TU Munich, 16-20 September 2013, RB Summer School

An introduction to geometrical parametrizations for the applications of reduced order modelling: learning by examples FUNDAMENTALS [RHP, 2008, ARCME, Vol.15, 229-275]

Gianluigi Rozza

Collaboration Network

MOX (A. Quarteroni, F. Ballarin, P. Pacciarini)

EPFL (T. Lassila, F. Negri, P. Chen, D. Forti)

MIT (A.T. Patera, D.B.P. Huynh, C.N. Nguyen)

SISSA (A. Manzoni, D. Devaud), U. Konstanz (L. Iapichino)

Outline

Simple Elliptic μPDEs: Setting

Problem Scope: Geometry

Problem Scope: Bilinear Forms

Working Examples: TBlock
AMass
EBlock3D

Simple Elliptic μPDEs

Statement

Given $\boldsymbol{\mu} \in \mathcal{D} \subset \mathbb{R}^P$,

evaluate $s^{\mathrm{e}}(\boldsymbol{\mu}) = \ell(u^{\mathrm{e}}(\boldsymbol{\mu}))$ †

where $u^{\mathrm{e}}(\boldsymbol{\mu}) \in X^{\mathrm{e}}(\Omega)$ satisfies

$$a(u^{\mathrm{e}}(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = f(v), \quad \forall\, v \in X^{\mathrm{e}}\,.$$

---

† Here $^{\mathrm{e}}$ refers to "exact."

Simple Elliptic μPDEs

Statement

Definitions and ...

μ: input parameter; P-tuple

$\mathcal{D}$: parameter domain;

s^{e}: output;

ℓ: linear bounded output functional;

u^{e}: field variable;

X^{e}: function space $(H_0^1(\Omega))^\nu \subset X^{\mathrm{e}} \subset (H^1(\Omega))^\nu$;

Simple Elliptic μPDEs

Statement

...Hypotheses

$$\left.\begin{array}{rl} a(\,\cdot\,,\cdot\,;\mu): & \text{bilinear,} \\ & \text{continuous,} \\ & \text{symmetric,} \\ & \text{coercive form, } \forall\,\mu \in \mathcal{D}; \\ f: & \text{linear bounded functional.} \end{array}\right\} \mu\text{PDE}$$

COMPLIANT case: $\ell = f$ (and a symmetric).

Simple Elliptic μPDEs

Statement

Affine Parameter Dependence†

Definition:

$$a(w, v; \mu) = \sum_{q=1}^{Q} \Theta^q(\mu)\, a^q(w, v)$$

where for $q = 1, \ldots, Q$

$\Theta^q\colon\ \mathcal{D} \to \mathbb{R},$ μ-*dependent* functions ;

$a^q\colon\ X^{\mathrm{e}} \times X^{\mathrm{e}} \to \mathbb{R},$ μ-*independent* forms .

---

†In fact, *broadly applicable* to many instances of property *and* geometry parametric variation.

Simple Elliptic μPDEs

FE Approximation

Galerkin Projection

Given $\mu \in \mathcal{D} \subset \mathbb{R}^P$,

evaluate $s^{\mathcal{N}}(\mu) = f(u^{\mathcal{N}}(\mu))$ †

where $u^{\mathcal{N}}(\mu) \in X^{\mathcal{N}} \subset X^{\mathrm{e}}$ satisfies $a^{\mathcal{N}}$

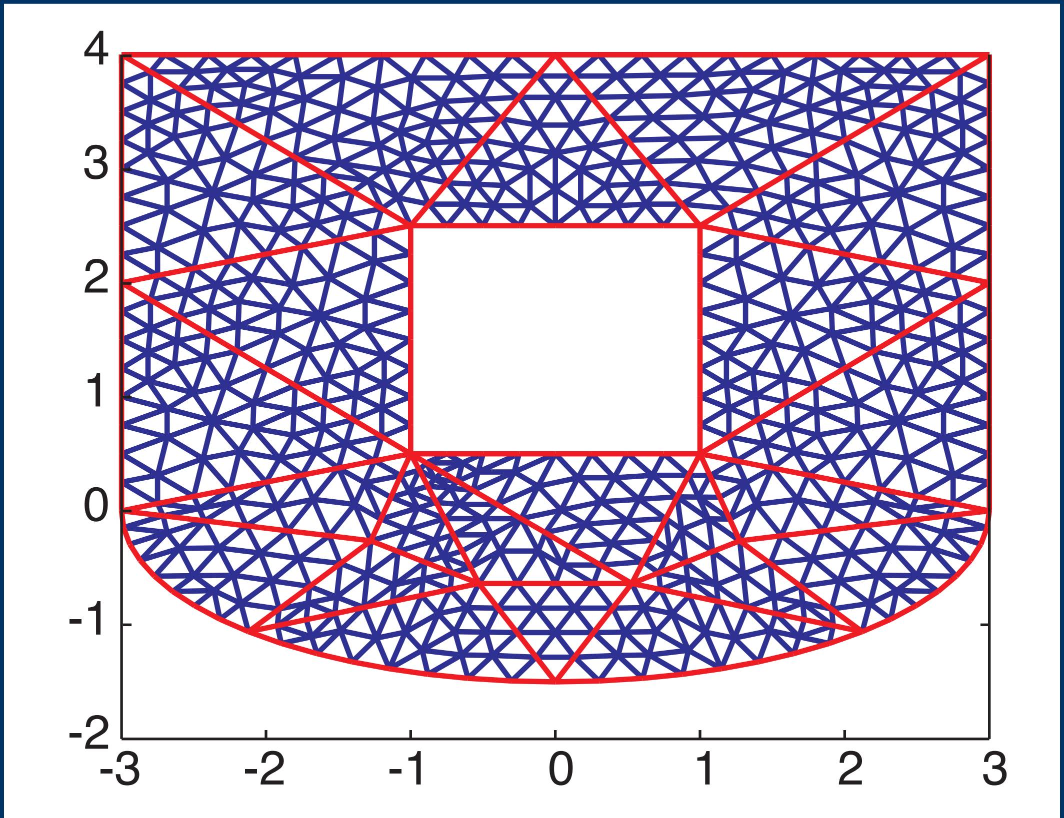
$$a(u^{\mathcal{N}}(\mu), v; \mu) = f(v), \quad \forall\, v \in X^{\mathcal{N}}.$$

†Here $X^{\mathcal{N}}$ is a sequence of FE approximation spaces indexed by $\dim(X^{\mathcal{N}}) = \mathcal{N}$.

Simple Elliptic μPDEs

FE Approximation

Typical Triangulation



Goal

For *any* $\varepsilon_{\rm des} > 0$, evaluate ACCURACY

$$\mu \in \mathcal{D} \rightarrow s_N^{\mathcal{N}}(\mu) \;\; (\approx s^{\mathcal{N}}(\mu))$$

that *provably* achieves desired accuracy RELIABILITY

$$|s^{\mathcal{N}}(\mu) - s_N^{\mathcal{N}}(\mu)| \leq \varepsilon_{\rm des}$$

but at (very low) marginal cost $\partial t_{\rm comp}$[†] EFFICIENCY

independent of $\mathcal{N}$ as $\mathcal{N} \rightarrow \infty$.

[†] $\partial t_{\rm comp}$: time to perform *one additional certified* evaluation $\mu \rightarrow s_N^{\mathcal{N}}(\mu)$.

Simple Elliptic μPDEs

Goal

Relevance

Real-Time Context (parameter estimation, . . .):

$$t_0\colon\quad \mu \qquad \to \qquad t_0 + \partial t_{\rm comp}\colon\quad s_N^{\mathcal{N}}(\mu)\ .$$

"need" "response"

Many-Query Context (dynamic simulation, . . .):

$$t_{\rm comp}(\mu_j \to s_N^{\mathcal{N}}(\mu_j),\ j = 1, \ldots, J)$$

$$= \partial t_{\rm comp}\, J \text{ as } J \to \infty\ .$$

Problem "Scope": Geometry

Domain Decomposition

Definition

Original Domain $\boldsymbol{\Omega}_{\mathrm{o}}(\boldsymbol{\mu})$, $\quad$ $\boldsymbol{u}_{\mathrm{o}}^{\mathrm{e}} \in \boldsymbol{X}_{\mathrm{o}}^{\mathrm{e}}(\boldsymbol{\Omega}_{\mathrm{o}}(\boldsymbol{\mu}))$

$$\overline{\boldsymbol{\Omega}}_{\mathrm{o}}(\boldsymbol{\mu}) = \bigcup_{k=1}^{K_{\mathrm{dom}}} \overline{\boldsymbol{\Omega}}_{\mathrm{o}}^{k}(\boldsymbol{\mu}) \ ;$$

reference domain $\boldsymbol{\Omega}$, $\quad$ $\boldsymbol{u}^{\mathrm{e}} \in \boldsymbol{X}^{\mathrm{e}}(\boldsymbol{\Omega})$

$$\overline{\boldsymbol{\Omega}} = \bigcup_{k=1}^{K_{\mathrm{dom}}} \overline{\boldsymbol{\Omega}}^{k} \ ,$$

common configuration

where $\boldsymbol{\Omega} = \boldsymbol{\Omega}_{\mathrm{o}}(\boldsymbol{\mu}_{\mathrm{ref}})$ for $\boldsymbol{\mu}_{\mathrm{ref}} \subset \boldsymbol{\mathcal{D}}^{\dagger}$.

---

†Connectivity requirement: subdomain intersections must be an entire edge, a vertex, or null.

Problem "Scope": Geometry

Domain Decomposition

Building Blocks

For $\mathbf{\Omega}^k$, $\mathbf{\Omega}_\mathrm{o}^k(\boldsymbol{\mu})$ we choose in $\mathbf{R}^{2\dagger}$,

(Parallelograms — by hand); EBlock3D
Triangles;
Elliptical Triangles*; and
Curvy Triangles*.

---

†In $\mathbf{R}^3$, we choose Parallelepipeds (and in theory Tetrahedra).

Problem "Scope": Geometry

Affine Mappings

Local

Require $\forall \mu \in \mathcal{D}$

$$\overline{\Omega}^k_{\mathrm{o}}(\mu) = \mathcal{T}^{\mathrm{aff},k}(\overline{\Omega}^k; \mu)\ , 1 \le k \le K_{\mathrm{dom}}\ ,$$

where

$$\mathcal{T}^{\mathrm{aff},k}(x; \mu) = C^{\mathrm{aff},k}(\mu) + G^{\mathrm{aff},k}(\mu) x\ ,$$

is an invertible affine mapping from $\overline{\Omega}^k$ onto $\overline{\Omega}^k_{\mathrm{o}}(\mu)$.

Problem "Scope": Geometry

Affine Mappings

Global

Further require $\forall \mu \in \mathcal{D}$

$$\mathcal{T}^{\mathrm{aff},k}(x;\mu) = \mathcal{T}^{\mathrm{aff},k'}(x;\mu), \quad \forall\, x \in \overline{\Omega}^k \cap \overline{\Omega}^{k'}, \\ 1 \le k, k' \le K_{\mathrm{dom}}\,,$$

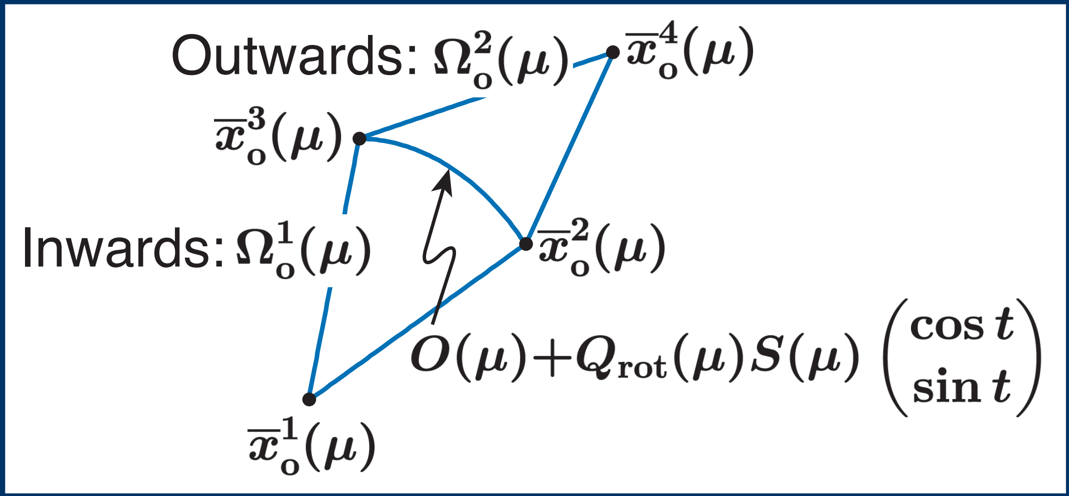
to ensure a *continuous* piecewise-affine global mapping $\mathcal{T}^{\mathrm{aff}}(\,\cdot\,;\mu)$ from $\overline{\Omega}$ onto $\overline{\Omega}_{\mathrm{o}}(\mu)^{\dagger}$.

†It follows that for $w_{\mathrm{o}} \in H^1(\Omega_{\mathrm{o}}(\mu)),\ w_{\mathrm{o}} \circ \mathcal{T}^{\mathrm{aff}} = H^1(\Omega)$.

Problem "Scope": Geometry

Elliptical Triangles

Definition



$$\begin{aligned} O(\mu) &= (x_{o1}^{\rm cen}, x_{o2}^{\rm cen})^{\rm T} \\ Q_{\rm rot}(\mu) &= \begin{pmatrix} \cos\phi(\mu) & -\sin\phi(\mu) \\ \sin\phi(\mu) & \cos\phi(\mu) \end{pmatrix} \\ S(\mu) &= \operatorname{diag}(\rho_1(\mu), \rho_2(\mu)) \end{aligned}$$

Problem "Scope": Geometry

Elliptical Triangles

Constraints

Given $\overline{x}_{\mathrm{o}}^{2}(\mu), \overline{x}_{\mathrm{o}}^{3}(\mu)$, find $\overline{x}_{\mathrm{o}}^{1}(\mu), \overline{x}_{\mathrm{o}}^{4}(\mu)$ $\quad (\Rightarrow \mathcal{T}^{\mathrm{aff},1\&2})$

$$\left.\begin{array}{ll} (i) & \text{produce desired elliptical arc} \\ (ii) & \text{satisfy internal angle criterion} \end{array}\right\} \forall \mu \in \mathcal{D};$$

conditions ensure *continuous invertible* mappings.

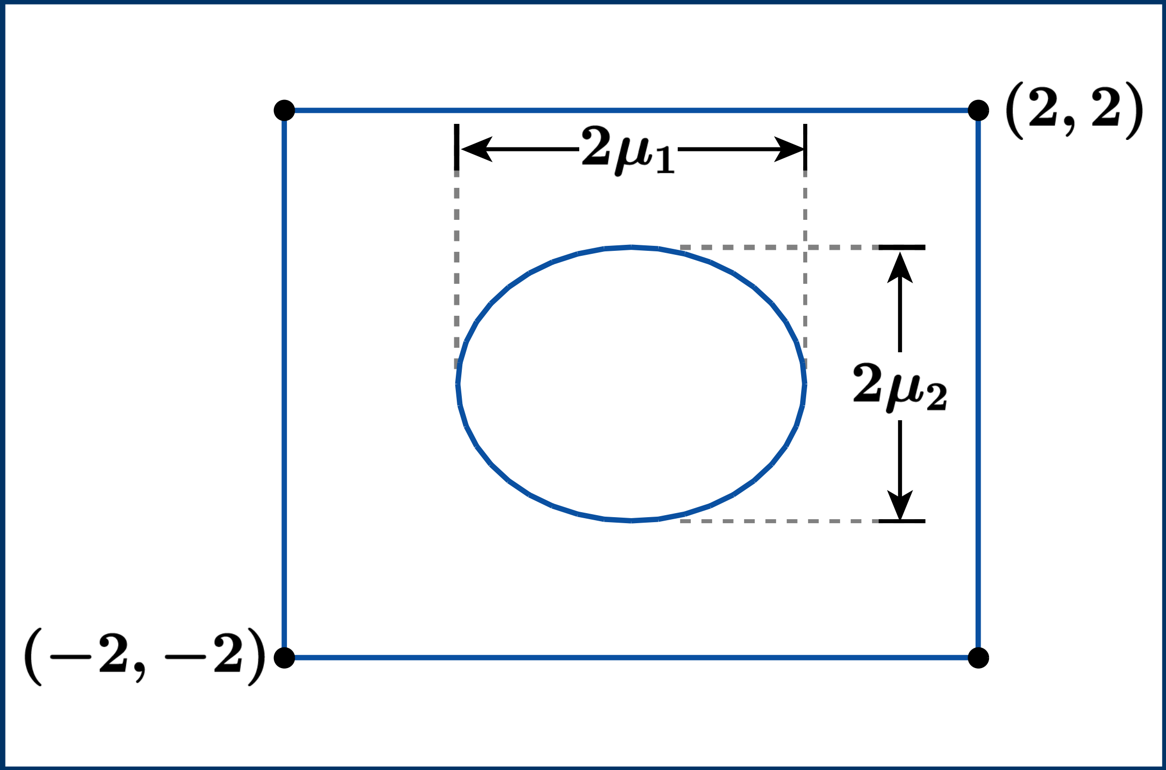
---

† Explicit recipes for admissible $x_{\mathrm{o}}^{1}(\mu)$ (Inwards case) and $x_{\mathrm{o}}^{4}(\mu)$ (Outwards case) are readily obtained.

Problem “Scope”: Geometry

Elliptical Triangles

Triangulation: ‘CinS’...

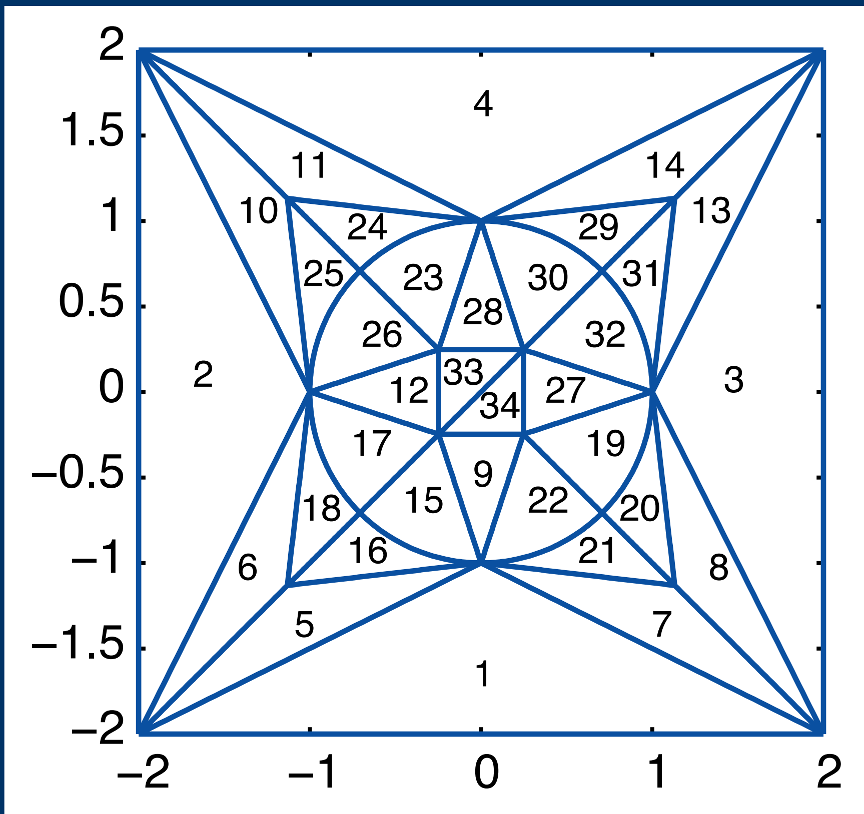


$$\Omega_o(\mu):\ \mu = (\mu_1, \mu_2, \ldots) \subset \mathcal{D} \equiv [0.8, 1.2]^2 \times \ldots$$

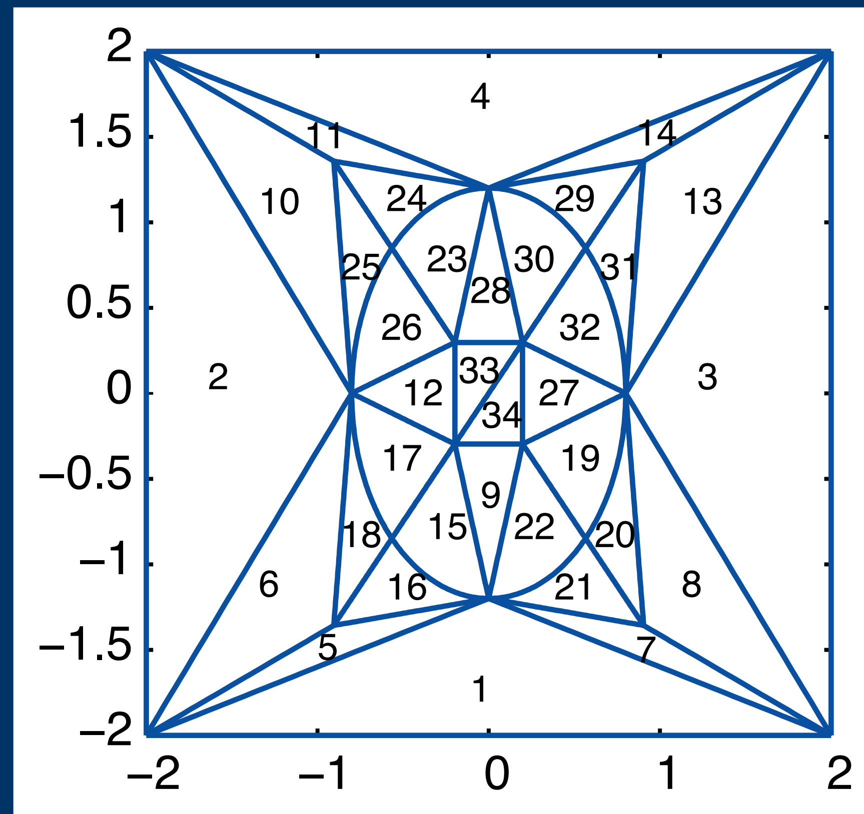
Problem “Scope”: Geometry

Elliptical Triangles

...Triangulation: ‘CinS’



$$\Omega = \Omega_o(\mu_{ref} = (1, 1))$$

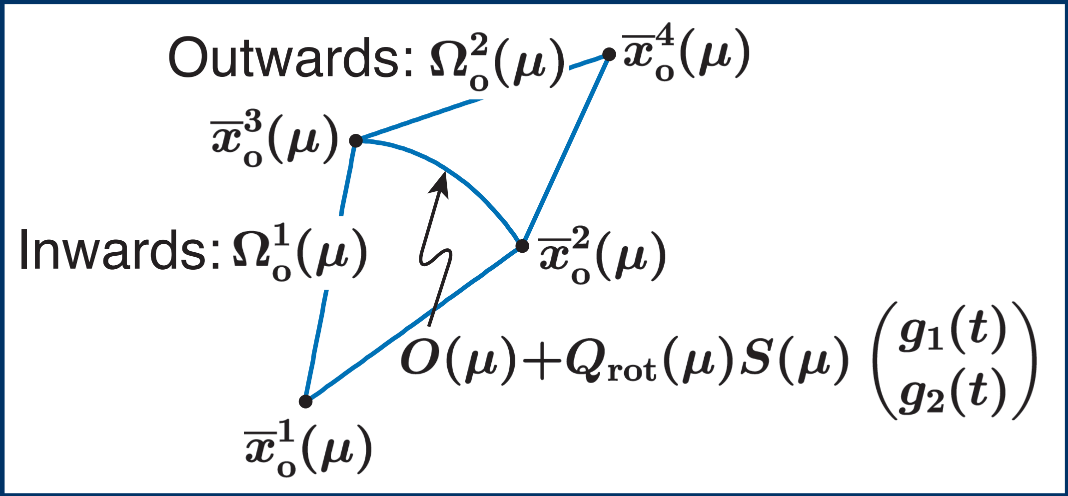


$$\Omega_o(\mu = (0.8, 1.2))$$

Problem "Scope": Geometry

Curvy Triangles

Definition



$$O(\mu) = (x_{o1}^{\text{cen}}, x_{o2}^{\text{cen}})^{\text{T}}$$

$$Q_{\text{rot}}(\mu) = \begin{pmatrix} \cos\phi(\mu) & -\sin\phi(\mu) \\ \sin\phi(\mu) & \cos\phi(\mu) \end{pmatrix}$$

$$S(\mu) = \text{diag}(\rho_1(\mu), \rho_2(\mu))$$

Problem “Scope”: Geometry

Curvy Triangles

Constraints

Given $\overline{x}_{\rm o}^2(\mu), \overline{x}_{\rm o}^3(\mu)$, find $\overline{x}_{\rm o}^1(\mu), \overline{x}_{\rm o}^4(\mu)$ $\quad (\Rightarrow \mathcal{T}^{\rm aff,1\&2})$

$$\left.\begin{array}{ll} (i) & \text{produce desired curvy arc} \\ (ii) & \text{satisfy internal angle criterion} \end{array}\right\} \forall \mu \in \mathcal{D};$$

conditions ensure *continuous invertible* mappings.

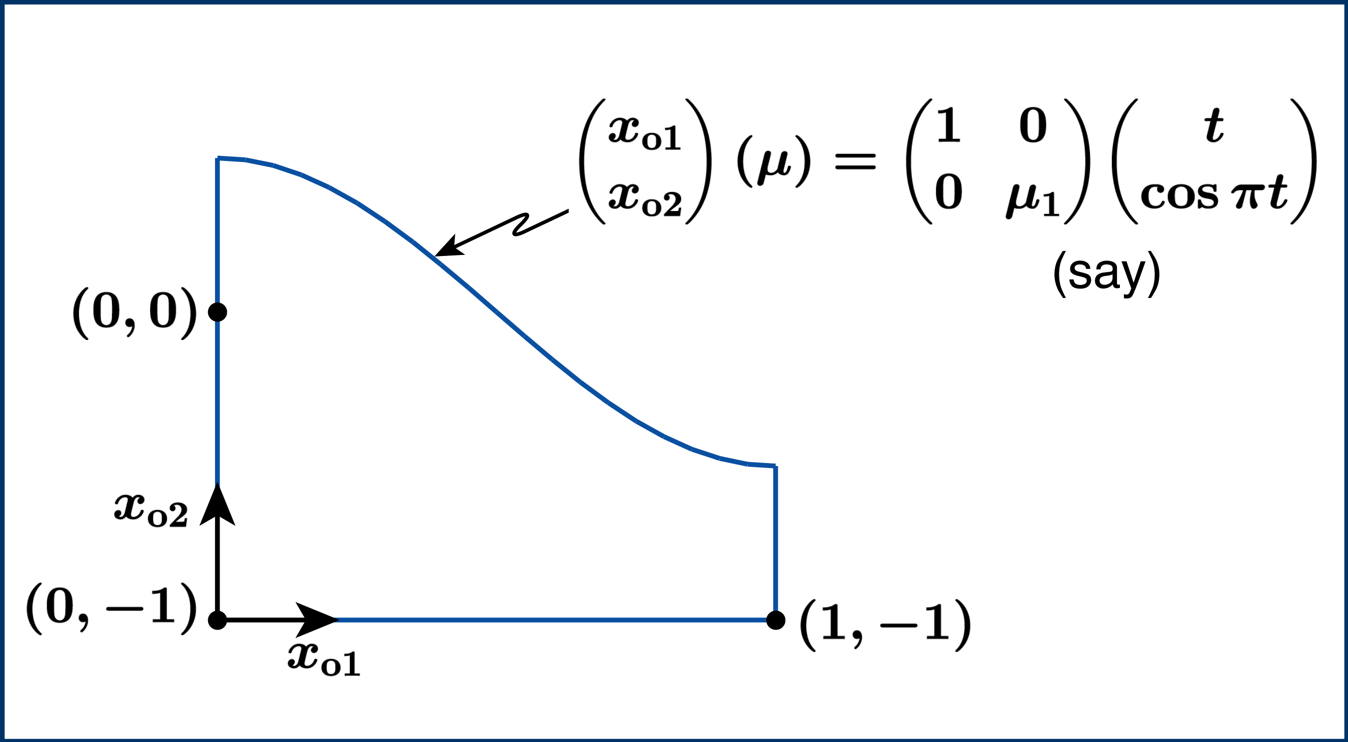
---

†Quasi-explicit recipes for admissible $\overline{x}_{\rm o}^1(\mu)$ and $\overline{x}_{\rm o}^4(\mu)$ can (sometimes) be obtained in the convex/concave case.

Problem “Scope”: Geometry

Curvy Triangles

Triangulation: ‘Cosine’...

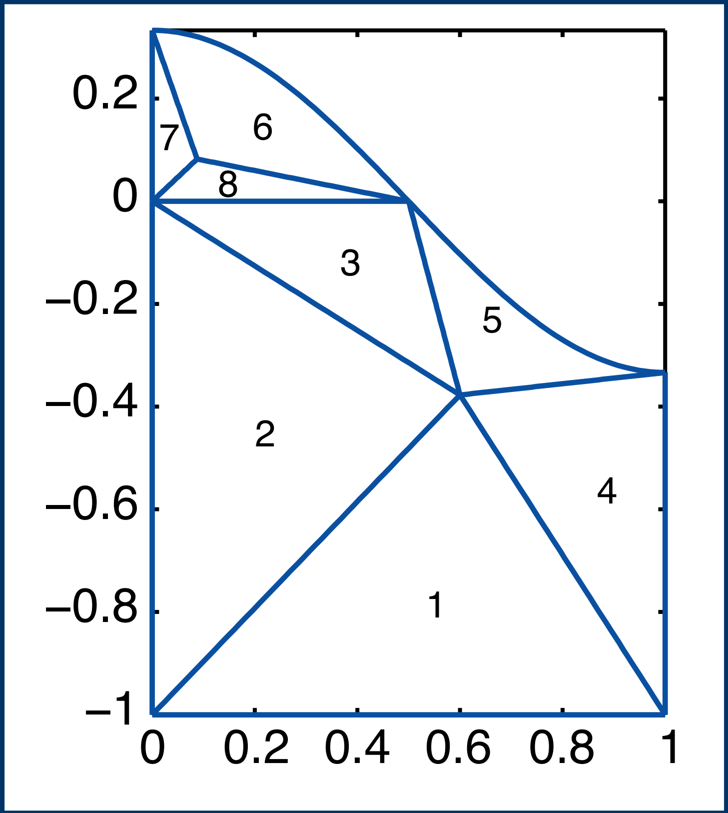


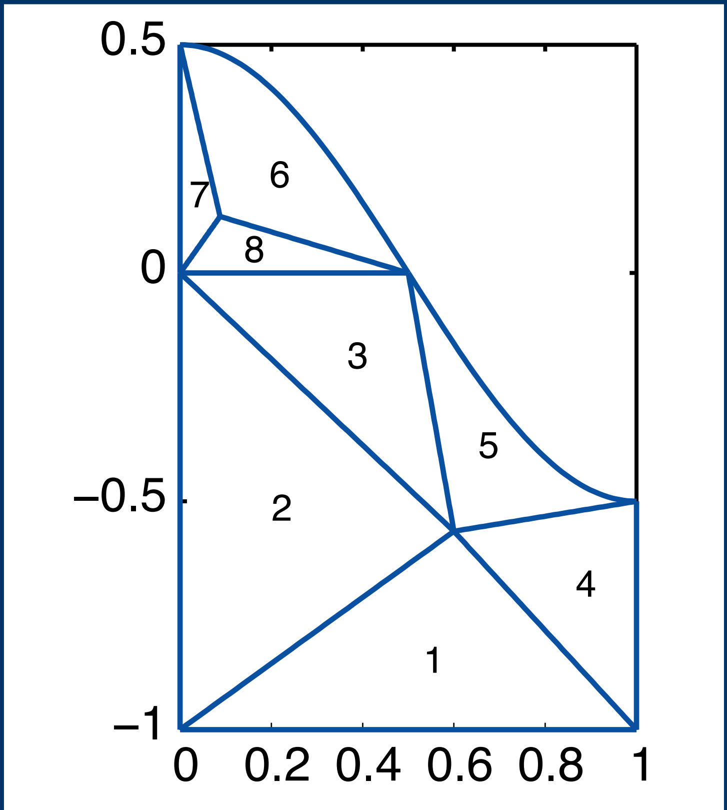
$$\Omega_o(\mu): \ \mu = (\mu_1, \ldots) \subset \mathcal{D} \equiv [\tfrac{1}{6}, \tfrac{1}{2}] \times \ldots$$

Problem “Scope”: Geometry

Curvy Triangles

...Triangulation: ‘Cosine’





$\Omega = \Omega_o(\mu_{\text{ref}} = \frac{1}{3})$

$\Omega_o(\mu = \frac{1}{2})$

Problem Scope: Bilinear Form

Transformation

Original Domain ($\mathbb{R}^2$)

For $w, v \in H^1(\Omega_\mathrm{o}(\mu))$[†] $\qquad u^\mathrm{e}_\mathrm{o}(\mu) \in H^1_0(\Omega_\mathrm{o}(\mu))$

$$a_\mathrm{o}(w, v; \mu) = \sum_{k=1}^{K_\mathrm{dom}} \int_{\Omega^k_\mathrm{o}(\mu)} \begin{bmatrix} \frac{\partial w}{\partial x_{\mathrm{o}1}} & \frac{\partial w}{\partial x_{\mathrm{o}2}} & w \end{bmatrix} \mathcal{K}^k_{\mathrm{o}ij}(\mu) \begin{bmatrix} \frac{\partial v}{\partial x_{\mathrm{o}1}} \\ \frac{\partial v}{\partial x_{\mathrm{o}2}} \\ v \end{bmatrix}$$

where $\mathcal{K}^k_\mathrm{o}\colon \mathcal{D} \to \mathrm{R}^{3\times 3}$, SPD for $1 \le k \le K_\mathrm{dom}$

(note $\mathcal{K}^k_\mathrm{o}$ affine in x_o is also permissible).

[†] We consider the scalar case; the vector case (linear elasticity) admits an analogous treatment.

Problem Scope: Bilinear Form

Transformation

Reference Domain

For $w, v \in H^1(\Omega)$ $\qquad u^{\mathrm{e}}(\mu) \in H_0^1(\Omega)$

$$a(w, v; \mu) = \sum_{k=1}^{K_{\mathrm{dom}}} \int_{\Omega^k} \begin{bmatrix} \frac{\partial w}{\partial x_1} & \frac{\partial w}{\partial x_2} & w \end{bmatrix} \mathcal{K}_{ij}^k(\mu) \begin{bmatrix} \frac{\partial v}{\partial x_1} \\ \frac{\partial v}{\partial x_2} \\ v \end{bmatrix}$$

$$\mathcal{K}^k(\mu) = |\det G^{\mathrm{aff},k}(\mu)| D(\mu) \mathcal{K}_{\mathrm{o}}^k(\mu) D^T(\mu),$$ and

$$D(\mu) = \begin{pmatrix} (G^{\mathrm{aff},k})^{-1} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem Scope: Bilinear Form

Transformation

Affine Form

Expand

$$a(w, v; \mu) = \underbrace{\mathcal{K}^1_{11}(\mu)}_{\Theta^1(\mu)} \underbrace{\int_{\Omega^1} \frac{\partial w}{\partial x_1} \frac{\partial v}{\partial x_1}}_{a^1(w,v)} + \ldots$$

with as many as $Q = 4K$ terms.

We (Maple) can often greatly reduce the requisite Q.

Problem Scope: Bilinear Form

Transformation

Achtung!

Many interesting problems are *not* affine (or require Q very large).

For example,

$\mathcal{K}^k_{\mathrm{o}}(x;\mu)$ for general x dependence; and

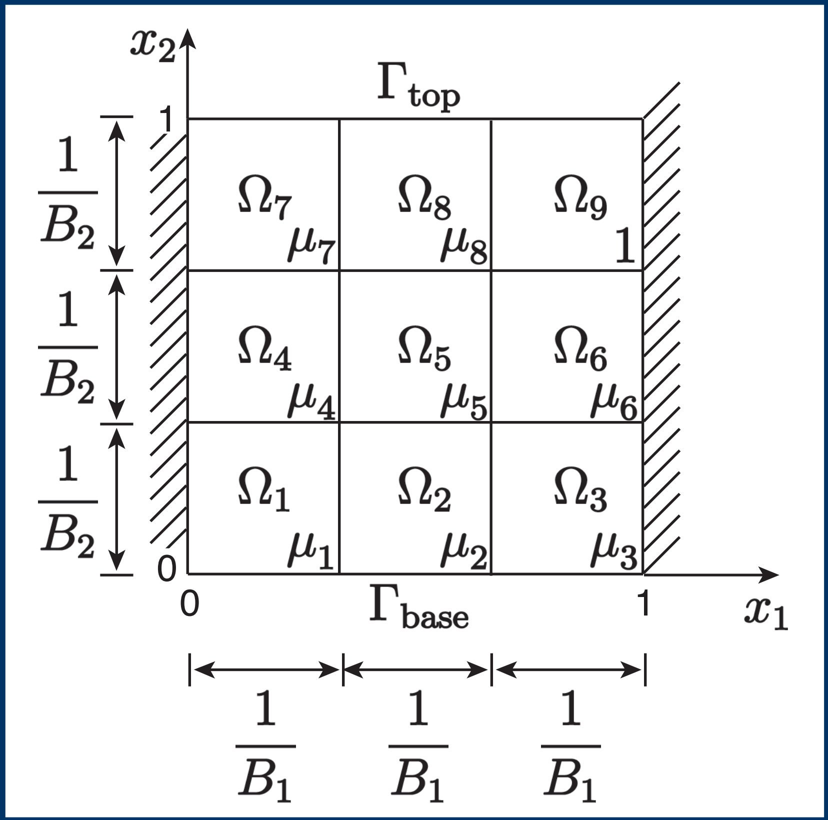
nonzero Neumann conditions on curvy $\partial\Omega$;

yield non-affine $a(\,\cdot\,,\,\cdot\,;\mu)$.

Working Examples

T(hermal)Block: Theory

Geometry



$$\overline{\boldsymbol{\Omega}} = \cup_{i=1}^{B_1 B_2} \overline{\boldsymbol{\Omega}}_i$$

Working Examples

T(hermal)Block: Theory

Problem Statement...

Given $\mu \equiv (\mu_1, \ldots, \mu_P) \in \mathcal{D} \equiv [\mu^{\min}, \mu^{\max}]^P$ †

evaluate $s^{\mathrm{e}}(\mu) = f(u^{\mathrm{e}}(\mu))$

where $u^{\mathrm{e}}(\mu) \in X^{\mathrm{e}} \equiv \{v \in H^1(\Omega) \mid v|_{\Gamma_{\mathrm{top}}} = 0\}$

satisfies $a(u^{\mathrm{e}}(\mu), v; \mu) = f(v), \quad \forall\, v \in X^{\mathrm{e}}$.

---

†Here $P = B_1 B_2 - 1$; we require $0 < \mu^{\min} < \mu^{\max} < \infty$.

Working Examples

T(hermal)Block: Theory

...Problem Statement

Here

$$f(v) \equiv f^{\mathrm{Neu}}(v) \equiv \int_{\Gamma_{\mathrm{base}}} v \; ,$$

and symmetric, coercive

$$a(w, v; \mu) = \sum_{i=1}^{P} \mu_i \int_{\Omega_i} \nabla w \cdot \nabla v + \int_{\Omega_{P+1}} \nabla w \cdot \nabla v \; ,$$

where $\overline{\Omega} = \cup_{i=1}^{P+1} \overline{\Omega}_i$.

Working Examples

T(hermal)Block: Theory

Affine Representation

We obtain $P = B_1 B_2 - 1$

$$a(w, v; \mu) = \sum_{q=1}^{Q=P+1} \Theta^q(\mu)\, a^q(w, v)$$

for

$$\Theta^q(\mu) = \mu_q,\ 1 \le q \le P, \quad \text{and} \quad \Theta^{P+1} = 1\,,$$

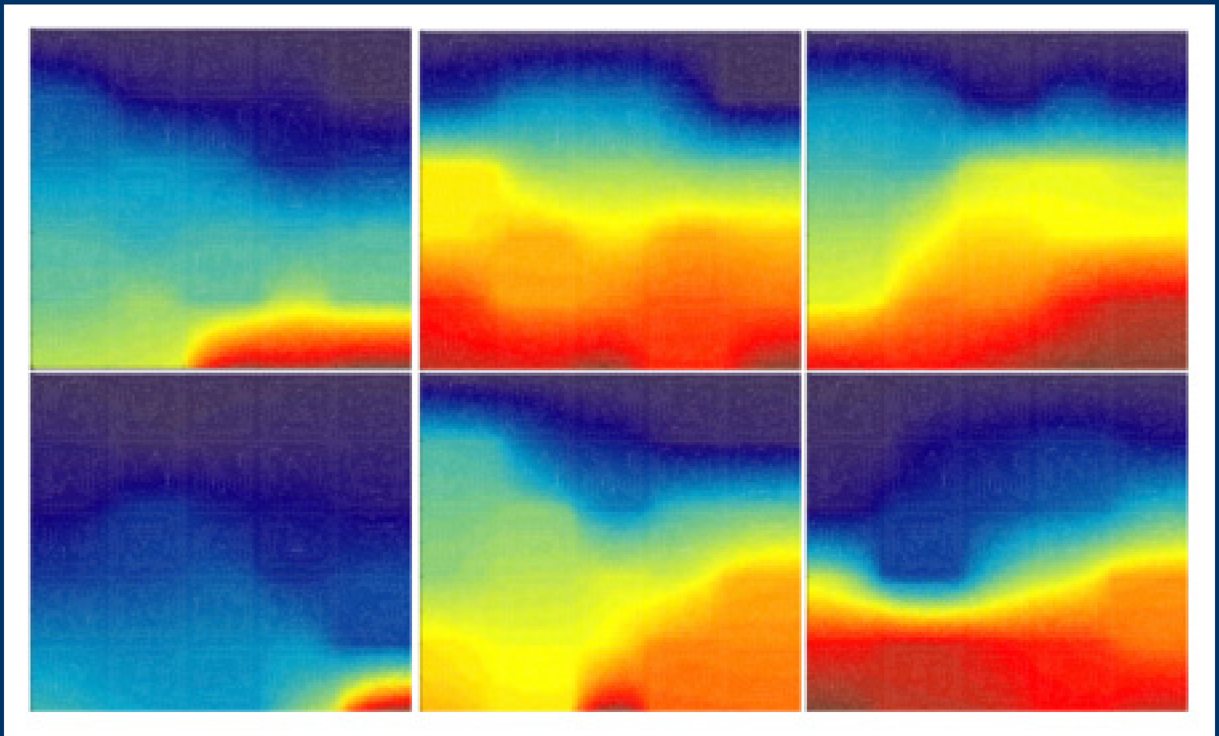
and

$$a^q(w, v) = \int_{\Omega_q} \nabla w \cdot \nabla v,\ 1 \le q \le P + 1\,.$$

Working Examples

T(hermal)Block: Theory

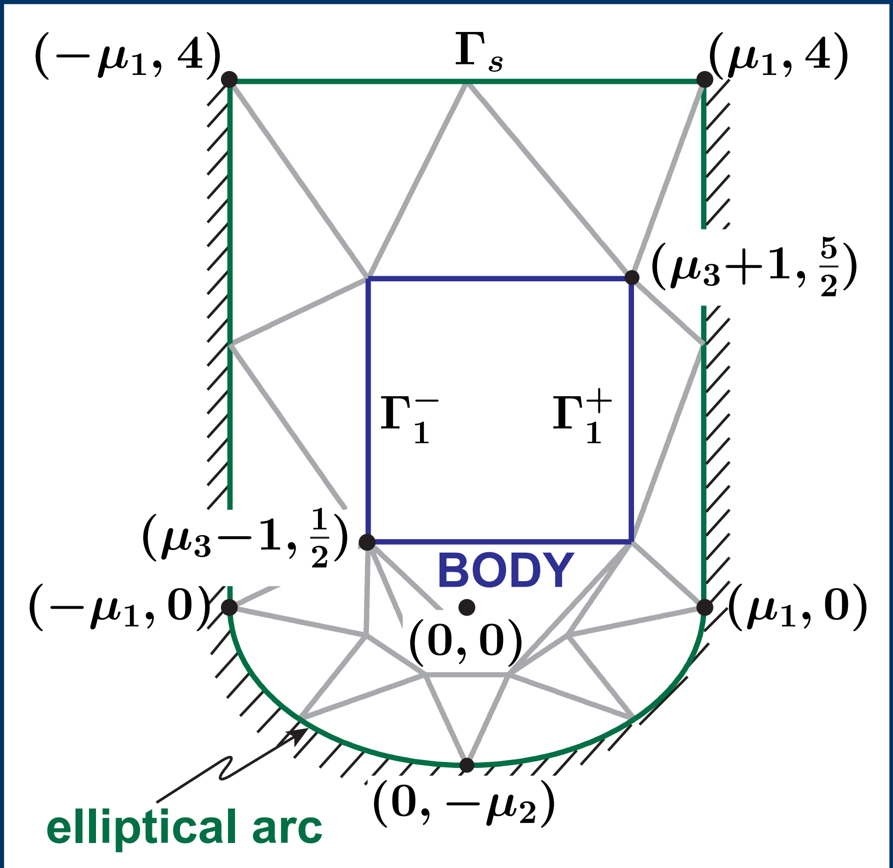
Representative Solutions



Working Examples

A(dded)Mass: Practice

Geometry...

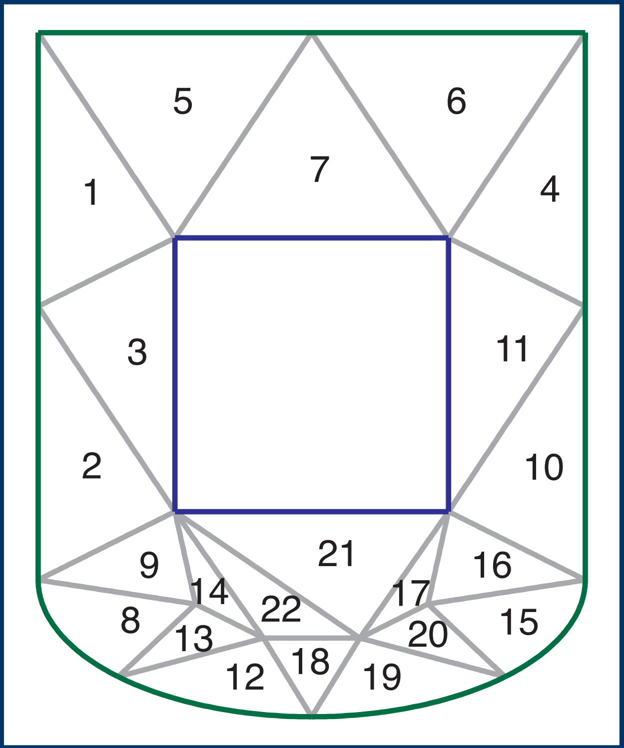


$$\Omega_o(\mu = (2.0, 1.2, .25)) \qquad = \mathcal{T}^{\text{aff}}(\Omega; \mu)$$

Working Examples

A(dded)Mass: Practice

...Geometry



$$\Omega \qquad = \Omega_o(\mu_{\text{ref}} = (2, 1, 0))$$

Working Examples

A(dded)Mass: Practice

Problem Statement...

Given $\mu \equiv (\mu_1, \mu_2, \mu_3) \in \mathcal{D}$ [†]

evaluate $s^{\mathrm{e}} = f(u^{\mathrm{e}}(\mu))$, ADDED MASS

where $u^{\mathrm{e}}(\mu) \in X^{\mathrm{e}} \equiv \{v \in H^1(\Omega) \,\big|\, v|_{\Gamma_s} = 0\}$ satisfies

$$a(u^{\mathrm{e}}(\mu), v; \mu) = f(v), \quad \forall\, v \in X^{\mathrm{e}} .$$

[†]Here $\mathcal{D} = [1.5, 3] \times [0.5, 1.5] \times [-0.35, 0.35]$;
for Demo, $\mathcal{D}$ shall be further restricted.

Working Examples

...Problem Statement

Here

$$f(v) = \int_{\Gamma_1^+} v - \int_{\Gamma_1^-} v \,,$$

and symmetric, coercive

$$a(w, v; \mu) = \int_\Omega \frac{\partial w}{\partial x_i} \; \underbrace{\kappa_{i\,j}(\mu)}_{\text{SPD}} \; \frac{\partial v}{\partial x_j} \,,$$

where $\kappa_{i\,j}(\mu)$ is induced by $\mathcal{T}^{\mathrm{aff}}(\,\cdot\,;\mu)$.

Working Examples

A(dded)Mass: Practice

Affine Representation...

We obtain $Q = 34$

$$a(w, v; \mu) = \sum_{q=1}^{Q} \Theta^q(\mu)\, a^q(w, v) \;,$$

where the

> *piecewise affine geometry mapping*, and
> *bilinear form affine representation*

are generated by symbolic manipulation.

Working Examples

A(dded)Mass: Practice

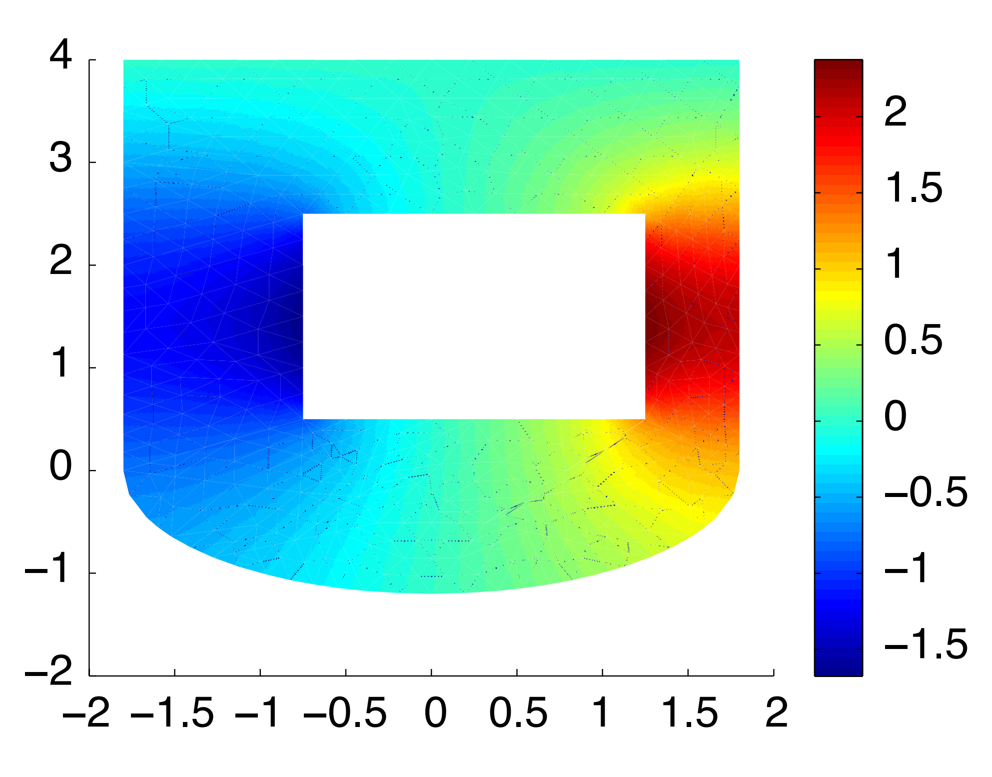
...Affine Representation

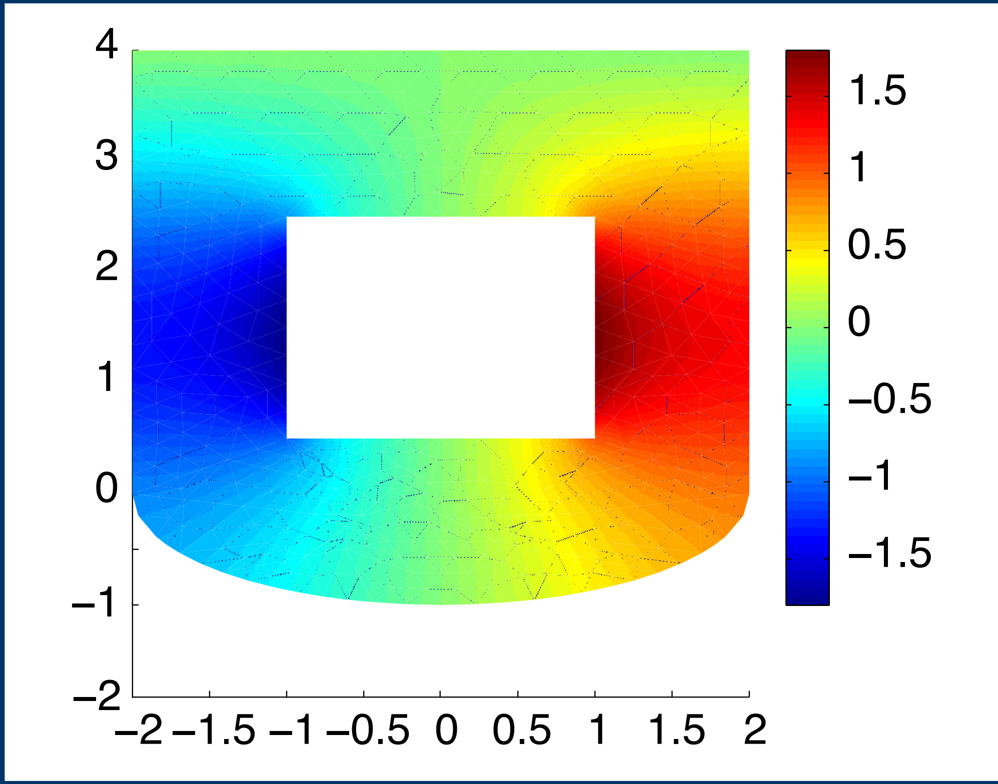
q	$\Theta^q(\mu)$	$a^q(w,v)$
22	$\frac{\mu_1 - 1 + \mu_3}{2}$	$\int_{\Omega_1} \frac{\partial w}{\partial x_2}\frac{\partial v}{\partial x_2} d\Omega + \int_{\Omega_2} \frac{\partial w}{\partial x_2}\frac{\partial v}{\partial x_2} d\Omega + \int_{\Omega_3} \frac{\partial w}{\partial x_2}\frac{\partial v}{\partial x_2} d\Omega$
25	$\frac{\mu_1}{3}$	$\int_{\Omega_5} \frac{\partial w}{\partial x_2}\frac{\partial v}{\partial x_2} d\Omega + \int_{\Omega_6} \frac{\partial w}{\partial x_2}\frac{\partial v}{\partial x_2} d\Omega$
28	$\frac{2}{\mu_1 - 1 + \mu_3}$	$\int_{\Omega_1} \frac{\partial w}{\partial x_1}\frac{\partial v}{\partial x_1} d\Omega + \int_{\Omega_2} \frac{\partial w}{\partial x_1}\frac{\partial v}{\partial x_1} d\Omega + \int_{\Omega_3} \frac{\partial w}{\partial x_1}\frac{\partial v}{\partial x_1} d\Omega$
32	$\frac{2}{3}(1 + \mu_3 - \frac{1}{3}\mu_1)$	$\int_{\Omega_6} \frac{\partial w}{\partial x_1}\frac{\partial v}{\partial x_2} d\Omega + \int_{\Omega_6} \frac{\partial w}{\partial x_2}\frac{\partial v}{\partial x_1} d\Omega$

Working Examples

A(dded)Mass: Practice

Representative Solutions

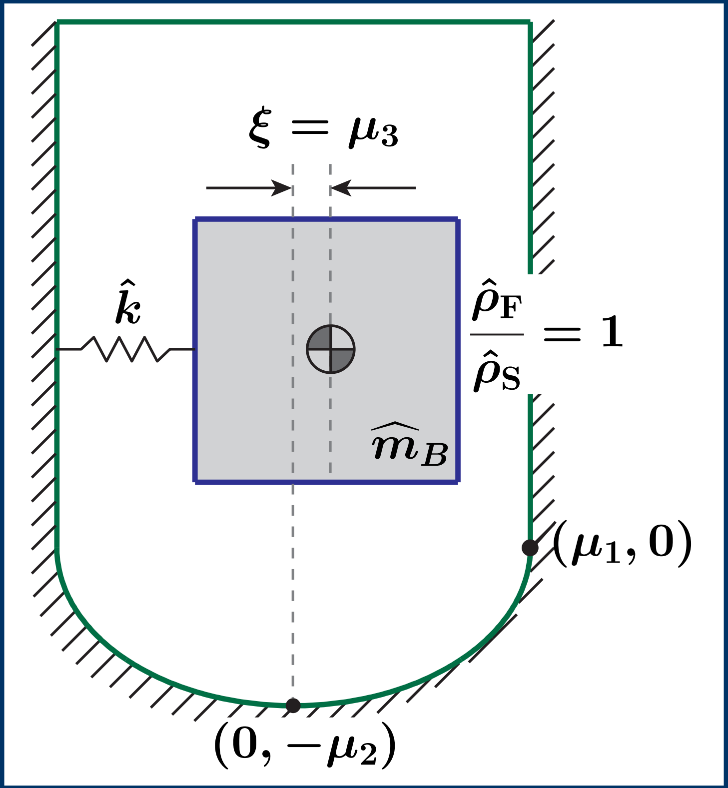




Working Examples

A(dded)Mass: Practice

Application: Oscillator† . . .



† (Gross) Assumptions: "small amplitude," inviscid, incompressible flow.

Given μ_1, μ_2: Many-Query

$$\xi(\hat{t} = 0) = \xi_0, \qquad \dot{\xi}(\hat{t} = 0) = \dot{\xi}_0\ ,$$

$$\left(1 + \frac{s^{\mathrm{e}}(\mu_1, \mu_2, \mu_3 = \xi)^{\dagger}}{4}\right)\ddot{\xi} + \frac{\hat{k}}{\widehat{m}_B}\,\xi = 0, \quad 0 < \hat{t} < \hat{t}_f\ .$$

Note the added mass $s^{\mathrm{e}} \to 4.754$ as $\mu_1 \to \infty$, $\mu_2 \to \infty$.

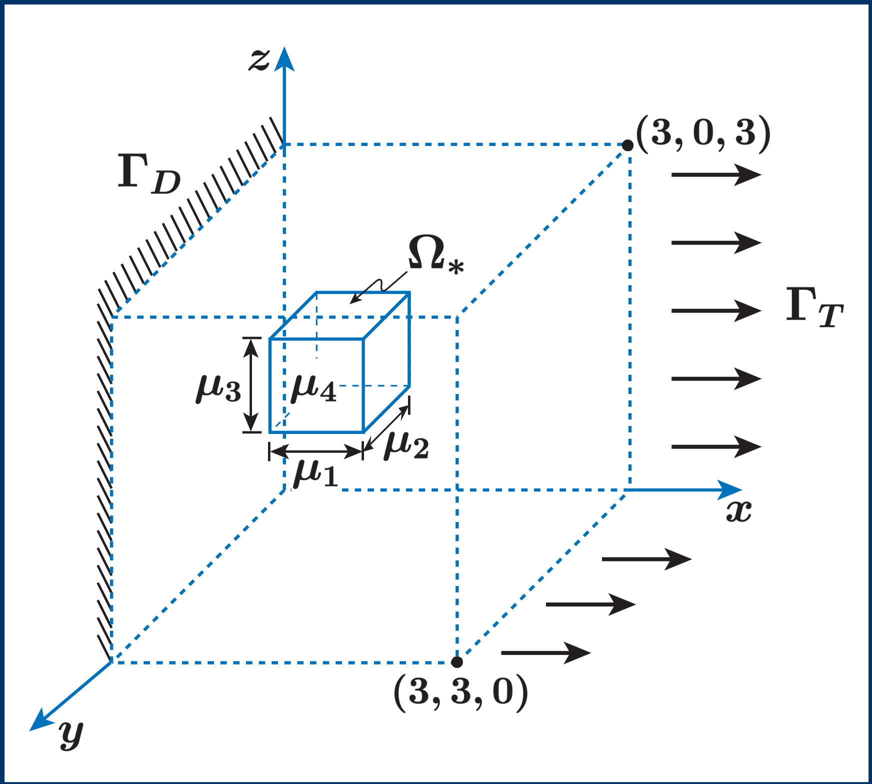
---

† For $|\xi|$ small, the approximation $s^{\mathrm{e}}(\mu_1, \mu_2, 0)$ is perhaps sufficient — but also less interesting for our methods.

Working Examples

E(lastic)Block3D

Geometry



Geometry:

$$\mu_G = \{\mu_1, \mu_2, \mu_3\}$$

Young's Modulus:

$$\mu_E = \{\mu_4\}$$

$$\Omega_o(\mu_G = (0.8, 0.8, 0.8)) = \mathcal{T}^{\text{aff}}(\Omega = \Omega_o(\mu_{G,\text{ref}} = (1, 1, 1)); \mu_G)$$

Working Examples

E(lastic)Block3D

Problem Statement...

Given $\mu \equiv (\mu_1, \mu_2, \mu_3, \mu_4) \in \mathcal{D}$ †

evaluate $s^{\mathrm{e}} = f(u^{\mathrm{e}}(\mu))$, DISPLACEMENT

for $u^{\mathrm{e}}(\mu) \in X^{\mathrm{e}} \equiv \{v \in (H^1(\Omega))^3 \,|\, v|_{\Gamma_D} = 0\}$

$$a(u^{\mathrm{e}}(\mu), v; \mu) = f(v), \quad \forall\, v \in X^{\mathrm{e}} .$$

† Here $\mathcal{D} = [0.5, 2] \times [0.5, 2] \times [0.5, 2] \times [0.1, 10]$.

Working Examples

E(lastic)Block3D

...Problem Statement

Here
$$f(v) = \int_{\Gamma_T} v_1 \;,$$
and
$$a(w, v; \mu) = \sum_{m=1}^{27} \int_{\Omega^m} \frac{\partial w_i}{\partial x_j} \; C_{i\,j\,k\,l}(\mu) \; \frac{\partial v_k}{\partial x_l}$$
where
$$C_{i\,j\,k\,l}(\mu_{\mathrm{ref}}) = \lambda^1 \delta_{ij}\delta_{kl} + \lambda^2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})^{\dagger}.$$

---

†Here λ^1 and λ^2 (Lamè constants) depend only on ν (Poisson ratio) $= 0.30$ and Young mod.

Working Examples

E(lastic)Block3D

Affine Representation...

We obtain $Q_a = 48$, $Q_f = 9$

$$a(w, v; \mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu)\, a^q(w, v)\ ,$$

and

$$f(v; \mu) = \sum_{q=1}^{Q_f} \Theta_f^q(\mu)\, f^q(v)\ ;$$

in this case f also depends (affinely) on μ.

Working Examples

E(lastic)Block3D

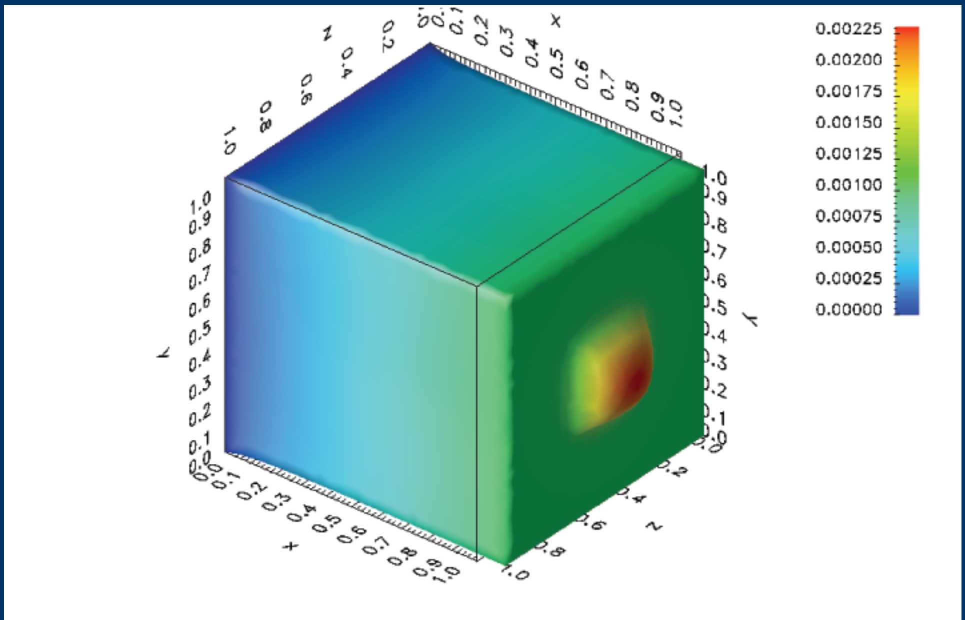
...Affine Representation

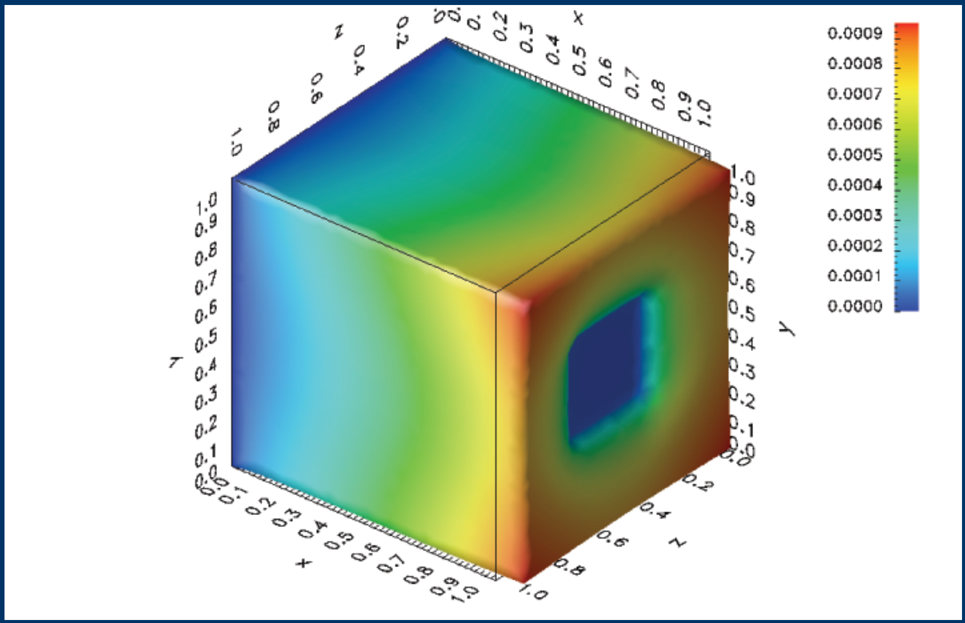
q	$\Theta_a^q(\mu)$	$a^q(w,v)$
1	$\frac{\mu_2\mu_3\mu_4}{\mu_1}$	$\int_{\Omega_*}((2\lambda^2+\lambda^1)\frac{\partial w_1}{\partial x_1}\frac{\partial v_1}{\partial x_1}+\lambda^2(\frac{\partial w_2}{\partial x_2}\frac{\partial v_2}{\partial x_2}+\frac{\partial w_3}{\partial x_3}\frac{\partial v_3}{\partial x_3}))$
2	$\frac{\mu_1\mu_3\mu_4}{\mu_2}$	$\int_{\Omega_*}((2\lambda^2+\lambda^1)\frac{\partial w_2}{\partial x_2}\frac{\partial v_2}{\partial x_2}+\lambda^2(\frac{\partial w_1}{\partial x_2}\frac{\partial v_1}{\partial x_2}+\frac{\partial w_3}{\partial x_2}\frac{\partial v_3}{\partial x_2}))$
3	$\frac{\mu_1\mu_2\mu_4}{\mu_3}$	$\int_{\Omega_*}((2\lambda^2+\lambda^1)\frac{\partial w_3}{\partial x_3}\frac{\partial v_3}{\partial x_3}+\lambda^2(\frac{\partial w_1}{\partial x_3}\frac{\partial v_1}{\partial x_3}+\frac{\partial w_2}{\partial x_3}\frac{\partial v_2}{\partial x_3}))$
4	$\mu_1\mu_4$	$\int_{\Omega_*}(\lambda^1(\frac{\partial w_2}{\partial x_2}\frac{\partial v_3}{\partial x_3}+\frac{\partial w_3}{\partial x_3}\frac{\partial v_2}{\partial x_2})+\lambda^2(\frac{\partial w_2}{\partial x_3}\frac{\partial v_3}{\partial x_2}+\frac{\partial w_3}{\partial x_2}\frac{\partial v_2}{\partial x_3}))$

Working Examples

E(lastic)Block3D

Representative Solutions





$$\mu_4 = 0.2 \qquad\qquad \mu_4 = 10$$

$$\mu_1 = \mu_2 = \mu_3 = 1.0$$

Outline

Convergence: $P = 1$

Convergence: $P > 1$

TBlock

AMass

EBlock3D

Reduced Basis Approximation

Preliminaries

Inner Products & Norms

Define, $\forall\, w, v \in X^{\mathrm{e}}$ $\qquad X^{\mathcal{N}} \subset X^{\mathrm{e}}$

$$\left.\begin{aligned} ((w,v))_{\mu} &\equiv a(w,v;\mu) \\ |||w|||_{\mu} &\equiv ((w,w))_{\mu}^{1/2} \end{aligned}\right\} \text{energy}$$

and, given $\overline{\mu} \in \mathcal{D}$

$$\left.\begin{aligned} (w,v)_X &\equiv ((w,v))_{\overline{\mu}} + \tau (w,v)_{L^2(\Omega)} \\ \|w\|_X &\equiv (w,w)_X^{1/2} \end{aligned}\right\} X\,.$$

Reduced Basis Approximation

Formulation

Spaces

Nested Samples:

$$S_N = \{\mu^1 \in \mathcal{D}, \ldots, \mu^N \in \mathcal{D}\},\ 1 \le N \le N_{\max}\ .$$

Hierarchical Spaces: Lagrange

$$W_N^{\mathcal{N}} = \mathrm{span}\{u^{\mathcal{N}}(\mu^n),\ 1 \le n \le N\},\ 1 \le N \le N_{\max}\ .$$

Orthonormal Basis:

$$\{\zeta^{\mathcal{N}\,n}\}_{1 \le n \le N_{\max}} = \text{G-S}\left(\{u^{\mathcal{N}}(\mu^n)\}_{1 \le n \le N_{\max}};\, (\,\cdot\,,\,\cdot\,)_X\right).$$

Reduced Basis Approximation

Formulation

Galerkin Projection...

Optimality:

$$|||u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu)|||_\mu \leq \inf_{w \in W_N^{\mathcal{N}}} |||u^{\mathcal{N}}(\mu) - w|||_\mu \; ;$$

best combination of snapshots.

Note also:

$$s^{\mathcal{N}}(\mu) - s_N^{\mathcal{N}}(\mu) \equiv |||u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu)|||_\mu^2 \; ;$$

output converges as square.

Reduced Basis Approximation

Formulation

...Galerkin Projection

$$s^{\mathcal{N}}(\mu) - s_N^{\mathcal{N}}(\mu) \equiv |||u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu)|||_{\mu}^2 \;;$$

$$s^{\mathcal{N}}(\mu) = f(u^{\mathcal{N}}(\mu)); s_N^{\mathcal{N}}(\mu) = f(u_N^{\mathcal{N}}(\mu));$$

$$s^{\mathcal{N}}(\mu) - s_N^{\mathcal{N}}(\mu) = f(u^{\mathcal{N}}(\mu)) - f(u_N^{\mathcal{N}}(\mu)) =$$
$$= a(v, u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu); \mu);$$

$$e(\mu) = u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu);$$

$$a(v, e(\mu); \mu) = a(e(\mu), v; \mu) = a(e(\mu), e(\mu); \mu);$$

$$a(e(\mu), e(\mu); \mu) = |||u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu)|||_{\mu}^2.$$

Reduced Basis Approximation

Formulation

Discrete Equations[†]

Express
$$u_N(\mu) = \sum_{j=1}^{N} u_{N\,j}(\mu)\,\zeta^j ;$$

then
$$s_N(\mu) \equiv f(u_N(\mu)) = \sum_{j=1}^{N} u_{N\,j}(\mu)\,f(\zeta^j)$$

where *well-conditioned*
$$\sum_{j=1}^{N} a(\zeta^j,\zeta^i;\mu)\,u_{N\,j} = f(\zeta^i), \quad 1 \le i \le N\,.$$

[†]We suppress $^{\mathcal{N}}$: $\mathcal{N}$ is *fixed* for *computational purposes*.

Reduced Basis Approximation

OFFLINE-ONLINE Procedure

Evaluation of $s_N(\mu)$ — *GIVEN* $u_{N\,j}$, $1 \le j \le N$

OFFLINE: Compute ζ^j, $1 \le j \le N$;

Form/Store $f(\zeta^j)$, $1 \le j \le N$. $\quad O(\mathcal{N})$

ONLINE: Perform sum

$$s_N(\mu) = \sum_{j=1}^{N} u_{N\,j}(\mu)\; f(\zeta^j) - O(N)\;.$$

Reduced Basis Approximation

OFFLINE-ONLINE Procedure[†]

Evaluation of $u_{N\,j}(\mu),\ 1 \le j \le N \ldots$

For $a(w, v; \mu)$ affine,

$$\sum_{j=1}^{N} a(\zeta^j, \zeta^i; \mu)\, u_{N\,j} = f(\zeta^i), \quad 1 \le i \le N$$

$$\Downarrow$$

$$\sum_{j=1}^{N} \Big(\sum_{q=1}^{Q} \Theta^q(\mu)\, a^q(\zeta^j, \zeta^i) \Big)\, u_{N\,j} = f(\zeta^i),\ 1 \le i \le N\ .$$

[†]Often (re-)invented: [B], [IR], [MMOPR].

Reduced Basis Approximation

OFFLINE-ONLINE Procedure

. . . Evaluation of $u_{N\,j}(\mu),\ 1 \le j \le N$ . . .

OFFLINE: Form/Store $a^q(\zeta^j, \zeta^i),\ \ 1 \le i, j \le N^{\dagger}_{\max},$
$1 \le q \le Q. \qquad O(\mathcal{N})$

ONLINE: Form $\sum_{q=1}^{Q} \Theta^q(\mu)\, a^q(\zeta^j, \zeta^i),\ 1 \le i, j \le N$
— $O(QN^2)$;

Solve for $u_{N\,j}(\mu),\ 1 \le j \le N$ — $O(N^3)$.

$^\dagger N_{\max}$ chosen to satisfy specified error tolerance.

Reduced Basis Approximation

OFFLINE-ONLINE Procedure

. . . Evaluation of $u_{N\,j}(\mu),\ 1 \le j \le N$

Note $a^q(\zeta^j, \zeta^i)$ $\qquad 1 \le i, j \le N_{\max}$

$$= a^q\Big(\sum_{k=1}^{\mathcal{N}} \zeta_k^j\, \phi_k^{\mathrm{FE}}, \sum_{k'=1}^{\mathcal{N}} \zeta_{k'}^i\, \phi_{k'}^{\mathrm{FE}} \Big)$$

$$= \sum_{k=1}^{\mathcal{N}} \sum_{k'=1}^{\mathcal{N}} \zeta_k^j\, a^q(\phi_k^{\mathrm{FE}}, \phi_{k'}^{\mathrm{FE}})\, \zeta_{k'}^i$$

$$= \underline{Z}_{N_{\max}}\, \underline{A}^{\mathrm{FE}\,q}\, \underline{Z}_{N_{\max}} .$$

Sample/Space Strategies

Preliminaries

General "Reduced Model"

Given $\mu \in \mathcal{D}$,

evaluate $s_N^{\mathcal{N}}(\mu) = f(u_N^{\mathcal{N}}(\mu))$,

where $u_N^{\mathcal{N}}(\mu) \in X_N^{\mathcal{N}} \subset X^{\mathcal{N}}$ satisfies $\dim(X_N^{\mathcal{N}}) = N$ †

$$a(u_N^{\mathcal{N}}(\mu), v; \mu) = f(v), \ \forall\, v \in X_N^{\mathcal{N}} .$$

---

†Here $X_N^{\mathcal{N}}$ may be a hierarchical or non-hierarchical Lagrange ($W_N^{\mathcal{N}}$) or non-Lagrange RB space (Taylor, Hermite), or even a "non-RB" (non-$\mathcal{M}^{\mathcal{N}}$) space (Kolmogorov).

"Train" sample:

$$\Xi_{\text{train}} \subset \mathcal{D} \subset \mathbb{R}^P; \qquad |\Xi_{\text{train}}| = n_{\text{train}} \ (\gg 1) \ .$$

"Test" sample:

$$\Xi_{\text{test}} \subset \mathcal{D} \subset \mathbb{R}^P; \qquad |\Xi_{\text{test}}| = n_{\text{test}} \ (\gg 1) \ .$$

Sample/Space Strategies

Given $\Xi \subset \mathcal{D}$, $y\colon \mathcal{D} \longrightarrow \mathbb{R}$,

$$\|y\|_{L^\infty(\Xi)} \equiv \operatorname*{ess\,sup}_{\mu \in \Xi} |y(\mu)| \ ,$$

$$\|y\|_{L^2(\Xi)} \equiv \left(|\Xi|^{-1} \sum_{\mu \in \Xi} y^2(\mu) \right)^{1/2} .$$

Given $z\colon \mathcal{D} \longrightarrow X^{\mathcal{N}}$ (or X^{e})

$$\|z\|_{L^\infty(\Xi;X)} \equiv \operatorname*{ess\,sup}_{\mu \in \Xi} \|z(\mu)\|_X \ ,$$

$$\|z\|_{L^2(\Xi;X)} \equiv \left(|\Xi|^{-1} \sum_{\mu \in \Xi} \|z(\mu)\|_X^2 \right)^{1/2} .$$

Sample/Space Strategies

3. Greedy

...Actual Method

Here, for $N = 1, \ldots$

$$\|u^{\mathcal{N}}(\mu) - u^{\mathcal{N}}_{W^{\mathcal{N}}_N}(\mu)\|_X \le \Delta_N(\mu), \quad \forall\, \mu \in \mathcal{D}:$$

$\Delta_N(\mu)$ is a sharp, *inexpensive*[†] *a posteriori* error *bound* for $\|u^{\mathcal{N}}(\mu) - u^{\mathcal{N}}_{W^{\mathcal{N}}_N}(\mu)\|_X$.

Greedy only computes actual (*winning* candidate) snapshots.

[†] Marginal cost (= average asymptotic cost) is *independent* of $\mathcal{N}$.

Sample/Space Strategies

3. Greedy$^{\mathrm{en}}$ . . .

Given Ξ_{train}, $S_1 = \{\mu^1\}$, $W_1^{\mathcal{N}} = \mathrm{span}\{u^{\mathcal{N}}(\mu^1)\}$,

[for $N = 2, \ldots, N_{\max}$:

$$\begin{aligned} \mu^N &= \arg\max_{\mu \in \Xi_{\mathrm{train}}} \omega_{N-1}^{-1}(\mu)\, \Delta_{N-1}^{\mathrm{en}}(\mu) \quad \dagger \\ S_N &= S_{N-1} \cup \mu^N; \\ W_N^{\mathcal{N}} &= W_{N-1}^{\mathcal{N}} + \mathrm{span}\{u^{\mathcal{N}}(\mu^N)\}.] \end{aligned}$$

---

†Typically, $\omega_N(\mu) = |||u_N^{\mathcal{N}}(\mu)|||_\mu$ (or $\omega_N(\mu) = 1$).

Sample/Space Strategies

. . . Greedy$^{\mathrm{en}}$

Here, for $N = 1, \ldots$

$$|||u^{\mathcal{N}}(\mu) - u^{\mathcal{N}}_{W^{\mathcal{N}}_N}(\mu)|||_\mu \le \Delta^{\mathrm{en}}_N(\mu), \quad \forall\, \mu \in \mathcal{D}\text{:}$$

$\Delta^{\mathrm{en}}_N(\mu)$ is a sharp, *inexpensive*[†]
a posteriori error *bound* for $|||u^{\mathcal{N}}(\mu) - u^{\mathcal{N}}_{W^{\mathcal{N}}_N}(\mu)|||_\mu$.

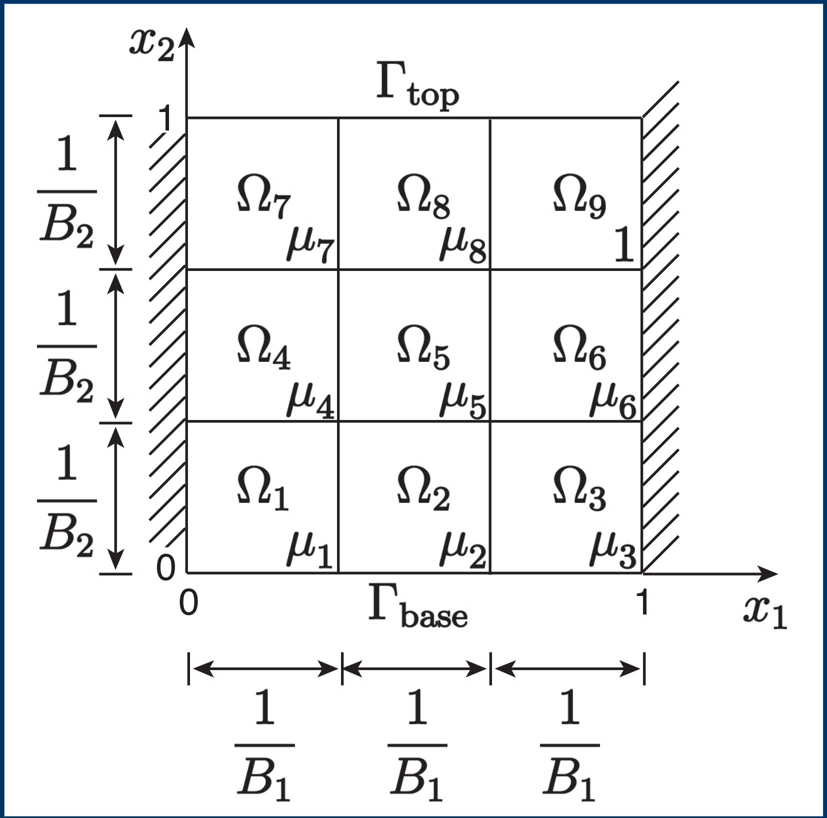
Greedy$^{\mathrm{en}}$ only computes actual (*winning* candidate) snapshots.

[†]Marginal cost (= average asymptotic cost) is *independent* of $\mathcal{N}$.

Convergence: $P > 1$

Numerics: TBlock-$(3,3)$

Geometry

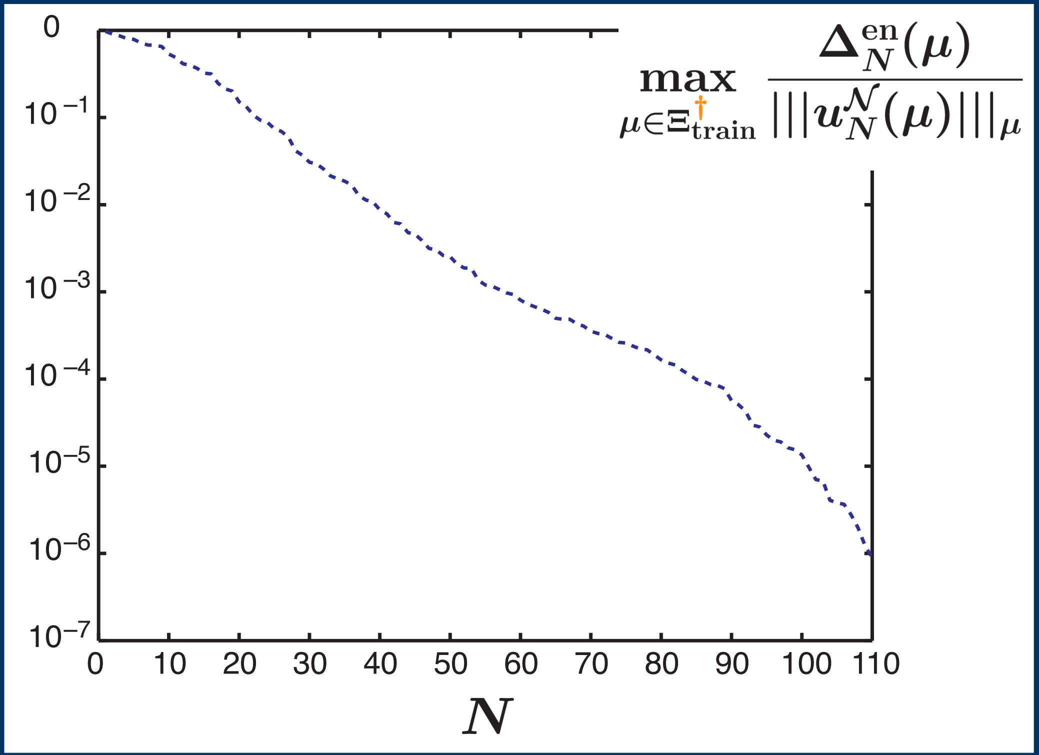


$$\overline{\Omega} = \cup_{i=1}^{B_1 B_2} \overline{\Omega}_i$$

Convergence: $P > 1$

Numerics: TBlock-$(3,3)$

Greedy$^{\mathrm{en}}$: RB Energy Error

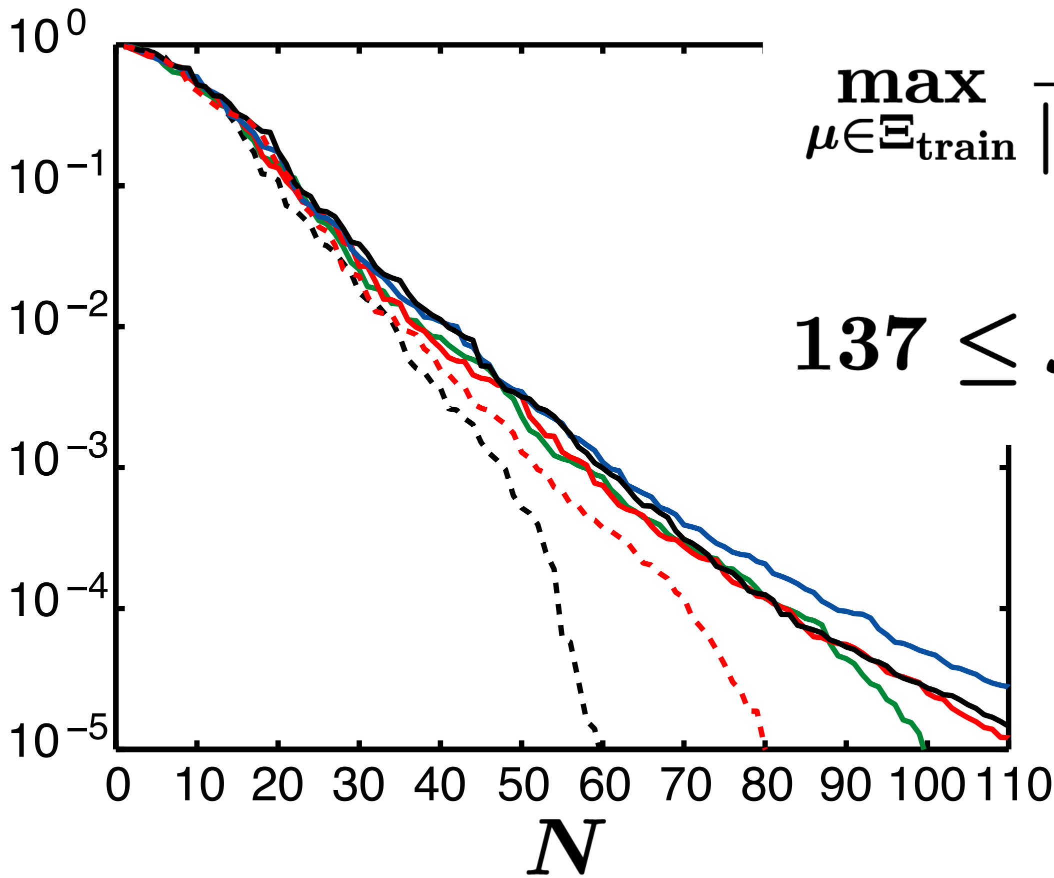


†Here Ξ_{train} is a Monte Carlo sample in $\ln \mu$ of size $n_{\mathrm{train}} = 5000\ (\gg N)$; note $|||u^{\mathcal{N}}(\mu) - u_N^{\mathcal{N}}(\mu)|||_\mu \le \Delta_N^{\mathrm{en}}(\mu)$, and $|||u_N^{\mathcal{N}}(\mu)|||_\mu \le |||u^{\mathcal{N}}(\mu)|||_\mu$.

Convergence: $P > 1$

Numerics: TBlock-$(3,3)$

Effect of $X^{\mathcal{N}}$



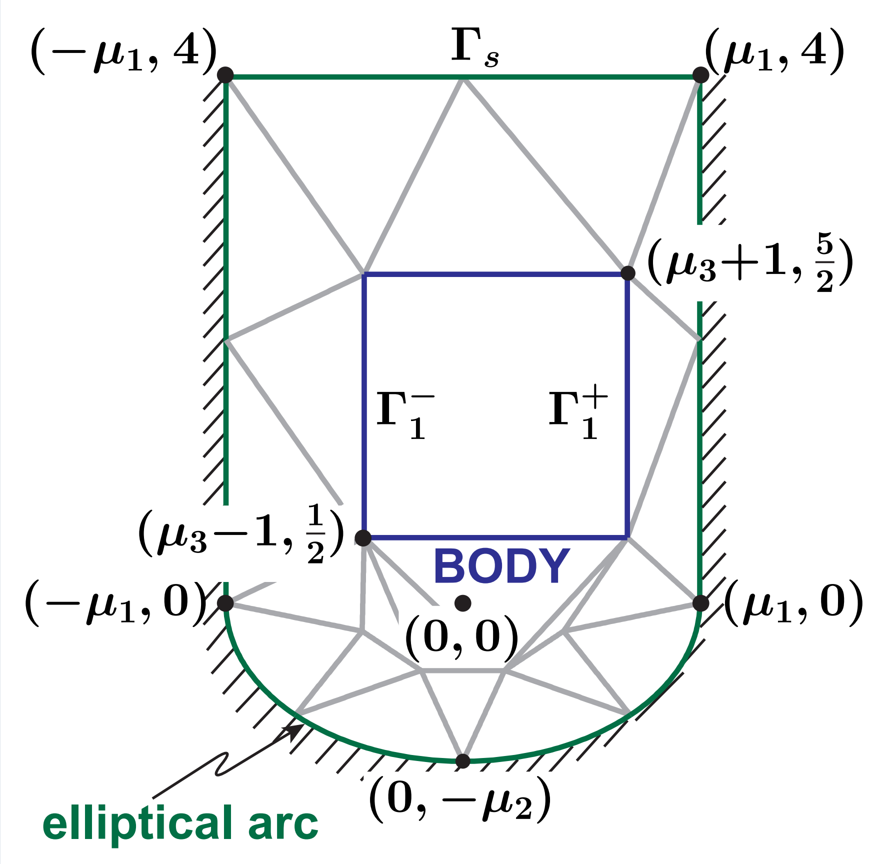
$$\max_{\mu \in \Xi_{\text{train}}} \frac{\Delta_N^{\text{en}}(\mu)}{|||u_N^{\mathcal{N}}(\mu)|||_\mu}$$

$$137 \leq \mathcal{N} \leq 6808$$

Convergence: $P > 1$

Numerics: AMass

Geometry

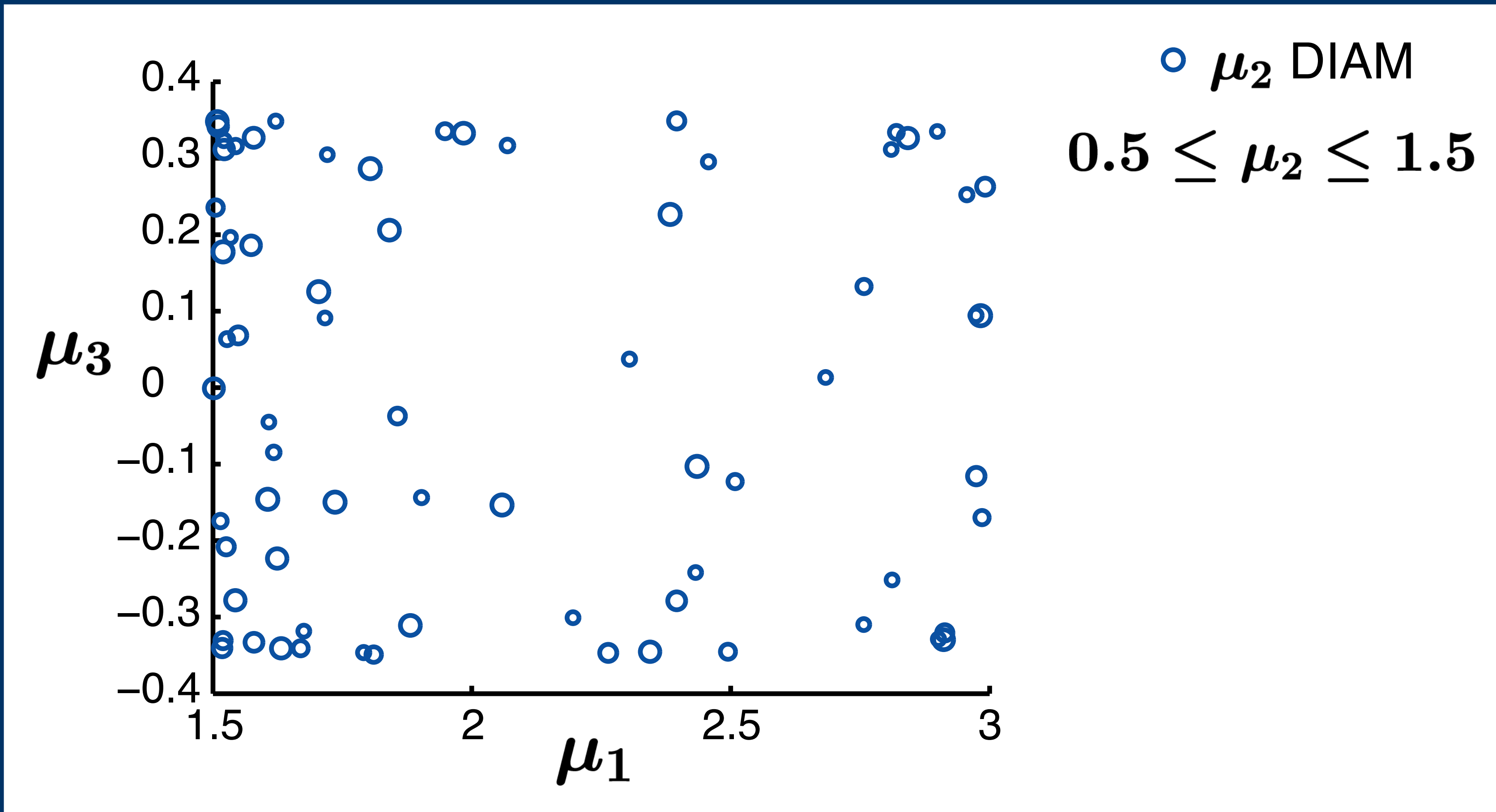


$$\Omega_o(\mu = (2.0, 1.2, .25)) \qquad = \mathcal{T}^{\mathrm{aff}}(\Omega; \mu)$$

Convergence: $P > 1$

Numerics: AMass

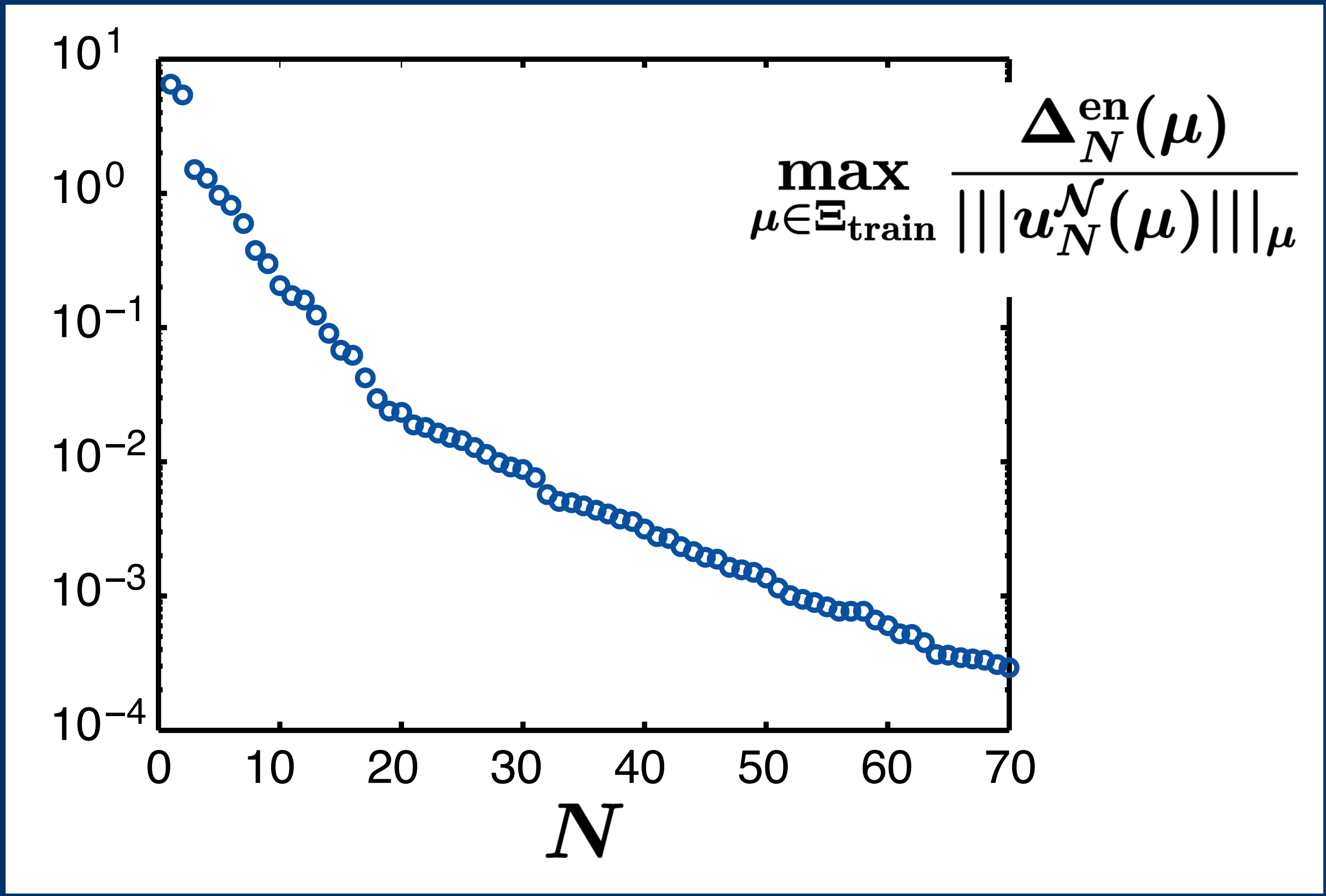
Greedy$^{\mathrm{en}}$: Sample



Convergence: $P > 1$

Numerics: AMass

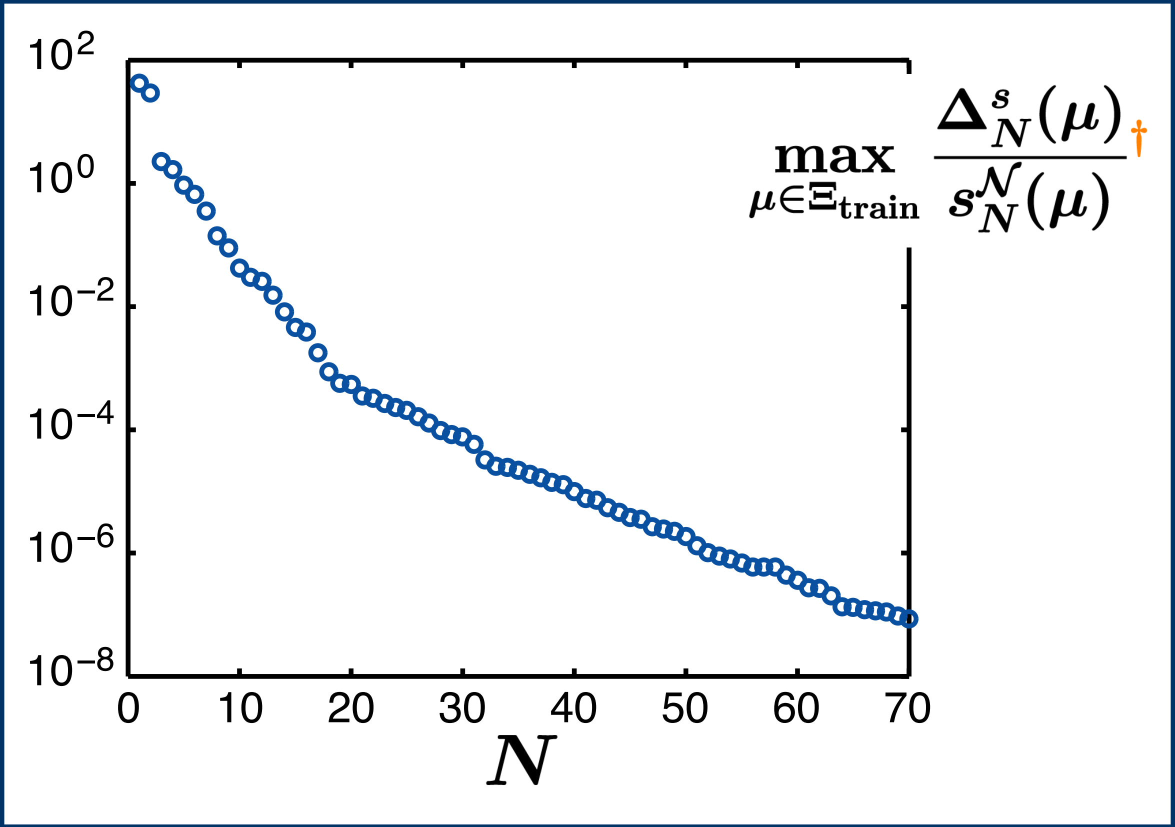
Greedy$^{\mathrm{en}}$: RB Energy Error



Convergence: $P > 1$

Numerics: AMass

Greedy$^{\mathrm{en}}$: RB Output Error

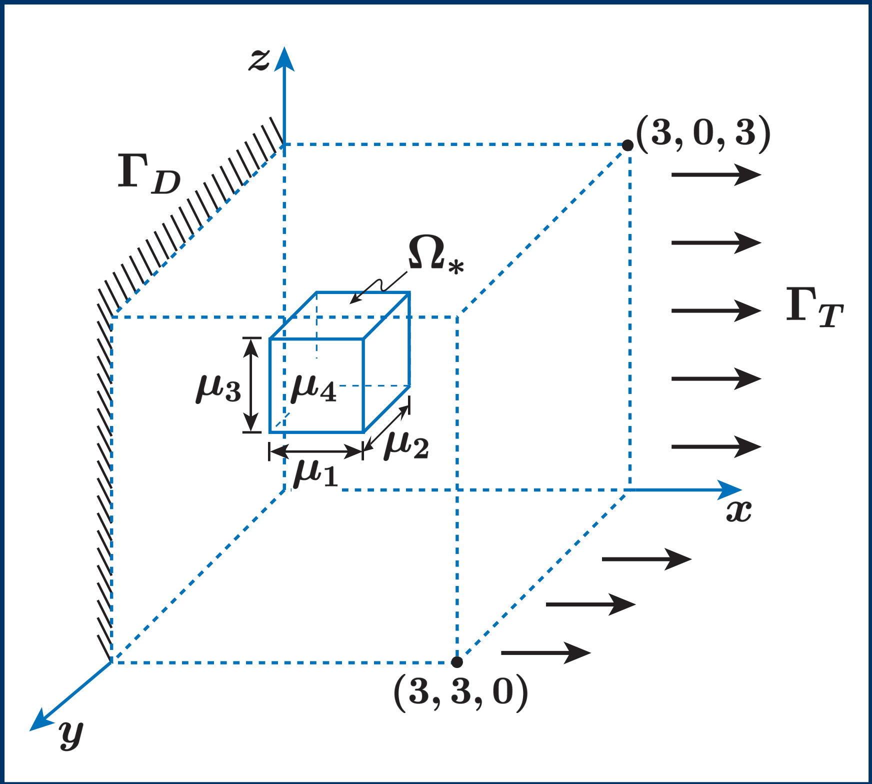


†Note $|s^{\mathcal{N}}(\mu) - s_N^{\mathcal{N}}(\mu)| \leq \Delta_N^s(\mu)$ and $s_N^{\mathcal{N}}(\mu) \leq s^{\mathcal{N}}(\mu)$.

Convergence: $P > 1$

Numerics: EBlock3D

Geometry



Geometry:

$$\mu_G = \{\mu_1, \mu_2, \mu_3\}$$

Young's Modulus:

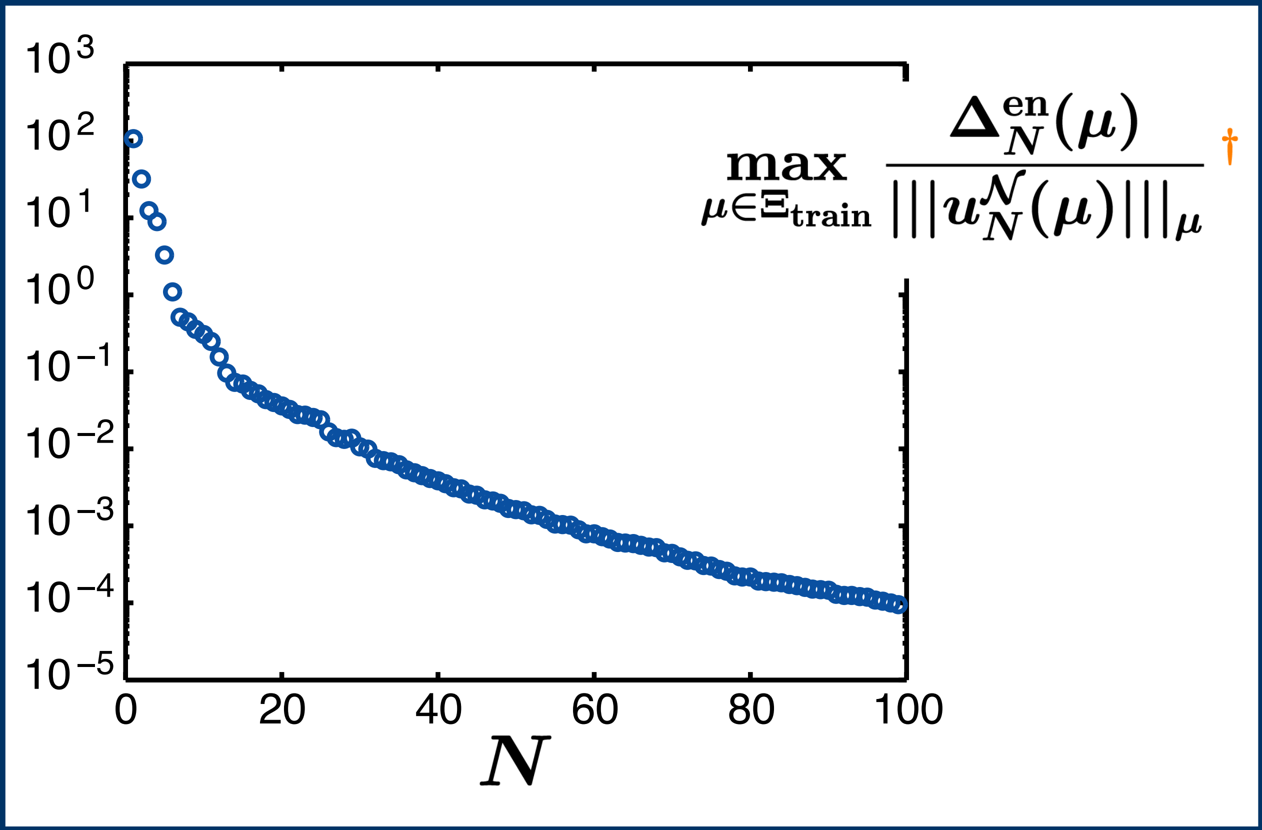
$$\mu_E = \{\mu_4\}$$

$$\Omega_o(\mu_G = (0.8, 0.8, 0.8)) = \mathcal{T}^{\mathrm{aff}}(\Omega = \Omega_o(\mu_{G,\mathrm{ref}} = (1,1,1)); \mu_G)$$

Convergence: $P > 1$

Numerics: EBlock3D

Greedy$^{\mathrm{en}}$: RB Energy Error



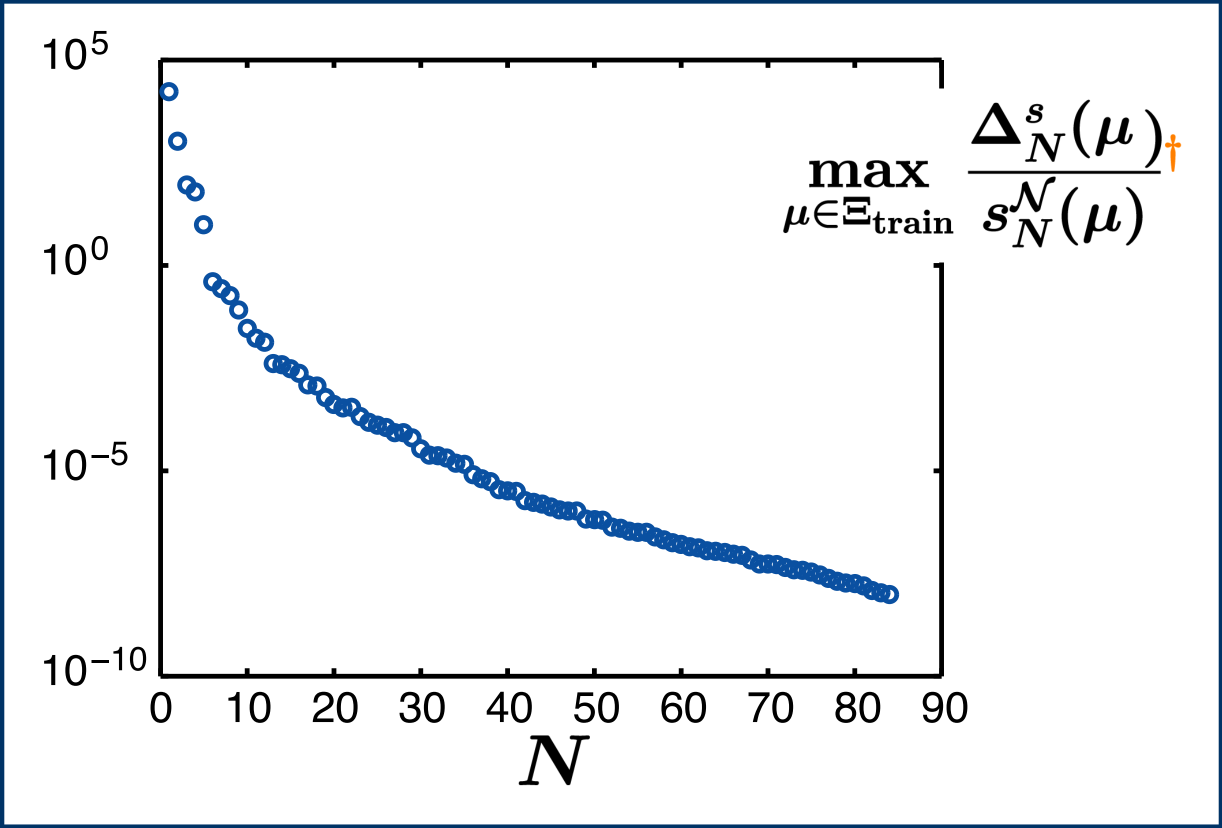
---

† We discuss computational details and performance subsequently.

Convergence: $P > 1$

Numerics: EBlock3D

Greedy$^{\mathrm{en}}$: RB Output Error



†Note $|s^{\mathcal{N}}(\mu) - s_N^{\mathcal{N}}(\mu)| \leq \Delta_N^s(\mu)$, and $s_N^{\mathcal{N}}(\mu) \leq s^{\mathcal{N}}(\mu)$.