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Approximation of stability factors: the successive constraint method (SCM)

Reduced basis methods for non-coercive problems

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Collaboration Network

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Coercivity Lower Bound†

Objective

Require $\alpha^{\mathcal{N}}_{\mathrm{LB}}\colon \mathcal{D} \longrightarrow \mathbb{R}$ such that

$$0 < \alpha^{\mathcal{N}}_{\mathrm{LB}}(\mu) \leq \alpha^{\mathcal{N}}(\mu), \quad \forall\, \mu \in \mathcal{D}\ ,$$

$$\partial t_{\mathrm{comp}}(\mu \longrightarrow \alpha^{\mathcal{N}}_{\mathrm{LB}}(\mu)) \text{ is } O(1)\ ,$$

where

$$\alpha^{\mathcal{N}}(\mu) = \inf_{w \in X^{\mathcal{N}}} \frac{a(w, w; \mu)}{\|w\|^2_X} \quad (\geq \alpha^{\mathrm{e}}_0,\ \forall\, \mu \in \mathcal{D})\ .^{\dagger}$$

---

†We consider symmetric a; extension to non-symmetric a is simple.

Coercivity Lower Bound

Reformulation

Affine Parameter Dependence

Recall

$$a(w, w; \mu) = \sum_{q=1}^{Q} \Theta^q(\mu)\, a^q(w, w)\ ;$$

hence

$$\alpha^{\mathcal{N}}(\mu) = \inf_{w \in X^{\mathcal{N}}} \mathcal{J}^{\mathrm{obj}}(\mu; w)$$

where

$$\mathcal{J}^{\mathrm{obj}}(\mu; w) \equiv \sum_{q=1}^{Q} \Theta^q(\mu)\, \frac{a^q(w, w)}{\|w\|_X^2}\ .$$

Coercivity Lower Bound

Reformulation

“Pseudo”-Linear Form

Express

$$\alpha^{\mathcal{N}}(\mu) = \inf_{y \in \mathcal{Y}} \mathcal{J}^{\mathrm{obj}}(\mu; y)$$

where

$$\mathcal{J}^{\mathrm{obj}}(\mu; y) \equiv \sum_{q=1}^{Q} \Theta^q(\mu)\, y_q$$

$$\mathcal{Y} = \Big\{ y \in \mathbb{R}^Q \;\Big|\; \exists\, w_y \in X^{\mathcal{N}} \text{ s.t. } y_q = \frac{a^q(w_y, w_y)}{\|w_y\|_X^2},\ 1 \le q \le Q \Big\}.$$

Coercivity Lower Bound

Bounds†

Set $\mathcal{Y}_{\mathrm{LB}}$ ...

Introduce

$$\mathcal{B} = \Pi_{q=1}^{Q} \left[\inf_{w \in X^{\mathcal{N}}} \frac{a^q(w,w)}{\|w\|_X^2}, \sup_{w \in X^{\mathcal{N}}} \frac{a^q(w,w)}{\|w\|_X^2} \right]$$

$$\mathcal{C}_J = \{\mu_{\mathrm{SCM}}^1 \in \mathcal{D}, \ldots, \mu_{\mathrm{SCM}}^J \in \mathcal{D}\}$$

and, given $\mu \in \mathcal{D}$,

$\mathcal{C}_J^{M,\mu} = \{M$ points in $\mathcal{C}_J$ closest to $\mu\}$.

†We consider the Successive Constraint Method (SCM). [HRSP]

Coercivity Lower Bound

Bounds

. . . Set $\mathcal{Y}_{\mathrm{LB}}$ . . . ...

Define $\mathcal{Y}_{\mathrm{LB}}(\mu; \mathcal{C}_J, M)$:

$$\mathcal{Y}_{\mathrm{LB}}(\mu) \equiv \Bigg\{ y \in \mathbb{R}^Q \;\Big|\; \textbf{(I)} \;\; y \in \mathcal{B} \text{ , and}$$

$$\textbf{(II)} \;\; \sum_{q=1}^{Q} \Theta^q(\mu')\, y_q > \alpha^{\mathcal{N}}(\mu'),\; \forall\, \mu' \in \mathcal{C}_J^{M,\mu} \Bigg\}.$$

Lemma 3.1. Given $\mathcal{C}_J \subset \mathcal{D},\;\; M \in \mathbb{N}$,

$$\mathcal{Y} \subset \mathcal{Y}_{\mathrm{LB}}(\mu; \mathcal{C}_J, M),\; \forall\, \mu \in \mathcal{D}\;. \qquad \square$$

Coercivity Lower Bound

Bounds

...... Set $\mathcal{Y}_{\mathrm{LB}}$ ...

Proof: For any $y \in \mathcal{Y}$, $\exists\, w_y \in X^{\mathcal{N}}$ such that

$$y_q = \frac{a^q(w_y, w_y)}{\|w_y\|_X^2}, \quad 1 \le q \le Q :$$

$$\inf_{w \in X^{\mathcal{N}}} \frac{a^q(w,w)}{\|w\|_X^2} \le \underbrace{\frac{a^q(w_y,w_y)}{\|w_y\|_X^2}}_{y_q} \le \sup_{w \in X^{\mathcal{N}}} \frac{a^q(w,w)}{\|w\|_X^2} \quad ; \quad \mathrm{(I)}$$

$$\sum_{q=1}^{Q} \Theta^q(\mu) \underbrace{\frac{a^q(w_y,w_y)}{\|w_y\|_X^2}}_{y_q} = \frac{a(w_y,w_y;\mu)}{\|w_y\|_X^2}$$

$$\ge \alpha^{\mathcal{N}}(\mu), \quad \forall\, \mu \in \mathcal{D} \, . \quad \mathrm{(II)}$$

Coercivity Lower Bound

Let

$$\alpha^{\mathcal{N}}_{\mathrm{LB}}(\mu; \mathcal{C}_J, M) = \min_{y \in \mathcal{Y}_{\mathrm{LB}}(\mu; \mathcal{C}_J, M)} \mathcal{J}^{\mathrm{obj}}(\mu; y) \; ;$$

a *linear optimization* problem (LP).

Proposition 3.2. Given $\mathcal{C}_J \subset \mathcal{D}, \; M \in \mathbb{N}$,

$$\alpha^{\mathcal{N}}_{\mathrm{LB}}(\mu) \leq \alpha^{\mathcal{N}}(\mu), \; \forall \, \mu \in \mathcal{D} \; . \qquad \square$$

Coercivity Lower Bound

Bounds

...Lower Bound

Proof:

$$
\begin{aligned}
\alpha^{\mathcal{N}}_{\mathrm{LB}}(\mu) &= \min_{y \in \mathcal{Y}_{\mathrm{LB}}(\mu)} \mathcal{J}^{\mathrm{obj}}(\mu; y) \\
&\leq \min_{y \in \mathcal{Y}} \mathcal{J}^{\mathrm{obj}}(\mu; y) \qquad \text{Lemma 3.1: } \mathcal{Y} \subset \mathcal{Y}_{\mathrm{LB}} \\
&= \alpha^{\mathcal{N}}(\mu) \; .
\end{aligned}
$$

Coercivity Lower Bound

Bounds

Set $\mathcal{Y}_{\mathrm{UB}}$

Define

$$\mathcal{Y}_{\mathrm{UB}}(\mu; \mathcal{C}_J, M) = \{y^\star(\mu') \mid \mu' \in \mathcal{C}_J^{M,\mu}\}$$

where

$$y^\star(\mu) = \arg\inf_{y \in \mathcal{Y}} \mathcal{J}^{\mathrm{obj}}(\mu; y) \ ;$$

clearly $\mathcal{Y}_{\mathrm{UB}} \subset \mathcal{Y}$.

Coercivity Lower Bound

Bounds

Upper Bound

Let

$$\alpha_{\mathrm{UB}}^{\mathcal{N}}(\mu; \mathcal{C}_J, M) = \min_{y \in \mathcal{Y}_{\mathrm{UB}}(\mu; \mathcal{C}_J, M)} \mathcal{J}^{\mathrm{obj}}(\mu; y) \; ;$$

a simple *enumeration* exercise.

Proposition 3.3. Given $\mathcal{C}_J \subset \mathcal{D}, \; M \in \mathbb{N}$,

$$\alpha_{\mathrm{UB}}^{\mathcal{N}}(\mu) \geq \alpha^{\mathcal{N}}(\mu), \; \forall \, \mu \in \mathcal{D} \, . \qquad \square$$

Coercivity Lower Bound

Greedy Selection: $\mathcal{C}_J$

Procedure

Given $\Xi_{\mathrm{train}}(\mathrm{SCM})$, $\varepsilon_{\mathrm{SCM}} \in [0,1]$, M

$$\texttt{While} \quad \max_{\mu \in \Xi_{\mathrm{train}}} \left[\frac{\alpha^{\mathcal{N}}_{\mathrm{UB}}(\mu;\mathcal{C}_J) - \alpha^{\mathcal{N}}_{\mathrm{LB}}(\mu;\mathcal{C}_J)}{\alpha^{\mathcal{N}}_{\mathrm{UB}}(\mu;\mathcal{C}_J)} \right] > \varepsilon_{\mathrm{SCM}}:$$

$$\mu^{J+1}_{\mathrm{SCM}} = \arg \max_{\mu \in \Xi_{\mathrm{train}}} \left[\frac{\alpha^{\mathcal{N}}_{\mathrm{UB}}(\mu;\mathcal{C}_J) - \alpha^{\mathcal{N}}_{\mathrm{LB}}(\mu;\mathcal{C}_J)}{\alpha^{\mathcal{N}}_{\mathrm{UB}}(\mu;\mathcal{C}_J)} \right];$$

$$\mathcal{C}_{J+1} = \mathcal{C}_J \cup \mu^{J+1}_{\mathrm{SCM}};$$

$$J \leftarrow J+1;$$

$\texttt{end.}$ $\quad \texttt{Set}$ $J_{\max} = J$.

Coercivity Lower Bound

Greedy Selection: $\mathcal{C}_J$

Convergence

If a is *parametrically* coercive,

$$\begin{aligned} \Theta^q(\mu) &> 0, \ \forall\, \mu \in \mathcal{D}\,, \\ a^q(w,w) &\geq 0, \ \forall\, w \in X,\ 1 \leq q \leq Q\,, \end{aligned}$$

$J = 1$ suffices to ensure $\alpha_{\mathrm{LB}}^{\mathcal{N}}(\mu) > 0,\ \forall\, \mu \in \mathcal{D}$.

Generally, continuity of $\Theta^{\cdot}$ ensures finite $J_{\max}$ such that tolerance is satisfied: but $J_{\max}(P)$?

Coercivity Lower Bound

Offline-Online Procedure

Offline

In Greedy, perform

$J_{\max}\,\mathrm{LP}(Q, M) \Rightarrow \mathcal{C}_{J_{\max}}$;

$2Q + J_{\max}$ eigenproblems[†] over $X^{\mathcal{N}}$
$\Rightarrow$ (I) $\mathcal{B}$ and (II) $\{\alpha^{\mathcal{N}}(\mu') \mid \mu' \subset \mathcal{C}_{J_{\max}}\} \Rightarrow \mathcal{Y}_{\mathrm{LB}}$;

$J_{\max}\, Q$ inner products over $X^{\mathcal{N}} \Rightarrow \mathcal{Y}_{\mathrm{UB}}$.

[†] Eigenproblems efficiently treated by Lanczos method.

Coercivity Lower Bound

Offline-Online Procedure

Online

Given $\mu \in \mathcal{D}$, perform

sort over $\mathcal{C}_{J_{\max}} \Rightarrow \mathcal{C}^{M,\mu}_{J_{\max}}$;

$(M+1)\,Q$ evaluations $\mu' \rightarrow \Theta^{\cdot}(\mu')$;

M *look-ups* $\mu' \rightarrow \alpha^{\mathcal{N}}(\mu')$;

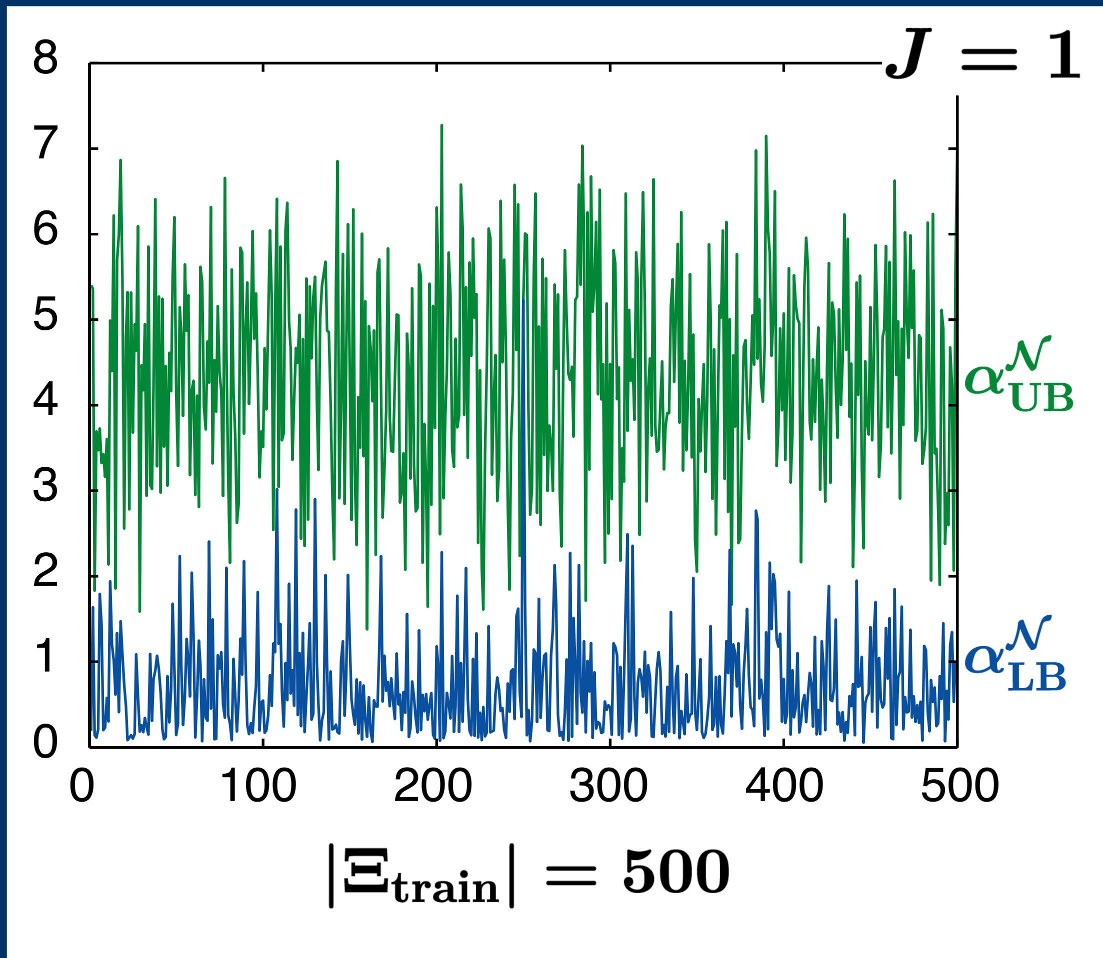
$\mathbf{LP}\ (Q, M) \qquad \rightarrow \alpha^{\mathcal{N}}_{\mathrm{LB}}(\mu)$.

Cost *independent* of $\mathcal{N}$.

Coercivity Lower Bound

Numerical Results

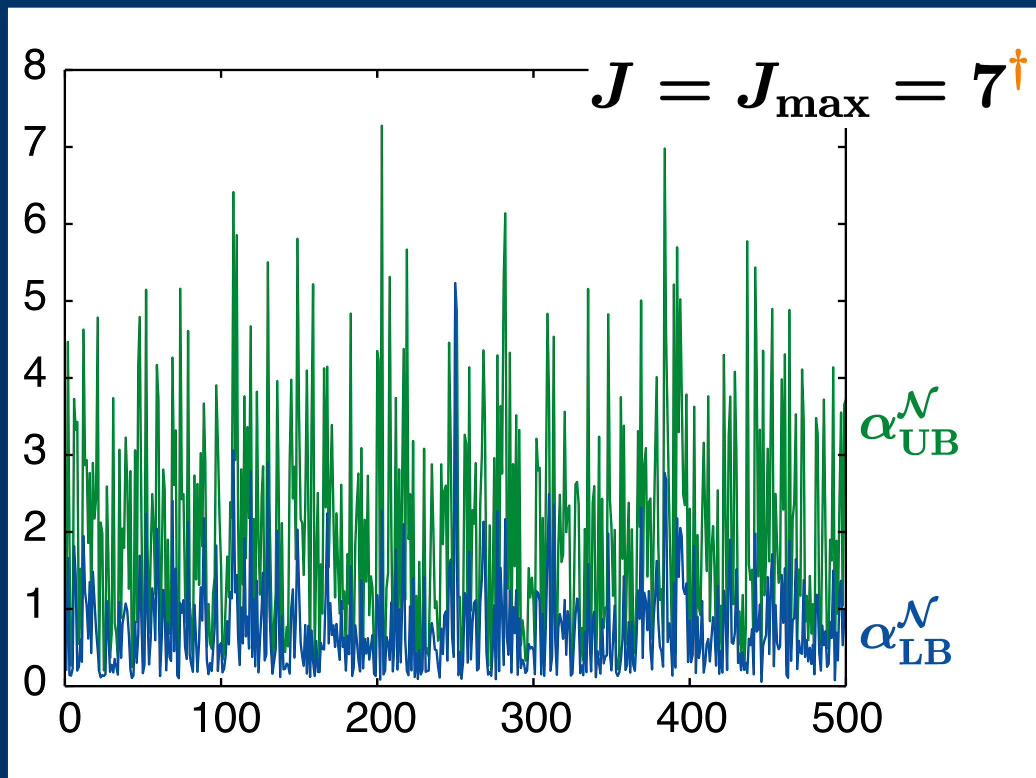
TBlock-$(3,3)$



Coercivity Lower Bound

Numerical Results

TBlock-$(3,3)$

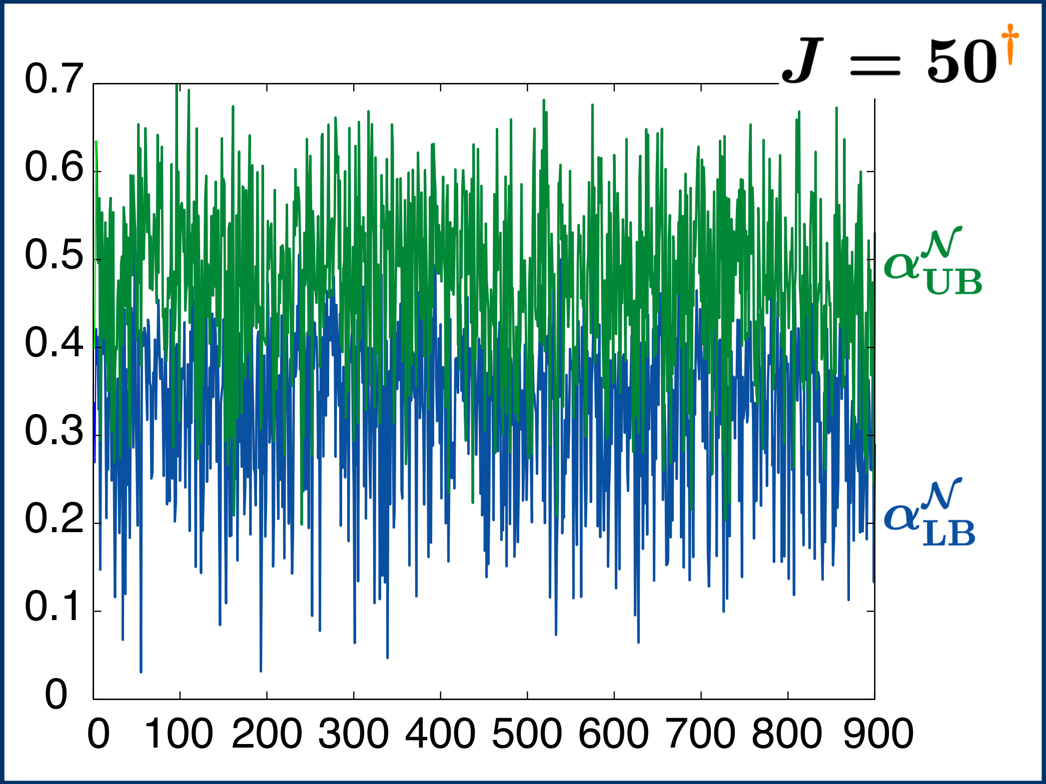


† Here $|\Xi_{\mathrm{train}}| = 500,\ \varepsilon_{\mathrm{SCM}} = 0.75,\ M = 64$;
note TBlock-$(3,3)$ is *parametrically coercive*.

Coercivity Lower Bound

Numerical Results

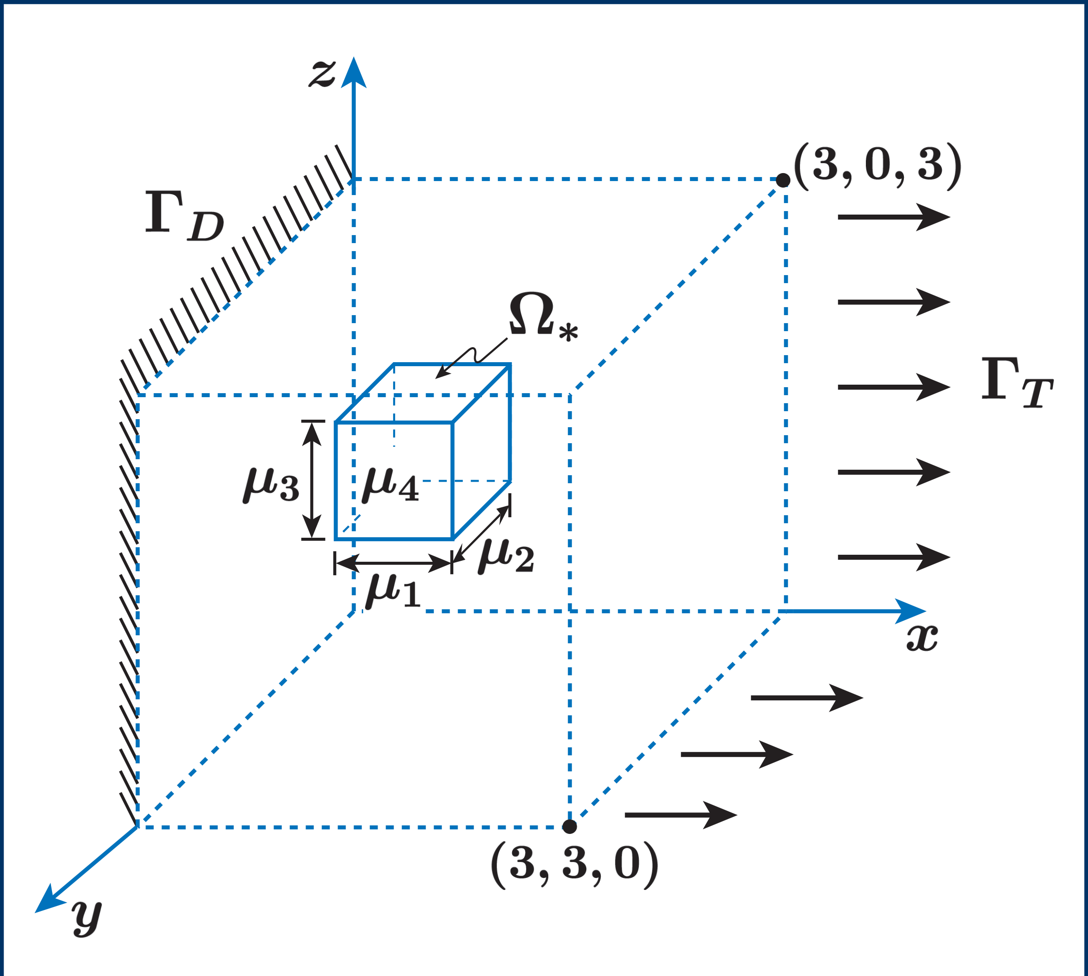
AMass



[†] Here $|\Xi_{\mathrm{train}}| = 900,\ \varepsilon_{\mathrm{SCM}} = 0.75,\ M = 8 :\ J_{\max} = 63.$

Computational Performance

EBlock3D



Computational Performance

FE Approximations

Definitions

Introduce

$$
\begin{aligned}
X^{\mathcal{N}_{\mathrm{vf}}},\ \mathcal{N}_{\mathrm{vf}} &= 106{,}754\ ,\\
X^{\mathcal{N}_{\mathrm{f}}},\ \mathcal{N}_{\mathrm{f}} &= 26{,}952\ ,\\
X^{\mathcal{N}_{\mathrm{c}}},\ \mathcal{N}_{\mathrm{c}} &= 6{,}315\ ;
\end{aligned}
$$

assume that X^{e} "$=$" $X^{\mathcal{N}_{\mathrm{vf}}}$.

Computational Performance

FE Approximations

Output Errors

For "typical" $\mu \in \mathcal{D}$,

$$\varepsilon^{\mathrm{FE},\mathcal{N}_{\mathrm{f}}} \equiv \frac{|s^{\mathcal{N}_{\mathrm{vf}}}(\mu) - s^{\mathcal{N}_{\mathrm{f}}}(\mu)|}{|s^{\mathcal{N}_{\mathrm{vf}}}(\mu)|} \approx 0.01$$

$$\varepsilon^{\mathrm{FE},\mathcal{N}_{\mathrm{c}}} \equiv \frac{|s^{\mathcal{N}_{\mathrm{vf}}}(\mu) - s^{\mathcal{N}_{\mathrm{c}}}(\mu)|}{|s^{\mathcal{N}_{\mathrm{vf}}}(\mu)|} \approx 9.10^{-2}$$

note the output $s^{\mathcal{N}_{\mathrm{vf}}}$ is $O(1)$.

Computational Performance

RB Approximations

Introduce
$$\Delta^{s,\mathrm{rel}}_{N,\max} = \max_{\mu\in\Xi_{\mathrm{train}}} \frac{\Delta^{s}_{N}(\mu)}{s^{\mathcal{N}}_{N}(\mu)}$$

$\mathrm{RB}^{\mathrm{f}}\colon u^{\mathcal{N}_{\mathrm{f}}}_{N}(\mu),\ 1 \le N \le N^{\mathrm{f}}_{\max}$; choose
$$N^{\mathrm{f}}_{\max} \mid \Delta^{s,\mathrm{rel}}_{N,\max} \le \frac{1}{4}\,\varepsilon^{\mathrm{FE},\mathcal{N}_{\mathrm{f}}}\ .$$

$\mathrm{RB}^{\mathrm{c}}\colon u^{\mathcal{N}_{\mathrm{c}}}_{N}(\mu),\ 1 \le N \le N^{\mathrm{c}}_{\max}$; choose
$$N^{\mathrm{c}}_{\max} \mid \Delta^{s,\mathrm{rel}}_{N,\max} \le \frac{1}{4}\,\varepsilon^{\mathrm{FE},\mathcal{N}_{\mathrm{c}}}\ .$$

Computational Performance

Cost Comparison

Metrics

Define

$$t_{\mathrm{comp}}^{\mathrm{Offline}}(\mathcal{N}_\bullet) = \frac{\text{Offline time for } \mathbf{RB}^{\mathcal{N}_\bullet}\textbf{: SCM \& RB Greedy}}{\text{time to evaluate } \mu \rightarrow s^{\mathcal{N}_\bullet}};$$

$$t_{\mathrm{comp}}^{\mathrm{Online}}(\mathcal{N}_\bullet, N) = \frac{\text{Online time for } \mathbf{RB}^{\mathcal{N}_\bullet}\textbf{: } \partial t_{\mathrm{comp}}(\mu \rightarrow s_N^{\mathcal{N}_\bullet}, \Delta_N^s)}{\text{time to evaluate } \mu \rightarrow s^{\mathcal{N}_\bullet}\ \dagger}.$$

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†Note the FE solver is minimum fill-in Cholesky:
for larger $\mathcal{N}$, PCG would clearly perform better.

We obtain

$$\mathcal{N}_{\mathrm{f}}\colon\ t^{\mathrm{Offline}}_{\mathrm{comp}}(\mathcal{N}_{\mathrm{f}}) = 53.72\ ,$$

break-even

$$t^{\mathrm{Online}}_{\mathrm{comp}}(\mathcal{N}_{\mathrm{f}}, N^{\mathrm{f}}_{\mathrm{max}}) = 0.0018\ ;$$

†

$$\mathcal{N}_{\mathrm{c}}\colon\ t^{\mathrm{Offline}}_{\mathrm{comp}}(\mathcal{N}_{\mathrm{c}}) = 554.4\ ,$$

$$t^{\mathrm{Online}}_{\mathrm{comp}}(\mathcal{N}_{\mathrm{c}}, N^{\mathrm{c}}_{\mathrm{max}}) = 0.027\ .$$

†Note roughly 70% of the Online time is associated with $\Delta^s_N(\mu)$.

Software

Components

Offline: Symbolic

Problem Definition (User);

Geometric Transformation:

$$\Omega^k,\ \Omega_{\mathrm{o}}^k(\mu),\ \mathcal{T}^{\mathrm{aff},k}(\,\cdot\,;\mu),\ 1 \le k \le K_{\mathrm{dom}}\ ;$$

Affine Representation: f, ℓ

$$\Theta^q(\mu),\ a^q(w,v),\ 1 \le q \le Q\ .$$

Software

Components

Offline: Numeric

Finite Element Matrices: $\boldsymbol{f}, \boldsymbol{\ell}$

$$a^q(\phi_k^{\mathrm{FE}}, \phi_{k'}^{\mathrm{FE}}),\ 1 \le k, k' \le \mathcal{N},\ 1 \le q \le Q\ ;$$

SCM Greedy:

$$\mathcal{C}_{J_{\max}}, \mathcal{B}, \{\alpha^{\mathcal{N}}(\mu') \mid \mu' \in \mathcal{C}_{J_{\max}}\}, \ldots\ ;$$

RB Greedy (*Primal*, Dual) $\{\zeta^{N\,n}\}_{1 \le n \le N_{\max}}$

$$a^q(\zeta^j, \zeta^i),\ 1 \le i, j \le N_{\max},\ 1 \le q \le Q\ ,$$

$$(\mathcal{L}_n^q, \mathcal{L}_{n'}^{q'}),\ 1 \le n, n' \le N_{\max},\ 1 \le q, q' \le Q\ .$$

Software

Components

Online

1. `Online_RB` (`probname`, $\boldsymbol{\mu}$, `outputname`, ...): $\star$ preload data

   $$\boldsymbol{\mu} \longrightarrow s_N^{\mathcal{N}}(\boldsymbol{\mu}),\ \Delta_N^s(\boldsymbol{\mu})\ .$$

2. `Vis_RB` (`probname`, $\boldsymbol{\mu}$):[†]

   $$\boldsymbol{\mu} \longrightarrow \boldsymbol{\Omega},\ u_N^{\mathcal{N}}(x;\boldsymbol{\mu}) \text{ for all } x \text{ in } \boldsymbol{\Omega}_{\mathrm{o}}(\boldsymbol{\mu})\ .$$

[†]Note: The cost of `Vis_RB` is *not* independent of $\mathcal{N}$:

find $N_{\min}$ to achieve $\|u^{\mathcal{N}}(\mu) - u^{\mathcal{N}}_{N_{\min}}\|_X \le \Delta^{\star}_{N_{\min}}(\mu) = \varepsilon^X_{\mathrm{des}}$;

expand $u^{\mathcal{N}}_{N_{\min}}(x;\mu) = \sum_{n=1}^{N_{\min}} u^{\mathcal{N}}_{N_{\min}\,n}(\mu)\,\zeta^{N\,n}(x) — O(N_{\min}\mathcal{N})$.

Extensions: the NONs

Menu

*Non*compliant outputs, $\ell \neq f$;

*Non*symmetric forms, $a(\,\cdot\,, v; \mu)$;

*Non*coercive forms, $a(\,\cdot\,, v; \mu)$;

*Non*affine forms, $a(\,\cdot\,, v; \mu)$;

Extensions: the NONs

Menu

*Non*linear	*quadratic*	outputs, $\mathcal{Q}(\,\cdot\,,\,\cdot\,)$;
*Non*bilinear	trilinear	forms, $a(\,\cdot\,,\,\cdot\,, v; \mu)$;
*Non*bilinear	general	forms, $a(\,\cdot\,, v; \mu)$;
*Non*elliptic	parabolic	equations;
*Non*elliptic	hyperbolic	equations.

Extensions: the NONs

Noncompliant: $\ell \neq f$ [†]

Introduce duals

$$
\begin{aligned}
a(v, \psi^{\mathrm{e}}(\mu); \mu) &= -\ell(v), \ \forall\, v \in X^{\mathrm{e}}\,, \\
a(v, \psi^{\mathcal{N}}(\mu); \mu) &= -\ell(v), \ \forall\, v \in X^{\mathcal{N}}\,, \\
a(v, \psi^{\mathcal{N}}_{N_{\mathrm{du}}}(\mu); \mu) &= -\ell(v), \ \forall\, v \in W^{\mathcal{N},\mathrm{du}}_{N_{\mathrm{du}}}\,,
\end{aligned}
$$

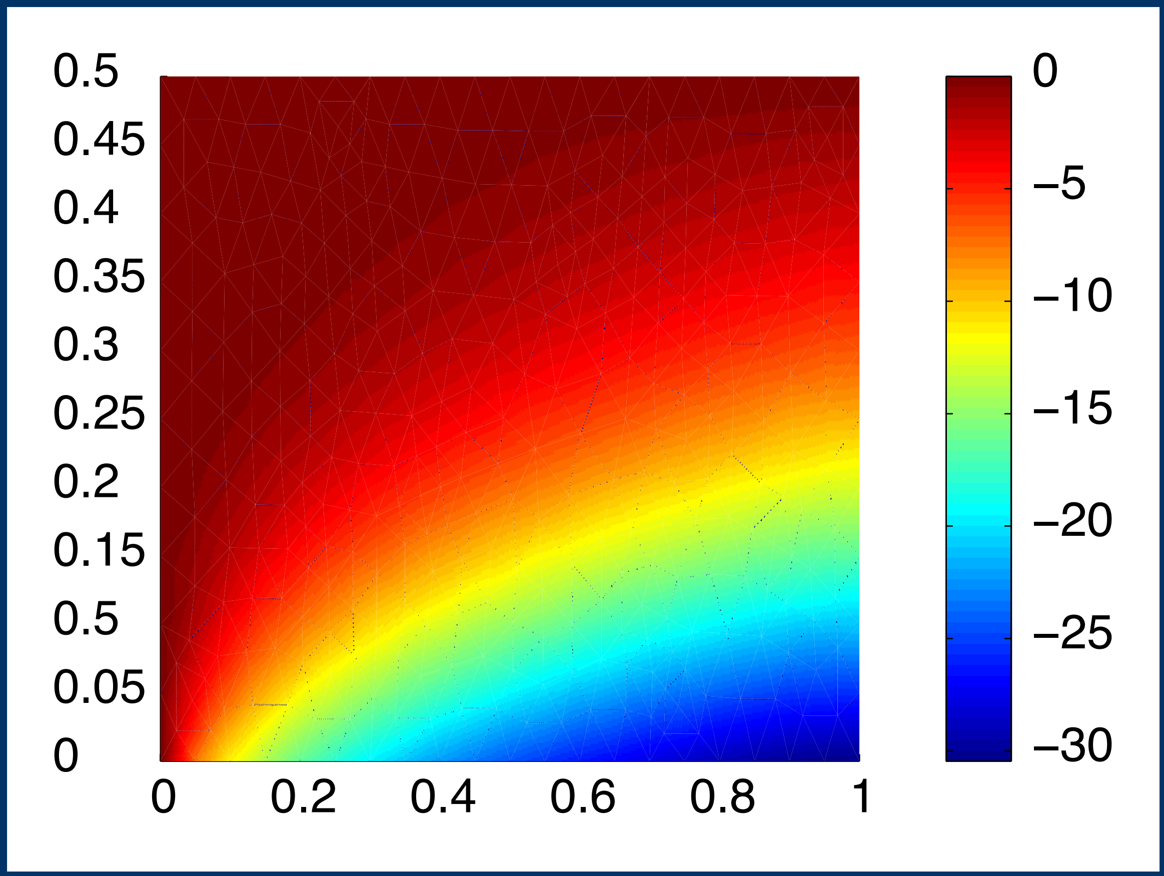
to improve *output convergence* (af fixed cost),
output bound effectivity.

[†] Or a non-symmetric.

Extensions: the NONs

Nonsymmetric

Example: adv-diff ...

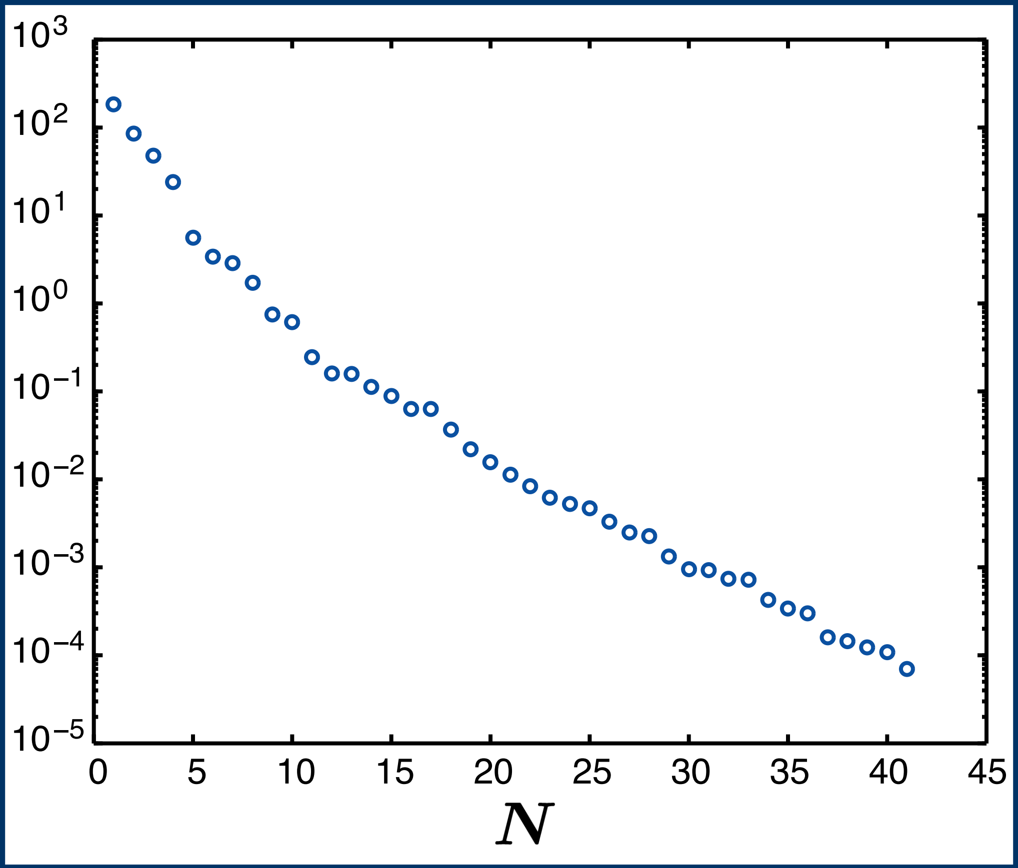


$$\mu = (\mu_1, \mu_2) = (\mathrm{Pe}, L) \in \mathcal{D} = [0.1, 100] \times [1, 100]$$

Extensions: the NONs

Nonsymmetric

Example: adv-diff ...



†

† Issues: stabilization; bound effectivity.

Noncoercive Problems

Inf-Sup Elements

Supremizer

We are given a bilinear form $a : X^1 \times X^2 \rightarrow \mathbb{R}$. Then

$$\beta_{\text{inf-sup}} = \inf_{w \in X^1} \sup_{v \in X^2} \frac{a(w,v)}{||w||_{X^1} ||v||_{X^2}};$$

we can also say that for any $w \in X^1$ there exists a v^* in X^2 (the inner supremizer) such that

$$a(w, v^*(w)) \geq \beta ||w||_{X^1} ||v^*||_{X^2}.$$

Note that $\beta \geq 0$ (if β is negative, just switch sign of v), however it is not necessarily true that $\beta > 0$.

Noncoercive Problems

Inf-Sup Elements

Supremizer

We introduce the inner supremizing operator $T : X^1 \to X^2$ as the following linear operator:

$$(Tw, v)_{X^2} = a(w, v), \qquad \forall v \in X^2;$$

why is $v = Tw$ the supremizer of $a(w,v)/||v||_{X^2}$?

Note

$$a(w, Tw) = (Tw, Tw)_{X^2}, \qquad w \text{ given},$$

so for $v = Tw$

$$\frac{a(w,v)}{||v||_{X^2}} = \frac{||Tw||^2_{X^2}}{||Tw||_{X^2}} = ||Tw||_{X^2}.$$

But by Cauchy-Schwarz inequality, for any $v \in X^2$

$$\frac{a(w,v)}{||v||_{X^2}} = \frac{(Tw,v)_{X^2}}{||v||_{X^2}} \leq \frac{||Tw||_{X^2}||v||_{X^2}}{||v||_{X^2}} \leq ||Tw||_{X^2},$$

which proves the result.

Note Tw is simply our $v^*(w)$ of earlier. Hence, for any $w \in X^1$,

$$a(w,Tw) \geq \beta ||w||_{X^1}||Tw||_{X^2}.$$

We can also develop an alternative expansion for β:

$$\beta = \inf_{w \in X^1} \frac{\left(\sup_{v \in X^2} \frac{a(w,v)}{||v||_{X^2}} \right)}{||w||_{X^1}} \overset{(v=Tw)}{=} \inf_{w \in X^1} \frac{||Tw||_{X^2}}{||w||_{X^1}},$$

or

$$\beta^2 = \inf_{w \in X^1} \frac{(Tw, Tw)_{X^2}}{||w||^2_{X^1}}$$

which is in fact a Rayleigh quotient.

Noncoercive Problems

Inf-Sup Elements

Singular Value

Note that β is just a fancy singular value. To see this, choose $\boldsymbol{X}^1 = \boldsymbol{X}^2 = \mathbb{R}^n$, and

$$\boldsymbol{a}(\mathrm{w}, \mathrm{v}) = \mathrm{v}^T \underline{\boldsymbol{A}} \mathrm{w},$$

for $\underline{\boldsymbol{A}} \in \mathbb{R}^{n \times n}$. It follows (for the usual Euclidean inner product) that

$$\begin{array}{ccc} (\boldsymbol{T}\mathrm{w}, \mathrm{v}) & = & \boldsymbol{a}(\mathrm{w}, \mathrm{v}) \\ \downarrow & & \downarrow \\ \mathrm{v}^T \underline{(\boldsymbol{T}\mathrm{w})} & = & \mathrm{v}^T \underline{\boldsymbol{A}} \mathrm{w} \end{array}$$

so $\underline{(T\mathrm{w})} = \underline{A}\mathrm{w}$.
Thus

$$\beta^2 = \inf_{\mathrm{w}\in\mathbb{R}^n} \frac{\mathrm{w}^T \underline{A}^T \underline{A}\mathrm{w}}{\mathrm{w}^T\mathrm{w}},$$

and hence β is the smallest singular value of $\underline{A}$.

Noncoercive Problems

Abstract Problem

Approximation

Find $u(\mu) \in X$ such that

$$a(u(\mu), v; \mu) = f(v) \qquad \forall\, v \in X$$

and

$$s(\mu) = \ell(u(\mu)),$$

where $\beta(\mu) > 0$, $\forall \mu \in \mathcal{D}$, with

$$\beta(\mu) = \inf_{w \in X} \sup_{v \in X} \frac{a(w, v; \mu)}{||w||_X ||v||_X}.$$

Noncoercive Problems

Abstract Problem

Approximation

We further assume that $a(\cdot, \cdot; \mu)$ is affine,

$$a(w, v; \mu) = \sum_{q=1}^{Q} \Theta^q(\mu) a^q(w, v).$$

We now denote our supremizer as $T^\mu : X \to X$, where $(T^\mu w, v)_X = a(w, v; \mu), \qquad \forall\, v \in X$

Note from our affine assumption it follows that

$$T^\mu w = \sum_{q=1}^{Q} \Theta^q(\mu) T^q w,$$

where $(T^q w, v)_X = a^q(w, v), \qquad \forall\, v \in X.$

Noncoercive Problems

Approximation

Petrov-Galerkin

We assume we are given two subspaces $\tilde{X}^1 \subset X$, $\tilde{X}^2 \subset X$. Then $\tilde{u}(\mu) \in \tilde{X}^1$ satisfies

$$a(\tilde{u}(\mu), v, \mu) = f(v), \quad \forall v \in \tilde{X}^2,$$

and

$$\tilde{s}(\mu) = \ell(\tilde{u}(\mu)).$$

We define

$$\tilde{\beta}(\mu) = \inf_{w \in \tilde{X}^1} \sup_{v \in \tilde{X}^2} \frac{a(w, v; \mu)}{||w||_X ||v||_X}$$

Noncoercive Problems

Approximation

Petrov-Galerkin

Our supremizer operator is then given by
$\tilde{T}^\mu : \tilde{X}^1 \longrightarrow \tilde{X}^2$

$$(\tilde{T}^\mu w, v)_X = a(w, v; \mu), \quad \forall v \in \tilde{X}^2.$$

It follows that, for any $w \in \tilde{X}^1$,

$$a(w, \tilde{T}^\mu w; \mu) \geq \tilde{\beta}(\mu) ||w||_X || \, T^\mu w||_X$$

We pursue here just a Primal approximation, however we can readily extend the approach to a Primal-Dual formulation as described for coercive problems.

Noncoercive Problems

Approximation

A Priori Theory

We know that

$$a(u(\mu), v; \mu) = f(v), \quad \forall v \in X$$

$$a(\tilde{u}(\mu), v; \mu) = f(v), \quad \forall v \in \tilde{X}^2$$

and hence

$$a(u - \tilde{u}, v; \mu) = 0, \quad \forall v \in \tilde{X}^2 (\subset X)$$

which is the usual (Petrov-)Galerkin orthogonality relationship.

We can write, for any $\tilde{w} \in \tilde{X}^1$,

$$\tilde{\beta}||\tilde{u}-\tilde{w}||_X||\tilde{T}^\mu(\tilde{u}-\tilde{w})||_X \leq a(\tilde{u}-\tilde{w},\tilde{T}^\mu(\tilde{u}-\tilde{w});\mu)$$

$$= a((\tilde{u}-\tilde{w})+(u-\tilde{u}),\underbrace{\tilde{T}^\mu(\tilde{u}-\tilde{w})};\mu)$$

(T^μ must be member of $\tilde{X}^2$, hence can not use stabler T^μ)

$$= a(u - \tilde{w}, \tilde{T}^{\mu}(\tilde{u} - \tilde{w}); \mu)$$

$$\leq \gamma ||u - \tilde{w}||_X ||\tilde{T}^{\mu}(\tilde{u} - \tilde{w})||_X$$

so

$$||\tilde{u} - \tilde{w}||_X \leq \frac{\gamma}{\tilde{\beta}} ||u - \tilde{w}||_X,$$

and hence

$$||u - \tilde{u}||_X \leq \inf_{\tilde{w} \in \tilde{X}^1} (||u - \tilde{w}||_X + ||\tilde{u} - \tilde{w}||_X)$$

$$(1 + \frac{\gamma}{\tilde{\beta}}) \inf_{w \in \tilde{X}^1} ||u - \tilde{w}||_X$$

Note it is not necessarily the case that $\tilde{\beta} \geq \beta$ or even $\tilde{\beta} > 0$ ($\tilde{\beta}$ may tend to zero as $\tilde{X}^1, \tilde{X}^2$ are refined); in this sense, noncoercive problems are much more difficult than coercive problems.
We observe that approximation is provided by $\tilde{X}^1$ and stability (through $\tilde{\beta}$) by $\tilde{X}^2$.

$$\tilde{X}^1 = \tilde{X}^2 = W_N$$

Introduce

$$W_N = span\left\{u(\mu^n_{pr}), 1 \leq n \leq N\right\}, \quad 1 \leq N \leq N_{max}$$

Then $u_N(\mu) \in W_N$ satisfies

$$a(u_N(\mu), v; \mu) = f(v), \quad \forall v \in W_N,$$

and

$$s_N(\mu) = \ell(u_N(\mu)).$$

Noncoercive Problems

RB Approximation

Galerkin

If we define

$$\beta_N(\mu) \equiv \inf_{w \in W_N} \sup_{v \in W_N} \frac{a(w, v; \mu)}{||w||_X ||v||_X},$$

then

$$||u - u_N||_X \leq \left(1 + \frac{\gamma}{\beta_N}\right) \inf_{w_N \in W_N} ||u - w_N||_X$$

(and $|s - s_N| \leq ||\ell||_{(X^{\mathcal{N}})'} ||u - u_N||_X$).

In practice this often works very well. In theory, however, it is not in general possible to ensure

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RB Approximation

Galerkin

$\beta_N \geq \beta(\mu)$ and thus in principle we could (though typically do not) observe $\beta_N \longrightarrow \mathbf{0}$ as $N \longrightarrow \infty$.

Noncoercive Problems

RB Approximation

Petrov-Galerkin

$$\tilde{X}^1 = W_N, \tilde{X}^2 = V_N^\mu$$

Introduce

$$W_N = span\left\{u(\mu_{pr}^n), 1 \le n \le N\right\}, 1 \le N \le N_{max}$$

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and

$$V_N^\mu = span\left\{T^\mu u(\mu_{pr}^n), 1 \le n \le N\right\}, 1 \le N \le N_{max}$$

Note V_N^μ is parameter dependent.

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Then $u_N(\mu) \in W_N$ satisfies

$$a(u_N(\mu), v; \mu) = f(v), \quad \forall v \in V_N^\mu,$$

and

$$s_N(\mu) = \ell(u_N(\mu)).$$

If we define

$$\beta_N(\mu) \equiv \inf_{w \in W_N} \sup_{v \in V_N^\mu} \frac{a(w, v; \mu)}{||w||_X ||v||_X},$$

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then

$$||u - u_N||_X \le \left(1 + \frac{\gamma}{\beta_N}\right) \inf_{w_N \in W_N} ||u - w_N||_X,$$

(and $|s - s_N| \le ||\ell||_{(X^{\mathcal{N}})'} ||u - u_N||_X$.) But in this case we can show that $\beta_N(\mu) \ge \beta(\mu), \forall \mu \in \mathcal{D}$.
To wit

$$\beta_N(\mu) \ge \inf_{w \in W_N} \frac{a(w, T^\mu w; \mu)}{||w||_X ||T^\mu w||_X} \qquad T^\mu : X^{\mathcal{N}} \to X^{\mathcal{N}}$$

since for any $w \in W_N, T^\mu w \in V_N^\mu$. But

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$a(w, T^\mu w; \mu) = (T^\mu w, T^\mu w)_X$ and hence

$$\beta_N(\mu) = \inf_{w \in W_N} \frac{||T^\mu w||_X}{||w||_X} \geq \inf_{w \in X} \frac{||T^\mu w||_X}{||w||_X} = \beta(\mu),$$

given that $W_N \subset X$.

Hence this Petrov-Galerkin scheme is guaranteed to be stable.

Re Offline-Online, we note that if

$$W_N = span\,\{\zeta^n, 1 \leq n \leq N\}$$

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then

$$V_N^{\mu} = span \left\{ \sum_{q=1}^{Q} \Theta^q(\mu) T^q \zeta^n, 1 \le n \le N \right\}$$

and hence

$$a(u_N(\mu), v; \mu) = \cdots$$

$$a \left(\sum_{j=1}^{N} u_{Nj}(\mu) \zeta^j, \sum_{q'=1}^{Q} \Theta^{q'}(\mu) T^{q'} \zeta^i; \mu \right) 1 \le i \le N$$

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Petrov-Galerkin

$$\underbrace{\sum_{j=1}^{N}\left(\sum_{q=1}^{Q}\sum_{q'=1}^{Q}\Theta^q(\mu)\Theta^{q'}(\mu)\,\overbrace{a^q(\zeta^j, T^{q'}\zeta^i)}^{\text{stored}}\right)u_{Nj}(\mu)}$$

$1 \leq i \leq N \quad O(Q^2N^2)$ Online operations.

(not particular onerous since there is already a $O(Q^2N^2)$ operation associated with a posteriori error bound.)

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A Posteriori Error Estimation

We know that

$$a(u - u_N, v; \mu) = r(v; \mu), \forall v \in X$$

$$= (\hat{e}(\mu), v)_X, \forall v \in X$$

where

$$r(v; \mu) = f(v) - a(u_N, v; \mu)$$

Here u_N can be either our Galerkin or Petrov-Galerkin approximation.

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A Posteriori Error Estimation

It thus follows that

$$\beta(\mu)||u - u_N||_X||T^\mu(u - u_N)||_X \leq$$

$$a(u - u_N, T^\mu(u - u_N); \mu) =$$

$$= (\hat{e}(\mu), T^\mu(u - u_N))_X$$

$$\leq ||\hat{e}(\mu)||_X||T^\mu(u - u_N)||_X$$

or

$$||u - u_N||_X \leq \frac{||\hat{e}(\mu)||_X}{\beta(\mu)}$$

Thus, for $\beta_{LB}(\mu)$ a positive lower bound for $\beta(\mu)$, and

$$\Delta_N(\mu) \equiv \frac{||\hat{e}(\mu)||_X}{\beta_{LB}(\mu)},$$

we obtain

$$||u - u_N||_X \leq \Delta_N(\mu)$$

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(and also $|s - s_N| \leq ||l||_{(X^{\mathcal{N}})'} \Delta_N(\mu)$: a Primal-Dual approach / result is also possible).

Re Offline-Online, the calculation of $||\hat{e}(\mu)||_X$ is identical to the coercive case. It only remains to construct $\beta_{LB}(\mu)$ by the SCM.

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SCM for $\beta_{LB}(\mu)$

We recall that

$$\beta^2(\mu) = \inf_{w \in X^{\mathcal{N}}} \frac{(T^\mu w, T^\mu w)_X}{||w||_X^2}$$

but since a is affine,

$$\beta^2(\mu) = \inf_{w \in X^{\mathcal{N}}} \sum_{q=1}^{Q} \sum_{q'=1}^{Q} \Theta^q(\mu)\Theta^{q'}(\mu) \frac{(T^q w, T^{q'} w)_X}{||w||_X^2}$$

$$= \inf_{w \in X^{\mathcal{N}}} \sum_{q=1}^{Q} \sum_{q'=q}^{Q} (2 - \delta_{qq'})\Theta^q(\mu)\Theta^{q'}(\mu) \frac{(T^q w, T^{q'} w)_X}{||w||_X^2}$$

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SCM for $\beta_{LB}(\mu)$

Hence

$$\beta^2(\mu) = \underbrace{\inf_{w \in X^{\mathcal{N}}} \sum_{q=1}^{\hat{Q}} \hat{\Theta}^q(\mu) \frac{\hat{a}^q(w,w)}{||w||_X^2}}_{\text{apply standard SCM}}$$

where

$$\begin{array}{c} (2-\delta_{q'q''})\Theta^{q'}(\mu)\Theta^{q''}(\mu) \\ 1 \le q' < q'' \le Q \end{array} \longmapsto \begin{array}{c} \hat{\Theta}^q(\mu) \\ 1 \le q \le \hat{Q} \equiv \frac{Q(Q+1)}{2} \end{array}$$

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SCM for $\beta_{LB}(\mu)$

$$\frac{1}{2}\Big((T^{q'}w, T^{q''}v)_X + (T^{q'}v, T^{q''}w)_X\Big)$$
$$1 \le q' < q'' \le Q$$

$$\longmapsto \quad \begin{array}{c} \hat{a}^q(w,v) \\ 1 \le q \le \hat{Q} \equiv \frac{Q(Q+1)}{2} \end{array}$$