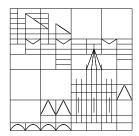
Universität Konstanz Fachbereich Mathematik und Statistik Prof. Dr. Stefan Volkwein Martin Gubisch, Oliver Lass 17.09.2013

Summerschool Reduced Basis Methods 2013 Exercise Sheet



PART A: POD FOR DYNAMICAL SYSTEMS

POD Galerkin ansatz. Consider the linear parabolic differential equation

$$M\dot{y}(t) + Ay(t) = f(t) + Bu(t), \qquad My(0) = y_{\circ}.$$

- 1. Find a linear and bounded operator $S: U \to Y$ and $\hat{y} \in Y$ such that $y = Su + \hat{y}$.
- 2. Complete line 7 and implement the function Rom in the algorithm

Algorithm 1 (PodGalerkin)

Require: $M, A, f, y_o, B, u, \Delta t, \ell_{\text{max}}$.

- 1: Solve $\hat{y} = \mathtt{State}(M, A, f, y_{\circ}, \Delta t)$
- 2: Solve $y_1 = \mathtt{State}(M, A, Bu, 0, \Delta t)$ and $y_2 = \mathtt{State}(M, A, f + Bu, y_o, \Delta t)$
- 3: Solve $\Psi_1 = \operatorname{Pod}(\Delta t, M, y_1, \ell_{\max})$ and $\Psi_2 = \operatorname{Pod}(\Delta t, M, y_2, \ell_{\max})$
- 4: **for** $\ell = 1, ..., \ell_{\text{max}}$ **do**
- 5: Determine $[M_1^{\ell}, A_1^{\ell}, B_1^{\ell}] = \text{Rom}(\Psi_1^{\ell}, M, A, B)$ and $[M_2^{\ell}, A_2^{\ell}, B_2^{\ell}, f_2^{\ell}, y_{\circ 2}^{\ell}] = \text{Rom}(\Psi_2^{\ell}, M, A, B, f, y_{\circ})$
- 6: Compute $x_1^{\ell} = \mathtt{State}(M_1^{\ell}, A_1^{\ell}, B_1^{\ell}u, 0, \Delta t)$ and $x_2^{\ell} = \mathtt{State}(M_2^{\ell}, A_2^{\ell}, f_2^{\ell} + B_2^{\ell}u, y_{\circ 2}^{\ell}, \Delta t)$
- 7: Determine the corresponding high-dimensional states $y_1^{\ell} = ..., y_2^{\ell} = ...$
- 8: Compute $e_1^{\ell} = \|y_2 y_1^{\ell}\|_Y$ and $e_2^{\ell} = \|y_2 y_2^{\ell}\|_Y$
- 9: end for
- 3. Visualize e_1, e_2 and the first few Pod elements of Ψ_1, Ψ_2 .

PART B: OPTIMAL CONTROL OF PDES

 ${\bf Optimization\ problem.}$ We consider the pde-constrained optimal control problem

$$\min J(y, u) = \frac{1}{2} \int_{\Theta} \|y(t) - y_Q(t)\|_H^2 dt + \frac{1}{2} \|y(T) - y_\Omega\|_H^2 + \frac{\sigma}{2} \|u\|_U^2$$

subject to

$$M\dot{y}(t) + Ay(t) = f(t) + Bu(t) \& My(0) = y_{\circ}, \qquad u_a(t) \le u(t) \le u_b(t).$$

Optimality system. An optimal control-state pair $(\bar{y}, \bar{u}) \in Y \times U$ is given by

$$M\dot{y}(t) + Ay(t) - f(t) - Bu(t) = 0,$$
 $y(0) = y_{\circ}$
 $-M\dot{p}(t) + Ap(t) + My(t) - y_{Q}(t) = 0,$ $Mp(T) = y_{\Omega} - My(T)$
 $u(t) - \mathbb{P}(\sigma^{-1}B^{*}p(t)) = 0$

where $\mathbb{P}(u) = \min(\max(u, u_a), u_b)$ is the canonical projection of U onto $[u_a, u_b]$.

- 4. Find a linear and bounded operator $T:U\to Y$ and $\hat{p}\in Y$ such that $p=Tu+\hat{p}$.
- 5. Define a selfmapping F on U such that the optimal control \bar{u} is a fixpoint of F.
- 6. Formulate a condition of the regularisation parameter σ such that the corresponding Banach fixpoint iteration admits a unique solution.

PART C: REDUCED ORDER MODELLUNG FOR OPTIMIZATION PROBLEMS

Optimization algorithm. We provide the following fixpoint strategy:

Algorithm 2 (SolverOpt)

```
Require: initial control u_{\circ}, desired exactness \varepsilon, maximal iterations k_{\max}, inhomogeneous component B^{\star}\hat{p}

1: Set k=0, u=u_{\circ}

2: repeat

3: Compute y_h=Su=\mathtt{State}(M,A,Bu,0,\Delta t)

4: Compute p_h=Tu=\mathtt{fliplr}(\mathtt{State}(M,A,-\mathtt{fliplr}(My_h),-My(T),\Delta t))

5: Evaluate u_+=\mathbb{P}(\sigma^{-1}(B^{\star}p_h+B^{\star}\hat{p}))

6: until \|u_+-u\|_U<\varepsilon or k=k_{\max}.

7: Set u=u_+ and k=k+1

8: Return optimal control u.
```

- 7. Design an algorithm which combines the model order reduction via POD with the provided optimization strategy.
- 8. Compute the error between the suboptimal controls u^{ℓ} and the optimal control u for $\ell = 1, ..., 15$. Visualize the result.