From a microscopic model to a macroscopic model with cross-diffusion in Population Dynamics

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Outline

Classical models

$u := u(t) \geq 0$ (and $v := v(t) \geq 0$) : densities of species at time $t \geq 0$.
$u_0 \geq 0$ (and v_0) : initial datum. Evolution of u (and v) ?

Logistic equation ($\sim$1840)

$$d_t u = ru(1-u).$$

- Competition for the ressources.

Lotka-Volterra system (1925)

$$\begin{cases} d_t u = u\,(r_1 - r_3 v)\,, \\ d_t v = -v\,(r_2 - r_4 u)\,. \end{cases}$$

- Predator-prey interaction.

Spatial models

$u := u(t,x) \geq 0$: density of species at time $t \geq 0$, space $x \in \Omega \subset \mathbb{R}^N$,
$u_0 := u_0(x) \geq 0$: initial datum.

Fisher-KPP equation (1938)

$$\partial_t u - D\Delta_x u = ru(1-u).$$

The diffusion term $D\Delta_x u$ (with $D > 0$) :

- has a homogenisation effect,
- can be seen as the "limit" of a random walk.

Cross diffusions

$u := u(t,x) \geq 0$ (resp. $v := v(t,x) \geq 0$) : density of species 1 (resp. 2) at time $t \geq 0$ and space $x \in \Omega$.

Triangular Shigesada-Teramoto-Kawasaki system (1979)

$$\begin{cases} \partial_t u - \Delta_x \left[Du + \mathbf{uv} \right] = u[1-u-v], \\ \partial_t v - \Delta_x v = v[1-v-u]. \end{cases}$$

Interactions between the two species :

- intraspecific and interspecific competitions,
- stress induced by the presence of species 2 :

$$\Delta_x [uv] = \underbrace{\nabla_x \cdot [u \nabla_x v]}_{\text{transport}} + \underbrace{\nabla_x \cdot [v \nabla_x u]}_{\text{Fickian diffusion}} .$$

Microscopic model

Iida-Mimura-Ninomiya system (2006)

$$\begin{cases} \partial_t u_A^\varepsilon - d_A\,\Delta_x u_A^\varepsilon = [1-(u_A^\varepsilon+u_B^\varepsilon)-v^\varepsilon]\,u_A^\varepsilon + \dfrac{1}{\varepsilon}\,[k(v^\varepsilon)\,u_B^\varepsilon - h(v^\varepsilon)\,u_A^\varepsilon], \\ \partial_t u_B^\varepsilon - d_B\,\Delta_x u_B^\varepsilon = [1-(u_A^\varepsilon+u_B^\varepsilon)-v^\varepsilon]\,u_B^\varepsilon - \dfrac{1}{\varepsilon}\,[k(v^\varepsilon)\,u_B^\varepsilon - h(v^\varepsilon)\,u_A^\varepsilon], \\ \partial_t v^\varepsilon - \Delta_x v^\varepsilon = [1-v^\varepsilon-(u_A^\varepsilon+u_B^\varepsilon)]\,v^\varepsilon, \end{cases}$$

- the species 1 exists in a quiet state A and a stressed state B ($d_B > d_A$),
- the stress is induced by the presence of the species 2,
- the rate of switch is of order $1/\varepsilon \gg 1$.

Microscopic model : acceleration of the switch

Equations for the densities of species

$$\begin{cases} \partial_t(u_A^\varepsilon + u_B^\varepsilon) - \Delta_x \left[(d_A \dfrac{u_A^\varepsilon}{u_A^\varepsilon + u_B^\varepsilon} + d_B \dfrac{u_B^\varepsilon}{u_A^\varepsilon + u_B^\varepsilon})(u_A^\varepsilon + u_B^\varepsilon) \right] \\ \qquad = \left[1 - (u_A^\varepsilon + u_B^\varepsilon) - v^\varepsilon\right](u_A^\varepsilon + u_B^\varepsilon), \\ \partial_t v^\varepsilon - \Delta_x v^\varepsilon = \left[1 - v^\varepsilon - (u_A^\varepsilon + u_B^\varepsilon)\right] v^\varepsilon. \end{cases}$$

Computation of the formal limit

If $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon) \to (u_A, u_B, v)$ (in a strong sense) when $\varepsilon \to 0$ then $h(v)u_A = k(v)u_B$, i. e. $\frac{u_A}{u_A + u_B} = \frac{k(v)}{h(v)+k(v)}$ and $\frac{u_B}{u_A + u_B} = \frac{h(v)}{h(v)+k(v)}$.

Microscopic model : acceleration of the switch

Equations for the densities of species at $\varepsilon = 0$

$$\begin{cases} \partial_t(u_A+u_B) - \Delta_x \left[\left(d_A \dfrac{k(v)}{h(v)+k(v)} + d_B \dfrac{h(v)}{h(v)+k(v)}\right)(u_A+u_B) \right] \\ \qquad\qquad = \left[1-(u_A+u_B)-v\right](u_A+u_B), \\ \partial_t v - \Delta_x v = \left[1-v-(u_A+u_B)\right] v. \end{cases}$$

With accurate choices of the functions h and k, the densities (u_A+u_B, v) satisfy the triangular Shigesada-Teramoto-Kawasaki system.

A more general cross-diffusion system

$$\left.\begin{aligned}\partial_t u - \Delta_x \left[Du + uG(v)\right] = u[1 - u^a - v^b] \qquad & \text{in } \mathbb{R}_+ \times \Omega,\\ \partial_t v - \Delta_x v = v[1 - v^c - u^d] \qquad & \text{in } \mathbb{R}_+ \times \Omega,\\ \nabla_x u(t,x) \cdot n(x) = \nabla_x v^\varepsilon(t,x) \cdot n(x) = 0 \qquad & \forall t \geq 0, x \in \partial\Omega,\\ u(0,x) = u_{in}(x), \qquad v(0,x) = v_{in}(x) \qquad & \forall x \in \Omega.\end{aligned}\right\} \tag{1}$$

Microscopic model

$$\left.\begin{aligned}
\partial_t u_A^\varepsilon - d_A\,\Delta_x u_A^\varepsilon &= [1-(u_A^\varepsilon+u_B^\varepsilon)^a-(v^\varepsilon)^b]\,u_A^\varepsilon+\frac{1}{\varepsilon}\,[k(v^\varepsilon)\,u_B^\varepsilon-h(v^\varepsilon)\,u_A^\varepsilon],\\
\partial_t u_B^\varepsilon - d_B\,\Delta_x u_B^\varepsilon &= [1-(u_A^\varepsilon+u_B^\varepsilon)^a-(v^\varepsilon)^b]\,u_B^\varepsilon-\frac{1}{\varepsilon}\,[k(v^\varepsilon)\,u_B^\varepsilon-h(v^\varepsilon)\,u_A^\varepsilon],\\
\partial_t v^\varepsilon - \Delta_x v^\varepsilon &= [1-(v^\varepsilon)^c-(u_A^\varepsilon+u_B^\varepsilon)^d]\,v^\varepsilon,\\
\nabla_x u_A(t,x)\cdot n(x) = \nabla_x u_B^\varepsilon(t,x)\cdot n(x) &= 0 \qquad \forall t\geq 0, x\in\partial\Omega,\\
\nabla_x v^\varepsilon(t,x)\cdot n(x) &= 0 \qquad \forall t\geq 0, x\in\partial\Omega,\\
u_A(0,x)=u_{A,in}(x),\quad u_B(0,x)=u_{B,in}(x)\quad v(0,x)&=v_{in}(x) \qquad \forall x\in\Omega.
\end{aligned}\right\} \tag{2}$$

Known results

Existence of nonnegative solutions $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon)$ of (2) is classical.

Results on the asymptotics

- for *a priori* uniformly bounded solutions [Iida Mimura Ninomiya 06],
- for stationnary solutions [Izuhara Mimura, 08],
- in dimension 1 [Conforto Desvillettes, 09],
- when the reaction term is Lipschitz continuous [Murakawa 12].

Main theorem : assumptions

Assumption A

- Ω is a smooth bounded domain of $\mathbb{R}^N$,
- $d_B > d_A > 0$, $a, b, c, d > 0$,
- h, k lie in $C^1(\mathbb{R}_+, \mathbb{R}_+)$ and are lower bounded by a positive constant,
- $u_{A,in}, u_{B,in}, v_{in} \geq 0$ such that $u_{A,in}, u_{B,in} \in L^{p_0}(\Omega)$, $v_{in} \in L^\infty(\Omega) \cap W^{2,1+p_0/d}(\Omega)$ for some $p_0 > 1$, and $\nabla_x u_{A,in} \cdot n(x) = \nabla_x u_{B,in} \cdot n(x) = \nabla_x v_{in} \cdot n(x) = 0$,
- $a > d$ or $(a \leq 1$ and $d \leq 2)$.

Theorem

Theorem (Desvillettes, T.)

Under Assumption A, When $\varepsilon \to 0$, $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon)$ *converges (up to a subsequence) for almost every* $(t,x) \in \mathbb{R}_+ \times \Omega$ *to a limit* (u_A, u_B, v) *lying in* $L^{q_0}([0,T]\times\Omega) \times L^{q_0}([0,T]\times\Omega) \times L^\infty([0,T]\times\Omega)$ *for all* $T > 0$. *Furthermore,*

$$h(v)\, u_A = k(v)\, u_B$$

and $(u := u_A + u_B, v)$ *is a weak solution of system* (1) *with*

$$D + G(v) = \frac{d_A k(v) + d_B h(v)}{h(v) + k(v)}$$

and initial data $u(0,\cdot) = u_{A,in} + u_{B,in}$, $v(0,\cdot) = v_{in}$.

Sketch of the proof

We fix $T > 0$ and consider a smooth nonnegative solution $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon)$.

- Estimates uniformly in ε,
- Convergence of the densities (compactness : Aubin's lemma),
- Vanishing of $h(v)u_A - k(v)u_B$.

Tool 1 : solve the equation of v^ε first

$$\partial_t v^\varepsilon - \Delta_x v^\varepsilon = [1 - (v^\varepsilon)^c - (u_A^\varepsilon + u_B^\varepsilon)^d]\, v^\varepsilon.$$

- Maximum principle : $0 \le v^\varepsilon \le C_T$.
- Properties of the heat kernel : for all $p > 1$,

$\|\partial_t v^\varepsilon\|_{L^p} + \|\nabla_x^2 v^\varepsilon\|_{L^p} \le C_T(1 + \|(u_A^\varepsilon + u_B^\varepsilon)^d\|_{L^p})$.

Tool 2 : Duality lemma (case $a \leq 1$, $d \leq 2$)

Lemma ([Pierre Schmitt, 2000])

If $0 < m_0 \leq M(t,x) \leq m_1$ *and* $u_{in} \in L^2(\Omega)$ *then any solution* $u \geq 0$ *of*

$$\begin{cases} \partial_t u - \Delta_x(Mu) \leq K \ \textit{in } [0,T] \times \Omega, \\ u(0,x) = u_{in}(x) \ \textit{in } \Omega, \\ \nabla_x(Mu)(t,x) \cdot n(x) = 0 \ \textit{on } [0,T] \times \partial\Omega, \end{cases}$$

satisfies

$$\|u\|_{L^2([0,T]\times\Omega)} \leq C_T \left(\|u_{in}\|_{L^2(\Omega)} + K\right).$$

- The total density of species 1 satisfies uniformly in ε :

$$\|u_A^\varepsilon + u_B^\varepsilon\|_{L^2} \leq C_T.$$

Tool 3 : Entropy

For any $p > 1$, let

$$\mathscr{E}^\varepsilon(t) := \int_\Omega h(v^\varepsilon)^{p-1} \frac{(u_A^\varepsilon)^p}{p}(t) + \int_\Omega k(v^\varepsilon)^{p-1} \frac{(u_B^\varepsilon)^p}{p}(t)$$

- This functional does *not* increase *too much*,
- the terms in $O(\frac{1}{\varepsilon})$ have a (good) sign,
- consequences : estimates for $u_A^\varepsilon, u_B^\varepsilon$ in Sobolev spaces (uniformly in ε)

$+$ estimates for $k(v^\varepsilon)\, u_B^\varepsilon - h(v^\varepsilon)\, u_A^\varepsilon$,

Known results

Existence of local (in time) solution [Amann, 90]. Global solutions ?

Quadratic case $a = b = c = d = 1$

- in small dimension ($N = 2$: [Lou Ni Wu, 98]),
- when $G(v) = d_G v$ with $d_G > 0$ small [Choi Lui Yamada, 03],
- in presence of self-diffusion [Choi Lui Yamada, 04].

Case $a > d$

Global strong solutions (for smooth initial data) [Yamada, 95].

Main theorem : assumptions

Assumption B

- Ω is a smooth bounded domain of $\mathbb{R}^N$,
- $D > 0$, $a, b, c, d > 0$,
- $G := G(v) \geq 0$ is C^1 on $\mathbb{R}_+$,
- $u_{in} \geq 0$, $v_{in} \geq 0$ such that $u_{in} \in L^{p_0}(\Omega)$, $v_{in} \in L^\infty(\Omega) \cap W^{2,1+p_0/d}(\Omega)$ for some $p_0 > 1$, and $\nabla_x u_{in} \cdot n(x) = \nabla_x v_{in} \cdot n(x) = 0$,
- $(a > d)$ or $(a \leq 1$ and $d \leq 2)$.

Main theorem

Theorem

Under assumption B, there exists a global weak nonnegative solution (u, v) of (1) lying in $L^{q_0}([0,T]\times\Omega)\times L^{\infty}([0,T]\times\Omega)$ for all $T>0$.

Main theorem

Theorem

Under assumption B, there exists a global weak nonnegative solution (u, v) of (1) lying in $L^{q_0}([0,T]\times\Omega)\times L^\infty([0,T]\times\Omega)$ for all $T > 0$.

Proof

The function G and parameter D are given. It suffices to define d_B, d_A, h and k such that

$$D + G(v) = \frac{d_A k(v) + d_B h(v)}{h(v) + k(v)}.$$

Results

Main results

- Convergence of the solutions of the approximating system,
- Corollary : existence of weak solutions for the cross-diffusion system.

Other results in the case $a < d$

- Classical solutions for the cross-diffusion system,
- Stability in L^2, and then uniqueness.

Perspectives

- Non-triangular case (with Desvillettes, Lepoutre and Moussa),
- Self diffusions.