HJ equations. Reachability analysis. Optimal control problems

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Outline

1. Introduction
2. The viscosity Notion
3. Controlled systems. Optimal control problems
4. Value functions. HJB equations
5. Comparison principle
6. Parabolic HJB equations. Stochastic control problems
7. HJ equation on multidomains. Transmission conditions
8. Link between HJB equations and optimal control problems

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Some steady equations

- $\lambda v(x) + H(x, Dv(x)) = 0 \quad x \in \mathbb{R}^d$

- $\begin{cases} H(x, Dv(x)) = 0 & x \in \Omega \\ v(x) = 0 & x \in \partial\Omega \end{cases}$ where $\Omega \subset \mathbb{R}^d$

Time-dependant equations

- $\begin{cases} \partial_t v(t,x) + H(t, x, Dv(t,x)) = 0 & x \in \mathbb{R}^d,\ t > 0 \\ v(0,x) = v_o(x) & x \in \mathbb{R}^d \end{cases}$

- $\begin{cases} \min\left(\partial_t v(t,x) + H(t, x, Dv(t,x)), v(t,x) - g(t,x)\right) = 0 \\ v(0,x) = v_o(x) \end{cases}$

Hamiltonian H

- $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a Lipschitz continuous function
- In the sequel, we will consider the particular case where the Hamiltonian H is defined by

$$H(x,p) := \sup_{q \in F(x)} \left(-q \cdot p - \ell(x,q)\right)$$

where $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is a multi-valued function, and $\ell : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$.

Some examples

- Eikonal equation:

$$\begin{cases} \partial_t v(t,x) + c(x)\|Dv(t,x)\| = 0 \quad x \in \mathbb{R}^d,\ t > 0 \\ v(0,x) = v_o(x) \quad x \in \mathbb{R}^d \end{cases}$$

Here the velocity $c(\cdot) \geq 0$, and the Hamiltonian is defined by

$$H(x,p) := c(x)\|p\| = \max_{q \in \mathbb{B}(0,1)} (-q \cdot p).$$

- Advection equation with an obstacle:

$$\begin{cases} \min\left(\partial_t v(t,x) - \vec{a}(x) \cdot Dv(t,x), v(t,x) - g(t,x)\right) = 0 \\ v(0,x) = v_o(x) \quad x \in \mathbb{R}^d \end{cases}$$

the Hamiltonian is defined by $H(x,p) := \vec{a}(x) \cdot p$.

Hamilton-Jacobi-Bellman (HJB for short) equations

- Notion of solutions for HJB equations
- Existence of solutions: Value function of optimal control problems
- Uniqueness results: Comparison principle

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Definition.

Let $\Omega \subset \mathbb{R}^M$, and $\mathcal{F} : \Omega \times \mathbb{R} \times \mathbb{R}^M \to \mathbb{R}$ be continuous. Consider the HJB equation:

$$\mathcal{F}(x, u(x), Du(x)) = 0 \quad \text{in } \Omega. \tag{1}$$

(i) A function $u \in USC(\Omega)$ is a viscosity **sub-solution** of (7) if for any $\varphi \in C^1(\Omega)$ and any local maximum point $x_0 \in \Omega$ of $u - \varphi$,

$$\mathcal{F}(x_0, u(x_0), D\varphi(x_0)) \leq 0.$$

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(ii) A function $u \in LSC(\Omega))$ is a viscosity **super-solution** of (7) if for any $\varphi \in C^1(\Omega)$ and any local minimum point $x_0 \in \Omega$ of $u - \varphi$,

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(iii) A function $u \in C(\Omega)$ is a viscosity **solution** of (7) if it is a sub- and super-solution.

Vanishing Viscosity Limits

Lemma

Let u_ε be a sequence of smooth solutions to the **viscous** equations:

$$\mathcal{F}(x, u_\epsilon(x), Du_\epsilon(x)) - \varepsilon \Delta u_\epsilon(x) = 0 \qquad x \in \Omega.$$

If $u_\varepsilon \to u$ uniformly on Ω, when $\varepsilon \to 0+$, then u is a viscosity solution of

$$\mathcal{F}(x, u(x), Du(x)) = 0.$$

Idea of the proof - 1:

- u_ε is a viscosity sub-solution to the viscous equation.
  Indeed, let $\psi \in C^1(\Omega)$, and let $x \in \Omega$ a local maximum of $u_\varepsilon - \psi$.
  Then
  $$Du_\varepsilon(x) = D\psi(x), \qquad \Delta u_\varepsilon(x) \leq \Delta\psi(x).$$
  Therefore,
  $$\mathcal{F}(x, u_\varepsilon(x), D\psi(x)) - \varepsilon\Delta\psi(x) \leq \mathcal{F}(x, u_\epsilon(x), Du_\epsilon(x)) - \varepsilon\Delta u_\epsilon(x) = 0.$$

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- Let $\varphi \in C^1(\Omega)$, $x \in \Omega$ is a local maximum of $u - \varphi$.
  $\forall \rho > 0$, $\exists \delta \leq \rho$ and there exists $\psi \in C^1(\Omega)$ s.t.
  $$\|\psi - \varphi\|_{C^1(\Omega)} \leq \delta$$
  $u_\varepsilon - \psi$ has a local maximum x_ε inside the ball $B(x, \rho)$.

Idea of the proof - 2:

- Extract a convergent subsequence $x_\varepsilon \to \bar{x} \in B(x, \rho)$. By passing to the limit in the inequality

$$\mathcal{F}(x_\varepsilon, u_\varepsilon(x_\varepsilon), D\psi(x_\varepsilon)) - \varepsilon\Delta\psi(x_\varepsilon) \leq 0,$$

we get:

$$\mathcal{F}(\bar{x}, u(\bar{x}), D\psi(\bar{x})) \leq 0.$$

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- Choosing ρ small enough, we have $|\bar{x} - x|$ and $|D\psi(\bar{x}) - D\varphi(x)|$ as small as we want. So by continuity of $\mathcal{F}$, we finally obtain:

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- The supersolution can be proved in the same way.

A more general stability result

- A general stability theorem holds for viscosity solutions, without additional requirements on the derivatives.

Theorem

Consider a sequence of continuous functions $(u_n)_{n\geq 1}$, where u_n is a viscosity sub- solutions (resp. super-solutions) to

$$\mathcal{F}_n(x, u_n(x), Du_n(x)) = 0, \quad x \in \Omega.$$

Assume $\mathcal{F}_n \to \mathcal{F}$ uniformly on compact subsets of $\Omega \times \mathbb{R}^d \times \mathbb{R}^d$, when $n \to +\infty$ and $u_n \to u$ uniformly on Ω.
Then u is a viscosity sub-solution (a super-solution) of

$$\mathcal{F}(x, u(x), Du(x)) = 0, \quad x \in \Omega.$$

➤ The stability results are based on the following "key ingredient":

Lemma

Let $v : \Omega \to \mathbb{R}$ be a upper semi-continuous function that achieves a *strict* local maximum at $\bar{x} \in \Omega$. Let $(v_n)_n$ be a sequence of upper semi-continuous function on Ω.
Assume that $\limsup_{\substack{z \to \bar{x} \\ n \to +\infty}} v_n(z) = v(x_0)$. Then there exists a sequence $(x_n)_n$ in Ω such that for every $n \geq 1$, x_n is a local maximum of v_n and

$$\lim_{n \to +\infty} x_n = \bar{x}, \qquad \lim_{n \to +\infty} v(x_n) = v(\bar{x}).$$

Semi-relaxed limits

Let $(u_n)_n$ be a sequence of functions on Ω.

- Assume that $(u_n)_n$ are locally uniformly bounded from above Define the *upper relaxed limit* by:

$$u^*(x) := \limsup_{x_n \to x} u_n(x_n).$$

- Assume that $(u_n)_n$ are locally uniformly bounded from below. Define the *lower relaxed limit* by:

$$u_*(x) := \liminf_{x_n \to x} u_n(x_n)).$$

Theorem (Stability by semi-relaxed limits)

Consider a sequence of usc (resp. lsc) functions $(u_n)_{n\geq 1}$, where u_n is a viscosity sub- solutions (resp. super-solutions) to

$$\mathcal{F}_n(x, u_n(x), Du_n(x)) = 0, \quad x \in \Omega.$$

Assume $\mathcal{F}_n \to \mathcal{F}$ uniformly on compact subsets of $\Omega \times \mathbb{R}^d \times \mathbb{R}^d$, when $n \to +\infty$ and $(u_n)_n$ is locally uniformly bounded from above (resp. from below) on Ω.
Then u^* is a viscosity sub-solution (resp u_* is a super-solution) of

$$\mathcal{F}(x, u(x), Du(x)) = 0, \quad x \in \Omega.$$

Viscosity notion: An equivalent definition.

Let $\Omega \subset \mathbb{R}^M$, and $\mathcal{F} : \Omega \times \mathbb{R} \times \mathbb{R}^M \to \mathbb{R}$ be continuous. Consider the HJB equation:

$$\mathcal{F}(x, u(x), Du(x)) = 0 \quad \text{in } \Omega, \tag{2}$$

(i) A function $u \in USC(\Omega)$ is a viscosity sub-solution of (7) if

$$\mathcal{F}(x, u(x), q) \leq 0 \quad \forall q \in D^+u(x).$$

$$q \in D^+u(x) \Longleftrightarrow u(y) \leq u(x) + q \cdot (y - x) + o(|y - x|),$$

$$q \in D^-u(x) \Longleftrightarrow u(y) \geq u(x) + q \cdot (y - x) + o(|y - x|).$$

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Lemma

Suppose $u \in C(\Omega)$.

(i) $q \in D^+u(x)$ if and only if there exists $\varphi \in C^1(\Omega)$ such that $D\varphi(x) = q$ and x is a local minimum of $u - \varphi$.

(ii) $q \in D^-u(x)$ if and only if there exists $\varphi \in C^1(\Omega)$ such that $D\varphi(x) = q$ and x is a local maximum of $u - \varphi$.

Some remarks

- If u is differentiable at $x \in \Omega$, then $D^- u(x) = D^+ u(x) = \{\nabla u(x)\}$.
- A viscosity solution u of the HJ equation satisfies the equality

$$\mathcal{F}(x, u(x), \nabla u(x)) = 0$$

  in the classical sense whenever u is differentiable at x.
- A Lipschitz continuous viscosity solution u is also a solution in the *"almost everywhere"* sense. (Indeed, by Rademacher's theorem, a Lipschitz continuous function is differentiable a.e.)

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Controlled Systems

$$\begin{cases} \dot{y}_x(s) \in F(y_x(s)), & s \in (0,1), \\ y_x(0) = x, \end{cases}$$

- **(H1)** $F : \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ is a usc set-valued function

  $\forall x_0 \in \mathbb{R}^d, \forall \varepsilon > 0, \exists \eta > 0,$
  $\|x - x_0\| \leq \eta \Longrightarrow F(x) \subset F(x_0) + \varepsilon B.$

Controlled Systems

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- **(H1)** $F : \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ is a usc set-valued function, with <u>convex</u> compact nonempty images, and F has a linear growth

  $\exists k > 0, \forall x \in \mathbb{R}^d, \sup_{v \in F(x)} \|v\| \leq k(1 + \|x\|).$

Controlled Systems

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- **(H2)** $F : \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ is Lipschitz continuous

  $\exists C > 0$, s.t. $F(y_2) \subset F(y_1) + C\|y_2 - y_1\|B, \ \ \forall y_1, y_2 \in \mathbb{R}^d$

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- **(H2)** $F : \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ is Lipschitz continuous
- Example: $F(x) := f(x, U)$; where $U \subset \mathbb{R}^m$ compact, $f : \mathbb{R}^d \times U \to \mathbb{R}^d$ is Lipschitz, bounded, and $f(x, U)$ is convex for any $x \in \mathbb{R}^d$.

Consider the set of admissible trajectories

$$\mathcal{S}_{[0,\tau]}(x) := \{y_x \mid \dot{y}_x(s) \in F(y_x(s)) \ \text{ on}(0,\tau), y_x(0) = x\}$$

- Under **(H1)**, $\mathcal{S}_{[0,\tau]}(x)$ is a compact subset of $C([0,\tau])$.
- Under **(H1)-(H2)**, the set-valued function $x \rightsquigarrow \mathcal{S}_{[0,\tau]}(x)$ is Lipschitz continuous,

$$\exists L > 0, \mathcal{S}_{[0,\tau]}(x) \subset \mathcal{S}_{[0,\tau]}(z) + L|x-z|\mathcal{B}_{W^{1,1}} \quad \forall x, z \in \mathbb{R}^d.$$

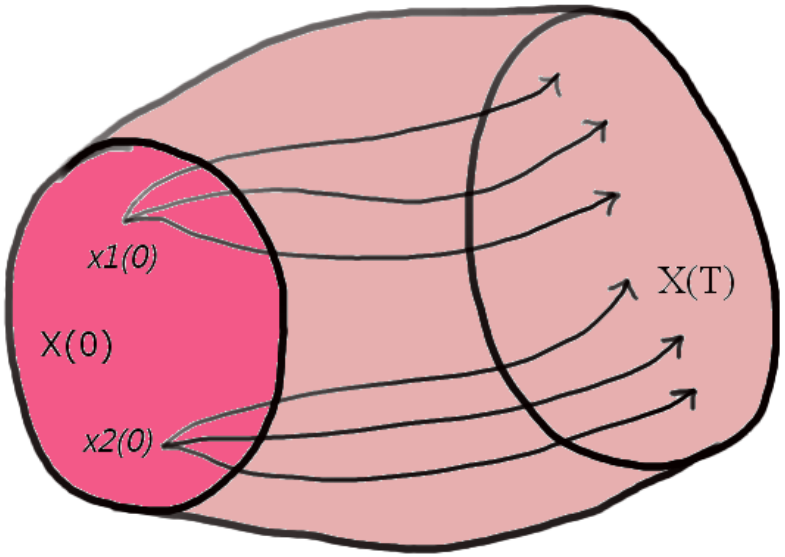
Reachable (or Attainable) set

- The reachable set $\mathcal{R}(x;t)$ from x at time t is the set of all points of the form $y_x(\tau)$, where $y_x \in \mathcal{S}_{[0,t]}(x)$:

$$\mathcal{R}(x;t) := \{y_x(\tau) \mid y_x \in \mathcal{S}_{[0,t]}(x), \tau \in [0,t]\}.$$

- The reachable set from X is defined by:

$$\mathcal{R}_X(t) := \cup_{x \in X} \mathcal{R}(x;t).$$



Some properties of the reachable sets

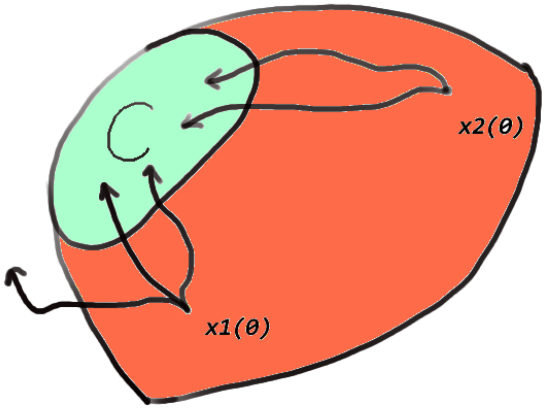
- If the set-valued F is usc with non-empty compact images (not necessarily convex images) then $\mathcal{R}_{\mathcal{C}}(t)$ is not necessarily closed.
- Assume (H1), then for every $t \geq 0$, $\mathcal{R}_{\mathcal{C}}(t)$ is a closed set.

In all the sequel, we will assume that F satisfies at least (H1).

Let $\mathcal{C}$ be a closed target set (in our examples, $\mathcal{C}$ is safe)

Capture Basin (or Backward reachable set)

- The Capture Basin $C^{\mathcal{C}}_{\sharp t}$, at time t, is the set of all initial positions x from which a trajectory $y_x \in \mathcal{S}_{[0,t]}(x)$ can reach the target $\mathcal{C}$.



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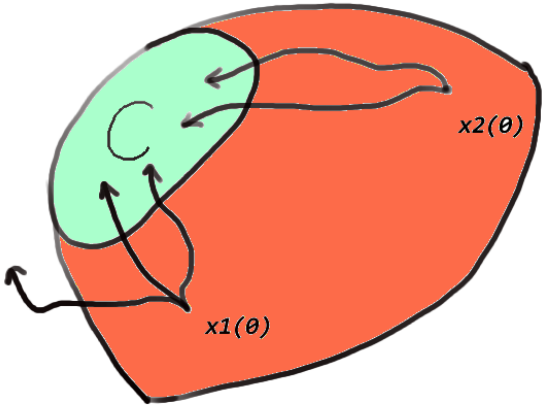
Capture Basin (or Backward reachable set)

- The Capture Basin $\mathcal{R}_t^{\mathcal{C}}$, before time t, is the set of all initial positions x from which a trajectory $y_x \in \mathcal{S}_{[0,t]}(x)$ can reach the target $\mathcal{C}$ before t.

$$\mathcal{R}_t^{\mathcal{C}} := \left\{ x \in \mathbb{R}^d, \exists \tau \in [0,t], \exists y_x \in \mathcal{S}_{[0,\tau]}(x),\ y_x(\tau) \in \mathcal{C} \right\}$$



Optimization-based controller design

- Minimum-time control problem:

$$\mathcal{T}(x) = \inf \left\{ t; \;\; y_x(t) \in \mathcal{C}, \; y_x \in \mathcal{S}_{[0,t]}(x) \right\}.$$

Optimization-based controller design

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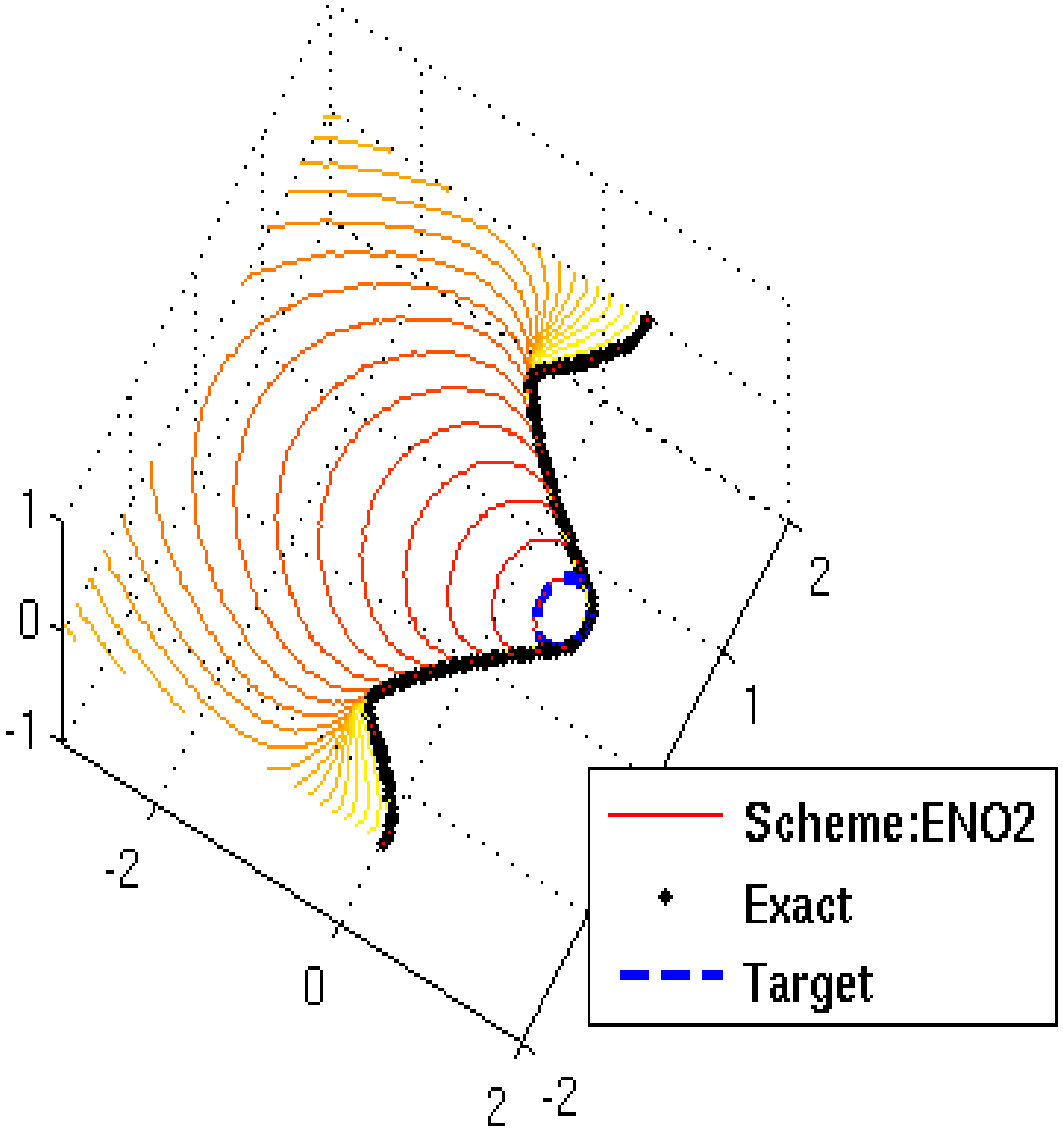
- The sublevels of the minimum function $\mathcal{T}$ correspond to the Capture Basins of the target $\mathcal{C}$:

$$C^{\mathcal{C}}_{\sharp t} = \{x \in \mathbb{R}^d \mid \mathcal{T}(x) = t\} \tag{3}$$

$$\mathcal{R}^{\mathcal{C}}_{t} = \{x \in \mathbb{R}^d \mid \mathcal{T}(x) \leq t\}. \tag{4}$$

Zermelo problem



Distance function. Level-set appraoch

Consider the function $\Phi(x) = d_{\mathcal{C}}(x)$.

- A Mayer's problem:

$$V(x,t) = \inf_{y_x \in \mathcal{S}_{[0,t]}(x)} \Phi(y_x(t))$$

- A Mayer's problem with extended trajectories

$$\widehat{V}(x,t) = \inf_{y_x \in \widehat{\mathcal{S}}_{[0,t]}(x)} \Phi(y_x(t))$$

where $\widehat{\mathcal{S}}_{[0,t]}(x)$ is the set of trajectories satisfying:

$$\dot{y} \in \widehat{F}(y(t)),\ y(0) = x,$$

with $\widehat{F}(x) = [0,1] \cdot F(x)$.

Level set approach

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- The minimum time function $\mathcal{T} : \mathbb{R}^d \to \mathbb{R}^+ \cup \{+\infty\}$ is lsc. Moreover, we have:

$$\mathcal{T}(x) = \inf\{t \geq 0; V(x,t) \leq 0\} = \min\{t \geq 0; \widehat{V}(x,t) \leq 0\}.$$

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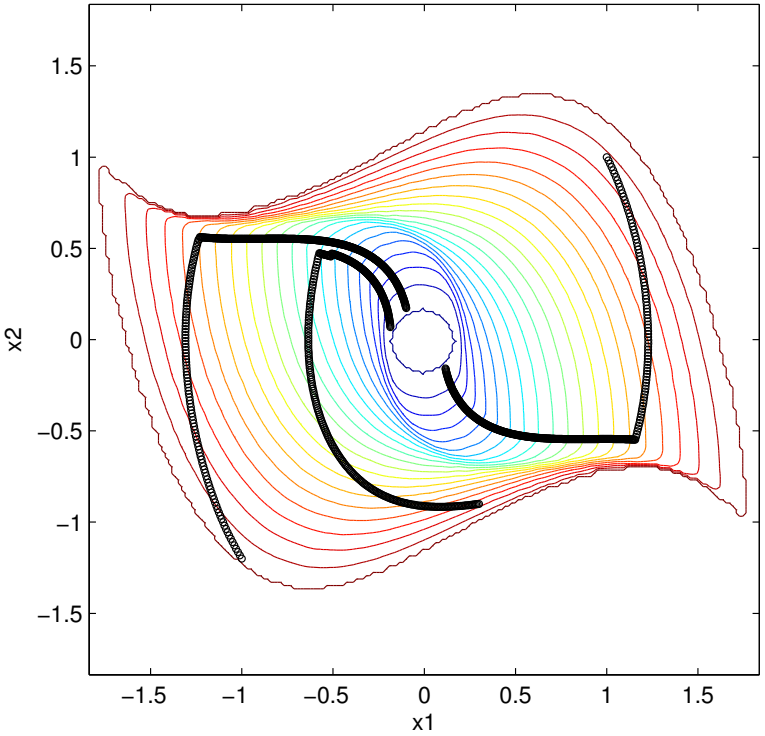
- Φ can be any function satisfying

$$\Phi(x) \leq 0 \Longleftrightarrow x \in \mathcal{C}.$$

Van der Pol Problem :

$$\begin{cases} \dot{y}_1(t) = y_2 \\ \dot{y}_2(t) = -y_1 + y_2(1 - y_1^2) + a(t) \\ a(t) \in [-1, 1] \end{cases}$$

$$\Phi(y) = d_{\mathcal{C}}(x)$$



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Now, consider the following control problems:

- Mayer's problem:

$$V(x,t) = \inf_{y_x \in \mathcal{S}_{[0,t]}(x)} \Phi(y_x(t))$$

- Time minimum problem ($\mathcal{C}$ closed set in $\mathbb{R}^d$):

$$\mathcal{T}(x) = \inf \left\{ t; \;\; y_x(t) \in \mathcal{C}, \; y_x \in \mathcal{S}_{[0,t]}(x) \right\}$$

- Supremum cost:

$$V^{\infty}(x,t) = \inf_{y_x \in \mathcal{S}_{[0,t]}(x)} \Phi(y_x(t)) \bigvee \sup_{\theta \in [0,t]} g(y_x(\theta))$$

Assume (H1)-(H2) hold, and $g : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous.

- If $\Phi : \mathbb{R}^d \to \mathbb{R}$ is lsc (resp. Lipschitz), then V and V^∞ are lsc (resp. Lipschitz).
- When the target $\mathcal{C}$ is closed, the minimum time function $\mathcal{T}$ is lsc.

Mayer Problem

$$
\begin{aligned}
V(x,t) &= \min_{y_x \in \mathcal{S}_{[0,h]}(x)} V(y_x(h), t-h) \qquad h \in (0,t), \\
V(x,0) &= \Phi(x)
\end{aligned}
$$

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V(x,0) &= \Phi(x)
\end{aligned}
$$

Minimum time problem:

$$
\begin{aligned}
\mathcal{T}(x) &= \min_{y_x \in \mathcal{S}_{[0,h]}(x)} \mathcal{T}(y_x(h)) + h \qquad h < \mathcal{T}(x),\ x \notin \mathcal{C}, \\
\mathcal{T}(x) &= 0 \quad x \in \mathcal{C};
\end{aligned}
$$

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Supremum cost

$$\begin{aligned} V^\infty(x,t) &= \min_{y_x \in \mathcal{S}_{[0,h]}(x)} V^\infty(y_x(h), t-h) \bigvee \sup_{\theta \in [0,h]} g(y_x(\theta)) \\ V^\infty(x,0) &= \Phi(x) \bigvee g(x); \end{aligned}$$

$$V(x,t) = \min_{y_x \in \mathcal{S}_{[0,t]}(x)} \Phi(y_x(t))$$

$$\begin{aligned} V(x,t) &= \min_{y_x \in \mathcal{S}_{[0,h]}(x)} V(y_x(h), t-h) \quad h \in (0,t). \\ V(x,0) &= \Phi(x) \end{aligned}$$

- ***Suboptimality:***

$$\forall y_x \in \mathcal{S}_{[0,t]}(x), \quad s \longmapsto V(y_x(s), t-s) \text{ is increasing,}$$

- **Superoptimality**

$$\exists y_x^* \in \mathcal{S}_{[0,t]}(x), \quad s \longmapsto V(y_x^*(s), t-s) \text{ is constant}$$

Next, we derive the Hamilton-Jacobi-Bellman equation (HJB), which is an infinitesimal version of the DPP.

If the value function is C^1 in a neighborhood of (x, t), then

- $\partial_t V(x,t) + H(x, D_x V(x,t)) = 0, \quad x \in \mathbb{R}^d,\ t > 0;$
- $H(x, D\mathcal{T}(x)) = 1, \quad x \notin \mathcal{C},\ \mathcal{T}(x) < +\infty;$
- $\min(\partial_t V^\infty(x,t) + H(x, DV^\infty(x,t)), V^\infty(x,t) - g(x)) = 0,$
  $x \in \mathbb{R}^d, t > 0;$

where $H(x,q) := \max_{p \in F(x)}(-p.q)$.

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- $\min(\partial_t V^\infty(x,t) + H(x, DV^\infty(x,t)), V^\infty(x,t) - g(x)) = 0,$

  $V^\infty(x,0) = \Phi(x) \vee g(x)$ HJB-VI ineqation

where $H(x,q) := \max_{p \in F(x)}(-p.q)$.

Let $u \in C_b^1(\mathbb{R}^d \times [0, T])$. We say that u is a classical verification function if it satisfies

$$\begin{aligned} &\partial_t u(x,t) + H(x, D_x u(x,t)) \leq 0 \quad x \in \mathbb{R}^d, t \in [0, T], \\ &u(x, 0) = \Phi(x); \end{aligned}$$

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Let $u \in C_b^1(\mathbb{R}^d \times [0, T])$. We say that u is a classical verification function if it satisfies

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Proposition

Let $y_x^* \in \mathcal{S}_{[0,T]}(x)$ be a feasible arc. Assume that there exists u a classical verification function, s.t.

$$u(x, T) = \Phi(y_x^*(T)),$$

then y_x^* is optimal for x on $[0, T]$.

Proof.

▷ We take any $y_x \in \mathcal{S}_{[0,T]}(x)$, we have:

$$\frac{d}{ds}[u(y_x(s), T-s)] = -\partial_t u(y_x(s), T-s) + \dot{y}_x(s) \cdot Du(y_x(s), T-s) \geq 0.$$

Which means that $s \longmapsto u(y_x(s), T-s)$ is increasing:

$$u(x,T) \leq u(y_x(T), 0) = \Phi(y_x(T)).$$

Therefore, $u(x,T) \leq V(x,T)$.

▷ $u(x,T) = \Phi(y_x^*(T)) \geq V(x,T)$.

▷ Therefore $u(x,T) = \Phi(y_x^*(T)) = V(x,T)$, and y_x^* is optimal.

Assume $F = f(x, A)$, A is a compact set in $\mathbb{R}^m$.

Proposition

Let $y_x^* \in \mathcal{S}_{[0,T]}(x)$ be a feasible arc, and let $a^*(\cdot) \in A$ be the corresponding control strategy:

$$\dot{y}_x^*(t) = f(y_x^*(t), a^*(t)) \quad \text{a.e in } [0, T].$$

Assume that there exists u a classical verification function, s.t.

$$\partial_t u(y_x^*(t), t) - f(y_x^*(t), a^*(t)).Du(y_x^*(t), t) = 0 \quad \text{for a.e } t \in [0, T],$$

then y_x^* is optimal for x on $[0, T]$.

Proof.

- We know that $s \longmapsto u(y_x(s), T-s)$ is increasing:

$$u(x,T) \leq u(y_x(T),0) = \Phi(y_x(T)).$$

  Therefore, $u(x,T) \leq V(x,T)$.

- $u(y_x^*(s), T-s) = u(x,T) = Cst = u(y_x^*(T),0) = \Phi(y_x^*(T)) \geq V(x,T)$.

- Therefore $u(x,T) = \Phi(y_x^*(T)) = V(x,T)$, and y_x^* is optimal.

If we take the value function V itself as a verification function (i.e, if V is smooth), then a necessary and sufficient condition of optimality is:

$$\begin{aligned} -f(y_x^*(t), a^*(t)) \cdot DV(y_x^*(t), t) &= \max_{a \in A}(-f(y_x^*(t), a) \cdot DV(y_x^*(t), t)) \\ &= H(y_x^*(t), DV(y_x^*(t), t)) \end{aligned}$$

for a.e $0 < t < T$.

Under the (irrealistic!) assumption that V is smooth, we consider the multivalued function:

$$\Psi(x,t) \quad := \text{argmax}_{a\in A}(-f(x,a)\cdot DV(x,t)).$$

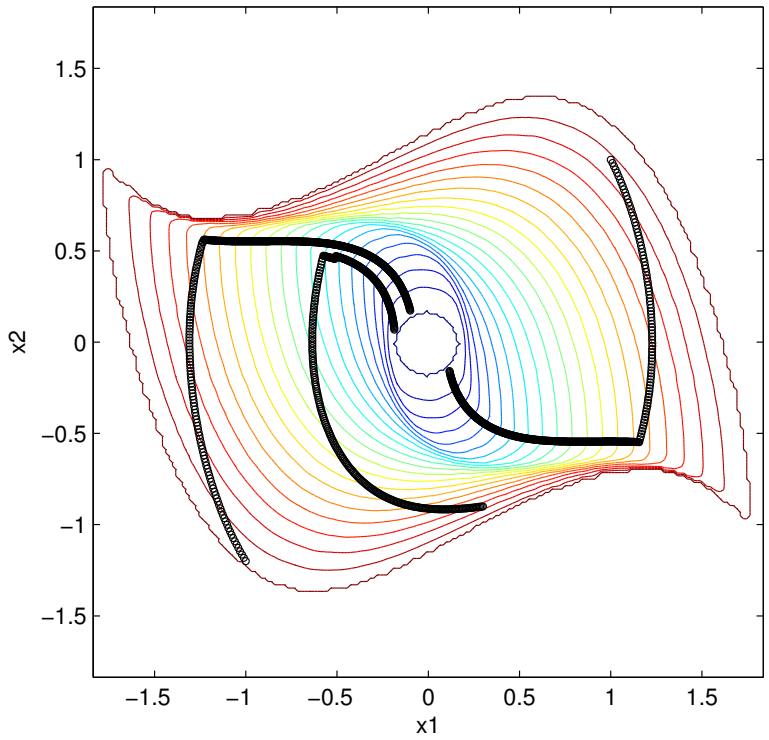
Let $y_x^* \in \mathcal{S}_{[0,t]}(x)$. y_x^* is optimal for x on $[0,T]$ if and only if $y_x^*(0) = x$, and

$$\dot{y}_x^*(t) \in f(y_x^*(t), \Psi(y_x^*(t),t)) \quad t \in (0,T).$$

Van der Pol Problem :

$$\begin{cases} \dot{y}_1(t) = y_2 \\ \dot{y}_2(t) = -y_1 + y_2(1 - y_1^2) + a(t) \\ a(t) \in [-1, 1] \end{cases}$$

$$\Phi(y) = \|y\| - r_0$$



The value function satisfies a Hamilton-Jacobi-Bellman equation (HJB) in a **viscosity sense**.

- $\partial_t V(x,t) + H(x, D_x V(x,t)) = 0, \quad x \in \mathbb{R}^d,\ t > 0;$
- $H(x, D\mathcal{T}(x)) = 1, \quad x \notin \mathcal{C},\ \mathcal{T}(x) < +\infty;$
- $\min(\partial_t V^\infty(x,t) + H(x, DV^\infty(x,t)), V^\infty(x,t) - g(x)) = 0,$
  $x \in \mathbb{R}^d, t > 0;$

where $H(x,q) := \max_{p \in F(x)}(-p.q)$.

The value function satisfies a Hamilton-Jacobi-Bellman equation (HJB) in a **viscosity sense**.

- $\partial_t V(x,t) + H(x, D_x V(x,t)) = 0, \quad x \in \mathbb{R}^d,\ t > 0;$
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where $H(x,q) := \max_{p \in F(x)}(-p.q)$.

Theorem

Assume that the value function is continuous. Then it is a viscosity solution of the corresponding HJB equation.

Proof for the Minimum time problem. Let $\varphi \in C^1$ is such that $\mathcal{T} - \varphi$ has a local maximum at $x \notin \mathcal{C}$, then for any $y_x \in \mathcal{S}_{[0,s]}(x)$ we have:

$$\varphi(x) - \varphi(y_x(s)) \leq \mathcal{T}(x) - \mathcal{T}(y_x(s)) \leq s$$

for small s. Therefore,

$$\sup_{p \in F(x)} (-p \cdot D\varphi(x)) = H(x, D\varphi) \leq 1,$$

so $\mathcal{T}$ is a subsolution. With similar arguments, we prove that $\mathcal{T}$ is a supersolution as well.

- The boundedness assumption on V and V^∞ is not restrictive. It is verified whenever Φ and g are bounded.
- The continuity of V and V^∞ holds whenever Φ and g are continuous.
- The uniqueness results for the HJB equation associated to the minimal time problem is not a trivial task.

Assume $\mathcal{C}$ is a closure of smooth domain, and let η_x be the normal to $\mathcal{C}$. If

$$\min_{p \in F(x)} p \cdot \eta_x < 0, \quad \forall x \in \partial\mathcal{C},$$

then the minimal time function $\mathcal{T}$ is continuous.

The continuity of $\mathcal{T}$ requires a *controllability* assumption of the system around the target. This important property is not satisfied in several examples, such as the Zermelo problem.

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$$\begin{cases} \partial_t V(x,t) + H(x, DV(x,t)) = 0 & x \in \mathbb{R}^d,\ t > 0 \\ V(x,0) = \Phi(x) & x \in \mathbb{R}^d \end{cases}$$

where $H(x,p) = \max_{q \in F(x)} (-q \cdot p)$.

- A viscosity solution is given by the value function:

$$V(x,t) = \min_{y_x \in \mathcal{S}_{[0,t]}(x)} \Phi(y_x(t)).$$

- Regularity and growth properties of V can be derived directly from the definition of V. Moreover, V satisfies:
  - Sub-optimality: $V(x,t) \leq V(y_x(h), t-h),\ \forall y_x \in \mathcal{S}_{[0,h]}(x)$.
  - Super-optimality: $\exists y_x^* \in \mathcal{S}_{[0,t]}(x),\ V(x,t) = V(y_x^*(h), t-h)$.

Comparison Principle: A geometric argument (nonsmooth analysis)

Proposition

Let $u : \mathbb{R}^d \times [0, T] \to \mathbb{R}$. (i) Assume u is usc. Then u is a viscosity sub-solution iff u satisfies the sub-optimality principle.

(ii) Assume u is lsc. Then, u is a viscosity super-solution iff u satisfies the super-optimality principle.

Idea of the proof

Let $U : (0, T) \to \mathbb{R}$ a usc viscosity sub-solution of:

$$-U'(t) \leq 0 \; t \in (0, T),$$

then $U(t) \leq U(t - s)$ for every $s \in [0, t]$, for every $0 \leq t \leq T$.

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- For every $p \in D^+U(t)$, we have $-p \leq 0$
- Since u is usc: $p \in D^+U(t) \longleftrightarrow (p, 1) \in \mathcal{N}_{Hp(U)}(t, U(t))$.

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- Since u is usc: $p \in D^+U(t) \longleftrightarrow (p, 1) \in \mathcal{N}_{Hp(U)}(t, U(t))$.
- We conclude (by Rockafellar's Horizontal theorem) that:

$$(p, q) \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} \leq 0, \quad \forall \, (p, q) \in \mathcal{N}_{Hp(U)}(t, U(t)). \tag{5}$$

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- For a given $t > 0$, consider the ODE:

$$\dot{Z}(s) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad s \in (0, t), \qquad Z(0) = \begin{pmatrix} t \\ U(t) \end{pmatrix}.$$

By (5) and the invariance principle, we conclude that

$$Z(s) \in Hp(U) \quad \forall s \in (0, t).$$

Comparison principle: Doubling variable techniques (Kruzhkov's method for conservation laws)

Theorem

Let Ω be a bounded open set of $\mathbb{R}^d$. Let $u_1, u_2 \in BUC(\overline{\Omega})$ be respectively, viscosity sub- and super-solutions of

$$u + H(x, Du(x)) = 0 \quad on\, \Omega,$$

and $u_1 \leq u_2$ on $\partial\Omega$. Moreover, assume that $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is uniformly continuous w.r.t the x-variable

$$|H(x_1, p) - H(x_2, p)| \leq \omega\left(|x_1 - x_2|(1 + |p|)\right),$$

where $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a continous non-deacring function satisfying $\omega(0) = 0$. Then $u_1(x) \leq u_2(x)$ for every $x \in \overline{\Omega}$.

Proof - 1: Case where u_1, u_2 are smooth (C^1 functions)

- Assume $u_1 - u_2$ attains a maximum at $x_0 \in \Omega$. Then $p := Du_1(x_0) = Du_2(x_0)$ and:

$$u_1(x_0) + H(x_0, p) \leq 0 \leq u_2(x_0) + H(x_0, p).$$

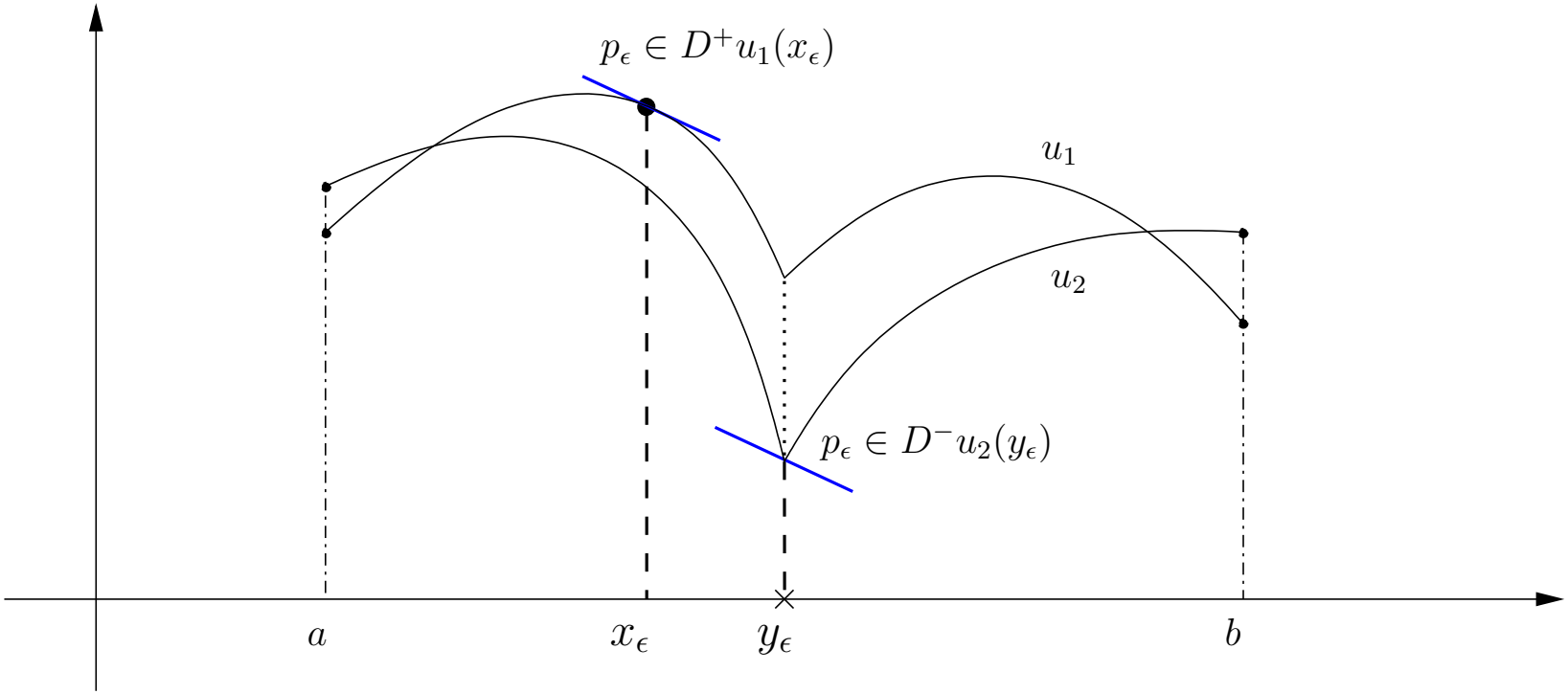
- We conclude that $u_1(x_0) \leq u_2(x_0)$.

Proof - 2: the general case

Assume $u_1 - u_2$ attains a positive maximum at $x_0 \in \Omega$.

- It may happen that $D^+u_1(x_0) \cap D^-u_2(x_0) = \emptyset$. In this case, the HJB equation doesn't give any information at x_0.
- The doubling variables idea consists of building two sequences (x_ϵ), (y_ϵ) in a neighbourhood of x_0, such that

$$D^+u_1(x_\epsilon) \cap D^-u_2(y_\epsilon) \neq \emptyset$$



$\exists p_\epsilon \in D^+u_1(x_\epsilon) \bigcap D^-u_2(y_\epsilon)$

Proof - 2: the general case

- Assume $u_1 - u_2$ attains a positive maximum at $x_0 \in \Omega$. Set

$$\delta := u_1(x_0) - u_2(x_0) > 0.$$

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- Define the function of two variables:

$$\Psi_\varepsilon(x, y) := u_1(x) - u_2(y) - \frac{|x - y|^2}{2\varepsilon}.$$

Proof - 2: the general case

- Assume $u_1 - u_2$ attains a positive maximum at $x_0 \in \Omega$. Set

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- Define the function of two variables:

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- Ψ attains its global maximum $(x_\varepsilon, y_\varepsilon)$ on $\overline{\Omega} \times \overline{\Omega}$, and we have:

$$\Psi(x_\varepsilon, y_\varepsilon) \geq \Psi(x_0, x_0) \Longrightarrow \Psi(x_\varepsilon, y_\varepsilon) \geq \delta > 0. \tag{6}$$

- Let $M > 0$ s.t $\|u_1\|_\infty + \|u_2\|_\infty \leq M$. We get:

$$0 < \Psi(x_\varepsilon, y_\varepsilon) \leq M - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon},$$

wich implies that: $|x_\varepsilon - y_\varepsilon| \leq \sqrt{2M\varepsilon}$.

Moreover, $\dfrac{x_\varepsilon - y_\varepsilon}{\varepsilon} \to 0$, when $\varepsilon \to 0$.

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wich implies that: $|x_\varepsilon - y_\varepsilon| \leq \sqrt{2M\varepsilon}$.

Moreover, $\dfrac{x_\varepsilon - y_\varepsilon}{\varepsilon} \to 0$, when $\varepsilon \to 0$.

- Since $u_2 \in \mathrm{BUC}(\overline{\Omega})$, there exists $\varepsilon' > 0$ s.t.:

$$|u_2(x) - u_2(y)| \leq \delta/2 \qquad \text{whenever } |x - y| \leq \sqrt{2M\varepsilon'}.$$

For every $\varepsilon < \varepsilon'$, we have: $x_\varepsilon, y_\varepsilon \notin \partial\Omega$. Indeed, if $x_\varepsilon \in \partial\Omega$, then we would have:

$$\begin{aligned} \Psi(x_\varepsilon, y_\varepsilon) &\leq (u_1(x_\varepsilon) - u_2(x_\varepsilon)) + |u_2(x_\varepsilon) - u_2(y_\varepsilon)| - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \\ &\leq 0 + \delta/2 + 0 \end{aligned}$$

in contradiction with (6).

- We can check that $p_\varepsilon := \dfrac{x_\varepsilon - y_\varepsilon}{\varepsilon} \in D^+ u_1(x_\varepsilon) \cap D^- u_2(y_\varepsilon)$

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- Finally, by using the assumption on H, it comes:

$$\delta \leq \Psi(x_\varepsilon, y_\varepsilon) \leq u_1(x_\varepsilon) - u_2(y_\varepsilon) \leq \omega(|x_\varepsilon - y_\varepsilon|(1 + p_\varepsilon)|).$$

This yields a contradiction when ε goes to 0.

Time dependent case.

Theorem

Let $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous:

$$|H(x,p) - H(y,p)| \leq C(|x-y|(1+|p|)),$$
$$|H(x,p_1) - H(x,p_2)| \leq C|p_1 - p_2|.$$

Let u_1, u_2 are resp., a sub- and super-solution of

$$\partial_t u + H(x, Du) = 0 \quad x \in \mathbb{R}^d,\ t \in (0,T]$$

Assume that $u_1(x,0) \leq u_2(x,0)$, then $u_1 \leq u_2$ on $\mathbb{R}^d \times [0,T]$.

Idea of the proof.

- A new difficulty: the domain is unbounded. For this, we define the *doubling-variable* function as:

$$\Psi_\varepsilon(x,t,y,s) := u_1(x,t) - u_2(y,s) - \varepsilon(|x|^2 + |y|^2) - \frac{(|t-s|^2 + |x-y|^2)}{2\varepsilon^2}.$$

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$$\begin{cases} \partial_t V(x,t) + H(x, DV(x,t), D^2 V(x,t)) = 0 \quad x \in \mathbb{R}^d, \ t > 0 \\ V(x,0) = \Phi(x) \quad x \in \mathbb{R}^d \end{cases}$$

where

$$H(x,p,Q) = \max_{a \in A}(-\ell(x,a) - b(x,a) \cdot p - \frac{1}{2}\mathrm{Tr}(\sigma(x,a)\sigma^{\mathsf{T}}(x,a) \cdot Q)).$$

$\sigma(\cdot)$ is a mapping $\mathbb{R}^d \times A$ into the space of $n \times r$ matrices, $b : \mathbb{R}^d \times A \to \mathbb{R}^d$.

Definition.

Let $\Omega \subset \mathbb{R}^m$, and $\mathcal{F} : \Omega \times \mathbb{R} \times \mathbb{R}^m \times \mathcal{S}_{m\times m} \to \mathbb{R}$ be continuous and satisfying $\mathcal{F}(x, r, p, X) \leq \mathcal{F}(x, r, p, Y)$ when $X \geq Y$.

Consider the HJB equation:

$$\mathcal{F}(x, u(x), Du(x), D^2u(x)) = 0 \quad \text{in } \Omega. \tag{7}$$

(i) A function $u \in USC(\Omega)$ is a viscosity **sub-solution** of (7) if for any $\varphi \in C^2(\Omega)$ and any local maximum point $x_0 \in \Omega$ of $u - \varphi$,

$$\mathcal{F}(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0.$$

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(iii) A function $u \in C(\Omega)$ is a viscosity **solution** of (7) if it is a sub- and super-solution.

- The notion of viscosity can also be defined by an adequate concept of "super-" and "sub-" derivatives (of order 2), called "semi-jets".
- The viscosity solution is the limit of classical solutions of *viscous* equations.
- As for first order equations, a general stability property holds, without any requirement about the convergence of derivatives.

TIme-dependent HJ equation

- The associated control problem:

$$
\begin{aligned}
V(x,t) \quad = \quad & \mathrm{Min}\,\mathbb{E}\left[\Phi(y(T)) + \int_0^t \ell(y(s),\alpha(s))\,ds\right]; \\
& dy(s) = b(y(s),\alpha(s))ds + \sigma(y(s),\alpha(s))dw_s, \\
& y(0) = x, \\
& \alpha(s) \in A, \quad s \in [0,t].
\end{aligned}
$$

- V satisfies a DPP:

$$
\begin{aligned}
V(x,t) \quad &= \quad \min_{\alpha(]0,h[)\subset A} \mathbb{E}\left[V(y_x^\alpha(h), t-h) + \int_0^h \ell(y_x^\alpha(s),\alpha(s))\,ds\right], \\
V(x,0) \quad &= \quad \Phi(x).
\end{aligned}
$$

- A general comparison principle can be stated, as for first order equations, under Lipschitz continuity of the Hamiltonian.

References on viscosity notion for HJ equations

- G. Barles, *Solutions de Viscosité des Equations de Hamilton-Jacobi*, Springer, 1994.
- M. Bardi and I. Capuzzo Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton- Jacobi-Bellman Equations*, Birkhäuser, 1997.
- M. Crandall and P. L. Lions, Viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.*, 277(1983), 1-42.
- M. Crandall, H. Ishii and P. L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc.*, 27(1992), 1-67.
- W.H. Fleming and H.M. Soner, *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag, New-York, 1993.
- S. N. Kruskov, First order quasilinear equations in several independant variables. *Math. USSR-Sbornik* 10(1970), 217-243.
- P.L. Lions, *Generalized solution of Hamilton-Jacobi equations*, Pitman, London, 1982.

Extensions and active fields

- The comparison principle has been extended to unbounded viscosity solutions, with polynomial or exponential growth (ref: Ishii'92)
- Boundary conditions can be also considered in the viscosity framework (ref: Barles, Capuzzo-Dolceta, Lions, Ishii, ...)
- Some existence and uniqueness results have been extended to Hamilton-Jacobi equations with L^p data (ref: Cafarrelli, Cardaliaguet, Souganidis, ...)
- Hamilton-Jacobi equations in infinite dimension setting have been investigated by (ref: Barbu, Cannarsa, Gozzi, Soner, ...)
- Homogenization and Ergodic control problems (ref: Alvarez, Bardi, Achdou, Cardaliaguet, Siconolfi, Souganidis ...)
- ....

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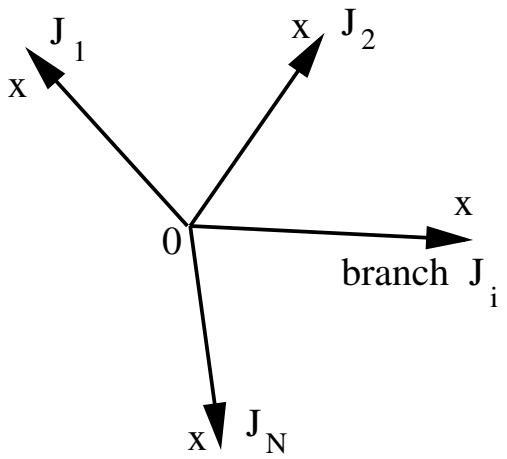
A junction condition on O

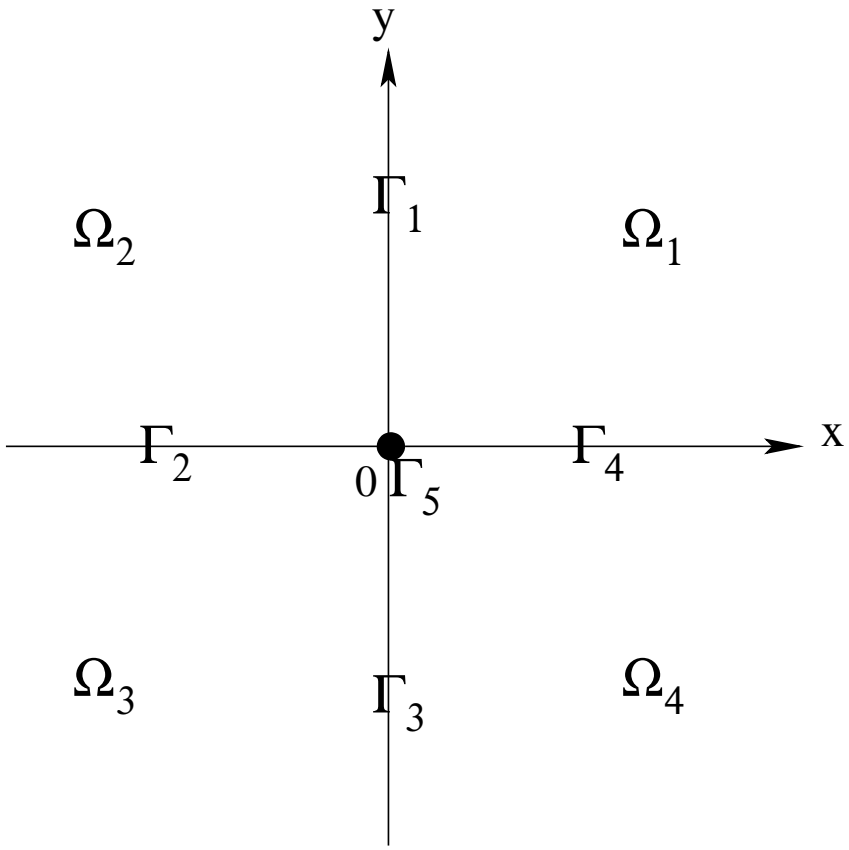


$$\begin{cases} u_t^i + H_i(u_x^i) = 0, & x > 0, \quad i = 1, ..., N \\ u^i = u^j =: u, & x = 0, \\ u_t + F(u_x^1, ..., u_x^N) = 0, & x = 0 \end{cases}$$

Junction conditions on multi-domains

- $\mathbb{R}^d$ is divided into several open disjoint subdomains $(\Omega_i)_{i=1,\dots,m}$ with

$$\mathbb{R}^d = \overline{\Omega}_1 \cup \cdots \cup \overline{\Omega}_m.$$



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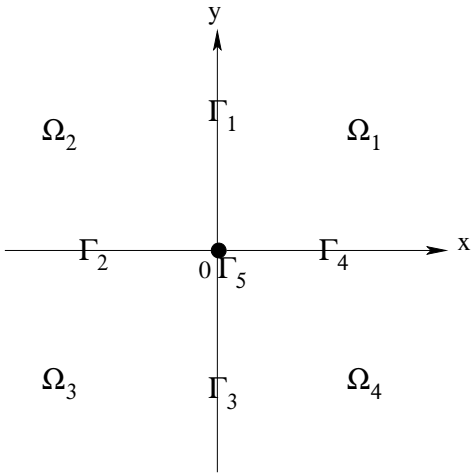

$$\begin{cases} \partial_t u_i(t,x) + H_i(x, D_x u_i(t,x)) = 0 & t \in (0,T),\ x \in \Omega_i, \\ u_i(0,x) = \varphi(x) & x \in \mathbb{R}^d, \end{cases}$$

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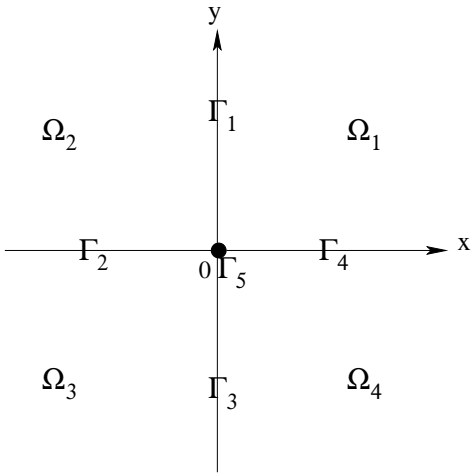

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$u = u_1, \dots, u_m$

$\partial_t u(t,x) + F(x, D_x u_1(t,x), \dots, D_x u_m(t,x)) = 0 \quad \text{on } \cup_j \Gamma_j$

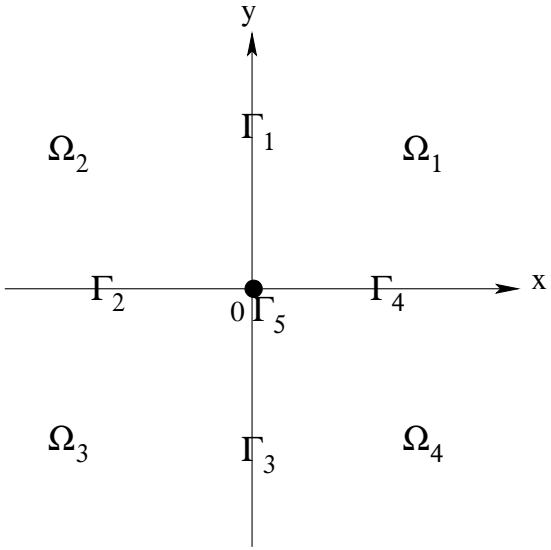
Motivation & Position of the problem

- data transmission and traffic management (Garavello-Piccoli'2006, Engel et al.'08, Lebacque'05
- blood circulation
- supply chains (Göttlich , Herty , Klar 2005)
- gas pipelines

- multiprocess systems without transition cost
- multiple thermostatic hybrid systems (Bagagiolo et al.'2012)
- Numerical purpose: decomposition of domains

Position of the problem

➢ $\mathbb{R}^d = \left(\bigcup_{i=1}^{m} \Omega_i\right) \bigcup \left(\bigcup_{j=1}^{\ell} \Gamma_j\right),$



- For each $i = 1, \dots, m$, consider the system of HJB equations:

$$\begin{cases} \partial_t u_i(t,x) + H_i(x, D_x u_i(t,x)) = 0 & t \in (0,T),\ x \in \Omega_i, \\ u(0,x) = \varphi(x) & x \in \mathbb{R}^d, \end{cases} \tag{8}$$

where $T > 0$ is a fixed final time and φ is a given regular function.

- Each H_i is a Lipschitz continuous Hamiltonian of Bellman form:

$$H_i(x,p) := \sup_{v \in F_i(x)} \{-v \cdot p - L_i(x,v)\}.$$

where F_i is a Lipscitz set-valued function on Ω_i.

Junction conditions

Question: what conditions of transition between the subdomains Ω_i should be considered to ensure the existence and *uniqueness* of solution u for the system (8)?

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For simplicity, we shall take $L \equiv 0$.

Junction conditions

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Consider the equation

$$\partial_t u(t,x) + H_F(x, D_x u(t,x)) = 0, \quad x \in \Omega = \mathbb{R}^d$$
$$u(0,x) = \varphi(x),$$

where $H_F(x,q) = \sup_{p \in F(x)}\{-p \cdot q\}$.

Definition: viscosity solution

A continuous function u is subsolution if for every $\Phi \in C^1((0,T) \times \Omega)$ such that
$u \leq \Phi$ (resp. $u \geq \Phi$) with equality at $(t_o, x_o) \in \Omega$, then:

$$\partial_t \Phi(t_o, x_o) + H_F(x_o, D_x \Phi(t_o, x_o)) \leq 0 \quad (\text{resp.} \geq 0).$$

Mayer's optimal control problems

- Assume: $\mathbb{R}^d = \Omega$
- Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a regular function and $F : \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ be a Lipschitz continuous multi-function with closed convex images. For an initial data $(t, x) \in [0, T] \times \mathbb{R}^d$, consider the differential inclusion:

$$\begin{cases} \dot{y}(s) \in F(y(s)) & s \in (0, t), \\ y(0) = x. \end{cases} \tag{9}$$

  The set $\mathcal{S}_{[0,t]}(x)$ of all trajectories satisfying (9) is closed in $W^{1,1}(0, t)$.
- And the value function $v : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is defined by

$$v(t, x) := \inf\{\varphi(y(t)) : y(\cdot) \text{ satisfies (9)}\}.$$

Classical Dynamical Programming Principle

For any $t \in [0, T]$, $x \in \mathbb{R}^d$, $h \in [0, t]$, we have

$$v(t, x) = \min_{y \in \mathcal{S}_{[0,t]}(x)} \{v(t - h, y(h))\}.$$

More precisely,

- The super-optimality: There exists an optimal trajectory $\bar{y} \in \mathcal{S}_{[0,t]}(x)$ such that

$$v(t, x) \geq (=) \, v(t - h, \bar{y}(h));$$

- The sub-optimality: For any trajectory $y \in \mathcal{S}_{[0,t]}(x)$, we have:

$$v(t, x) \leq v(t - h, y(h)).$$

Characterization

In the classical case where F is Lipschitz continuous, consider the Hamiltonian

$$H_F(x,p) = \sup_{q \in F(x)} \{-q \cdot p\}.$$

For any Lipschitz continuous function $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, we have

- u satisfies the super-optimality $\Leftrightarrow$ u satisfies

$$\partial_t u(t,x) + H_F(x, Du(t,x)) \geq 0;$$

- u satisfies the sub-optimality $\Leftrightarrow$ u satisfies

$$\partial_t u(t,x) + H_F(x, Du(t,x)) \leq 0.$$

Classical case

The value function v defined by:

$$v(t,x) := \min_{y \in \mathcal{S}_{[0,t]}(x)} \varphi(y(t)),$$

is the unique viscosity solution of

$$\begin{aligned} &\partial_t v(t,x) + H_F(x, Dv(t,x)) = 0 \\ &v(0,x) = \varphi(x). \end{aligned}$$

Consider the cross structure in $\mathbb{R}^2$

$$\begin{aligned} &\partial_t u(t,x) + H_i(x, Du(t,x)) = 0 \quad (0,T) \times \Omega_i \\ &u(0,x) = \varphi(x), \end{aligned}$$

with $H_i(x,p) = \sup_{v \in F_i(x)}(-v \cdot p)$.

- Idea: introduce a global optimal control problem and investigate the equation satisfied by the value function on the interfaces.

Main difficulty:

- How to define the dynamics on the interfaces which allow that the trajectories can either transit between the domains or stay on the interfaces?
- A global Lipschitz continuous Hamiltonian could not be expected due to the singularity of the domain.

Literature on non-Lipschitz Hamiltonians

- Hamilton-Jacobi equations with discontinuous Hamiltonians: Ishii'89.
- Hamilton-Jacobi-Bellman equations with discontinuous Lagrangians: Soravia'02.
- Hamilton-Jacobi equations with measurable Hamiltonians: Camilli-Siconolfi'03.
- Infinite horizon optimal control problem on two-domains: Barles-Briani-Chasseigne'12.
- Hamilton-Jacobi approach to junction problems on network: Achdou-Camilli-Cutri-Tchou'12, Imbert-Monneau-Zidani'12.

Assumptions

We are interested in the Hamiltonians H_i of the following Bellman form:

$$H_i(x,q) = \sup_{p \in F_i(x)} \{-p \cdot q\},$$

where $F_i : \overline{\Omega}_i \rightsquigarrow \mathbb{R}^2$ are multifunctions satisfying the following: for any $x \in \overline{\Omega}_i$,

- $F_i(x)$ is a nonempty, convex and compact set;
- F_i is Lipschitz continuous with respect to Hausdorff metric;
- A controllability assumption. For example: $\exists\, \delta > 0$ so that $B(0,\delta) \subset F_i(x)$.

Filippov regularization for the dynamics

To define the global dynamics, we consider the following Filippov regularization of the multi-functions $(F_i)_{i=1,\dots,4}$:

$$FC(x) := \begin{cases} F_i(x) & \text{if } x \in \Omega_i, \\ \overline{co}(F_i(x) : x \in \overline{\Omega}_i) & \text{otherwise.} \end{cases}$$

Then consider the differential inclusion

$$\begin{cases} \dot{y}(s) \in FC(y(s)) & s \in (0,t), \\ y(0) = x. \end{cases} \tag{10}$$

Remark: FC is the smallest upper semi-continuous envelope of $(F_i)_{i=1,\dots,4}$. Moreover,

$$FC(x) = \bigcap_{\varepsilon>0} \overline{co}\left\{F_i(y) : \|y-x\| < \varepsilon, y \in \overline{\Omega}_i\right\}.$$

Global optimal control problem

For any $t \in [0, T]$, $x \in \mathbb{R}^2$, we set

$$S_{[0,t]}(x) := \{y(\cdot) | y(t) = x, \ \ \dot{y}(s) \in FC(y(s)) \text{ for } s \in (0, t)\}$$

as the set of trajectories. Consider the value function

$$v(t, x) := \inf_{y(\cdot) \in S_{[0,t]}(x)} \{\varphi(y(t))\}.$$

The upper semi-continuity and the convexity of FC imply that $S_{[0,t]}(x)$ is compact, then the "inf" is in fact "min".

Remark: Here, v is the value function of a control problem of hybrid systems without transition cost $\Longrightarrow$ Zeno effect

Dynamical Programming Principle

For any $t \in [0, T]$, $x \in \mathbb{R}^2$, $h \in [0, t]$, we have

$$v(t, x) = \min_{y(\cdot) \in S_{[t,T]}(x)} \{v(t - h, y(h))\}.$$

More precisely,

- The super-optimality: $\exists\, \bar{y} \in S_{[t,T]}(x)$ such that

$$v(t, x) \geq (=)\, v(t - h, \bar{y}(h));$$

- The sub-optimality: $\forall\, y \in S_{[t,T]}(x)$ such that

$$v(t, x) \leq v(t - h, y(h)).$$

Problem of characterization

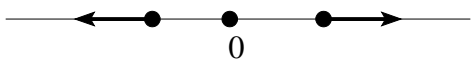
FC is only upper semi-continuous, then the characterization of the sub-optimality fails, i.e. v does not satisfy

$$\partial_t v(t,x) + \sup_{p \in FC(x)} \{-p \cdot Dv(t,x)\} \leq 0.$$

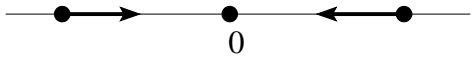
From a point view of trajectories: the set of dynamics $FC(x)$ is too large and may contain some useless dynamics which are never used by the trajectories.

Examples in dimension 1 with one interface $\{0\}$:

- The case "pull-pull":
  $F_1(x) = [-1, \frac{1}{2}]$ for $x < 0$, $F_2(x) = [-\frac{1}{2}, 1]\}$ for $x > 0$. The convexification on 0 is $FC(0) = [-1, 1]$.



- The case "push-push":
  $F_1(x) = \{1\}$ for $x < 0$, $F_2(x) = \{-1\}$ for $x > 0$. The convexification on 0 is also $FC(0) = [-1, 1]$ which is not the reasonable set of dynamics.



We need to consider only the "useful" dynamics!

Essential dynamics

Some notations:

- $\mathcal{T}_{\overline{\Omega}_i}(x)$ is the tangent cone of $\overline{\Omega}_i$ on x,
- $\mathcal{T}_\Gamma$ the tangent space of Γ on x.

Introduce the **essential dynamic** multifunction $F^E : \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ (ref. Barnard-Wolenski'13, Rao-HZ'13):

- For $x \in \Omega_i$, $F^E(x) := F_i(x)$;
- For $x \in \Gamma_j$,

$$F^E(x) = (FC(x) \cap \mathcal{T}_{\Gamma_j}(x)) \bigcup \bigcup_{i|x\in\overline{\Omega}_i} \left(F_i(x) \cap \mathcal{T}_{\overline{\Omega}_i}(x)\right).$$

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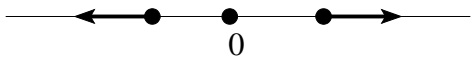
- $F_i(x) \cap \mathcal{T}_{\overline{\Omega}_i}(x)$: the dynamics in $F_i(x)$ are inward to $\overline{\Omega}_i$,
- $F(x) \cap \mathcal{T}_{\Gamma_j}(x)$: the dynamics in $F(x)$ are tangent to Γ_j.

Examples in dimension 1

- Example of "pull-pull":



$$F_1(0) \cap \mathcal{T}_{\overline{\Omega}_1}(0) = \{-1\},\ F_2(0) \cap \mathcal{T}_{\overline{\Omega}_2}(0) = \{1\},\ F(0) \cap \mathcal{T}_{\Gamma}(0) = \{0\},$$

$$F^E(0) = \{-1, 0, 1\}.$$

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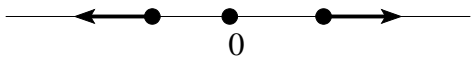

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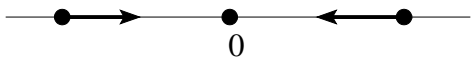
$$F^E(0) = \{-1, 0, 1\}.$$

- Example of "push-push":



$$F_1(0) \cap \mathcal{T}_{\overline{\Omega}_1}(0) = F_2(0) \cap \mathcal{T}_{\overline{\Omega}_2}(0) = \emptyset,\ F(0) \cap \mathcal{T}_{\Gamma}(0) = \{0\},$$

$$F^E(0) = \{0\}.$$

Examples in dimension 2

$\Omega_1 = \{(x,y)|x < 0, y \in \mathbb{R}\}$, $\Omega_2 = \{(x,y)|x > 0, y \in \mathbb{R}\}$,
$\Gamma = \{(0,y)|y \in \mathbb{R}\}$.

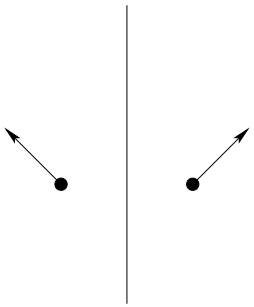


Figure : *

Original dynamics:
$F_1 = \{(-1,1)\}$,
$F_2 = \{(1,1)\}$.

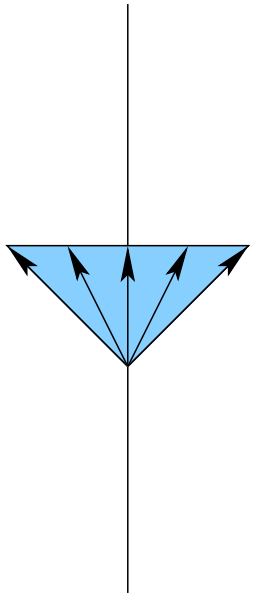


Figure : *

Convexified dynamics

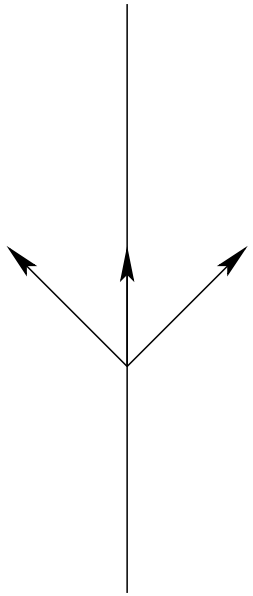


Figure : *

Essential dynamics

Properties of F^E

- $F^E(x) \subset FC(x)$.
- The set of trajectories:
  $\{y \mid \dot{y}(s) \in FC(y(s)); y(t) = x\} = \{y \mid \dot{y}(s) \in F^E(y(s)); y(t) = x\}$.
- For any $p \in F^E(x)$, there exists $y \in S_{[t;T]}(x)$ such that $y(t) = x$ and $\dot{y}(t) = p$.

Equations on the interface

Consider the new Essential Hamiltonian

$$H^E(x,q) := \sup_{p \in F^E(x)} \{-p \cdot q\}.$$

And here is the existence result:

Theorem (Rao-HZ'12)

The value function v is Lipschitz continuous and is the viscosity solution of the system of HJB equations:

$$\begin{cases} \partial_t u(t,x) + H_i(x, Du(t,x)) = 0 & t \in (0,T),\ x \in \Omega_i, \\ \\ u(0,x) = \varphi(x) & x \in \mathbb{R}^2. \end{cases}$$

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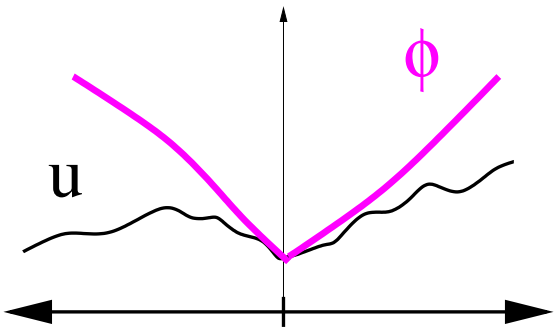
Lipschitz continuity comes from the controllability assumption!

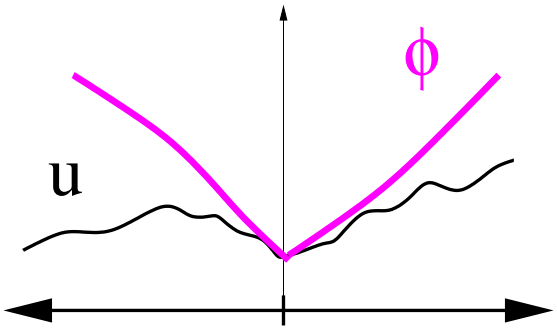


Definition: viscosity solution (for $d = 1$)

A continuous function u is subsolution if for every $\Phi \in C^0((0,T) \times \mathbb{R}^d)$ such that
$\Phi \in C^1((0,T) \times \mathcal{M})$, for $\mathcal{M}^r = \mathbb{R}^+$, $\mathcal{M}^l = \mathbb{R}^-$ or $\mathcal{M}^O = \{0\}$,

$u \leq \Phi$ (resp. $u \geq \Phi$) with equality at $(t_o, x_o) \in \Omega$, then:

$$\partial_t \Phi(t_o, x_o) + \sup_{k=l,r,O} H_k(x_o, D_x^k \Phi(t_o, x_o)) \leq 0 \quad (\text{resp. } \geq 0),$$



Definition: viscosity solution (for $d \geq 1$)

A continuous function u is subsolution if for every $\Phi \in C^0((0,T) \times \mathbb{R}^d)$ such that

$\Phi \in C^1((0,T) \times \mathcal{M})$, for any $\mathcal{M} = \Omega_i$ or $\mathcal{M} = \Gamma_j$,
$u \leq \Phi$ (resp. $u \geq \Phi$) with equality at $(t_o, x_o) \in \Omega$, then:

$$\partial_t \Phi(t_o, x_o) + \sup_{k \in \mathbb{I}(x_o)} H_k(x_o, D_x^k \Phi(t_o, x_o)) \leq 0 \quad (\text{resp. } \geq 0),$$

where $\mathbb{I}(x_o) := \{i \mid x \in \overline{\Omega}_i\} \cup \{j \mid x \in \Gamma_j\}$.

Super-solution: Idea of the proof (I)

The value function v is viscosity super-solution of:

$$\begin{cases} \partial_t u(t,x) + H^E(x, Du(t,x)) \geq 0 & t \in (0,T),\ x \in \mathbb{R}^d, \\ u(0,x) \geq \varphi(x) & x \in \mathbb{R}^d. \end{cases}$$

- for every $(t,x) \in (0,T) \times \mathbb{R}^d$, there exists $\bar{y}_x \in \mathcal{S}_{[0,t]}(x)$:

$$v(t,x) \geq v(t-h, \bar{y}(h)) \quad \forall h \in (0,t).$$

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- if $v \geq \Phi$ with equality at (t_o, x_o), then

$$\Phi(t_o, x_o) \geq \Phi(t_o - h, \bar{y}_{x_o}(h)) \quad \forall h \in (0, t_o).$$

Super-solution: Idea of the proof (II)

- On the other hand, for any $x \in \mathbb{R}^d$, and every $y \in \mathcal{S}_{[0,t]}(x)$, we have (whenever the limit exists):

$$\lim_n \frac{y(h_n) - x}{h_n} \in \mathrm{co}F^E(x).$$

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- By separation Theorem, we get:

$$\partial_t \Phi(t_o, x_o) + H^E(x_o, D\Phi(t_o, x_o)) \geq 0.$$

Sub-solution:Idea of the proof

Theorem (Rao-Siconolfi-HZ'13)

The value function v is viscosity sub-solution of the system of HJB equations:

$$\begin{cases} \partial_t u(t,x) + H^E(x, Du(t,x)) \leq 0 & t \in (0,T),\ x \in \mathbb{R}^d, \\ u(0,x) \leq \varphi(x) & x \in \mathbb{R}^d. \end{cases}$$

- for every $(t,x) \in (0,T) \times \mathbb{R}^d$, for every $\bar{y}_x \in \mathcal{S}_{[0,t]}(x)$:

$$v(t,x) \leq v(t-h, \bar{y}(h)) \quad \forall h \in (0,t).$$

- if $v \leq \Phi$ with equality at (t_o, x_o), then

$$\Phi(t_o, x_o) \leq \Phi(t_o - h, \bar{y}_{x_o}(h)) \quad \forall h \in (0, t_o).$$

- Therefore

$$\partial_t \Phi(t_o, x_o) + H^E(x_o, D\Phi(t_o, x_o)) \leq 0.$$

The uniqueness result

A strong comparison principle:

Theorem

Let u_1 and u_2 be respectively a Lipschitz continuous sub-solution and super-solution of (2)*, then we have*

$$u_1(t,x) \leq u_2(t,x), \ \textit{for } t \in [0,T], x \in \mathbb{R}^d.$$

Main steps of the proof:

- The subsolution u_1 satisfies the sub-optimality $\Rightarrow u_1 \leq v$;
- The supersolution u_2 satisfies the super-optimality $\Rightarrow u_2 \geq v$;

First step: proof for the characterization of super-optimality

Since $F^E(x) \subset FC(x)$, the super-solution inequation

$$\partial_t u(t,x) + sup_{p \in F^E(x)}(p \cdot Du(t,x)) \geq 0$$

implies that

$$\partial_t u(t,x) + sup_{p \in FC(x)}(p \cdot Du(t,x)) \geq 0$$

Then the following of the proof is classical since FC is upper semi-continuous.

Second step: The characterization of the sub-optimality is more technical ...

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  - cross the interface and pass to another manifold: continuity of the function
  - take a sliding mode and fellow the same interface for a while: Junction conditions include the behavior of the trajectories on each interface

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- For a trajectory starting in Ω_i and living in the same manifold : NO Problem
- When a trajectory reaches an interface, it can
  - cross the interface and pass to another manifold: continuity of the function
  - take a sliding mode and fellow the same interface for a while: Junction conditions include the behavior of the trajectories on each interface
- The set of trajectories contains also some trajectories with more complicated behavior: Crucial difficulty

An explicit example in 1d.

The multi-domains: $\Omega_1 = \{x : x < 0\}$, $\Omega_2 = \{x : x > 0\}$, $\Gamma = \{0\}$.
The dynamics: $F_1 = [-\frac{1}{2}, 1]$, $F_2 = [-1, \frac{1}{2}]$.
$T > 0$ is a given final time, the final cost function $\varphi(x) = x$.

$$v_1(t,x) := \min\{y_{t,x}(T)\} = \begin{cases} x - \frac{1}{2}(T-t) & x \leq 0, \\ -\frac{1}{2}(T-t-x) & 0 \leq x \leq T-t, \\ x - (T-t) & x \geq T-t. \end{cases}$$

At point $(t,x) = (0,0)$, $\partial_t v_1(0,0) = \frac{1}{2}$, $Dv_1(0,0^-) = 1$, $Dv_1(0,0^+) = \frac{1}{2}$, $D^+ v_1(0,0) = [\frac{1}{2}, 1]$.

An explicit example in 1d

The dynamics: $F_1 = [-\frac{1}{2}, 1]$, $F_2 = [-1, \frac{1}{2}]$.
The convexification: $FC(0) = [-1, 1]$.
The essential dynamics: $F^E(0) = [-\frac{1}{2}, \frac{1}{2}]$.

$$-\partial_t v_1(0,0) + \max_{p \in F^E(0)} \{-p \cdot D^+ v_1(0,0)\} = 0 \leq 0,$$

while

$$-\partial_t v_1(0,0) + \max_{p \in FC(0)} \{-p \cdot D^+ v_1(0,0)\} = \frac{1}{2} > 0.$$

The subsolution property fails for FC which is larger than F^E.

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The multi-domains: $\Omega_1 = \{x : x < 0\}$, $\Omega_2 = \{x : x > 0\}$, $\Gamma = \{0\}$.
The dynamics: $F_1 = [-\frac{1}{2}, 1]$, $F_2 = [-1, \frac{1}{2}]$.
$T > 0$ is a given final time, the final cost function $\varphi(x) = -x$.

$$v_2(t,x) := \min\{-y_{t,x}(T)\} = \begin{cases} -[x + \frac{1}{2}(T-t)] & x \geq 0, \\ -\frac{1}{2}(x + T - t) & t - T \leq x \leq 0, \\ -[x + T - t] & x \leq t - T. \end{cases}$$

At point $(t,x) = (0,0)$, $\partial_t v_2(0,0) = \frac{1}{2}$, $Dv_1(0,0^-) = -1$,
$Dv_1(0,0^+) = -\frac{1}{2}$, $D^- v_1(0,0) = [-1, -\frac{1}{2}]$.

An explicit example in 1d

The dynamics: $F_1 = [-\frac{1}{2}, 1]$, $F_2 = [-1, \frac{1}{2}]$.
The tangent dynamics: $F_\Gamma(0) = \{0\}$.
The essential dynamics: $F^E(0) = [-\frac{1}{2}, \frac{1}{2}]$.

$$-\partial_t v_2(0,0) + \max_{p \in F^E(0)} \{-p \cdot D^- v_2(0,0)\} = 0 \geq 0,$$

while

$$-\partial_t v_2(0,0) + \max_{p \in F_\Gamma(0)} \{-p \cdot D^- v_1(0,0)\} = -\frac{1}{2} < 0.$$

The supersolution property fails for F_Γ which is smaller than F^E.

Extensions (Rao-HZ'13, Rao-Siconolfi-HZ '14)

- Weaker controllability assumptions
- Equations with Lagrangians (running cost):

$$H_i(x,p) = \sup_{\alpha \in A}\{-p \cdot f_i(x,\alpha) - \ell_i(x,\alpha)\}.$$

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Thank you for your attention.