# Evolution reaction-diffusion systems with positivity and mass control: Global existence, Singular perturbations, $L^\infty, L^p, L^1, L^2$ approaches

Michel Pierre

Ecole Normale Supérieure de Rennes
and Institut de Recherche Mathématique de Rennes, France

Summerschool
*Analysis and Applications of Partial Differential Equations*
Graz, Austria, September 08-11, 2014

# Goals of the talk:

- (1) To understand global existence in time for reaction-diffusion systems which have two main properties:
  - *positivity of the solutions is preserved*
  - *the total mass of the solution is controlled*
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- (2) To apply the same $L^2$-estimates to the description of fast-reaction limits in some chemical systems and to existence questions for some cross-diffusion systems.

# An easy O.D.E.

$$\begin{cases} u' = -u\,v^{\beta}, \\ v' = \quad u\,v^{\beta} \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0, \\ u_0, v_0 \textit{ given in } [0, \infty), \end{cases}$$

where $u, v : [0, T) \to \boldsymbol{R}$ are the unknown functions. Here $\beta \geq 1$. Local existence of a nonnegative unique solution on a maximal interval $[0, T^*)$ is well-known due to the $C^1$-property of $(u, v) \to uv^{\beta}$. Moreover $u \geq 0, v \geq 0$ and

$$(u + v)'(t) = 0 \Rightarrow (u + v)(t) = u_0 + v_0,$$

so that: $\sup_{t \in [0, T^*)} |u(t)| + |v(t)| < +\infty,$

and therefore

$$T^* = +\infty$$

# What happens when diffusion is added?

$$\begin{cases} \partial_t u - d_1 \Delta u = -uv^\beta \ in\ Q_T = (0,T) \times \Omega \\ \partial_t v - d_2 \Delta v = \ \ uv^\beta \ in\ Q_T = (0,T) \times \Omega \\ \partial_\nu u = \partial_\nu v = 0 \ on\ \Sigma_T = (0,T) \times \partial\Omega, \\ u(0) = u_0 \geq 0, \ \ v(0) = v_0 \geq 0. \end{cases}$$

Here $\Omega \subset \boldsymbol{R}^N$, regular. The total mass is preserved:

$$\int_\Omega \partial_t (u+v) - \int_\Omega \Delta(d_1 u + d_2 v) = 0.$$

$$\partial_\nu (d_1 u + d_2 v) = 0 \ on\ \partial\Omega \Rightarrow \int_\Omega \Delta(d_1 u + d_2 v) = 0.$$

$$\int_\Omega (u+v)(t) = \int_\Omega u_0 + v_0$$

Insufficient for global existence!

# Local existence for reaction-diffusion systems with $L^\infty$-data

$$(S)\begin{cases} \partial_t u - d_1 \Delta u = -uv^\beta \text{ on } Q_T \\ \partial_t v - d_2 \Delta v = uv^\beta \text{ on } Q_T \\ \partial_\nu u = \partial_\nu v = 0 \text{ on } \Sigma_T, \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0. \end{cases}$$

- **Theorem ($L^\infty$-approach):** Let $u_0, v_0 \in L^\infty(\Omega)$, $u_0 \geq 0, v_0 \geq 0$. Then, there exist a maximum time $T^* > 0$ and $(u, v)$ unique classical nonnegative solution of (S) on $[0, T^*[$. Moreover,

$$\sup_{t \in [0, T^*[} \left\{ \|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} \right\} < +\infty \Rightarrow [T^* + \infty].$$

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- By maximum principle: $\|u(t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}$. But, what about $v(t)$?
- If $d_1 = d_2$: $\partial_t(u+v) - d_1\Delta(u+v) = 0$,

$$\Rightarrow \|u(t) + v(t)\|_{L^\infty(\Omega)} \leq \|u_0 + v_0\|_{L^\infty(\Omega)},$$

$\Rightarrow \quad T^* = +\infty$ !

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## What if $d_1 \neq d_2$ ?

*Remark:* here $\int_\Omega (u+v)(t) = \int_\Omega u_0 + v_0$, that is

$$\sup_{t \in [0, T^*[} \left\{ \|u(t)\|_{L^1(\Omega)}, \|v(t)\|_{L^1(\Omega)} \right\} \leq \|u_0\|_{L^1(\Omega)} + \|v_0\|_{L^1(\Omega)}.$$

How does this estimate help for global existence? Very frequent situation in applications !

# Same question for the general family of systems:

$$\begin{cases} \forall i = 1, ..., m \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in } Q_T \\ \partial_\nu u_i = 0 & \text{on } \Sigma_T \\ u_i(0, \cdot) = u_i^0(\cdot) \geq 0. \end{cases}$$

$d_i > 0,\ f_i : [0, \infty)^m \to \boldsymbol{R}$ of class $C^1$ where

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- **(M):** $\sum_{1 \leq i \leq m} f_i \leq 0$ or more generally
- **(M')** $\forall r \in [0, \infty[^m,\ \sum_{1 \leq i \leq m} a_i f_i(r) \leq C[1 + \sum_{1 \leq i \leq m} r_i]$ for some $a_i > 0$

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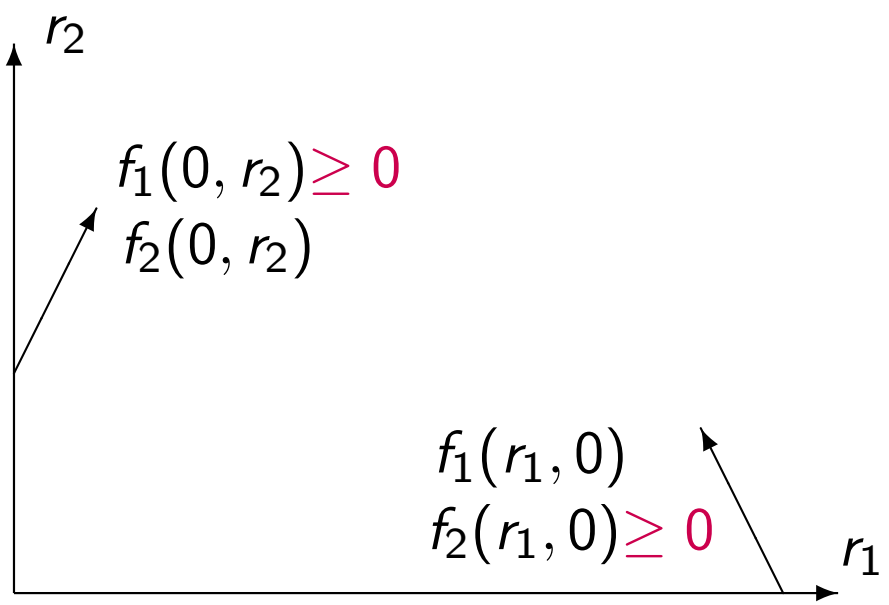
- **(P) Preservation of Positivity** (quasipositivity): $\forall i = 1, ..., m$ $\forall r = (r_1, ..., r_m) \in [0, \infty[^m,\ f_i(r_1, ..., r_{i-1}, 0, r_{i+1}, ..., r_m) \geq 0.$



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  $\forall r \in [0, +\infty[^m, \quad f_i(r_1, ..., r_{i-1}, 0, r_{i+1}, ..., r_m) \geq 0.$
- **(M):** $\sum_{1 \leq i \leq m} f_i(r_1, ..., r_m) \leq 0 \Rightarrow$ **'Control of the Total Mass':**

$$\forall t \geq 0, \quad \int_\Omega \sum_{1 \leq i \leq r} u_i(t, x) dx \leq \int_\Omega \sum_{1 \leq i \leq r} u_i^0(x) dx.$$

Add up, integrate on $\Omega$, use $\int_\Omega \Delta u_i = \int_{\partial\Omega} \partial_\nu u_i = 0$:

$$\int_\Omega \partial_t [\sum u_i(t)] dx = \int_\Omega \sum_i f_i(u) dx \leq 0.$$

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- Remark: same with **(M')**

QUESTION:

*What about Global Existence of solutions*

*under assumption* **(P)+(M)***??*

*or more generally* **(P)+ (M')** *??*

**Remarks:** Global existence holds for the associated ODE.
Global existence holds for the full system if all the $d_i$ are equal
since then, by maximum principle
$\| \sum_i u_i(t) \|_{L^\infty(\Omega)} \leq \| \sum_i u_i(0) \|_{L^\infty(\Omega)}$.

# Explicit examples with property **(P)+(M) or (M')**

**"Chemical morphogenetic process ("Brusselator", R. Lefever-I. Prigogine-G. Nicolis)**

$$\begin{cases} \partial_t u - d_1 \Delta u = -uv^2 + b\,v \\ \partial_t v - d_2 \Delta v = uv^2 - (b+1)\,v + a \\ u_{|\partial\Omega} = b/a, \;\; v_{|\partial\Omega} = a, \\ a, b, d_1, d_2 > 0. \end{cases}$$

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**See also: Glycolosis model–Gray-Scott models**

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- **Exothermic combustion in a gas**

$$\begin{cases} \partial_t Y - \mu \Delta Y = -H(Y,T) \\ \partial_t T - \lambda \Delta T = q\,H(Y,T), \end{cases}$$

$Y = \text{concentration of a reactant}, \ T = \text{temperature},$

# Explicit examples with **(P)+(M')**

- **Lotka-Volterra Systems**

$$\forall i = 1...m, \quad \partial_t u_i - d_i \Delta u_i = e_i u_i + u_i \sum_{1 \leq j \leq m} p_{ij} u_j,$$

with $e_i, p_{ij} \in \boldsymbol{R}$ such that for some $a_i > 0$.

$$\forall w \in [0, \infty)^m, \ \sum_{i,j=1}^{m} a_i w_i p_{ij} w_j \leq 0, \quad [\Rightarrow (\mathbf{M}')].$$

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- **Diffusive epidemic models: SIR**

$S$=Susceptibles= can be infected
$I$=Infectives=infected and transmit disease
$R$=Removed=immune; $P = S + I + R$

$$\begin{cases} S_t - \nabla \cdot d_1(x) \nabla S = bP - (m + kP)S - g(S, I) \\ I_t - \nabla \cdot d_2(x) \nabla I = -(m + kP)I + g(S, I) - \lambda I \\ R_t - \nabla \cdot d_3(x) \nabla R = -(m + kP)R + \lambda I \end{cases}$$

May be coupled with an extra variable: $S = S(t, x, age)$....

# Elementary chemical reactions: a simple example

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- Whence the full system of O.D.E.:
$$\begin{aligned} \tfrac{d}{dt}u_1 &= k^- u_3 - k^+ u_1 u_2 \\ \tfrac{d}{dt}u_2 &= k^- u_3 - k^+ u_1 u_2 \\ \tfrac{d}{dt}u_3 &= -k^- u_3 + k^+ u_1 u_2. \end{aligned}$$

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- $$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -k^+ u_1 u_2 + k^- u_3 \\ \partial_t u_2 - d_2 \Delta u_2 = -k^+ u_1 u_2 + k^- u_3 \\ \partial_t u_3 - d_3 \Delta u_3 = \quad k^+ u_1 u_2 - k^- u_3 \end{cases}$$
  Note :$f_1 + f_2 + 2f_3 = 0$ and positivity is preserved.

# A quadratic model

$$U_1 + U_2 \underset{k^-}{\overset{k^+}{\rightleftharpoons}} U_3 + U_4$$

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -k^+ u_1 u_2 + k^- u_3 u_4 \\ \partial_t u_2 - d_2 \Delta u_2 = -k^+ u_1 u_2 + k^- u_3 u_4 \\ \partial_t u_3 - d_3 \Delta u_3 = k^+ u_1 u_2 - k^- u_3 u_4 \\ \partial_t u_4 - d_4 \Delta u_4 = k^+ u_1 u_2 - k^- u_3 u_4 \end{cases}$$

Note: $f_1 + f_2 + f_3 + f_4 = 0$ and positivity is preserved.

# Superquadratic reaction-diffusion systems.

- A general chemical reaction:

$$p_1 U_1 + p_2 U_2 + ... + p_m U_m \underset{k^-}{\overset{k^+}{\rightleftharpoons}} q_1 U_1 + q_2 U_2 + ... + q_m U_m,$$

$p_i, q_i =$nonnegative integers.

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$$\partial_t u_i - d_i \Delta u_i = (p_i - q_i) \left( k^- \Pi_{j=1}^m u_j^{q_j} - k^+ \Pi_{j=1}^m u_j^{p_j} \right), \forall i = 1...m.$$

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- Here $\sum_i m_i p_i = \sum_i m_i q_i$ for some $m_i \in (0, \infty), i = 1...m$.
  This implies **(M')**: $\sum_{i=1}^m m_i f_i = 0$.

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- Here $\sum_i m_i p_i = \sum_i m_i q_i$ for some $m_i \in (0, \infty), i = 1...m$. This implies **(M')**: $\sum_{i=1}^m m_i f_i = 0$.
- Global existence in general ?

# Models in electromigration (Nernst-Planck)

$$\begin{cases} \partial_t c_i - d_i \, div\,(\nabla c_i + z_i c_i \nabla \Phi) = f_i(c) \text{ in } Q, \\ -\Delta \Phi = \sum_{i=1}^m z_i c_i \text{ in } Q, \\ +\text{initial and boundary conditions.} \end{cases}$$

$c_i = c_i(t,x)$= concentration of ionized species
with charge number $z_i \in \boldsymbol{R}$
$\Phi$ is the electrical potential
The nonlinearity $f_i$ have the same structure (reversible chemical reactions).
see Amann-Renardy, Gajewski-Glitzsky-Gröger-Hünlich, Choi-Lui, Biler-Dolbeault, Hebisch-Nadzieja, Bothe-Fischer-Saal, Bothe-Fischer-P.-Rolland,...

# Models with degenerate diffusion

- **Modelization of pollutants transfer in atmospher ($N = 3$):** W. Fitzgibbon-M. Langlais-J. Morgan,R. Texier-Picard-MP:

$$\begin{cases} \partial_t \phi_i = d_i \, \partial^2_{zz} \phi_i + \omega \cdot \nabla \phi_i + f_i(\phi) + g_i, \ \forall i = 1...20, \\ + \text{Bdy and initial conditions} \end{cases}$$

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- **The reaction terms:**

$$\begin{cases}
\begin{array}{rcl}
f_1(\phi) & = & -k_1\phi_1 + k_{22}\phi_{19} + k_{25}\phi_{20} + k_{11}\phi_{13} + k_9\phi_{11}\phi_2 + k_3\phi_5\phi_2 \\
 & + & k_2\phi_2\phi_4 - k_{23}\phi_1\phi_4 - k_{14}\phi_1\phi_6 + k_{12}\phi_{10}\phi_2 - k_{10}\phi_{11}\phi_1 - k_{24}\phi_{19}\phi_1, \\
f_2(\phi) & = & k_1\phi_1 + k_{21}\phi_{19} - k_9\phi_{11}\phi_2 - k_3\phi_5\phi_2 - k_2\phi_2\phi_4 - k_{12}\phi_{10}\phi_2 \\
f_3(\phi) & = & k_1\phi_1 + k_{17}\phi_4 + k_{19}\phi_{16} + k_{22}\phi_{19} - k_{15}\phi_3 \\
f_4(\phi) & = & -k_{17}\phi_4 + k_{15}\phi_3 - k_{16}\phi_4 - k_2\phi_2\phi_4 - k_{23}\phi_1\phi_4 \\
f_5(\phi) & = & 2k_4\phi_7 + k_7\phi_9 + k_{13}\phi_{14} + k_6\phi_7\phi_6 - k_3\phi_5\phi_2 + k_{20}\phi_{17}\phi_6 \\
f_6(\phi) & = & 2k_{18}\phi_{16} - k_8\phi_9\phi_6 - k_6\phi_7\phi_6 + k_3\phi_5\phi_2 - k_{20}\phi_{17}\phi_6 - k_{14}\phi_1\phi_6 \\
f_7(\phi) & = & -k_4\phi_7 - k_5\phi_7 + k_{13}\phi_{14} - k_6\phi_7\phi_6 \\
f_8(\phi) & = & k_4\phi_7 + k_5\phi_7 + k_7\phi_9 + k_6\phi_7\phi_6 \\
f_9(\phi) & = & -k_7\phi_9 - k_8\phi_9\phi_6 \\
f_{10}(\phi) & = & k_7\phi_9 + k_9\phi_{11}\phi_2 - k_{12}\phi_{10}\phi_2 \\
f_{11}(\phi) & = & k_{11}\phi_{13} - k_9\phi_{11}\phi_2 + k_8\phi_9\phi_6 - k_{10}\phi_{11}\phi_1 \\
f_{12}(\phi) & = & k_9\phi_{11}\phi_2 \\
f_{13}(\phi) & = & -k_{11}\phi_{13} + k_{10}\phi_{11}\phi_1 \\
f_{14}(\phi) & = & -k_{13}\phi_{14} + k_{12}\phi_{10}\phi_2 \\
f_{15}(\phi) & = & k_{14}\phi_1\phi_6 \\
f_{16}(\phi) & = & -k_{19}\phi_{16} - k_{18}\phi_{16} + k_{16}\phi_4 \\
f_{17}(\phi) & = & -k_{20}\phi_{17}\phi_6 \\
f_{18}(\phi) & = & k_{20}\phi_{17}\phi_6 \\
f_{19}(\phi) & = & -k_{21}\phi_{19} - k_{22}\phi_{19} + k_{25}\phi_{20} + k_{23}\phi_1\phi_4 - k_{24}\phi_{19}\phi_1 \\
f_{20}(\phi) & = & -k_{25}\phi_{20} + k_{24}\phi_{19}\phi_1 .
\end{array}
\end{cases}$$

# Back to the model example: what about $L^\infty$-estimates?

- $$(S)\begin{cases} \partial_t u - d_1 \Delta u = -uv^\beta \text{ on } Q_T \\ \partial_t v - d_2 \Delta v = uv^\beta \text{ on } Q_T \\ \partial_\nu u = \partial_\nu v = 0 \text{ on } \Sigma_T, \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0. \end{cases}$$

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  - What happens when $d_1 \neq d_2$?

# A general $L^p$-approach

- $$(S)\begin{cases} \partial_t u - d_1\Delta u = -uv^\beta \ on\ Q_T \\ \partial_t v - d_2\Delta v = uv^\beta \ on\ Q_T \\ \partial_\nu u = \partial_\nu v = 0 \ on\ \Sigma_T, \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0. \end{cases}$$

$$\partial_t v - d_2\Delta v = -[\partial_t u - d_1\Delta u],\ u \in L^\infty(Q_{T^*}).$$

$$FORMALLY : v = -[\partial_t - d_2\Delta]^{-1}(\partial_t - d_1\Delta)\, u\, (= \mathcal{A}u).$$

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- Therefore

$$\|v\|_{L^\infty(Q_{T^*})} < +\infty \ and\ T^* = +\infty.$$

# The proof of the $L^p$-estimate by duality

- $$\partial_t v - d_2 \Delta v \leq -\left[\partial_t u - d_1 \Delta u\right], \quad v \geq 0,$$
  implies the existence of $C = C(p, T, \Omega, u_0, v_0)$ such that:
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- $\Rightarrow \left|\int_{Q_T} v\Theta\right| \leq C\|\Theta\|_{L^{p'}(Q_T)} \ \Rightarrow L^p(Q_T)$-estimate on $v$ by duality.

# Extensions and limits of the $L^p$-approach

- The same approach provides global existence
  - for the "Brusselator", for the epidemic models SIR
  - for the $3 \times 3$ system

$$U_1 + U_2 \quad \underset{k^-}{\overset{k^+}{\rightleftharpoons}} \quad U_3 \ : \begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -k^+ u_1 u_2 + k^- u_3 \\ \partial_t u_2 - d_2 \Delta u_2 = -k^+ u_1 u_2 + k^- u_3 \\ \partial_t u_3 - d_3 \Delta u_3 = \ \ k^+ u_1 u_2 - k^- u_3 \end{cases}$$

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- More generally it applies to $m \times m$ systems if there exists a triangular invertible matrix $Q$ with nonnegative entries such that

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- Can be used for general systems with only **(P)+(M)** when the $d_i$ are close to each other.

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$$\begin{cases} \partial_t u - d_1 \Delta u = -uv^\beta \ \textit{in } Q_T \\ \partial_t v - d_2 \Delta v = \ \ uv^\beta \ \textit{in } Q_T \\ u = 1, \ \partial_\nu v = 0 \ \textit{on } \Sigma_T. \end{cases}$$

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- Extends to Wentzell type boundary conditions, like

$$\begin{cases} \partial_t u_i - d_i \Delta u_i = f_i(u) \ in\ Q_T \\ \sigma \partial_t u_i + d_i \partial_\nu u_i - \delta_i \Delta_{\partial\Omega} u_i = g_i(u) \ on\ \Sigma_T \end{cases}$$

with $\sigma, \delta_i \geq 0$ and "good $g_i$'s. [G. Goldstein, J. Goldstein, M. Meyries, M.P.]

# Extensions and limits of the $L^p$-approach

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$$\begin{cases} \partial_t u - d_1 \Delta u = -uh(v) \text{ in } Q_T \\ \partial_t v - d_2 \Delta v = uh(v) \text{ in } Q_T \end{cases}$$

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- The case $h(v) = e^v$ can be reached for this particular system by using Orlicz spaces, rather than $L^p$. There is also a different method based on the use of a specific Lyapunov function which works with systems with more specific stucture {K. Masuda, J.I. Kanel, A. Haraux, A. Youkana, A. Barabanova, M. Kirane, S. Kouachi, S. Benachour, B. Rebiai,...}

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- Still the system

$$\begin{cases} \partial_t u - d_1 \Delta u = -ue^{v^2} \text{ in } Q_T \\ \partial_t v - d_2 \Delta v = ue^{v^2} \text{ in } Q_T \end{cases}$$

  remains open.

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- and even not to the "better" system with $\lambda \in [0, 1[$

$$\begin{cases} \partial_t u - d_1 \Delta u = \lambda u^3 v^2 - u^2 v^3 \ in \ Q_T, \\ \partial_t v - d_2 \Delta v = u^2 v^3 - u^3 v^2 \ in \ Q_T \end{cases}$$

where $f(u, v) + g(u, v) \leq 0$ and also $f(u, v) + \lambda g(u, v) \leq 0$

# Finite time $L^\infty$-blow up may appear!

- $$\begin{cases} \partial_t u - d_1 \Delta u = f(u,v) \text{ in } Q_T, \\ \partial_t v - d_2 \Delta v = g(u,v) \text{ in } Q_T \end{cases}$$

Theorem: (D. Schmitt, MP) One can find polynomial nonlinearities $f, g$ satisfying **(P)** and

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and for which there exists $T^* < +\infty$ with

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- The blow up is similar to $u(t,x) = \frac{1}{(T^*-t)^2+|x|^2}$ which is solution of $\partial_t u - \Delta u = g(t,x)u^2$ with $g \in L^\infty, N \geq 4$. The solution goes out of $L^\infty(\Omega)$ at $t = T^*$, but still exists for $t > T^*$.$- - - >$ Incomplete blow up !

# Idea of the proof of the "possible blow up" Theorem

- Look for solutions of the form

$$u(t,x) = \frac{a(T^* - t) + b|x|^2}{[T^* - t + |x|^2]^\gamma}, \quad v(t,x) = \frac{c(T^* - t) + d|x|^2}{[T^* - t + |x|^2]^\gamma},$$

Find $a, b, c, d, d_1, d_2 > 0, \gamma > 1, N \geq 1$ so that $u, v$ be solutions of a **(P)+(M)** system.

# Idea of the proof of the "possible blow up" Theorem

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$$u(t,x) = \frac{a(T^* - t) + b|x|^2}{[T^* - t + |x|^2]^\gamma}, \quad v(t,x) = \frac{c(T^* - t) + d|x|^2}{[T^* - t + |x|^2]^\gamma},$$

  Find $a, b, c, d, d_1, d_2 > 0, \gamma > 1, N \geq 1$ so that $u, v$ be solutions of a **(P)+(M)** system.
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  Find $a, b, c, d, d_1, d_2 > 0, \gamma > 1, N \geq 1$ so that $u, v$ be solutions of a **(P)+(M)** system.
- There are examples even in dimension $N = 1$.
- By choosing $N$ large enough, we can obtain blow up with nonlinearities $f(u,v), g(u,v)$ with growth $2 + \epsilon, \epsilon > 0$ as small as we want.

CONCLUSION at this stage:

Look rather for *weak solutions* which are allowed to go out of $L^\infty(\Omega)$ from time to time or even often.

We ask the nonlinearities to be at least in $L^1(Q_T)$.

$$f_i(u) \in L^1(Q_T) \ \ ?$$

# An $L^1$-approach

$$(S)\begin{cases} \forall i = 1, ..., m & \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in } Q_T \\ \partial_\nu u_i = 0 & \text{on } \Sigma_T \\ u_i(0, \cdot) = u_i^0(\cdot) \geq 0. & \end{cases}$$

- **$L^1$-Theorem.** Assume the two conditions **(P)+ (M')** hold. Assume moreover that the following a priori estimate holds:

$$\forall i = 1, ..., m, \int_{Q_T} |f_i(u)| \leq C.$$

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- **Proof:** via supersolutions and truncations techniques !

# Main ingredients in the proof of the $L^1$-theorem

- Truncating the $f_i \to f_i^n, u_i^0 \to (u_i^0)^n \mapsto$ global approximate solutions $u_i^n$ with $\|f_i^n(u^n)\|_{L^1(Q_T)}$ bounded independently of $n$

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- Compactness of the mapping $(g, w_0) \in L^1(Q_T)\times L^1(\Omega) \mapsto w \in L^1(Q_T)$ where

$$\partial_t w - d\Delta w = g \text{ on } Q_T,\ w(0,\cdot) = w_0,\ \partial_\nu w = 0 \text{ on } \partial\Omega.$$

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- We first prove that the limit $u_i$ is a supersolution.
- For this, we use the equation satisfied by $T_k\left(u_i^n + \eta \sum_{j\neq i} u_j^n\right)$ where $T_k(r) = \min\{r, k\}, \eta > 0$.

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- The limit $u_i$ is a supersolution by letting successively: $n \to \infty,\ \eta \to 0,\ k \to +\infty$.
- Main estimate for $\eta \to 0$ : $\int_{[u_i^n \leq k]} |\nabla u_i^n|^2 \leq C\, k$

# End of the proof of the $L^1$-theorem

- Since $u_i$ is a supersolution, we have

$$\partial_t u_i - d_i \Delta u_i = f_i(u) + \mu_i, \;\; 0 \leq \mu_i \; (= \textit{nonnegative measure}).$$

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- $$\sum_i [f_i(u) + \mu_i] \leq \sum_i f_i(u) \Rightarrow \mu_i \equiv 0 \; \forall i.$$

# $L^1$-Theorem applies to many situations

$$\begin{cases} \partial_t u - d_1 \Delta u = -u e^{v^2} \ \textit{in } Q_T \\ \partial_t v - d_2 \Delta v = u e^{v^2} \ \textit{in } Q_T \end{cases}$$

$$\int_\Omega u(T) + \int_{Q_T} u e^{v^2} = \int_\Omega u_0,$$

whence the $L^1(Q_T)$-estimate of the nonlinearity.

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- Open problem if $\lambda = 1$: $L^1$-estimate of the nonlinearity??

# $L^1$-Theorem applies to many situations

- More generally it applies if there exists an invertible matrix $Q$ with nonnegative entries such that

$$\forall r \in [0,\infty)^m,\ Q\, f(r) \leq [1 + \sum_{1 \leq i \leq m} r_i]\mathbf{b},$$

for some $\mathbf{b} \in \boldsymbol{R}^m, f = (f_1, ..., f_m)^t$.
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  (In other words, there are $m$ linearly independent inequalities for the $f_i$'s and not only one).
- Extends partially to electro-diffusion-reaction systems.

# A surprising a priori $L^2$-estimate for these systems

- $$(S)\begin{cases} \forall i = 1, ..., m & \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \textit{in } Q_T \\ \partial_\nu u_i = 0 & \textit{on } \Sigma_T \\ u_i(0, \cdot) = u_i^0(\cdot) \geq 0. & \end{cases}$$

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$$\forall i = 1, ..., m, \ \forall T > 0, \ \int_{Q_T} u_i^2 \leq C\,[1 + \sum_i \int_\Omega (u_i^0)^2]..$$

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- **Corollary of the $L^1$- and $L^2$-Theorems**: Assume **(P),(M')** and $f_i$ is at most quadratic. Then, System (S) has a global weak solution.
- Recall that nonlinearities are quadratic in many examples.

# Application to the quadratic chemical reaction:

- $$U_1 + U_2 \underset{k^-}{\overset{k^+}{\rightleftharpoons}} U_3 + U_4$$

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -k^+ u_1 u_2 + k^- u_3 u_4 \\ \partial_t u_2 - d_2 \Delta u_2 = -k^+ u_1 u_2 + k^- u_3 u_4 \\ \partial_t u_3 - d_3 \Delta u_3 = \phantom{-}k^+ u_1 u_2 - k^- u_3 u_4 \\ \partial_t u_4 - d_4 \Delta u_4 = \phantom{-}k^+ u_1 u_2 - k^- u_3 u_4 \end{cases}$$

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- Global existence of a weak solution
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- Open problem: does the solution blow up in $L^\infty(\Omega)$ in finite time or not??

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  - estimate on the size of the "blow-up set" when $N \geq 3$: Th. Goudon, A. Vasseur
- And also, strong solutions for (rather general) strongly subquadratic systems: J.I. Kanel–M. Caputo, A. Vasseur

# Idea of the proof of the $L^2$-estimate

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- We may even show that the mapping $W_0 \in L^2(\Omega) \to W \in L^2(Q_T)$ is compact where $\partial_t W - \Delta(AW) = 0, W(0) = W_0$.

# A proof of the linear $L^2$-estimate: by duality

Introduce the dual problem

$$\begin{cases} -\partial_t \psi - A\Delta\psi = \Theta \in C_0^\infty(Q_T)^+ \\ \psi(T) = 0,\ \partial_\nu \psi = 0\ on \Sigma_T \end{cases} \tag{1}$$

Then, from $\partial_t W - \Delta(A\,W) \leq 0$, we deduce

$$\int_{Q_T} W\,\Theta = \int_\Omega \psi(0)\,W_0 \leq \|\psi(0)\|_{L^2(\Omega)} \|W_0\|_{L^2(\Omega)}.$$

But multiplying (2) by $-\Delta\psi$ gives

$$\int_{Q_T} \Delta\psi\partial_t\psi + A(\Delta\psi)^2 = -\int_{Q_T} \Theta\Delta\psi$$

$$\int_{Q_T} \Delta\psi\partial_t\psi = -\int_{Q_T} \nabla\psi\partial_t\nabla\psi = -\frac{1}{2}\int_{Q_T} \partial_t|\nabla\psi|^2 = \frac{1}{2}\int_\Omega |\nabla\psi(0)|^2 \geq 0$$

# $L^2$-bound and even $L^2$-compactness !

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We deduce, for various $C = C(\underline{d}, \overline{d}, T)$:

$$\int_{Q_T} (\Delta\psi)^2 \leq C \int_{Q_T} \Theta^2,\ \int_{Q_T} (\partial_t \psi)^2 \leq C \int_{Q_T} \Theta^2,$$

$$\int_\Omega (\psi(0))^2 + \int_\Omega |\nabla\psi(0)|^2 \leq C \int_{Q_T} \Theta^2$$

$$\int_{Q_T} W\,\Theta = \int_\Omega W_0\,\psi(0) \leq C\|W_0\|_{L^2(\Omega)}\|\Theta\|_{L^2(Q_T)}.$$

$$\Rightarrow \|W\|_{L^2(Q_T)} \leq C\|W_0\|_{L^2(\Omega)}.$$

Even better: $W_0 \in L^2(\Omega) \to W \in L^2(Q_T)$ is compact ! since $\Theta \in L^2(Q_T) \to \psi(0) \in L^2(\Omega)$ is compact

# Three extensions of the $L^2$-estimate: (1)

- It extends to nonlinear diffusions of the form

$$\partial_t u_i - \nabla \cdot (d_i(u_i)\nabla u_i) = f_i(u), \ \underline{d} \leq d_i \leq \overline{d}.$$

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# Three extensions of the $L^2$-estimate: (2): $u_0 \in L^1(\Omega))$

- Recall:

$$\partial_t W - \Delta(A\,W) \leq 0 \Rightarrow \|W\|_{L^2(Q_T)} \leq C\|W_0\|_{L^2(\Omega)}$$

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- This allows to solve Systems of type **(P)+(M)** with quadratic reaction terms and with initial data in $L^1(\Omega)$ only.

# Three extensions of the $L^2$-estimate (3): A third one: $L^{2+\epsilon}$

(by J.A. Cañizo, L. Desvillettes, K. Fellner):

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- Allows global weak solutions for reaction terms growing faster than quadratic (growth depending on the dimension)
- Better results on asymptotic behaviors...

# Applications of the $L^2$-compactness to singular limits: **(1)**

- $$U_1 + U_2 \underset{k_1}{\overset{1}{\rightleftharpoons}} C \underset{1}{\overset{k_2}{\rightleftharpoons}} U_3 + U_4$$

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- The $L^p$-approach applies to this system so that global existence of classical solutions holds!

# Case of the O.D.E. system when $k_1 + k_2 \to +\infty$

- $$\begin{cases} \partial_t u_1 = -u_1 u_2 + k_1 c \\ \partial_t u_2 = -u_1 u_2 + k_1 c \\ \partial_t c = u_1 u_2 - (k_1 + k_2) c + u_3 u_4 \\ \partial_t u_3 = -u_3 u_4 + k_2 c \\ \partial_t u_4 = -u_3 u_4 + k_2 c, \end{cases}$$

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- <u>Quasi-steady state approximation:</u>
  "$\partial_t c = 0$" as "$k_1 + k_2 = +\infty$"
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  or
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# Case of the O.D.E. system when $k_1 + k_2 \to +\infty$

The reaction

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'tends' to the limit dynamics

$$U_1 + U_2 \underset{1-\alpha}{\overset{\alpha}{\rightleftharpoons}} U_3 + U_4$$

$+$ convergence of the solutions of the corresponding systems.

Note the boundary layer at $t = 0$: the new initial values are $u_1^0 + \alpha c^0,\ u_2^0 + \alpha c^0,\ u_3^0 + (1-\alpha)c^0,\ u_4^0 + (1-\alpha)c^0$.

# $k_1 + k_2 \to +\infty$ for the full system?

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- $\partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2 + \lim \frac{k_1}{k_1 + k_2}(u_1 u_2 + u_3 u_4)$
  or
  $\partial_t u_1 - d_1 \Delta u_1 = -\alpha u_1 u_2 + (1 - \alpha) u_3 u_4$
  with $\alpha = \lim_{k_1 + k_2 \to +\infty} \frac{k_2}{k_1 + k_2}$.

# $k_1 + k_2 \to +\infty$ for the full system?

- $$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2 + k_1 c \\ \partial_t u_2 - d_2 \Delta u_2 = -u_1 u_2 + k_1 c \\ \partial_t c - d_c \Delta c = u_1 u_2 - (k_1 + k_2) c + u_3 u_4 \\ \partial_t u_3 - d_3 \Delta u_3 = -u_3 u_4 + k_2 c \\ \partial_t u_4 - d_4 \Delta u_4 = -u_3 u_4 + k_2 c, \end{cases}$$
- The limit system may (formally) be obtained:

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -\alpha u_1 u_2 + (1-\alpha) u_3 u_4 \\ \partial_t u_2 - d_2 \Delta u_2 = -\alpha u_1 u_2 + (1-\alpha) u_3 u_4 \\ \partial_t u_3 - d_3 \Delta u_3 = \alpha u_1 u_2 - (1-\alpha) u_3 u_4 \\ \partial_t u_4 - d_4 \Delta u_4 = \alpha u_1 u_2 - (1-\alpha) u_3 u_4, \end{cases}$$

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- Again, formally the chemical reaction
$$U_1 + U_2 \underset{k_1}{\overset{1}{\rightleftharpoons}} C \underset{1}{\overset{k_2}{\rightleftharpoons}} U_3 + U_4$$
"tends" to the limit chemical reaction:
$$U_1 + U_2 \underset{1}{\overset{\alpha}{\rightleftharpoons}} \alpha \; U_3 + U_4$$

# The limit system

- **Theorem.** The solution $(u_1^k, u_2^k, c^k, u_3^k, u_4^k), k = (k_1, k_2)$ of the previous system converges as $k_1 + k_2 \to +\infty$ in $L^2(Q_T)^5$ for all $T > 0$ to $(u_1, u_2, 0, u_3, u_4)$ solution of

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -\alpha u_1 u_2 + \beta u_3 u_4 \\ \partial_t u_2 - d_2 \Delta u_2 = -\alpha u_1 u_2 + \beta u_3 u_4 \\ \partial_t u_3 - d_3 \Delta u_3 = \alpha u_1 u_2 - \beta u_3 u_4 \\ \partial_t u_4 - d_4 \Delta u_4 = \alpha u_1 u_2 - \beta u_3 u_4, \end{cases}$$

where $\alpha = \lim_{k_1+k_2\to\infty} \frac{k_2}{k_1+k_2}, \beta = 1 - \alpha$.

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  where $\alpha = \lim_{k_1+k_2\to\infty} \frac{k_2}{k_1+k_2}, \beta = 1 - \alpha$.
- **Remark:** Boundary layer at $t = 0$: the new initial values are $u_1^0 + \alpha c^0,\ u_2^0 + \alpha c^0,\ u_3^0 + (1-\alpha)c^0,\ u_4^0 + (1-\alpha)c^0$.

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- M. Bisi, F. Conforto, L. Desvillettes—D. Bothe, M.P.

# Steps the proof of the $L^2$-convergence

$$(S_k)\begin{cases} \partial_t u_1^k - d_1\Delta u_1^k = -u_1^k u_2^k + k_1 c^k \\ \partial_t u_2^k - d_2\Delta u_2^k = -u_1^k u_2^k + k_1 c^k \\ \partial_t c^k - d_c\Delta c^k = u_1^k u_2^k - (k_1+k_2)c^k + u_3^k u_4^k \\ \partial_t u_3^k - d_3\Delta u_3^k = -u_3^k u_4^k + k_2 c^k \\ \partial_t u_4^k - d_4\Delta u_4^k = -u_3^k u_4^k + k_2 c^k, \end{cases}$$

- $\partial_t(u_1^k+u_2^k+2c^k+u_3^k+u_4^k)-\Delta(d_1u_1^k+d_2u_2^k+2d_cc^k+d_3u_3^k+d_4u_4^k)=0,$ or, setting $W^k = u_1^k + u_2^k + 2c^k + u_3^k + u_4^k,$

$$\partial_t W^k - \Delta\left(A^k W^k\right) = 0,$$

$$\text{with}: \quad \min d_i \leq A^k \leq 2\max d_i.$$

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- $\partial_t(u_1^k+u_2^k+2c^k+u_3^k+u_4^k)-\Delta(d_1u_1^k+d_2u_2^k+2d_cc^k+d_3u_3^k+d_4u_4^k)=0,$ or, setting $W^k = u_1^k + u_2^k + 2c^k + u_3^k + u_4^k,$

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- This implies that $W^k$ is bounded in $L^2(Q_T)$ (for all $T$),

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- $\partial_t(u_1^k+u_2^k+2c^k+u_3^k+u_4^k)-\Delta(d_1u_1^k+d_2u_2^k+2d_cc^k+d_3u_3^k+d_4u_4^k)=0,$ or, setting $W^k = u_1^k + u_2^k + 2c^k + u_3^k + u_4^k,$

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$$\text{with}: \quad \min d_i \le A^k \le 2\max d_i.$$

- This implies that $W^k$ is bounded in $L^2(Q_T)$ (for all $T$),
- and so are $u_i^k, c^k$.

# Steps of the proof of the strong $L^2$-convergence

$$(S_k)\begin{cases} \partial_t u_1^k - d_1\Delta u_1^k = -u_1^k u_2^k + k_1 c^k \\ \partial_t u_2^k - d_2\Delta u_2^k = -u_1^k u_2^k + k_1 c^k \\ \partial_t c^k - d_c\Delta c^k = u_1^k u_2^k - (k_1+k_2)c^k + u_3^k u_4^k \\ \partial_t u_3^k - d_3\Delta u_3^k = -u_3^k u_4^k + k_2 c^k \\ \partial_t u_4^k - d_4\Delta u_4^k = -u_3^k u_4^k + k_2 c^k, \end{cases}$$

- The nonlinearities $u_1^k u_2^k, u_3^k u_4^k$ are bounded in $L^1(Q_T), \forall T$, thanks to the $L^2$-estimate

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- The nonlinearities $u_1^k u_2^k, u_3^k u_4^k$ are bounded in $L^1(Q_T), \forall T$, thanks to the $L^2$-estimate
- Integrating the equation in $c^k$ gives

$$\int_\Omega c^k(T) + \int_{Q_T} (k_1+k_2)c^k = \int_\Omega c^0 + \int_{Q_T} u_1^k u_2^k + u_3^k u_4^k.$$

# Steps of the proof of the strong $L^2$-convergence

$$(S_k)\begin{cases} \partial_t u_1^k - d_1\Delta u_1^k = -u_1^k u_2^k + k_1 c^k \\ \partial_t u_2^k - d_2\Delta u_2^k = -u_1^k u_2^k + k_1 c^k \\ \partial_t c^k - d_c\Delta c^k = u_1^k u_2^k - (k_1+k_2)c^k + u_3^k u_4^k \\ \partial_t u_3^k - d_3\Delta u_3^k = -u_3^k u_4^k + k_2 c^k \\ \partial_t u_4^k - d_4\Delta u_4^k = -u_3^k u_4^k + k_2 c^k, \end{cases}$$

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- All right-hand sides of the system are bounded in $L^1(Q_T)$: this implies that the sequences $(u_i^k)_k$ are compact in $L^1(Q_T)$ and $c^k \to 0$ in $L^1(Q_T)$...But, this is not enough to pass to the limit !!

# Steps of the proof of the strong $L^2$-convergence

- Recall that, with $W^k = \sum_i u_i^k + 2c^k$,

$$\partial_t W^k - \Delta(A^k W^k) = 0, \quad W^k(0) = W_0$$

where

$$0 < \underline{d} \leq A^k \leq \overline{d} < +\infty.$$

$$W^k \to W := \sum_i u_i \text{ a.e.}$$

# Steps of the proof of the strong $L^2$-convergence

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$$W^k \to W := \sum_i u_i \text{ a.e.}$$

- But, not only this implies the $L^2(Q_T)$-estimate on $W^k$, but it also implies the $L^2(Q_T)$-compactness of $W^k$.
(This is an extension of the previous compactness result to the case when $A^k$ is moving).

# Last steps of the proof of $L^2$-convergence

$$(S_k)\begin{cases} \partial_t u_1^k - d_1\Delta u_1^k = -u_1^k u_2^k + k_1 c^k \\ \partial_t u_2^k - d_2\Delta u_2^k = -u_1^k u_2^k + k_1 c^k \\ \partial_t c^k - d_c\Delta c^k = u_1^k u_2^k - (k_1+k_2)c^k + u_3^k u_4^k \\ \partial_t u_3^k - d_3\Delta u_3^k = -u_3^k u_4^k + k_2 c^k \\ \partial_t u_4^k - d_4\Delta u_4^k = -u_3^k u_4^k + k_2 c^k, \end{cases}$$

- The sequence $W^k = \sum_i u_i^k + 2c^k$ is compact in $L^2(Q_T)$.

# Last steps of the proof of $L^2$-convergence

$$(S_k)\begin{cases} \partial_t u_1^k - d_1 \Delta u_1^k = -u_1^k u_2^k + k_1 c^k \\ \partial_t u_2^k - d_2 \Delta u_2^k = -u_1^k u_2^k + k_1 c^k \\ \partial_t c^k - d_c \Delta c^k = u_1^k u_2^k - (k_1 + k_2) c^k + u_3^k u_4^k \\ \partial_t u_3^k - d_3 \Delta u_3^k = -u_3^k u_4^k + k_2 c^k \\ \partial_t u_4^k - d_4 \Delta u_4^k = -u_3^k u_4^k + k_2 c^k, \end{cases}$$

- The sequence $W^k = \sum_i u_i^k + 2c^k$ is compact in $L^2(Q_T)$.
- Since, for all $i$, $u_i^k \leq W^k$, and, up to a subsequence, $u_i^k$ converges a.e., the $L^2(Q_T)$-compactness of $u_i^k$ follows.

# Last steps of the proof of $L^2$-convergence

$$(S_k)\begin{cases} \partial_t u_1^k - d_1\Delta u_1^k = -u_1^k u_2^k + k_1 c^k \\ \partial_t u_2^k - d_2\Delta u_2^k = -u_1^k u_2^k + k_1 c^k \\ \partial_t c^k - d_c\Delta c^k = u_1^k u_2^k - (k_1+k_2)c^k + u_3^k u_4^k \\ \partial_t u_3^k - d_3\Delta u_3^k = -u_3^k u_4^k + k_2 c^k \\ \partial_t u_4^k - d_4\Delta u_4^k = -u_3^k u_4^k + k_2 c^k, \end{cases}$$

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- $c_k \to 0$ so that $\partial_t c^k - d_c\Delta c^k \to 0$, in the sense of distributions (only).

# Last steps of the proof of $L^2$-convergence

$$(S_k)\begin{cases} \partial_t u_1^k - d_1\Delta u_1^k = -u_1^k u_2^k + k_1 c^k \\ \partial_t u_2^k - d_2\Delta u_2^k = -u_1^k u_2^k + k_1 c^k \\ \partial_t c^k - d_c\Delta c^k = u_1^k u_2^k - (k_1+k_2)c^k + u_3^k u_4^k \\ \partial_t u_3^k - d_3\Delta u_3^k = -u_3^k u_4^k + k_2 c^k \\ \partial_t u_4^k - d_4\Delta u_4^k = -u_3^k u_4^k + k_2 c^k, \end{cases}$$

- The sequence $W^k = \sum_i u_i^k + 2c^k$ is compact in $L^2(Q_T)$.
- Since, for all $i$, $u_i^k \leq W^k$, and, up to a subsequence, $u_i^k$ converges a.e., the $L^2(Q_T)$-compactness of $u_i^k$ follows.
- $c_k \to 0$ so that $\partial_t c^k - d_c\Delta c^k \to 0$, in the sense of distributions (only).
- Same computations as for the O.D.E. to prove convergence toward the expected limit system. **QED**

# Applications of the $L^2$-estimate to singular limits: **(2)**

- (D. Bothe, MP, G. Rolland, '11)

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -k[u_1 u_2 - u_3] \\ \partial_t u_2 - d_2 \Delta u_2 = -k[u_1 u_2 - u_3] \\ \partial_t u_3 - d_3 \Delta u_3 = \phantom{-}k[u_1 u_2 - u_3] \\ U_1 + U_2 \underset{k}{\overset{k}{\rightleftharpoons}} U_3 \end{cases}$$

For fixed $k$: global existence of classical solutions $u^k$.

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- What is the limit kinetics when $k \to +\infty$?
- Estimates independent of $k$:

$$\sup_t \|u_i^k(t)\|_{L^1(\Omega)} \leq C, \ \forall T > 0, \ \|u_i^k\|_{L^2(Q_T)} \leq C.$$

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$$\sup_t \|u_i^k(t)\|_{L^1(\Omega)} \leq C, \ \forall T > 0, \ \|u_i^k\|_{L^2(Q_T)} \leq C.$$

- A main difficulty: what about $k[u_1 u_2 - u_3]$ ?

# Case $d_1 = d_2 = d_3 = d$

$\partial_t(u_1^k + u_2^k + 2u_3^k) - d\Delta(u_1^k + u_2^k + 2u_3^k) = 0$
and by maximum principle

$$\forall i, t,\ \|(u_1^k + u_2^k + 2u_3^k)(t)\|_{L^\infty(\Omega)} \leq \|u_1^0 + u_2^0 + 2u_3^0\|_{L^\infty(\Omega)}.$$

Moreover, it may be proved (D. Bothe) that, as $k \to +\infty$

$$\|k[u_1^k u_2^k - u_3^k]\|_{L^1(Q_T)} \leq C \textit{ independent of } k.$$

Then, it follows that the $u_i^k$ converge, at least in any $L^p(Q_T)$, $p < +\infty$, to the unique regular nonnegative solution of

$$\begin{cases} \left.\begin{array}{c} \partial_t(u_1 + u_3) - d\Delta(u_1 + u_3) = 0 \\ \partial_t(u_2 + u_3) - d\Delta(u_2 + u_3) = 0 \end{array}\right\} + \textit{ boundary cond.} \\ (u_1 + u_3)(0) = u_1^0 + u_3^0,\ (u_2 + u_3)(0) = u_2^0 + u_3^0, \\ u_1 u_2 = u_3. \end{cases}$$

# Case of different diffusions $d_1 \neq d_2 \neq d_3$

$$\begin{cases} \partial_t u_1^k - d_1 \Delta u_1^k = -k[u_1^k u_2^k - u_3^k] \\ \partial_t u_2^k - d_2 \Delta u_2^k = -k[u_1^k u_2^k - u_3^k] \\ \partial_t u_3^k - d_3 \Delta u_3^k = k[u_1^k u_2^k - u_3^k] \end{cases}$$

- A main difficulty: no a priori $L^1(Q_T)$-estimate on $k(u_1^k u_2^k - u_3^k)$ seems to be true !

# Case of different diffusions $d_1 \neq d_2 \neq d_3$

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- On the other hand, for $i = 1, 2$, we have

$$\begin{cases} \partial_t (u_i^k + u_3^k) - \Delta \left[ A_1^k (u_i^k + u_3^k) \right] = 0 \\ 0 < \min\{d_i, d_3\} \leq A_i^k := \frac{u_i^k + u_3^k}{d_i u_i^k + d_3 u_3^k} \leq \max\{d_i, d_3\} < +\infty. \end{cases}$$

# Case of different diffusions $d_1 \neq d_2 \neq d_3$

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- It follows that $u_i^k + u_3^k$ are bounded in $L^2(Q_T)$ for $i = 1, 2$.

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- It follows that $u_i^k + u_3^k$ are bounded in $L^2(Q_T)$ for $i = 1, 2$.
- If we knew that they converge pointwise, then we would deduce that they are compact in $L^2(Q_T)$ (previous result above).
- Even not enough to conclude! Need to know that, separately, the $u_i^k$ are compact in $L^2(Q_T)$. Convergence a.e. of each of them would be enough (by dominated convergence).
- The missing information will be given by the entropy inequality

# The entropy inequality (we drop the $k$)

- $$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -k[u_1 u_2 - u_3] \\ \partial_t u_2 - d_2 \Delta u_2 = -k[u_1 u_2 - u_3] \\ \partial_t u_3 - d_3 \Delta u_3 = k[u_1 u_2 - u_3] \end{cases}$$

  We set $\theta_i = u_i \log u_i - u_i$ and write the equation in $\theta_i$

- $$\partial_t \theta_i = \log u_i \, \partial_t u_i \; ; \; -\Delta \theta_i + \frac{|\nabla u_i|^2}{u_i} = -\log u_i \; \Delta u_i,$$

  $$\partial_t \theta_1 - d_1 \Delta \theta_1 + \frac{d_1 |\nabla u_1|^2}{u_1} = -k[u_1 u_2 - u_3] \log u_1,$$

- $$\sum_i (\partial_t - d_i \Delta)\, \theta_i + \frac{d_i |\nabla u_i|^2}{u_i} = -k[u_1 u_2 - u_3][\log(u_1 u_2) - \log u_3] \leq 0.$$

- Integrating leads to the bound

  $$\int_{Q_T} \sum_i \frac{d_i |\nabla u_i|^2}{u_i} + k[u_1 u_2 - u_3][\log \frac{u_1 u_2}{u_3}] \leq C \; (\textit{independent of } k) \; .$$

# Passing to the limit as $k \to \infty$

- Recall the estimates

$$\sup_t \|u_i(t)\|_{L^1(\Omega)} \leq C, \ \forall T > 0, \|u_i\|_{L^2(Q_T)} \leq C.$$

$$\int_{Q_T} \sum_i \frac{d_i |\nabla u_i|^2}{u_i} + k[u_1 u_2 - u_3][\log \frac{u_1 u_2}{u_3}] \leq C.$$

The last implies that each $\nabla \sqrt{u_i}$ is bounded in $L^2(Q_T)$.

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- Next, we use for $i = 1, 2$ the identity

$$\partial_t (u_i + u_3) - \Delta(d_i u_i + d_3 u_3) = 0$$

  to show that $\partial_t \sqrt{u_i + u_3} \in L^2\left(0, T; H^{-1}(\Omega)\right) + L^1(Q_T)$
  By Aubin-Simon type of compactness, we deduce that $u_i + u_3$ is compact in $L^1(Q_T)$ and therefore converges a.e. ...which implies they converge in $L^2(Q_T)$ thanks to our previous analysis.

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- We use the pointwise entropy inequality to prove that all three $u_i$ converge a.e.. Whence their convergence in $L^2(Q_T)$.

# A general convergence result

(D. Bothe, M.P., G. Rolland)

$$\begin{cases} \partial_t u_1^k - d_1 \Delta u_1^k = -k[u_1^k u_2^k - u_3^k] \\ \partial_t u_2^k - d_2 \Delta u_2^k = -k[u_1^k u_2^k - u_3^k] \\ \partial_t u_3^k - d_3 \Delta u_3^k = k[u_1^k u_2^k - u_3^k] \end{cases}$$

**Theorem.** Up to a subsequence, the $u_i^k$ converge in $L^2(Q_T), \forall T > 0$ to a weak nonnegative solution of

$$(Lim) \begin{cases} \left.\begin{array}{c} \partial_t(u_1 + u_3) - \Delta(d_1 u_1 + d_3 u_3) = 0 \\ \partial_t(u_2 + u_3) - \Delta(d_2 u_2 + d_3 u_3) = 0 \end{array}\right\} + \textit{ boundary cond.} \\ u_1 u_2 = u_3. \\ (u_1 + u_3)(0) = u_1^0 + u_3^0, \ (u_2 + u_3)(0) = u_2^0 + u_3^0, \end{cases}$$

# About the problem (Lim)

$$(Lim)\begin{cases} \left.\begin{array}{c}\partial_t(u_1+u_3)-\Delta(d_1u_1+d_3u_3)=0\\ \partial_t(u_2+u_3)-\Delta(d_2u_2+d_3u_3)=0\end{array}\right\}+\ \textit{boundary cond.}\\ u_1u_2=u_3.\\ (u_1+u_3)(0)=u_1^0+u_3^0,\ (u_2+u_3)(0)=u_2^0+u_3^0,\end{cases}$$

If we set, $w_1 := u_1+u_3, w_2 = u_2+u_3$, then it is equivalent to the $2\times 2$ cross-diffusion system

$$(Lim')\begin{cases} \left.\begin{array}{c}\partial_t w_1-\Delta\psi_1(w_1,w_2)=0\\ \partial_t w_2-\Delta\psi_2(w_1,w_2)=0\end{array}\right\}+\ \textit{boundary cond.}\\ w_1(0)=u_1^0+u_3^0,\ w_2(0)=u_2^0+u_3^0,\end{cases}$$

where $\psi=(\psi_1,\psi_2):[0,\infty[^2\to \boldsymbol{R}^2$ is $C^\infty$ and the Jacobian matrix $D\psi(w_1,w_2)$ satisfies the spectral conditions for this problem to have unique local classical solution (see H. Amann's theory).

# Open problems

*As a by-product of the existence of the limit on* $[0, \infty)$ *of the* $k$*-systems, we obtain existence of a global weak solution, but*

(1) Does it coincide with the (a priori local) classical solution?

*We can prove uniqueness of global weak solutions for some range of the diffusions* $[(d_1 - d_3)^2(d_2 - d_3)^2 < 16 d_1 d_2 d_3^2]$*. In this case, the answer is yes, but*

(2) It may a priori happen that the strong solution becomes (only) weak after some time.

(3) Does one have uniqueness of weak solutions for all values of the $d_i$'s?

# Applications of the $L^2$-compactness to some "relaxed" cross-diffusion systems: **(3)**

Classical conservative cross-diffusion systems may be written

$$\begin{cases} \partial_t u_i - \Delta[a_i(u)u_i] = 0, \ i = 1, ..., m \\ \partial_\nu(a_i(u)u_i) = 0, \ u_i(0) = u_i^0 \geq 0, \end{cases}$$

where, for instance,

$$a_i(u) = d_i + \sum_j d_{ij} u_j^p$$

[N. Shigesada, K. Kawasaki and E. Teramoto]. Local existence of strong solutions by Amann's theory, but not much about global existence except for $p = 1$ (see results and survey by A. Jüngel).

Interaction between species through motion, not through reaction
$\rightarrow\rightarrow$ Formation of "patterns like in Turing's instabilities"

# Applications of the $L^2$-compactness to some "relaxed" cross-diffusion systems: **(3)**

- Existence of solutions to the cross-diffusion system where $a_i : (0,\infty)^m \to [\underline{d},\infty)$ continuous (only), $\underline{d} > 0$:

$$\begin{cases} \partial_t u_i - \Delta[a_i(\tilde{u})u_i] = 0, \ i = 1, ..., I \\ \tilde{u}_i - \delta_i \Delta \tilde{u}_i = u_i, \qquad \delta_i > 0 \ , \\ \partial_\nu u_i = \partial_\nu \tilde{u}_i = 0, \ u_i(0) = u_i^0 \geq 0. \end{cases}$$

Model proposed by M. Bendahmane, Th. Lepoutre, A. Marrocco, B. Perthame (partial results in dimensions N=1,2).

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- This relaxed version takes into account that the intensity of the underlying brownian depends on the density of the whole population in a neighborhood of size $\delta_i$ of each point.

# A general global existence result

**THEOREM.** (Th. Lepoutre, MP, G. Rolland, '11 ): Existence of global solutions satisfying for all $T > 0, p < \infty$

$$u_i \in L^p(Q_T), \tilde{u}_i \in C^\alpha(Q_T) \cap L^p\left(0, T; W^{2,p}(Q_T)\right),$$

$$u_i(t) - \Delta\left[\int_0^t a_i(\tilde{u})u_i\right] = u_i^0.$$

$$\tilde{u}_i - \delta_i \Delta \tilde{u}_i = u_i$$

If, moreover, $a_i$ is locally Lipschitz continuous, the solution is classical, unique and

$$u_i \in L^\infty(Q_T), \partial_t u_i, \Delta(a_i(\tilde{u})u_i) \in L^p_{loc}\left((0, T]; L^p(\Omega)\right).$$

$$\partial_t u_i - \Delta\left(a_i(\tilde{u}_i)u_i\right) = 0.$$

# Step 1 of the proof: $L^2$-estimate

- We first truncate the nonlinearities $a_i(\cdot)$ and prove existence of a fixed point for the mapping

$$\mathcal{T} : v = (v_i)_{1 \le i \le m} \to u = (u_i)_{1 \le i \le m} \in X == \Pi_{i=1}^m X_i,$$

$$u_i \text{ weak solution of } \partial_t u_i - \Delta\left(a_i(\tilde{v}) u_i\right) = 0, \;\; u_i(0) = u_i^0$$

$$X_i = \{v_i \in L^2(Q_T); \partial_t \tilde{v}_i \in L^2(Q_T), \tilde{v}_i = (I - \delta_i \Delta)^{-1} v_i\}$$

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- First, we can solve **in** $L^2(Q_T)$ -with estimates- the linear problem

$$u_i(t) - \Delta \int_0^t A_i\, u_i = u_i^0,\ \partial_\nu u_i = 0,\ (*)$$

where $A_i \in L^\infty(Q_T), 0 < \underline{a} \le A_i \le \overline{a} < \infty$. Here $A_i := a_i(\tilde{v})$.

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- Next, the $L^2$ compactness together with the choice of $X_i$ implies that $\mathcal{T}$ is compact. Coupled with **uniqueness** of the weak solutions of (*), it follows that $\mathcal{T}$ is continuous.

# Step 2 of the proof: $\tilde{u} \in L^\infty$ !

- $$u_i(t) - \Delta \int_0^t a_i(\tilde{u}) u_i = u_i^0, \ \tilde{u}_i(t) - \delta_i \Delta \tilde{u}_i(t) = u_i(t)$$

  may be rewritten

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- Since $\tilde{u}_i \geq 0$, and thanks to Neumann bdy conditions:

$$\|\delta_i \tilde{u}_i + \int_0^t a_i(\tilde{u}) u_i\|_{L^\infty(\Omega)} \leq C \left[ \|u_i^0\|_{L^\infty(\Omega)} + \int_\Omega \left\{ \delta_i \tilde{u}_i + \int_0^t a_i(\tilde{u}) u_i \right\} \right].$$

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- We may bound $\int_{Q_T} a_i(\tilde{u}) u_i$ independently of the upper bound of $a_i$ (main point ! )
- It follows $\|\tilde{u}_i\|_{L^\infty(Q_T)} \leq C$. Thus, we get rid of the truncation of $a_i$.

# Step 3: Use of Krylov-Safonov estimates

- We apply the $C^\alpha$ estimates of Krylov-Safonov to $U_i = \int_0^t a_i(\tilde{u}) u_i$ which satisfies

$$\partial_t U_i - a_i(\tilde{u}) \Delta U_i = a_i(\tilde{u})\, u_i^0 \in L^\infty(Q_T),$$

where now $\underline{a} \leq a_i(\tilde{u}) \leq \overline{a}(T) < +\infty$.

$$\Rightarrow \|U_i\|_{C^\alpha(Q_T)} \leq C \ \text{ for some } \alpha \in (0,1).$$

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$$\partial_t w_i - \delta_i \Delta(\partial_t w_i) = a_i(\tilde{u}) u_i \leq C(T) u_i$$

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- $\Rightarrow$ $w_i$ is Lipschitz-continuous

$$\Rightarrow \|\delta_i \tilde{u}_i\|_{C^\alpha(Q_T)} \leq C.$$

# Step 4: Use the maximal $L^p$-regularity theory

- Recall that for $U_i(t) = \int_0^t a_i(\tilde{u}) u_i$

$$\partial_t U_i - a_i(\tilde{u}) \Delta U_i = a_i(\tilde{u})\, u_i^0 \in L^\infty(Q_T),$$

Now, we know that $a_i(\tilde{u})$ is continuous on $\overline{Q}_T$ and bounded from below. Therefore, we have $L^p$-maximal regularity. In particular,

$$\partial_t U_i = a_i(\tilde{u}) u_i \in L^p(Q_T) \text{ for all } p < +\infty.$$

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- And we get more if $a_i$ is locally Lipschitz :

$$\partial_t u_i, \Delta\left(a_i(\tilde{u}) u_i\right) \in L^p_{loc}((0, T]; L^p(\Omega)),\ \forall p < \infty.$$

# Again the $L^2$-approach for uniqueness !

Let $u, v$ be two solutions, $a_i = a_i(\tilde{u}), b_i = a_i(\tilde{v})$.

$$\partial_t(u_i - v_i) - \Delta\left[a_i(u_i - v_i) + v_i(a_i - b_i)\right] = 0.$$

This may be rewritten with $U_i = u_i - v_i, \tilde{U} = \tilde{u} - \tilde{v}$

$$\partial_t U_i - \Delta\left[a_i U_i + v_i A_i \cdot \tilde{U}\right] = 0,\ i = 1, ..., m,$$

$$A_i = \int_0^1 Da_i(t\tilde{u} + (1-t)\tilde{v})dt \in L^\infty(Q_T).$$

Proving $U \equiv 0$ is equivalent to solving the dual problem for any $F \in C_0^\infty(Q_T)^m$ (here $B_{ij} = v_j A_{ji}$):

$$\begin{cases} \psi_i, \partial_t\psi_i, \Delta\psi_i \in L^2(Q_T) \\ \partial_t\psi_i + a_i\Delta\psi_i + (I - \delta_i\Delta)^{-1}(B_i \cdot \Delta\psi) = F_i \\ \psi = (\psi_1, ..., \psi_m),\ \partial_\nu\psi_i = 0,\ \psi_i(T) = 0. \end{cases} \tag{3}$$