

Dispersive Estimates for Schrödinger Equations and Applications

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We begin by looking at the autonomous linear first-order system

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0,$$

where A is a given n by n matrix.

Then a straightforward calculation shows that the solution is given by

$$x(t) = \exp(tA)x_0,$$

where the exponential function is defined by the usual power series

$$\exp(tA) = \sum_{j=0}^{\infty} \frac{t^j}{j!} A^j,$$

which converges by comparison with the real-valued exponential function since

$$\left\| \sum_{j=0}^{\infty} \frac{t^j}{j!} A^j \right\| \leq \sum_{j=0}^{\infty} \frac{|t|^j}{j!} \|A\|^j = \exp(|t|\|A\|).$$

One of the basic questions concerning the solution is the long-time behavior: Does the solutions remain bounded for all times (**stability**) or does it even converge to zero (**asymptotic stability**)?

This question is usually answered by determining the spectrum (i.e. the eigenvalues) of A :

Theorem

A linear constant coefficient ODE is asymptotically stable iff all eigenvalues of A have negative real part. It is stable iff all eigenvalues have non-positive real part and for those with real part zero the algebraic and geometric multiplicities coincide.

This can easily be seen by resorting to the Jordan canonical form of A .

Proof.

If U maps A to Jordan canonical form $U^{-1}AU = J_1 \oplus \cdots \oplus J_m$, where $J = \alpha\mathbb{I} + N$ are the Jordan blocks, then

$$\exp(tA) = U \begin{pmatrix} \exp(tJ_1) & & \\ & \ddots & \\ & & \exp(tJ_m) \end{pmatrix} U^{-1},$$

with the exponential of one Jordan block given by

$$\exp(tJ) = e^{\alpha t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{k-1}}{(k-1)!} \\ & 1 & t & \ddots & \vdots \\ & & 1 & \ddots & \frac{t^2}{2!} \\ & & & \ddots & t \\ & & & & 1 \end{pmatrix}.$$

In particular, in the stable case one has

$$\|\exp(tA)\| \leq C, \quad t \geq 0,$$

and in the asymptotically stable case

$$\|\exp(tA)\| \leq Ce^{-\alpha t}, \quad t \geq 0,$$

for some $C \geq 1$ and some $\alpha > 0$. Hence one has a quite good qualitative understanding of this case.

We remark that the first case is typical for physical systems which are e.g. of Hamiltonian type. For such systems the **conservation of energy** implies stability but clearly this is incompatible with asymptotic stability. If one adds some friction to the system, one has **energy dissipation** and hence gets asymptotic stability.

Trying to apply these ideas to partial differential equations things get more tricky. To understand this consider the heat equation which can formally be written as

$$\dot{x}(t) = Ax(t), \quad A = \Delta,$$

where $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ is the usual Laplacian. Now the underlying space X will be a suitable function space (e.g., some $L^p(\mathbb{R}^n)$) and Δ is considered as a linear operator on this space. If A were a bounded operator on X we could define $\exp(tA)$ as before by using power series. This gives a continuous group of operators

$$t \mapsto T(t) = \exp(tA).$$

However, our operator is not bounded (this is already reflected by the fact that we need to restrict its domain to functions which can be differentiated at least twice in some suitable sense) and hence this simple approach does not work here. Moreover, the fact that a Gaussian initial condition concentrates and eventually becomes unbounded for negative times, shows that we cannot expect $T(t)$ to be defined for all $t \in \mathbb{R}$.

Nevertheless, there is a well-developed theory under what conditions on A the **abstract Cauchy problem**

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0,$$

in a Banach space X has a solution given by

$$x(t) = T(t)x_0,$$

where $T(t)$, $t \geq 0$, forms a strongly continuous semigroup of (bounded) operators.

A **strongly continuous operator semigroup** (also C_0 semigroup) is a family of bounded operators $T(t) : X \rightarrow X$, $t \geq 0$, such that

- ① $T(t)x \in C([0, \infty), X)$ for every $x \in X$ (strong continuity) and
- ② $T(0) = \mathbb{I}$, $T(t+s) = T(t)T(s)$ for every $t, s \geq 0$ (semigroup property).

Note: Requiring uniform continuity instead of strong continuity is equivalent to A being bounded.

In this setting A is called the **infinitesimal generator** of $T(t)$ and can be recovered via

$$Ax = \lim_{t \downarrow 0} \frac{1}{t} (T(t)x - x)$$

with domain precisely those $x \in X$ for which the above limit exists.

We are interested in bounded semigroups satisfying $\|T(t)\| \leq 1$ which are known as **semigroups of contraction**.

Theorem (Hille–Yosida theorem)

A linear (possibly unbounded) operator A is the infinitesimal generator of a contraction semigroup if and only if

- ① *A is densely defined, closed and*
- ② *the resolvent set of A contains the positive half line and for every $\lambda > 0$ we have*

$$\|(A - \lambda)^{-1}\| \leq \frac{1}{\lambda}.$$

Another version which is often easier to check is

Theorem (Lumer–Phillips theorem)

Let X be a complex Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. A linear (possibly unbounded) operator A is the infinitesimal generator of a contraction semigroup if and only if

- ① *A is densely defined, closed and*
- ② *the resolvent set of A contains the positive half line and A is **dissipative***

$$\operatorname{Re}\langle x, Ax \rangle \leq 0$$

for every x in the domain of A .

Note: The theorem also holds in Banach spaces if the scalar product is replaced by a duality section.

Note: In the finite dimensional case dissipativity implies that all eigenvalues have non-positive real part.

In our case the underlying partial differential equation will be linear with constant coefficients and just as in the ODE case we will be able to obtain an explicit formula for $T(t)$ and hence we will not further pursue these ideas here.

So far we have only looked at linear equations while most more realistic problems are nonlinear. However, in many situations nonlinear problems can be considered as perturbations of linear problems via

$$\dot{x}(t) = Ax(t) + g(t, x(t)).$$

In such an situation we can use our knowledge about the linear problem ($g \equiv 0$) to show that asymptotic stability persists for the perturbation under suitable assumptions:

Theorem

Suppose A is asymptotically stable

$$\|\exp(tA)\| \leq Ce^{-\alpha t}, \quad t \geq 0,$$

for some constants $C, \alpha > 0$. Suppose that

$$|g(t, x)| \leq g_0|x|, \quad |x| \leq \delta, \quad t \geq 0,$$

for some constant $0 < \delta \leq \infty$. Then, if $g_0C < \alpha$, the solution $x(t)$ starting at $x(0) = x_0$ satisfies

$$\|x(t)\| \leq Ce^{-(\alpha - g_0C)t} |x_0|, \quad |x_0| \leq \frac{\delta}{C}, \quad t \geq 0.$$

Proof.

Our point of departure is the following integral equation

$$x(t) = \exp(tA)x_0 + \int_0^t \exp((t-s)A)g(s, x(s))ds.$$

which follows from Duhamel's formula for the in homogenous problem and is clearly equivalent to our original Cauchy problem. Inserting our assumptions we obtain

$$|x(t)| \leq Ce^{-\alpha t}|x_0| + \int_0^t Ce^{-\alpha(t-s)}g_0|x(s)|ds$$

for $t \geq 0$ as long as $|x(s)| \leq \delta$ for $0 \leq s \leq t$.

Proof (Cont.).

Introducing $y(t) = |x(t)|e^{\alpha t}$ we get

$$y(t) \leq C|x_0| + \int_0^t Cg_0 y(s) ds$$

and Gronwall's inequality implies $y(t) \leq C|x_0|e^{Cg_0 t}$. This is the desired estimate

$$|x(t)| \leq C|x_0|e^{-(\alpha - Cg_0)t}, \quad 0 \leq t \leq T,$$

for solutions which satisfy $|x(t)| \leq \delta$ for $0 \leq t \leq T$. If we start with $|x_0| \leq \delta/C$ we have $|x(t)| \leq \delta$ for sufficiently small T (note $C \geq 1$) by continuity and the above estimate shows that this remains true for all times. □

First of all note that in applications the linear part Ax will capture the linear behavior of the right hand side at 0 and the remainder $g(t, x)$ will vanish faster than linear near 0. Hence in our assumption we will be able to make g_0 as small as we want at the expense of choosing δ small. In particular, asymptotic stability persists under nonlinear perturbations.

Secondly, it is important to emphasize that the two main ingredients for this proof were

- Duhamel's formula and
- a good qualitative understanding of the linear part.

In particular, it generalizes to situations where a corresponding estimate for the linear part is available (e.g. periodic equations).

The generalization to strongly continuous semigroups is a bit more tricky. First of all any solution of the inhomogenous abstract Cauchy problem

$$\dot{x}(t) = Ax(t) + g(t), \quad x(0) = x_0,$$

will still be given by Duhamel's formula

$$x(t) = T(t)x_0 + \int_0^t T(t-s)g(s)ds$$

provided g is integrable. However, the above formula might not be a (classical) solution of the abstract Cauchy problem and one speaks of a mild solution (it can be shown that one has uniqueness and hence there will be no classical solution in such a situation).

So solutions of our integral equation in the nonlinear situation might not be solutions of the original problem.

Concerning (uniform) **exponential stability of semigroups**

$$\|T(t)\| \leq Ce^{-\alpha t}$$

for some $C, \alpha > 0$ we mention:

Theorem (Gearhart–Prüss–Greiner)

Let X be a Hilbert space. Then a strongly continuous semigroup $T(t)$ is uniformly exponentially stable if and only if the resolvent set of the generator A contains the half-plane $\{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$ and satisfies

$$\sup_{\operatorname{Re}(z) > 0} \|(A - z)^{-1}\| < \infty.$$

Note: Since the resolvent $(A - z)^{-1}$ is analytic in z and $\|(A - z)^{-1}\| \rightarrow \infty$ as z approaches the spectrum, the condition implies that the spectrum (which is closed) must be contained in $\{z \in \mathbb{C} \mid \operatorname{Re}(z) < 0\}$. Since for bounded operators we have $\|(A - z)^{-1}\| \leq (|z| - \|A\|)^{-1}$ for $|z| > \|A\|$ this condition reduces to our previous one for finite matrices.

Finally we want to look at the conservative case. Unfortunately this case is much less stable under perturbations as the trivial example

$$\dot{x} = x^2$$

show. In fact, the solution is given by

$$x(t) = \frac{x_0}{1 - x_0 t}$$

which blows up at the finite positive time $t = \frac{1}{x_0}$ if $x_0 > 0$.

Theorem

Suppose A is stable

$$\|\exp(tA)\| \leq C, \quad t \geq 0,$$

for some constant $C \geq 1$. Suppose that

$$|g(t, x)| \leq g_0(t)|x|, \quad |x| \leq \delta, \quad t \geq 0,$$

for some constant $0 < \delta \leq \infty$ and some function $g_0(t)$ with $G_0 = \int_0^\infty g_0(t) < \infty$. Then the solution $x(t)$ starting at $x(0) = x_0$ satisfies

$$|x(t)| \leq C \exp(CG_0)|x_0|, \quad |x_0| \leq \frac{\delta}{C \exp(CG_0)}, \quad t \geq 0.$$

Proof.

As in the previous proof we start with

$$x(t) = \exp(tA)x_0 + \int_0^t \exp((t-s)A)g(s, x(s))ds$$

and using our estimates we obtain

$$|x(t)| \leq C|x_0| + \int_0^t Cg_0(s)|x(s)|ds.$$

Hence an application of Gronwall's inequality

$$|x(t)| \leq C|x_0| \exp \left(C \int_0^t g_0(s)ds \right) \leq C|x_0| \exp (CG_0)$$

finishes the proof.

Observe that in this case the linear evolution does not provide any decay and hence we had to assume that our perturbation provides the necessary decay. In finite dimension there is nothing much we can do. However, in infinite dimensions there are different norms and while one might be conserved, another one might decay!

To understand this let us look again at the heat equation whose solution in \mathbb{R}^n is given by

$$u(t, x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y) dy.$$

For positive initial condition g a simple application of Fubini shows $\|u(t, \cdot)\|_1 = \|g\|_1$ that the L^1 norm is conserved. Physically this corresponds to energy conservation. Moreover, Young's inequality shows $\|u(t, \cdot)\|_p \leq \|g\|_p$ for every $1 \leq p \leq \infty$. However, if we use different norms on both sides we get decay

$$\|u(t, \cdot)\|_\infty \leq \frac{1}{(4\pi t)^{n/2}} \|g\|_1.$$

Of course this decay is no longer exponential. In fact, we have equality for the fundamental solution of the heat equation and hence the above estimate is optimal.

What you should have remembered so far:

- Nonlinear equations can be treated as perturbations of linear problems by virtue of Duhamel's formula!
- This requires a good qualitative understanding of the underlying linear problem.
- For dissipative systems we have exponential decay in time for the underlying semigroup. For conservative systems we can still get polynomial decay of the semigroup if we chose the right norms.

To keep the technical problems at a minimum and to be able to focus on some of the main ideas we will use the one-dimensional discrete Schrödinger equation as our model:

$$i\dot{u}(t) = H_0 u(t), \quad t \in \mathbb{R},$$

where

$$(H_0 u)_n = -u_{n+1} + 2u_n - u_{n-1}, \quad n \in \mathbb{Z}.$$

We will investigate this operator and its associated group in the weighted spaces $\ell_\sigma^p = \ell_\sigma^p(\mathbb{Z})$, $\sigma \in \mathbb{R}$, associated with the norm

$$\|u\|_{\ell_\sigma^p} = \begin{cases} (\sum_{n \in \mathbb{Z}} (1 + |n|)^{p\sigma} |u(n)|^p)^{1/p}, & p \in [1, \infty), \\ \sup_{n \in \mathbb{Z}} (1 + |n|)^\sigma |u(n)|, & p = \infty. \end{cases}$$

Of course, the case $\sigma = 0$ corresponds to the usual $\ell_0^p = \ell^p$ spaces without weight.

Making the plane wave ansatz

$$u_n(t) = e^{-i(\theta n - \omega t)}$$

we obtain a solution if and only if the **dispersion relation**

$$\omega = 2(\cos(\theta) - 1)$$

holds. Hence different plane waves travel at different speeds which is known as **dispersion**. According to physical intuition the superposition of different plane waves will lead to destructive interference and thus to decay of wave packets.

Mathematically a wave packets is a (continuous superposition) of plane waves given by

$$u_n(t) = \int_{-\pi}^{\pi} c(\theta) e^{-i(\theta n - \omega(\theta)t)} d\theta$$

where $c(\theta)$ is the amplitude of the plane wave with frequency θ . Evaluating this formula at $t = 0$ shows that the initial condition

$$g_n = u_n(0) = \int_{-\pi}^{\pi} c(\theta) e^{-i\theta n} d\theta$$

is given by 2π the Fourier coefficients of $c(\theta)$:

$$c(\theta) = \sum_{k \in \mathbb{Z}} \frac{g_k}{2\pi} e^{i\theta k}.$$

Putting all together we obtain

$$u_n(t) = (\exp(-itH_0)g)_n = \sum_{m \in \mathbb{Z}} K_{n,k}(t)g_k,$$

where

$$K_{n,k}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-it(2-2\cos\theta)} e^{-i\theta|n-k|} d\theta.$$

Note: The last integral is Bessel's integral implying

$$K_{n,k}(t) = e^{i(-2t + \frac{\pi}{2}|n-k|)} J_{|n-k|}(2t),$$

where $J_\nu(z)$ denotes the Bessel function of order ν . But for our purpose the integral will be more suitable.

As $t \rightarrow \infty$ our integrand will oscillate faster and faster and hence is of oscillatory type. By the method of stationary phase it is well-known that the main contribution of such oscillatory integrals will come from the stationary phase points. By this method one can get precise asymptotes of the kernel along rays where $v = \frac{|n-k|}{t} = \text{const}$. However, we would rather like an estimate which is uniform in n and k . Such a quantitative estimate is provided by the [van der Corput lemma](#). To this end we look at the oscillatory integral

$$\int_a^b A(\theta) e^{it\phi(\theta)} d\theta,$$

where $A(\theta)$ is the amplitude and $\phi(\theta)$ is the phase (which is assumed to be real).

Lemma

Suppose that ϕ is real-valued and l times differentiable with $|\phi^{(l)}(\theta)| \geq 1$ for $\theta \in [a, b]$ with $\phi'(\theta)$ monotonic if $l = 1$. Then there is a universal constant c_l (independent of a , b , and ϕ) such that

$$\left| \int_a^b e^{it\phi(\theta)} d\theta \right| \leq \frac{c_l}{t^{1/l}}$$

for $t > 0$.

Note: If $|\phi^{(l)}(\theta)| \geq \delta > 0$ we can scale t and ϕ and the estimate reads

$$\left| \int_a^b e^{it\phi(\theta)} d\theta \right| \leq \frac{c_l}{(\delta t)^{1/l}}.$$

Proof.

Denote the integral by $I(t)$. We will use induction and begin with $l = 1$. Integration by parts shows

$$\begin{aligned} I(t) &= \int_a^b \frac{1}{it\phi'(\theta)} \left(\frac{d}{d\theta} e^{it\phi(\theta)} \right) d\theta \\ &= \frac{1}{it\phi'(\theta)} e^{it\phi(\theta)} \Big|_a^b - \frac{1}{it} \int_a^b \left(\frac{d}{d\theta} \frac{1}{\phi'(\theta)} \right) e^{it\phi(\theta)} d\theta \end{aligned}$$

and estimating the absolute value gives

$$\begin{aligned} t|I(t)| &\leq \frac{1}{|\phi'(b)|} + \frac{1}{|\phi'(a)|} + \int_a^b \left| \frac{d}{d\theta} \frac{1}{\phi'(\theta)} \right| d\theta \\ &= |\phi'(b)|^{-1} + |\phi'(a)|^{-1} + |\phi'(b)^{-1} - \phi'(a)^{-1}| \leq 2. \end{aligned}$$

This shows the case $l = 1$ with $c_1 = 2$.

Proof (Cont.).

Now suppose $|\phi^{(l+1)}(\theta)| \geq 1$. Then there can be at most one point $\theta_0 \in [a, b]$ with $\phi^{(l)}(\theta_0) = 0$ and we have $|\phi^{(l)}(\theta)| \geq \delta$ for $|\theta - \theta_0| \geq \delta$. If there is no such point we can scale ϕ such that we can apply the lemma for l . Now split $I(t) = I_1(t) + I_2(t)$ where I_1 is the integral over $(a, \theta_0 - \delta) \cup (\theta_0 + \delta, b)$ and I_2 the integral over $(\theta_0 - \delta, \theta_0 + \delta)$. Then

$$|I_1(t)| \leq \frac{2c_l}{(\delta t)^{1/l}} \quad \text{and} \quad |I_2(t)| \leq 2\delta,$$

where we have used the induction hypothesis for both parts of the first integral. Choosing $\delta = t^{-1/(l+1)}$, to balance both contributions, finishes the proof with $c_{l+1} = 2(1 + c_l)$. □

Theorem

For the time evolution of the free discrete Schrödinger equation the following dispersive decay estimate holds:

$$\|e^{-itH_0}\|_{\ell^1 \rightarrow \ell^\infty} = \mathcal{O}(t^{-1/3}), \quad t \rightarrow \infty.$$

Proof.

We set $\nu := |n - k|/t \geq 0$. Then $K_{n,k}(t)$ is an oscillatory integral with the phase function

$$\phi_\nu(\theta) = 2 - 2 \cos \theta + \nu \theta.$$

Then, since $\phi_\nu''(\theta) = 2 \cos(\theta)$ and $\phi_\nu'''(\theta) = -2 \sin(\theta)$, we can split our domain of integration into four intervals where either $|\phi_\nu''(\theta)| \geq \sqrt{2}$ or $|\phi_\nu'''(\theta)| \geq \sqrt{2}$. Applying the van der Corput lemma on each interval gives the desired claim. □

The above decay rate is “sharp” as can be seen from the following well-known asymptotics of the Bessel function

$$J_t(t) \sim t^{-1/3}, \quad t \rightarrow \infty.$$

In fact, the slowest decay happens precisely along the ray $v = 2$, where we have a degenerate phase point with $\theta'_2(-\pi/2) = \theta''_2(-\pi/2) = 0$ and where the van der Corput lemma with $l = 3$ is the best we can do. Away from this ray we can apply the van der Corput lemma with $l = 2$ or $l = 1$ and hence improve the decay rate. Moreover, this is the case when n and k are restricted to a finite region or, as we will show next, are sufficiently localized.

Theorem

For the time evolution of the free discrete Schrödinger equation the following dispersive decay estimate holds:

$$\|e^{-itH_0}\|_{\ell_\sigma^2 \rightarrow \ell_{-\sigma}^2} = \mathcal{O}(t^{-1/2}), \quad t \rightarrow \infty, \quad \sigma > 2/3.$$

Note: The claim holds for $\sigma > 1/2$ but the proof is significantly more complicated.

Proof.

The problematic direction which causes the slowest decay is $\nu = 2$, as pointed out before. Hence we distinguish the two cases $|n - k| \leq t$ and $|n - k| \geq t$. In the region $|n - k| \leq t$ have $\nu = |n - k|/t \leq 1$ and avoid this degenerate phase point. Hence we can split the domain of integration into the region $[-\pi, -\frac{2\pi}{3}] \cup [-\frac{\pi}{3}, \frac{\pi}{3}] \cup [\frac{3\pi}{2}, \pi]$ where $|\phi''_v(\theta)| \geq 1$ and $[-\frac{2\pi}{3}, -\frac{\pi}{3}] \cup [\frac{\pi}{3}, \frac{3\pi}{2}]$ where $|\phi'_v(\theta)| \geq \sqrt{3} - 1 > 0$. Applying the van der Corput lemma on each of these intervals we obtain the bound

$$\sup_{|n-k| \leq t} |K_{n,k}(t)| \leq Ct^{-1/2}.$$

On the other hand, our previous proof implies

$$\sup_{|n-k| \geq t} |K_{n,k}(t)| \leq Ct^{-1/3} = Ct^{-1/2}t^{1/6} \leq Ct^{-1/2}|n - k|^{1/6}.$$

Proof (Cont.).

From this the desired estimate follows since $K_{n,k} = |n - k|^\alpha$ is bounded in $\ell_\sigma^2 \rightarrow \ell_{-\sigma}^2$ iff $\tilde{K}_{n,k} = (1 + |n|)^{-\sigma} |n - k|^\alpha (1 + |k|)^{-\sigma}$ is bounded in ℓ^2 .

Finally,

$$\sum_{n,k} |\tilde{K}_{n,k}|^2 \leq \sum_{n,k} (1 + |n|)^{-2(\sigma - \alpha)} (1 + |k|)^{-2(\sigma - \alpha)} = \left(\sum_n (1 + |n|)^{-2(\sigma - \alpha)} \right)^2$$

shows that \tilde{K} is Hilbert–Schmid (and hence bounded) if $2(\sigma - \alpha) > 1$. For $\alpha = \frac{1}{6}$ we obtain $\sigma > \frac{3}{2}$. □

Now we would like to consider the perturbed Schrödinger equation

$$i\dot{u}(t) = Hu(t), \quad t \in \mathbb{R},$$

where

$$(Hu)_n = (H_0u)_n + q_nu_n, \quad n \in \mathbb{Z}.$$

We will assume that q_n is real-valued and decays at $|n| \rightarrow \infty$. In particular, $q \in \ell^\infty(\mathbb{Z})$ and thus H is a bounded self-adjoint operator in $\ell^2(\mathbb{Z})$. The main difference now is that the underlying difference equation $Hu = zu$ can no longer be solved explicitly!

Despite this lack of explicit solvability we will still be able to show qualitative features of the solutions. But first we will need a good formula for the time evolution.

We begin with some general remarks which are valid for an arbitrary self-adjoint operator H in a Hilbert space \mathfrak{H} .

The most important observation in this case that in order to understand the spectral features of H one needs to understand its **resolvent**

$$z \mapsto \mathcal{R}_H(z) = (H - z)^{-1}$$

which is an analytic map from the resolvent set $\rho(H) \subset \mathbb{C}$ to the set of bounded operators from \mathfrak{H} to itself. Since the spectrum of a self-adjoint operator is confined to the real line and hence the resolvent is analytic in the upper and lower half planes. Because of

$$\mathcal{R}_H(\bar{z}) = \mathcal{R}_H(z)^*$$

we can restrict our attention to the upper half space. Here we use the bar for complex conjugation and the star for the adjoint operator.

More specific, what needs to be understood is the limit of the resolvent as z approaches the real line. To see this recall [Stone's formula](#).

Theorem

Let H be self-adjoint. Then

$$\frac{1}{2\pi i} \int_{\lambda_1}^{\lambda_2} (\mathcal{R}_H(\lambda + i\varepsilon) - \mathcal{R}_H(\lambda - i\varepsilon)) d\lambda \xrightarrow{s} \frac{1}{2} (P_H([\lambda_1, \lambda_2]) + P_H((\lambda_1, \lambda_2)))$$

strongly. Here $P_H(\Omega) = \chi_\Omega(H)$ denotes the orthogonal projection onto the set $\Omega \subseteq \mathbb{R}$ associated with H (and $\chi_\Omega(\lambda)$ is the characteristic function of Ω).

Proof.

By the spectral theorem we have $f_n(H) \xrightarrow{s} f(H)$ provided $f_n(\lambda) \rightarrow f(\lambda)$ pointwise and $\|f_n\|_\infty \leq C$ is uniformly bounded. Hence the problem reduces to compute the limit of the associated functions:

$$\begin{aligned} \frac{1}{2\pi i} \int_{\lambda_1}^{\lambda_2} \left(\frac{1}{x - \lambda - i\varepsilon} - \frac{1}{x - \lambda + i\varepsilon} \right) d\lambda &= \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \frac{\varepsilon}{(x - \lambda)^2 + \varepsilon^2} d\lambda \\ &= \frac{1}{\pi} \left(\arctan \left(\frac{\lambda_2 - x}{\varepsilon} \right) - \arctan \left(\frac{\lambda_1 - x}{\varepsilon} \right) \right) \\ &\rightarrow \frac{1}{2} \left(\chi_{[\lambda_1, \lambda_2]}(x) + \chi_{(\lambda_1, \lambda_2)}(x) \right). \end{aligned}$$



In a similar fashion one establishes the following extension:

Theorem

Let H be self-adjoint. Then

$$\frac{1}{2\pi i} \int_{\mathbb{R}} f(\lambda) (\mathcal{R}_H(\lambda + i\varepsilon) - \mathcal{R}_H(\lambda - i\varepsilon)) d\lambda \xrightarrow{s} f(H)$$

for any bounded and continuous function $f \in C_b(\mathbb{R})$.

In particular,

$$\exp(-itH) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\mathbb{R}} e^{-it\lambda} (\mathcal{R}_H(\lambda + i\varepsilon) - \mathcal{R}_H(\lambda - i\varepsilon)) d\lambda$$

and everything boils down to understand the limits $\mathcal{R}_H(\lambda \pm i0)$.

For differential (or difference) equations the resolvent is an integral (summation) operator whose kernel is known as **Green's function**. For our discrete Schrödinger operator H this kernel is explicitly given by

$$\mathcal{R}_{n,m}(z) = \mathcal{R}_{m,n}(z) = \frac{f_n^+(z)f_m^-(z)}{W(f^+(z), f^-(z))}, \quad m \leq n,$$

where $f^\pm(z)$ are the (weak) solutions of the underlying difference equation $Hf = zf$ which are square summable near $\pm\infty$ and

$$W_n(f, g) = f_n g_{n+1} - f_{n+1} g_n$$

is the discrete Wronski determinant which can be easily seen to be independent of n if f and g both solve $Hf = zf$.

To proceed further we will assume that q decays sufficiently fast. $q \in \ell_1^1(\mathbb{Z})$ will be sufficient for our purpose. Under this assumption we assume that there exist solutions of $Hf = zf$ which asymptotically look like the unperturbed solutions:

$$f_n^\pm(\theta) \sim e^{\mp i n \theta}, \quad n \rightarrow \pm\infty.$$

Here θ and z are related via

$$2 - 2 \cos \theta = z$$

as before, normalized such that $\text{Im}(\theta) < 0$.

If q is compactly supported this is obvious. In the general case use Duhamel's formula to obtain an integral equations for the perturbed solutions in terms of the unperturbed ones. The trick here is to start the summation not at a finite initial point but (at first formally) at $\pm\infty$. Then show existence of solutions using the standard fix point iteration procedure for Volterra-type equations.

If q is compactly supported, then $f_n^\pm(\theta)$ will start out equal to the unperturbed solutions and will start to pick up multiples of shifted perturbed solutions as soon as we hit the support of q . This motivates the following ansatz for the Jost solutions

$$f_n^\pm(\theta) = e^{\mp i\theta n} \left(1 + \sum_{m=\pm 1}^{\pm\infty} B_{n,m}^\pm e^{\mp im\theta} \right),$$

where $B_{n,m}^\pm \in \mathbb{R}$. Moreover, inserting this into our equation one can get a corresponding difference equation for B^\pm and then use induction to show

$$|B_{n,m}^\pm| \leq C_n^\pm \sum_{k=n+\lfloor m/2 \rfloor}^{\pm\infty} |q_k|.$$

The estimate for B^\pm shows that the decay of B^\pm is directly related to the decay of q . Moreover, $f_n^\pm(\theta)$ is analytic for θ in the strip $\text{Im}(\theta) < 0$, $-\pi \leq \text{Re}(\theta) \leq \pi$ and continuously extends to the boundary $\text{Im}(\theta) = 0$, $-\pi \leq \text{Re}(\theta) \leq \pi$. If we require further decay of q we can also control derivatives on the boundary. For compact support we get even an analytic extension to $\text{Im}(\theta) > 0$.

Note: The limit $\text{Im}(\theta) \rightarrow 0$ corresponds to the limit $z \rightarrow [0, 4]$ which is precisely the continuous spectrum of H .

The case $\text{Im}(\theta) = 0$ corresponds to the continuous spectrum $z \in [0, 4]$. In particular, in this case the differential equation is real-valued and the complex conjugate of a solution will again be a solution. But then $f^\pm(\theta)$ and $\overline{f^\pm(\theta)} = f^\pm(-\theta)$ are four solutions of a second order equation which cannot be linearly independent! This leads to the [scattering relations](#)

$$T(\theta)f_m^\pm(\theta) = R^\mp(\theta)f_m^\mp(\theta) + f_m^\mp(-\theta), \quad \theta \in [-\pi, \pi],$$

Physically $f_m^\mp(-\theta)$ can be interpreted as an incoming (from $\mp\infty$) plane wave which splits into a reflected wave $R^\mp(\theta)f_m^\mp(\theta)$ and a transmitted wave $T(\theta)f_m^\pm(\theta)$. The [transmission and reflection coefficients](#) can be expressed in terms of Wronskians:

$$T(\theta) = \frac{2i \sin \theta}{W(\theta)}, \quad R^\pm(\theta) = \pm \frac{W^\pm(\theta)}{W(\theta)}.$$

with $W(\theta) = W(f^+(\theta), f^-(\theta))$ and $W^\pm(\theta) = W(f^\mp(\theta), f^\pm(-\theta))$.

Inserting everything into our abstract formula for $\exp(-itH)P_H([0, 4])$ we obtain the following formula for its kernel

$$K_{n,k}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-it\phi_v(\theta)} h_n^+(\theta) h_k^-(\theta) T(\theta) d\theta, \quad k \leq n,$$

where $\phi_v(\theta) = 2 - 2\cos\theta + v\theta$, $v = \frac{|n-k|}{t}$, is the phase function we already know from the free case and h^\pm capture the difference between the free and perturbed solutions:

$$f_n^\pm(\theta) = e^{\mp in} h_n^\pm(\theta).$$

If $q = 0$ we have $h_n^\pm(\theta) = 1$ as well as $T(\theta) = 1$ and we recover our formula for the free case. Note that here we have added the projector $P_H([0, 4])$ onto the continuous spectrum $[0, 4]$ and consequently ignored the contribution from any possible eigenvalues (which are stationary solutions and hence cannot exhibit decay).

However, in the perturbed case our oscillatory integral has a nontrivial amplitude and we cannot apply the van der Corput lemma directly. The key observation is that the amplitude is in the **Wiener algebra**, that is, its Fourier coefficients are summable. While for $h_n^\pm(\theta)$ this is immediate from summability of $B_{n,n+}^\pm$. This also shows that $W(\theta)$ is in the Wiener algebra and so will be $W(\theta)^{-1}$ by Wiener's theorem provided $W(\theta)$ does not vanish on $[-\pi, \pi]$. In fact energy conservation $|T|^2 + |R_\pm|^2 = 1$ shows that this can only happen on the boundary of the spectrum when $\sin(\theta) = 0$. In this case, when $W(\theta)$ vanishes at one of the boundaries, one speaks of a resonance. Moreover, varying just one value of q will immediately destroy a resonance and hence this situation is not generic. Nevertheless, one can still show that $T(\theta)$ is always in the Wiener algebra, but this is nontrivial.

Lemma

Consider the oscillatory integral

$$I(t) = \int_{-\pi}^{\pi} e^{it\phi(\theta)} A(\theta) d\theta,$$

where $\phi(\theta)$ is real-valued. If $|\phi^{(l)}(\theta)| \geq 1$ for some $l \geq 2$ and A is in the Wiener algebra, then

$$|I(t)| \leq \frac{c_l \|\hat{A}\|_{\ell^1}}{t^{1/l}},$$

where c_l is the same constant as in the van der Corput lemma.

Proof.

We insert $A(\theta) = \sum_{p \in \mathbb{Z}} \hat{A}_p e^{ip\theta}$, with $\hat{A}_p = (2\pi)^{-1} \int_{-\pi}^{\pi} A(\theta) e^{-ip\theta} d\theta$, and use Fubini to obtain

$$I(t) = \int_{-\pi}^{\pi} e^{it\phi(\theta)} \sum_{p \in \mathbb{Z}} \hat{A}_p e^{ip\theta} d\theta = \sum_{p \in \mathbb{Z}} \hat{f}_p I_{p/t}(t), \quad I_v(t) = \int_{-\pi}^{\pi} e^{it(\phi(\theta)+v\theta)} d\theta.$$

By the van der Corput lemma we have $|I_v(t)| \leq c_I t^{-1/s}$ and hence

$$|I(t)| \leq \sum_{p \in \mathbb{Z}} |\hat{f}_p| |I_{p/t}(t)| \leq \frac{c_I \|\hat{A}\|_{\ell^1}}{t^{1/I}}$$

as claimed. □

Lemma

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$$|I(t)| \leq \frac{c_l \|\hat{A}\|_{\ell^1}}{t^{1/l}},$$

where c_l is the same constant as in the van der Corput lemma.

A few remarks about this lemma:

- It cannot hold for $l = 1$ as the Fourier coefficients of an element in the Wiener algebra can have arbitrary slow decay (consider lacunary Fourier coefficients).
- One can show that it does not hold for continuous functions.
- This lemma is typically found for absolutely continuous functions (when it can be proved using integration by parts). Neither version implies the other.

As a consequence we obtain:

Theorem

Let $q \in \ell_1^1$. Then the following dispersive decay estimates hold

$$\|e^{-itH}P_c(H)\|_{\ell^1 \rightarrow \ell^\infty} = \mathcal{O}(t^{-1/3}), \quad t \rightarrow \infty,$$

and

$$\|e^{-itH}P_c(H)\|_{\ell_\sigma^2 \rightarrow \ell_{-\sigma}^2} = \mathcal{O}(t^{-1/2}), \quad t \rightarrow \infty, \quad \sigma > 1/2,$$

where $P_c(H) = P_H([0, 4])$ is the projection onto the continuous spectrum.

Note: The proof requires in fact one more trick! We need the Wiener norm of h_n^\pm to be uniformly bounded with respect to $n \in \mathbb{Z}$. However, this is only true for $n \in \mathbb{Z}_\pm$. To get it for our amplitude one has to use the scattering relations to turn $h_n^\pm(\theta)$ into linear combinations of $h_n^\mp(\theta)$ and $h_n^\mp(-\theta)$.

Out of these estimates one can get further estimates using some abstract techniques. First of all we can use interpolation:

By self-adjointness of H we obtain that $\exp(-itH) : \ell^2 \rightarrow \ell^2$ is unitary and in particular

$$\|e^{-itH}P_c(H)\|_{\ell^2 \rightarrow \ell^2} \leq 1.$$

Interpolating between this and our $\ell^1 \rightarrow \ell^\infty$ estimate the Riesz–Thorin theorem gives us the following estimate

$$\|e^{-itH}P_c(H)\|_{\ell^{p'} \rightarrow \ell^p} = \mathcal{O}(t^{-1/3(1/p' - 1/p)})$$

for any $p' \in [1, 2]$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

Next we look at average decay in an L^p sense instead of point wise estimates with respect to t . To this end we introduce the following space-time norms

$$\|F\|_{L_t^q \ell_n^p} = \left(\int \|F(t)\|_{\ell_n^p}^q \right)^{1/q}.$$

Then our previous estimates imply:

Theorem

Suppose $q \in \ell_1^1$. Then we have the following estimates:

$$\begin{aligned} \|e^{-itH} P_c(H) f\|_{L_t^q \ell_n^p} &\leq C \|f\|_{\ell_n^2}, \\ \left\| \int e^{-itH} P_c(H) F(s) ds \right\|_{\ell_n^2} &\leq C \|F\|_{L_t^{q'} \ell_n^{p'}}, \\ \left\| \int e^{-i(t-s)H} P_c(H) F(s) ds \right\|_{L_t^q \ell_n^p} &\leq C \|F\|_{L_t^{q'} \ell_n^{p'}}, \end{aligned}$$

where $p, q \geq 2$,

$$\frac{1}{q} + \frac{1}{3p} \leq \frac{1}{6},$$

and a prime denotes the corresponding dual index.

Proof.

To begin with we use the following variational characterization of our space-time norms:

$$\|F\|_{L_t^q \ell_n^p} = \sup_{\|G\|_{L_t^{q'} \ell_n^{p'}=1}} \left| \int \sum_n F(n, t) G(n, t) dt \right|$$

Then, using self-adjointness of H and Fubini

$$\int \sum_n (\exp(-itH) P_c f)_n F(n, t) dt = \sum_n f_n \int (\exp(-itH) P_c F(t))_n dt,$$

which shows that the first and second estimate are equivalent upon using the above characterization.

Proof (Cont.).

Similarly, using again self-adjointness

$$\begin{aligned} & \left\| \int \exp(-itH) P_c F(., t) dt \right\|_{\ell_n^2}^2 \\ &= \sum_n \int \exp(-itH) P_c F(n, t) dt \int \overline{\exp(-isH) P_c F(n, s)} ds \\ &= \sum_n \int F(n, t) \int \exp(-i(t-s)H) P_c \overline{F(n, s)} ds dt \end{aligned}$$

which shows that the second and the third estimate are equivalent with a similar argument as before.

Proof (Cont.).

Hence it remains to prove the last one. Applying Minkowski's inequality and our interpolation estimate we obtain

$$\begin{aligned} \left\| \int e^{-i(t-s)H} P_c(H) F(n, s) ds \right\|_{\ell_n^p} &\leq \int \|e^{-i(t-s)H} P_c(H) F(n, s)\|_{\ell_n^p} ds \\ &\leq C \int \frac{1}{|t-s|^\alpha} \|F(n, s)\|_{\ell_n^{p'}} ds, \end{aligned}$$

where $\alpha = (1/3)(1/p' - 1/p)$. Now taking the $\|\cdot\|_{L_t^q}$ norm on both sides and using the Hardy–Littlewood–Sobolev inequality finishes the proof. \square

Note: There is also an abstract result by

- M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), 955–980.

which applies here. Note that one can apply the *truncated version* since our kernels are bounded near $t = 0$.

An important physical model is the discrete nonlinear Schrödinger (dNLS) equation

$$i\dot{u}(t) = Hu(t) \pm |u(t)|^{2p}u(t), \quad t \in \mathbb{R},$$

which arises in many physical applications.

The above estimates are typically used to show solvability of this equation. However, in the discrete case the situation is simpler.

Theorem

The Cauchy problem for the discrete nonlinear Schrödinger equation has a unique global norm preserving solution in $\ell^2(\mathbb{Z})$.

Proof.

First of all we can regard the right-hand side as a C^1 vector field in the Hilbert space $\ell^2(\mathbb{Z})$. Hence the standard Picard–Lindelöf theorem shows existence of a unique global solution. To show that this solution is in fact global observe

$$\frac{d}{dt} \|u(t)\|_{\ell^2}^2 = 0.$$

Hence the ℓ^2 norm is preserved and cannot blow up. □

Note that in the above argument ℓ^2 can be replaced by weighted versions (as long as the weight is at most exponential) but the corresponding norms will no longer be preserved. However, using

$$\frac{d}{dt} |u_n(t)|^2 = 2\operatorname{Im}(u_n(\overline{u_{n+1} + u_{n-1}}))$$

plus Gronwall's inequality these norms can grow at most linearly.

Another application is stability of soliton solutions.

- P. G. Kevrekidis, D. E. Pelinovsky, and A. Stefanov, *Asymptotic stability of small bound states in the discrete nonlinear Schrödinger equation* SIAM J. Math. Anal. **41** (2009), 2010–2030.

Idea: Using nonlinear functional analysis there is a solution of the stationary dNLS equation of the type

$$e^{-it\omega} \phi_n(\omega)$$

bifurcating from an eigenvalue of the Schrödinger equation $H\phi = \omega\phi$.

Now make the ansatz

$$u(t) = e^{-i\theta(t)} (\phi(\omega(t)) + z(t)).$$

Then the linearized system for $v = \operatorname{Re}(z)$ and $w = \operatorname{Im}(z)$ can be shown to have double eigenvalue and the solution can be decomposed with respect to the generalized eigenvectors. Then, using the Strichartz and some additional estimates proved using them, one can show that the resulting system has a global solution provided the initial conditions are close to the stationary solution. Moreover, $\|z(t)\|_\infty \rightarrow 0$ and the parameters $\theta(t)$ and $\omega(t)$ are asymptotically linear, constant, respectively. The details are however quite tedious.