal**

Motivation II/II

Question:

- How large is the error introduced by the surrogate model approach?

Goal II/II

Develop *a posteriori* error bounds for the

- optimal control

$$\|u^*(\mu) - u_N^*(\mu)\|_{\mathcal{U}} \leq \Delta_N^u(\mu), \quad \forall \mu \in \mathcal{D},$$

- cost functional

$$|J^*(y^*, u^*; \mu) - J_N^*(y_N^*, u_N^*; \mu)| \leq \Delta_N^J(\mu), \quad \forall \mu \in \mathcal{D},$$

which are rigorous *and* (online-)efficient.

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Other Work: – POD a-posteriori error bounds [TV09]

– RB: parabolic [Ded10, Ded12], elliptic (SPP) [NRMQ12]

Error Definitions

$$\begin{array}{ccc} \nabla \mathcal{L}(y^*, u^*, p^*; \mu) = 0 & & \nabla \mathcal{L}_N(y_N^*, u_N^*, p_N^*; \mu) = 0 \\ \downarrow & & \downarrow \\ u^* & & u_N^* \\ \downarrow & & \downarrow \\ y^*(u^*) & & y_N^*(u_N^*) \\ \downarrow & & \downarrow \\ p^*\left(y^*(u^*)\right) & & p_N^*\left(y_N^*(u_N^*)\right) \end{array}$$

Given $\mu \in \mathcal{D}$, find $y^*(u^*) \in Y$ such that

$$a(y^*, \phi; \mu) = b(\phi) u^*, \quad \forall \phi \in Y,$$

and $p^*\left(y^*(u^*)\right) \in Y$ such that

$$a(\varphi, p^*; \mu) = (y^* - y_d, \varphi)_{L^2(\Omega)}, \quad \forall \varphi \in Y.$$

Error Definitions

$$\nabla \mathcal{L}(y^*, u^*, p^*; \mu) = 0 \quad \rightarrow \quad u^* \quad \rightarrow \quad y^*(u^*) \quad \rightarrow \quad p^*\left(y^*(u^*)\right)$$

$$\nabla \mathcal{L}_N(y_N^*, u_N^*, p_N^*; \mu) = 0 \quad \rightarrow \quad u_N^* \quad \rightarrow \quad y_N^*(u_N^*) \quad \rightarrow \quad p_N^*\left(y_N^*(u_N^*)\right)$$

- Optimality error

  state: $e^{y,*} = y^*(u^*) - y_N^*(u_N^*)$

Error Definitions

$$\nabla \mathcal{L}(y^*, u^*, p^*; \mu) = 0 \quad\quad \nabla \mathcal{L}_N(y_N^*, u_N^*, p_N^*; \mu) = 0$$
$$\downarrow \quad\quad \downarrow$$
$$u^* \quad\quad u_N^*$$
$$\downarrow \quad\quad \downarrow$$
$$y^*(u^*) \quad\quad y_N^*(u_N^*)$$
$$\downarrow \quad\quad \downarrow$$
$$p^*\left(y^*(u^*)\right) \quad\quad p_N^*\left(y_N^*(u_N^*)\right)$$

- Optimality error

  state: $e^{y,*} = y^*(u^*) - y_N^*(u_N^*)$

  adjoint: $e^{p,*} = p^*\left(y^*(u^*)\right) - p_N^*\left(y_N^*(u_N^*)\right)$

Error Definitions

$$
\begin{array}{c}
\nabla \mathcal{L}(y^*, u^*, p^*; \mu) = 0 \\
\downarrow \\
u^* \\
\downarrow \\
y^*(u^*) \\
\downarrow \\
p^*\left(y^*(u^*)\right)
\end{array}
\qquad
\begin{array}{ccc}
 & & \nabla \mathcal{L}_N(y_N^*, u_N^*, p_N^*; \mu) = 0 \\
 & & \downarrow \\
 & \swarrow & u_N^* \\
 & & \downarrow \\
y(u_N^*) & & y_N^*(u_N^*) \\
\downarrow & & \downarrow \\
p\left(y(u_N^*)\right) & & p_N^*\left(y_N^*(u_N^*)\right)
\end{array}
$$

- Optimality error

  state: $e^{y,*} = y^*(u^*) - y_N^*(u_N^*)$

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- Predictability error

Error Definitions

$$
\begin{array}{ccccc}
\nabla \mathcal{L}(y^*, u^*, p^*; \mu) = 0 & & & & \nabla \mathcal{L}_N(y_N^*, u_N^*, p_N^*; \mu) = 0 \\
\downarrow & & & & \downarrow \\
u^* & & & & u_N^* \\
\downarrow & & & \swarrow & \downarrow \\
y^*(u^*) & & {\color{red} y(u_N^*)} & & {\color{red} y_N^*(u_N^*)} \\
\downarrow & & \downarrow & & \downarrow \\
p^*\left(y^*(u^*)\right) & & p\left(y(u_N^*)\right) & & p_N^*\left(y_N^*(u_N^*)\right)
\end{array}
$$

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  state: $e^{y,*} = y^*(u^*) - y_N^*(u_N^*)$

  adjoint: $e^{p,*} = p^*\left(y^*(u^*)\right) - p_N^*\left(y_N^*(u_N^*)\right)$

- Predictability error

  state: ${\color{red} \tilde{e}^y = y_N^*(u_N^*) - y(u_N^*)}$,

Error Definitions

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$$\nabla \mathcal{L}_N(y_N^*, u_N^*, p_N^*; \mu) = 0 \;\rightarrow\; u_N^* \;\rightarrow\; y_N^*(u_N^*) \;\rightarrow\; p_N^*\left(y_N^*(u_N^*)\right)$$

$$u_N^* \;\rightarrow\; y(u_N^*) \;\rightarrow\; p\left(y(u_N^*)\right)$$

- Optimality error

  state: $e^{y,*} = y^*(u^*) - y_N^*(u_N^*)$

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  state: $\tilde{e}^y = y_N^*(u_N^*) - y(u_N^*)$,

  adjoint: $\tilde{e}^p = p_N^*\left(y_N^*(u_N^*)\right) - p\left(y(u_N^*)\right)$

Control Error Bound — Main Idea [GK11]

Given u_N^*, the error in the optimal control satisfies [TV09]

$$\|u^* - u_N^*\|_{\mathcal{U}} \leq \frac{1}{\lambda} \left\| \lambda(u_N^* - u_d) - b\left(p\left(y(u_N^*)\right); \mu\right) \right\|_{\mathcal{U}}$$

where $y(u_N^*) \in Y$ is the solution of

$$a(y, \phi; \mu) = b(\phi; \mu) u_N^*, \quad \forall \phi \in Y,$$

and $p\left(y(u_N^*)\right) \in Y$ is the solution of

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$$a(\varphi, p^*; \mu) = (y_d - y(u_N^*), \varphi)_{L^2(D)}, \quad \forall \varphi \in Y.$$

- Provides rigorous bound for error in the optimal control,
- Evaluation of bound requires a forward/backward "truth" solve

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$$\left. \pm b\left(p_N^*\left(y_N^*(u_N^*)\right); \mu\right) \right\|_{\mathcal{U}}$$

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where $y_N^*(u_N^*) \in Y_N$ is the solution of

$$a(y_N^*, \phi; \mu) = b(\phi; \mu) u_N^*, \quad \forall \phi \in Y_N.$$

and $p_N^*\left(y_N^*(u^*)\right) \in Y_N$ is the solution of

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$$\left. + b\left(p_N^*\left(y_N^*(u_N^*)\right) - p\left(y(u_N^*)\right); \mu\right) \right\|_{\mathcal{U}}$$

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Control Error Bound — Main Idea [GK11]

Given u_N^*, the error in the optimal control satisfies

$$\|u^* - u_N^*\|_{\mathcal{U}} \leq \frac{1}{\lambda} \Big\| \underbrace{\lambda(u_N^* - u_d) - b\left(p_N^*\left(y_N^*(u_N^*)\right); \mu\right)}_{0 \text{ from optimality}} + b\left(p_N^*\left(y_N^*(u_N^*)\right) - p\left(y(u_N^*)\right); \mu\right) \Big\|_{\mathcal{U}}$$

where $y_N^*(u_N^*) \in Y_N$ is the solution of

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Control Error Bound — Main Idea [GK11]

Given u_N^*, the error in the optimal control satisfies

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$$a(\varphi, p_N^*; \mu) = (y_d - y_N^*(u_N^*), \varphi)_{L^2(D)}, \quad \forall \varphi \in Y_N.$$

Predictability Error Bounds

Recall: $y(u_N^*) \in Y$ is the solution of

$$a(y, \phi; \mu) = b(\phi; \mu) u_N^*, \quad \forall \phi \in Y,$$

and $y_N^*(u_N^*) \in Y_N$ is the solution of

$$a(y_N^*, \phi; \mu) = b(\phi; \mu) u_N^*, \quad \forall \phi \in Y_N.$$

Lemma 1 – State

The primal predictability error, $\tilde{e}^y = y_N^*(u_N^*) - y(u_N^*)$, is bounded by

$$\|\tilde{e}^y\|_Y \le \tilde{\Delta}_N^y(\mu) \equiv \frac{\|r_y(\cdot;\mu)\|_{Y'}}{\alpha_{\mathrm{LB}}(\mu)}, \quad \forall \mu \in \mathcal{D}$$

where $r_y(\phi;\mu) = b(\phi;\mu) u_N^* - a(y_N^*, \phi; \mu), \ \forall \phi \in Y$.

Predictability Error Bounds

Recall: $p\left(y(u_N^*)\right) \in Y$ is the solution of

$$a(\varphi, p; \mu) = (y(u_N^*) - y_d, \varphi)_{L^2(D)}, \quad \forall \varphi \in Y,$$

and $p_N^*\left(y_N^*(u^*)\right) \in Y_N$ is the solution of

$$a(\varphi, p_N^*; \mu) = (y_d - y_N^*(u_N^*), \varphi)_{L^2(D)}, \quad \forall \varphi \in Y_N.$$

Lemma 2 – Adjoint

The dual predictability error, $\tilde{e}^p = p_N^*\left(y_N^*(u_N^*)\right) - p\left(y(u_N^*)\right)$, is bounded by

$$\|\tilde{e}^p\|_Y \le \tilde{\Delta}_N^p(\mu) \equiv \frac{1}{\alpha_{\mathrm{LB}}(\mu)}\left(\|r_p(\cdot;\mu)\|_{Y'} + C_D^2\tilde{\Delta}_N^y(\mu)\right), \ \forall \mu \in \mathcal{D}$$

where

$$r_p(\varphi;\mu) = (y_d - y_N^*, \varphi)_{L^2(D)} - a(\varphi, p_N^*;\mu), \ \forall \varphi \in Y,$$

and $C_D \equiv \sup_{v\in Y} \|v\|_{L^2(D)}/\|v\|_Y$.

Control Error Bound

Given u_N^*, the error in the optimal control satisfies

$$\|u^* - u_N^*\|_{\mathcal{U}} \leq \frac{1}{\lambda} \|b(\cdot;\mu)\|_{Y'} \|p_N^* \left(y_N^*(u_N^*)\right) - p \left(y(u_N^*)\right)\|_Y,$$

$$\leq \Delta_N^{u,*}(\mu) \equiv \frac{1}{\lambda} \|b(\cdot;\mu)\|_{Y'} \tilde{\Delta}_N^p(\mu).$$

where $y_N^*(u_N^*) \in Y_N$ is the solution of

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and $p_N^* \left(y_N^*(u^*)\right) \in Y_N$ is the solution of

$$a(\varphi, p_N^*; \mu) = (y_d - y_N^*(u_N^*), \varphi)_{L^2(D)}, \quad \forall \varphi \in Y_N.$$

- Derive error bound $\tilde{\Delta}_N^p(\mu)$ ("standard" error bound + propagation)
- Allows offline/online decomposition

Extensions and Remarks

- The error bounds can be extended to [KG13a, KG13b, Kï1]
  - multiple control inputs: $\mathcal{U} \equiv \mathbb{R}^m, \ m > 1$
  - control constraints: $u_{\mathrm{LB}} \le u \le u_{\mathrm{UB}}$
  - time-varying (parabolic) problems: $u \in L^2(0, T)$
  - distributed controls: $u \in L^2(D)$
- We construct integrated spaces Y_N using a Greedy procedure on the control error bound
- Dual approach [KG13a] to obtain superconvergent error bounds for
  - the control ($\mathcal{U} \equiv \mathbb{R}^m$), and
  - general (linear) output functionals of state and adjoint
- We can develop error bounds for the cost functional [BKR00]

Problem Statement

Given $\mu \in \mathcal{D}$, find $u^*(t) \in \mathcal{U}^e \equiv L^2(0,T;\mathbb{R})$ such that

$$u^*(t) = \arg\min_{u \in \mathcal{U}^e} J(y(\mu), u(\mu); \mu)$$

where $y^e(t;\mu) \in W(0,T;Y^e)$ satisfies

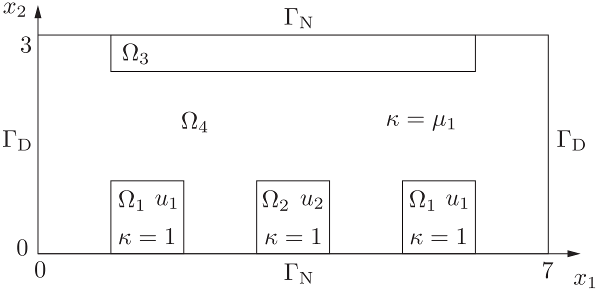
$$\frac{d}{dt} m(y^e(t), \phi) + a(y^e, \phi; \mu) = b(\phi)u(t), \quad \forall \phi \in Y^e, \text{ f.a.a. } t \in [0,T]$$

Cost Functional

$$J(y,u;\mu) = \frac{1}{2}\int_0^T \|y - y_d\|^2_{L^2(D)} dt + \frac{\sigma}{2}\|y(T) - y_d(T)\|^2_{L^2(D)} + \frac{\lambda}{2}\|u - u_d\|^2_{\mathcal{U}^e}$$

where $\lambda > 0$, $\sigma \geq 0$, and y_d (u_d) is the desired state (control input).

Model Problem



- Unsteady heat conduction in $[0, 4]$: $\boldsymbol{K = 200}$
- Control $\boldsymbol{u(t) = (u_1(t), u_2(t)) \in \mathcal{U} \equiv L^2(0, T; \mathbb{R}^2)}$
- Control $\boldsymbol{u_d = 0}$, state $\boldsymbol{y_d = (1.2 + \sin(\pi t))\chi_D(x)}$, $\boldsymbol{D = \Omega_3}$
- Regularization parameter: $\boldsymbol{\sigma = 0}$
- Input parameter: $\boldsymbol{\mu = (\kappa, \lambda) \in \mathcal{D} \equiv [0.5, 5] \times [0.1; 1]}$.

State Predictability Error and Bound

We present $|\Xi_{\text{test}}| = 40$

- Maximum relative predictability error $\epsilon^y_{N,\max,\text{rel}}$
- Maximum relative error bound $\tilde{\Delta}^y_{N,\max,\text{rel}}$
- Average effectivity $\bar{\eta}^y_N$

N	$\epsilon^y_{N,\max,\text{rel}}$	$\tilde{\Delta}^y_{N,\max,\text{rel}}$	$\bar{\eta}^y_N$
8	2.10E−1	2.99E−1	1.29E+0
16	1.32E−2	1.69E−2	1.24E+0
24	3.37E−3	4.39E−3	1.26E+0
32	5.94E−4	8.09E−4	1.23E+0
40	3.81E−4	5.22E−4	1.28E+0
48	7.40E−5	1.10E−4	1.26E+0
56	4.72E−5	7.24E−5	1.27E+0

Adjoint Predictability Error and Bound

We present $|\Xi_{\text{test}}| = 40$

- Maximum relative predictability error $\epsilon^p_{N,\max,\text{rel}}$
- Maximum relative error bound $\tilde{\Delta}^p_{N,\max,\text{rel}}$
- Average effectivity $\bar{\eta}^p_N$

N	$\epsilon^p_{N,\max,\text{rel}}$	$\tilde{\Delta}^p_{N,\max,\text{rel}}$	$\bar{\eta}^p_N$
8	$1.67E{-}1$	$4.47E{-}1$	$3.03E{+}0$
16	$9.07E{-}3$	$3.02E{-}2$	$4.01E{+}0$
24	$1.78E{-}3$	$7.90E{-}3$	$4.59E{+}0$
32	$3.56E{-}4$	$1.33E{-}3$	$3.83E{+}0$
40	$1.59E{-}4$	$8.47E{-}4$	$4.08E{+}0$
48	$3.33E{-}5$	$1.72E{-}4$	$3.87E{+}0$
56	$1.67E{-}5$	$9.24E{-}5$	$3.80E{+}0$

Control Error and Bound

We present $|\Xi_{\text{test}}| = 40$

- Maximum relative control error $\epsilon^{u,*}_{N,\max,\text{rel}}$ and bound $\tilde{\Delta}^{u,*}_{N,\max,\text{rel}}$
- Average effectivity $\bar{\eta}^{u,*}_N$

N	$\epsilon^{u,*}_{N,\max,\text{rel}}$	$\Delta^{u,*}_{N,\max,\text{rel}}$	$\bar{\eta}^{u,*}_N$
8	$6.26E{-}2$	$1.89E{+}0$	$5.01E{+}1$
16	$5.07E{-}3$	$1.28E{-}1$	$1.10E{+}2$
24	$5.58E{-}4$	$3.49E{-}2$	$2.47E{+}2$
32	$1.99E{-}5$	$8.67E{-}3$	$5.43E{+}2$
40	$6.04E{-}6$	$3.58E{-}3$	$2.49E{+}3$
48	$2.55E{-}7$	$1.06E{-}3$	$5.60E{+}3$
56	$1.52E{-}7$	$4.91E{-}4$	$4.86E{+}3$

Speed-up truth/RB ($N = 32$):

- OCP solution & bounds: ≈ 90

Problem Statement

Given $\mu \in \mathcal{D}$, find $u^* \in \mathcal{U}^{\mathrm{e}} \equiv L^2(\Omega)$ such that

$$u^* = \arg\min_{u \in \mathcal{U}^{\mathrm{e}}} J(y(\mu), u(\mu); \mu)$$

where $y^{\mathrm{e}}(\mu) \in Y^{\mathrm{e}}$ satisfies

$$a(y^{\mathrm{e}}, \phi; \mu) = b(\phi, u), \quad \forall \phi \in Y^{\mathrm{e}}.$$

Cost Functional: $J(y, u; \mu) = \frac{1}{2}\|y - y_d\|^2_{L^2(D)} + \frac{\lambda}{2}\|u - u_d\|^2_{\mathcal{U}^{\mathrm{e}}}$

Note: Additional reduction of control space $\mathcal{U}_M \subset \mathcal{U}$

- Set $\mathcal{U}_M = \mathrm{span}\{u^*(\mu^m),\ 1 \le m \le M\}$ and
  $Y_N = \mathrm{span}\{y^*(\mu^m), p^*(\mu^m), 1 \le m \le M\}$, where
  the parameters μ^m, $1 \le m \le M$, are picked by Greedy

Problem Statement

Given $\mu \in \mathcal{D}$, find $u^* \in \mathcal{U}^e \equiv L^2(\Omega)$ such that

$$u^* = \arg\min_{u \in \mathcal{U}^e} J(y(\mu), u(\mu); \mu)$$

where $y^e(\mu) \in Y^e$ satisfies

$$a(y^e, \phi; \mu) = b(\phi, u), \quad \forall \phi \in Y^e.$$

Cost Functional: $J(y, u; \mu) = \frac{1}{2}\|y - y_d\|^2_{L^2(D)} + \frac{\lambda}{2}\|u - u_d\|^2_{\mathcal{U}^e}$

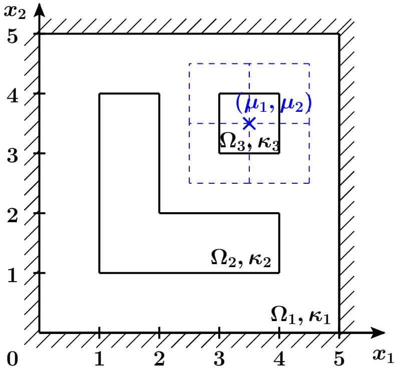
Note: Additional reduction of control space $\mathcal{U}_M \subset \mathcal{U}$

- Set $\mathcal{U}_M = \text{span}\{u^*(\mu^m),\ 1 \leq m \leq M\}$ and
  $Y_N = \text{span}\{y^*(\mu^m), p^*(\mu^m), 1 \leq m \leq M\}$, where
  the parameters $\mu^m,\ 1 \leq m \leq M$, are picked by Greedy

Disclaimer: Distributed OCP with control constraints

$\Rightarrow$ evaluation of $\Delta_N^{*,u}(\mu)$ incurs $\mathcal{N}$-dependent cost.

Model Problem

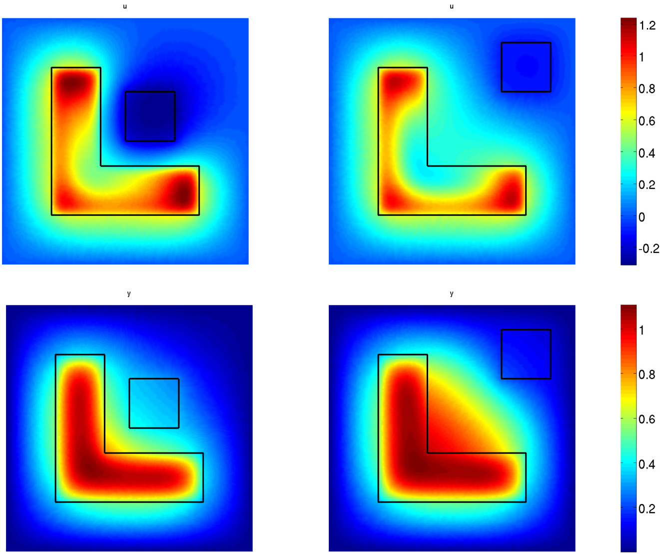


- Steady heat conduction with $\kappa_1 = 1,\ \kappa_2 = 0.2,\ \kappa_3 = 5$
- $Y \subset Y^e \equiv \{v \in H^1(\Omega), v|_{\partial\Omega} = 0\}$, $\dim(Y) = 18{,}117$
- Control $u_d = 0$, state $y_d = 1$ in Ω_2 and $y_d = 0$ in Ω_3
- Input parameter: $\mu = (\mu_1, \mu_2, \lambda) \in \mathcal{D} \equiv [3,4]^2 \times [0.1; 1]$.

Sample Solutions ($\lambda = 0.1$)

control

state



Control Error and Bound

We present $|\Xi_{\text{test}}| = 20$

- Maximum relative control error $\epsilon_{N,\max,\text{rel}}^{u,*}$ and bound $\Delta_{N,\max,\text{rel}}^{u,*}$
- Average effectivity $\bar{\eta}_N^{u,*}$

N	M	$\epsilon_{N,\max,\text{rel}}^{u,*}$	$\Delta_{N,\max,\text{rel}}^{u,*}$	$\bar{\eta}_N^{u,*}$
2	1	$4.89\,E-01$	$4.48\,E+01$	29.9
10	5	$9.69\,E-02$	$1.65\,E+00$	47.3
30	15	$3.98\,E-03$	$1.52\,E-01$	43.7
60	30	$3.32\,E-04$	$1.77\,E-02$	75.0
90	45	$7.97\,E-05$	$3.96\,E-03$	77.0

Speed-up truth/RB ($N = 90$):

- OCP solution & bounds: ≈ 100 (75% of ∂t_{Δ} for $\alpha_{\text{LB}}(\mu)$ via SCM)

References I

[BKR00] Roland Becker, Hartmut Kapp, and Rolf Rannacher. Adaptive finite element methods for optimal control of partial differential equations: Basic concept. *SIAM J Control Optim*, 39:113–132, 2000.

[Ded10] Luca Dedè. Reduced basis method and a posteriori error estimation for parametrized linear-quadratic optimal control problems. *SIAM J. Sci. Comput.*, 32(2):997–1019, 2010.

[Ded12] Luca Dedè. Reduced basis method and error estimation for parametrized optimal control problems with control constraints. *Journal of Scientific Computing*, 50(2):287–305, 2012.

[GK11] Martin A. Grepl and Mark Kärcher. Reduced basis a posteriori error bounds for parametrized linear-quadratic elliptic optimal control problems. *Comptes Rendus Mathematique*, 349(15-16):873–877, 2011.

[Kï1] Mark Kärcher. The reduced-basis method for parametrized linear-quadratic elliptic optimal control problems. Master's thesis, Technische Universität München, April 2011.

[KG13a] Mark Kärcher and Martin A. Grepl. A certified reduced basis method for parametrized elliptic optimal control problems. *ESAIM: Control, Optimisation and Calculus of Variations*, 2013. accepted for publication.

[KG13b] Mark Kärcher and Martin A. Grepl. Certified reduced basis solution of parametrized parabolic optimal control problems. Technical report, submitted to ESAIM: Mathematical Modelling and Numerical Analysis, 2013.

[NRMQ12] F. Negri, G Rozza, A. Manzoni, and A. Quarteroni. Reduced basis method for parametrized elliptic optimal control problems. Technical report, Submitted, 2012.

[TV09] F. Tröltzsch and S. Volkwein. POD a-posteriori error estimates for linear-quadratic optimal control problems. *Computational Optimization and Applications*, 44:83–115, 2009.