



Summerschool Reduced Basis Methods 2013
Exercise Sheet

PART A: POD FOR DYNAMICAL SYSTEMS

POD Galerkin ansatz. Consider the linear parabolic differential equation

$$M\dot{y}(t) + Ay(t) = f(t) + Bu(t), \quad My(0) = y_\circ.$$

1. Find a linear and bounded operator $S : U \rightarrow Y$ and $\hat{y} \in Y$ such that $y = Su + \hat{y}$.
2. Complete line 7 and implement the function `Rom` in the algorithm

Algorithm 1 (PodGalerkin)

Require: $M, A, f, y_\circ, B, u, \Delta t, \ell_{\max}$.

- 1: Solve $\hat{y} = \text{State}(M, A, f, y_\circ, \Delta t)$
 - 2: Solve $y_1 = \text{State}(M, A, Bu, 0, \Delta t)$ and $y_2 = \text{State}(M, A, f + Bu, y_\circ, \Delta t)$
 - 3: Solve $\Psi_1 = \text{Pod}(\Delta t, M, y_1, \ell_{\max})$ and $\Psi_2 = \text{Pod}(\Delta t, M, y_2, \ell_{\max})$
 - 4: **for** $\ell = 1, \dots, \ell_{\max}$ **do**
 - 5: Determine $[M_1^\ell, A_1^\ell, B_1^\ell] = \text{Rom}(\Psi_1^\ell, M, A, B)$ and $[M_2^\ell, A_2^\ell, B_2^\ell, f_2^\ell, y_{\circ 2}^\ell] = \text{Rom}(\Psi_2^\ell, M, A, B, f, y_\circ)$
 - 6: Compute $x_1^\ell = \text{State}(M_1^\ell, A_1^\ell, B_1^\ell u, 0, \Delta t)$ and $x_2^\ell = \text{State}(M_2^\ell, A_2^\ell, f_2^\ell + B_2^\ell u, y_{\circ 2}^\ell, \Delta t)$
 - 7: Determine the corresponding high-dimensional states $y_1^\ell = \dots, y_2^\ell = \dots$
 - 8: Compute $e_1^\ell = \|y_2 - y_1^\ell\|_Y$ and $e_2^\ell = \|y_2 - y_2^\ell\|_Y$
 - 9: **end for**
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3. Visualize e_1, e_2 and the first few Pod elements of Ψ_1, Ψ_2 .
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PART B: OPTIMAL CONTROL OF PDES

Optimization problem. We consider the pde-constrained optimal control problem

$$\min_{\Theta} J(y, u) = \frac{1}{2} \int_{\Theta} \|y(t) - y_Q(t)\|_H^2 dt + \frac{1}{2} \|y(T) - y_\Omega\|_H^2 + \frac{\sigma}{2} \|u\|_U^2$$

subject to

$$M\dot{y}(t) + Ay(t) = f(t) + Bu(t) \ \& \ My(0) = y_\circ, \quad u_a(t) \leq u(t) \leq u_b(t).$$

Optimality system. An optimal control-state pair $(\bar{y}, \bar{u}) \in Y \times U$ is given by

$$\begin{aligned} M\dot{y}(t) + Ay(t) - f(t) - Bu(t) &= 0, & y(0) &= y_\circ \\ -M\dot{p}(t) + Ap(t) + My(t) - y_Q(t) &= 0, & Mp(T) &= y_\Omega - My(T) \\ u(t) - \mathbb{P}(\sigma^{-1} B^* p(t)) &= 0 \end{aligned}$$

where $\mathbb{P}(u) = \min(\max(u, u_a), u_b)$ is the canonical projection of U onto $[u_a, u_b]$.

4. Find a linear and bounded operator $T : U \rightarrow Y$ and $\hat{p} \in Y$ such that $p = Tu + \hat{p}$.
 5. Define a selfmapping F on U such that the optimal control \bar{u} is a fixpoint of F .
 6. Formulate a condition of the regularisation parameter σ such that the corresponding Banach fixpoint iteration admits a unique solution.
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Optimization algorithm. We provide the following fixpoint strategy:

Algorithm 2 (SolverOpt)

Require: initial control u_o , desired exactness ε , maximal iterations k_{\max} , inhomogeneous component $B^*\hat{p}$

- 1: Set $k = 0$, $u = u_o$
 - 2: **repeat**
 - 3: Compute $y_h = Su = \text{State}(M, A, Bu, 0, \Delta t)$
 - 4: Compute $p_h = Tu = \text{fliplr}(\text{State}(M, A, -\text{fliplr}(My_h), -My(T), \Delta t))$
 - 5: Evaluate $u_+ = \mathbb{P}(\sigma^{-1}(B^*p_h + B^*\hat{p}))$
 - 6: **until** $\|u_+ - u\|_U < \varepsilon$ **or** $k = k_{\max}$.
 - 7: Set $u = u_+$ and $k = k + 1$
 - 8: Return optimal control u .
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7. Design an algorithm which combines the model order reduction via POD with the provided optimization strategy.
8. Compute the error between the suboptimal controls u^ℓ and the optimal control u for $\ell = 1, \dots, 15$. Visualize the result.