9/9/2014

ASYMPTOTIC BEHAVIOR OF MICROBIAL DEGRADATION DYNAMICS IN SOIL

NGUYEN NGOC DOANH
JOINT WORK WITH CUNG THE ANH

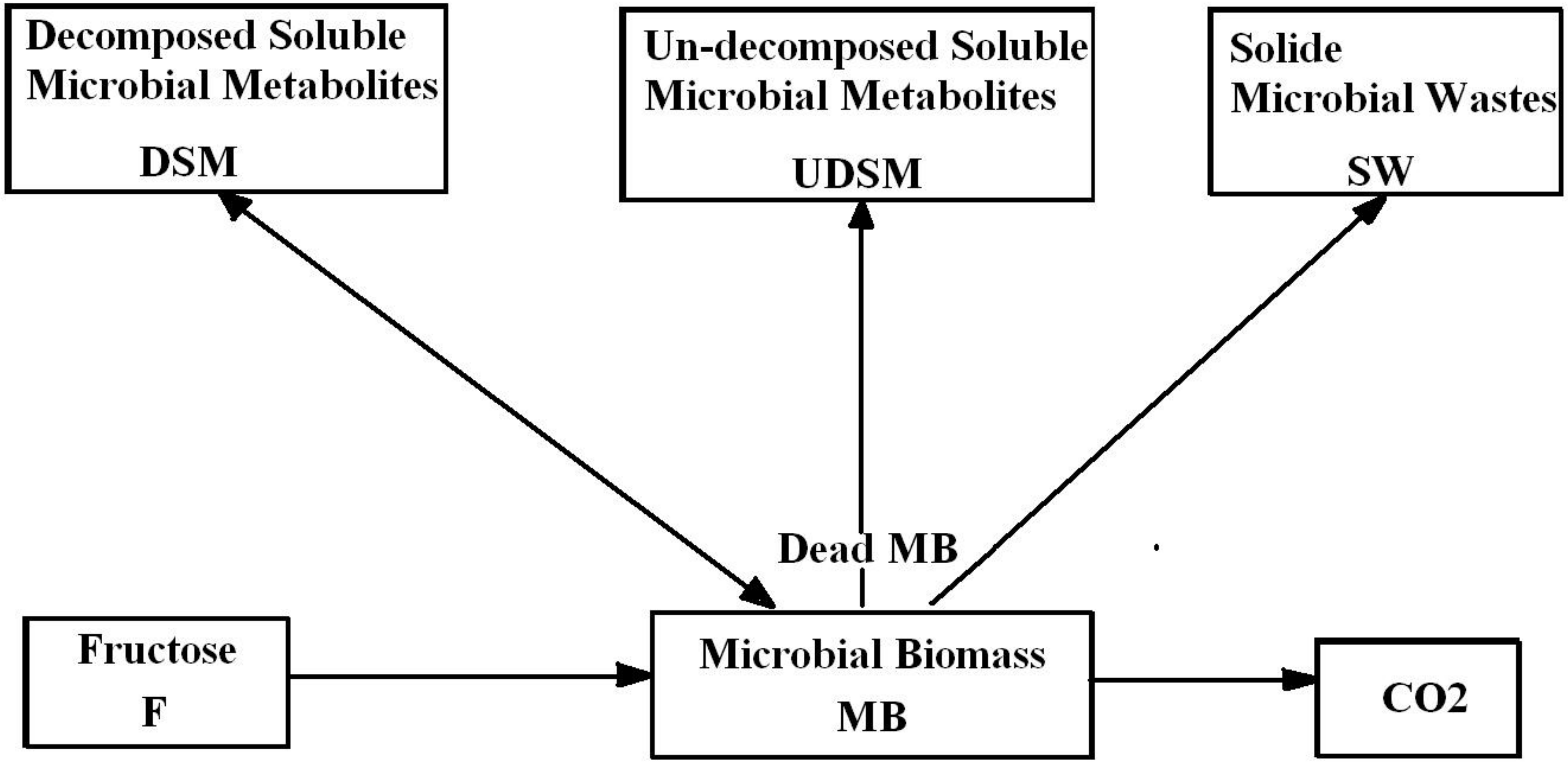
School of Applied Mathematics and Informatics

OUTLINE

- Introduction
- Existence and Uniqueness of global mild solutions
- Existence of a finite-dimensional global attractor
- Local ultimate boundedness of global solutions
- Conclusions and Perspectives

Biological problem

[Coucheyney E. 2009]



Decomposed Soluble Microbial Metabolites

DSM

Un-decomposed Soluble Microbial Metabolites

UDSM

Solide Microbial Wastes

SW

Dead MB

Fructose

F

Microbial Biomass

MB

CO2

The model

$$\begin{array}{l} \textbf{MB} \\[3em] \textbf{F} \\[3em] \textbf{SW} \\[3em] \textbf{UDSM} \\[3em] \textbf{DSM} \\[3em] \textbf{CO}_{\textbf{2}} \end{array} \quad \begin{cases} \dfrac{\partial b}{\partial t} = \\[2em] \dfrac{\partial f}{\partial t} = \\[2em] \dfrac{\partial m_1}{\partial t} = \\[2em] \dfrac{\partial m_2}{\partial t} = \\[2em] \dfrac{\partial n}{\partial t} = \\[2em] \dfrac{\partial c}{\partial t} = \end{cases}$$

The model

$$\begin{array}{ll}
\textbf{MB} & \left\{\begin{array}{ll}
\dfrac{\partial b}{\partial t} & = \\[2em]
\dfrac{\partial f}{\partial t} & = D_f \Delta f + r f \left(1 - \dfrac{f}{K}\right) - \dfrac{k f b}{K_b + f} \\[2em]
\dfrac{\partial m_1}{\partial t} & = \\[2em]
\dfrac{\partial m_2}{\partial t} & = \\[2em]
\dfrac{\partial n}{\partial t} & = \\[2em]
\dfrac{\partial c}{\partial t} & =
\end{array}\right.
\end{array}$$

Row labels: **MB**, **F**, **SW**, **UDSM**, **DSM**, **CO_2**

The model

$$\begin{matrix} \textbf{MB} \\[2em] \textbf{F} \\[2em] \textbf{SW} \\[2em] \textbf{UDSM} \\[2em] \textbf{DSM} \\[2em] \textbf{CO}_{\textbf{2}} \end{matrix} \quad \left\{ \begin{aligned} \frac{\partial b}{\partial t} &= D_b \Delta b + \frac{kfb}{K_b + f} \\ \frac{\partial f}{\partial t} &= D_f \Delta f + rf\left(1 - \frac{f}{K}\right) - \frac{kfb}{K_b + f} \\ \frac{\partial m_1}{\partial t} &= \\ \frac{\partial m_2}{\partial t} &= \\ \frac{\partial n}{\partial t} &= \\ \frac{\partial c}{\partial t} &= \end{aligned} \right.$$

The model

$$\begin{array}{l} \textbf{MB} \\[3em] \textbf{F} \\[3em] \textbf{SW} \\[3em] \textbf{UDSM} \\[3em] \textbf{DSM} \\[3em] \textbf{CO}_\textbf{2} \end{array} \left\{ \begin{array}{ll} \dfrac{\partial b}{\partial t} & = D_b \Delta b + \dfrac{kfb}{K_b + f} \qquad\qquad\qquad - \gamma b \\[2em] \dfrac{\partial f}{\partial t} & = D_f \Delta f + rf \left(1 - \dfrac{f}{K}\right) - \dfrac{kfb}{K_b + f} \\[2em] \dfrac{\partial m_1}{\partial t} & = \\[2em] \dfrac{\partial m_2}{\partial t} & = \\[2em] \dfrac{\partial n}{\partial t} & = \\[2em] \dfrac{\partial c}{\partial t} & = D_c \Delta c + \gamma b \end{array} \right.$$

The model

MB

F

SW

UDSM

DSM

CO_2

$$\begin{cases} \dfrac{\partial b}{\partial t} &= D_b \Delta b + \dfrac{kfb}{K_b + f} \qquad\qquad - \mu b - \gamma b \\ \dfrac{\partial f}{\partial t} &= D_f \Delta f + rf\left(1 - \dfrac{f}{K}\right) - \dfrac{kfb}{K_b + f} \\ \dfrac{\partial m_1}{\partial t} &= \\ \dfrac{\partial m_2}{\partial t} &= \\ \dfrac{\partial n}{\partial t} &= \\ \dfrac{\partial c}{\partial t} &= D_c \Delta c + \gamma b \end{cases}$$

The model

$$\begin{array}{l}
\textbf{MB}\vphantom{\dfrac{\partial b}{\partial t}} \\[3ex]
\textbf{F}\vphantom{\dfrac{\partial f}{\partial t}} \\[3ex]
\textbf{SW}\vphantom{\dfrac{\partial m_1}{\partial t}} \\[3ex]
\textbf{UDSM}\vphantom{\dfrac{\partial m_2}{\partial t}} \\[3ex]
\textbf{DSM}\vphantom{\dfrac{\partial n}{\partial t}} \\[3ex]
\textbf{CO}_{\textbf{2}}\vphantom{\dfrac{\partial c}{\partial t}}
\end{array}
\left\{\begin{array}{ll}
\dfrac{\partial b}{\partial t} & = D_b \Delta b + \dfrac{kfb}{K_b + f} \qquad\qquad - \mu b - \gamma b \\[3ex]
\dfrac{\partial f}{\partial t} & = D_f \Delta f + rf\left(1 - \dfrac{f}{K}\right) - \dfrac{kfb}{K_b + f} \\[3ex]
\dfrac{\partial m_1}{\partial t} & = \alpha \mu b \\[3ex]
\dfrac{\partial m_2}{\partial t} & = D_{m_2} \Delta m_2 + (1 - \alpha - \beta)\mu b \\[3ex]
\dfrac{\partial n}{\partial t} & = D_n \Delta n + \beta \mu b \\[3ex]
\dfrac{\partial c}{\partial t} & = D_c \Delta c + \gamma b
\end{array}\right.$$

The model

$$\begin{array}{ll}
\textbf{MB} & \left\{ \begin{aligned}
\frac{\partial b}{\partial t} &= D_b \Delta b + \frac{kfb}{K_b + f} + \frac{knb}{K_b + n} - \mu b - \gamma b \\
\textbf{F} \quad \frac{\partial f}{\partial t} &= D_f \Delta f + rf\left(1 - \frac{f}{K}\right) - \frac{kfb}{K_b + f} \\
\textbf{SW} \quad \frac{\partial m_1}{\partial t} &= \alpha \mu b \\
\textbf{UDSM} \quad \frac{\partial m_2}{\partial t} &= D_{m_2} \Delta m_2 + (1 - \alpha - \beta)\mu b \\
\textbf{DSM} \quad \frac{\partial n}{\partial t} &= D_n \Delta n + \beta \mu b - \frac{knb}{K_b + n} \\
\textbf{CO}_2 \quad \frac{\partial c}{\partial t} &= D_c \Delta c + \gamma b
\end{aligned} \right.
\end{array}$$

The model

MB
F
SW
UDSM
DSM
CO₂

$$\begin{cases} \dfrac{\partial b}{\partial t} &= D_b \Delta b + \dfrac{kfb}{K_b + f} + \dfrac{knb}{K_b + n} - \mu b - \gamma b \\ \dfrac{\partial f}{\partial t} &= D_f \Delta f + rf\left(1 - \dfrac{f}{K}\right) - \dfrac{kfb}{K_b + f} \\ \dfrac{\partial m_1}{\partial t} &= \alpha \mu b \\ \dfrac{\partial m_2}{\partial t} &= D_{m_2} \Delta m_2 + (1 - \alpha - \beta)\mu b \\ \dfrac{\partial n}{\partial t} &= D_n \Delta n + \beta \mu b - \dfrac{knb}{K_b + n} \\ \dfrac{\partial c}{\partial t} &= D_c \Delta c + \gamma b \end{cases} \tag{1}$$

The model

Dirichlet boundary condition

$$b(t,x) = f(t,x) = m_1(t,x) = m_2(t,x) = n(t,x) = c(t,x) = 0 \text{ on } [0,+\infty) \times \partial\Omega \tag{2}$$

Initial condition

$$\begin{aligned} b(0,x) = b_0(x),\ f(0,x) = f_0(x),\ m_1(0,x) = m_{10}(x), \\ m_2(0,x) = m_{20}(x),\ n(0,x) = n_0(x),\ c(0,x) = c_0(x) \text{ for } x \in \Omega. \end{aligned} \tag{3}$$

The model

$$\begin{array}{c} \\[5ex] \\[5ex] X \\[5ex] X \\[5ex] \\[5ex] X \end{array} \quad \begin{cases} \dfrac{\partial b}{\partial t} &= D_b \Delta b + \dfrac{kfb}{K_b + f} + \dfrac{knb}{K_b + n} - \mu b - \gamma b \\[2ex] \dfrac{\partial f}{\partial t} &= D_f \Delta f + rf\left(1 - \dfrac{f}{K}\right) - \dfrac{kfb}{K_b + f} \\[2ex] \dfrac{\partial m_1}{\partial t} &= \alpha \mu b \\[2ex] \dfrac{\partial m_2}{\partial t} &= D_{m_2} \Delta m_2 + (1 - \alpha - \beta)\mu b \\[2ex] \dfrac{\partial n}{\partial t} &= D_n \Delta n + \beta \mu b - \dfrac{knb}{K_b + n} \\[2ex] \dfrac{\partial c}{\partial t} &= D_c \Delta c + \gamma b \end{cases} \qquad (1)$$

The model

$$\begin{cases} \dfrac{\partial b}{\partial t} &= D_b \Delta b + \dfrac{kfb}{K_b + f} + \dfrac{knb}{K_b + n} - \mu b - \gamma b \\ \\ \dfrac{\partial f}{\partial t} &= D_f \Delta f + rf\left(1 - \dfrac{f}{K}\right) - \dfrac{kfb}{K_b + f} \\ \\ \dfrac{\partial n}{\partial t} &= D_n \Delta n + \beta \mu b - \dfrac{knb}{K_b + n} \end{cases} \tag{4}$$

Dirichlet boundary condition

$$b(t,x) = f(t,x) = n(t,x) = 0 \text{ on } [0,+\infty) \times \partial\Omega \tag{5}$$

Initial condition

$$b(0,x) = b_0(x),\ f(0,x) = f_0(x),\ n(0,x) = n_0(x),\ \text{for } x \in \Omega. \tag{6}$$

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Existence and Uniqueness of global mild solutions

$$\mathcal{H} = L^2(\Omega)^3,\ \mathcal{H}^+ = \text{positive functions in } \mathcal{H}$$

$$\mathcal{V} = (H_0^1(\Omega))^3,\ \mathcal{V}^+ = \text{positive functions in } \mathcal{V}$$

Theorem 1

For any initial value $u_0 = (b_0, f_0, n_0) \in \mathcal{V}^+$ given, there exists a unique positive global mild solution $u(t) = (b(t), f(t), n(t))$ of the problem *(4-6)*.

Existence and Uniqueness of global mild solutions

Sketch of proof

- **Positivity preserving:** Using Theorem 2.1 in [Efendiev, M.A. and Eberl, H.J. 2007]

- **Local Lipschitz:**

$$\begin{cases} F_1(b,f,n) &= \dfrac{kfb}{K_b+f} + \dfrac{knb}{K_b+n} - \mu b - \gamma b, \\ F_2(b,f,n) &= rf\left(1-\dfrac{f}{K}\right) - \dfrac{kfb}{K_b+f}, \\ F_3(b,f,n) &= \beta\mu b - \dfrac{knb}{K_b+n}. \end{cases}$$

Existence and Uniqueness of global mild solutions

Sketch of proof

- **Positivity preserving:** Using Theorem 2.1 in [Efendiev, M.A. and Eberl, H.J. 2007]

- **Local Lipschitz:**

$$
\begin{aligned}
|F_1(b,f,n)-F_1(\bar{b},\bar{f},\bar{n})|^2_{L^2} &= \left| \frac{kfb}{K_b+f} - \frac{k\overline{f}\overline{b}}{K_b+\overline{f}} + \frac{knb}{K_b+n} - \frac{k\overline{n}\overline{b}}{K_b+\overline{n}} - \mu(b-\bar{b}) - \gamma(b-\bar{b}) \right|^2_{L^2} \\
&\le c(k,\mu,\gamma)\left(\left| \frac{kfb}{K_b+f} - \frac{k\overline{f}\overline{b}}{K_b+\overline{f}} \right|^2_{L^2} + \left| \frac{knb}{K_b+n} - \frac{k\overline{n}\overline{b}}{K_b+\overline{n}} \right|^2_{L^2} + (\mu+\gamma)\left|b-\bar{b}\right|^2_{L^2} \right).
\end{aligned}
$$

Existence and Uniqueness of global mild solutions

Because

$$\begin{aligned}\left|\frac{kfb}{K_b+f}-\frac{k\overline{fb}}{K_b+\overline{f}}\right|_{L^2}^2 &= \left|\frac{kfb}{K_b+f}-\frac{kf\overline{b}}{K_b+f}+\frac{kf\overline{b}}{K_b+f}-\frac{k\overline{fb}}{K_b+\overline{f}}\right|_{L^2}^2\\ &\le c(k,\mu,\gamma)\left(\left|\frac{kf}{K_b+f}(b-\overline{b})\right|_{L^2}^2+\left|\frac{k\overline{b}K_b(f-\overline{f})}{(K_b+f)(K_b+\overline{f})}\right|_{L^2}^2\right)\\ &\le c(K_b,k,\mu,\gamma)\left(|b-\overline{b}|_{L^2}^2+|\overline{b}(f-\overline{f})|_{L^2}^2\right)\end{aligned}$$

yields, using Hölder's inequality,

$$\begin{aligned}\left|\frac{kfb}{K_b+f}-\frac{k\overline{fb}}{K_b+\overline{f}}\right|_{L^2}^2 &\le c(K_b,k,\mu,\gamma)\left(|b-\overline{b}|_{L^2}^2+|\overline{b}|_{L^4}^2|f-\overline{f}|_{L^4}^2\right)\\ &\le c(R,K_b,k,\mu,\gamma)\left(||b-\overline{b}||_{H_0^1}^2+||f-\overline{f}||_{H_0^1}^2\right).\end{aligned}$$

Similarly, we also obtain the following inequality

$$\left|\frac{knb}{K_b+n}-\frac{k\overline{n}\overline{b}}{K_b+\overline{n}}\right|_{L^2}^2 \le c(R,K_b,k,\mu,\gamma)\left(||b-\overline{b}||_{H_0^1}^2+||n-\overline{n}||_{H_0^1}^2\right).$$

Existence and Uniqueness of global mild solutions

So, we have

$$|F_1(b,f,n)-F_1(\overline{b},\overline{f},\overline{n})|_{L^2}^2 \leq c(R,K_b,k,\mu,\gamma)\left(||b-\overline{b}||_{H_0^1}^2+||f-\overline{f}||_{H_0^1}^2+||n-\overline{n}||_{H_0^1}^2\right).$$

Analogously,

$$|F_2(b,f,n)-F_2(\overline{b},\overline{f},\overline{n})|_{L^2}^2 \leq c(R,k,K_b,r,K)\left(||f-\overline{f}||_{H_0^1}^2+||b-\overline{b}||_{H_0^1}^2\right),$$

$$|F_3(b,f,n)-F_3(\overline{b},\overline{f},\overline{n})|_{L^2}^2 \leq c(R,k,K_b,\mu,\beta)\left(||n-\overline{n}||_{H_0^1}^2+||b-\overline{b}||_{H_0^1}^2\right).$$

Therefore,

$F = (F_1, F_2, F_3) : X^{1/2} = \mathcal{V}^+ \longrightarrow X = \mathcal{H}^+$ is locally Lipschitz.

Existence and Uniqueness of global mild solutions

Sketch of proof

- **Positivity preserving:** Using Theorem 2.1 in [Efendiev, M.A. and Eberl, H.J. 2007]

- **Local Lipschitz:**

local non-negative mild solution + priori estimate of solutions (see later on)

=> global non-negative mild solution

Reminder

Mild solution:

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}F(u(s))ds.$$

$$u := (b, f, n)^t,\ F(u) := (F_1(u), F_2(u), F_3(u))^t.$$

$$A := -\begin{pmatrix} D_b & 0 & 0 \\ 0 & D_f & 0 \\ 0 & 0 & D_n \end{pmatrix} \Delta_D$$

Reminder

The operator

$$A := -\begin{pmatrix} D_b & 0 & 0 \\ 0 & D_f & 0 \\ 0 & 0 & D_n \end{pmatrix} \Delta_D$$

is positive and self-adjoint in $\mathcal{H}$ and has compact inverse. Consider an orthonormal basis in $H_0^1(\Omega)$ consisting of eigenfunctions $e_j \in H_0^1(\Omega), j = 1, 2, ...,$ of the operator $-\Delta_D$ corresponding to the eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq ...., \lambda_j \longrightarrow \infty \text{ as } j \longrightarrow \infty.$$

For $u = (b, f, n)^t \in \mathcal{V}$, we have

$$u = \left(\sum_{j=1}^{\infty} \langle b, e_j \rangle e_j, \sum_{j=1}^{\infty} \langle f, e_j \rangle e_j, \sum_{j=1}^{\infty} \langle n, e_j \rangle e_j \right)^t,$$

$$Au = \left(\sum_{j=1}^{\infty} D_b \lambda_j \langle b, e_j \rangle e_j, \sum_{j=1}^{\infty} D_f \lambda_j \langle f, e_j \rangle e_j, \sum_{j=1}^{\infty} D_n \lambda_j \langle n, e_j \rangle e_j \right)^t.$$

Reminder

Hence $e^{-tA} : \mathcal{V} \to \mathcal{V}$ defined by

$$e^{-tA}u := \left(\sum_{j=1}^{\infty} e^{-D_b\lambda_j t}\langle b, e_j\rangle e_j, \sum_{j=1}^{\infty} e^{-D_f\lambda_j t}\langle f, e_j\rangle e_j, \sum_{j=1}^{\infty} e^{-D_n\lambda_j t}\langle n, e_j\rangle e_j \right)^t$$

is an analytic semigroup in $\mathcal{V}$ generated by the operator A.

We denote the fractional power spaces associated to A by

$$X^\alpha = \left(\mathcal{D}(A^\alpha), (.,.)_{X^\alpha}\right), \ \alpha \in \mathbb{R}.$$

The inner product in X^α is given by

$$(u, v)_{X^\alpha} = (A^\alpha u, A^\alpha v)_{X^0}, \ u, v \in \mathcal{D}(A^\alpha),$$

where

$$\mathcal{D}(A^\alpha) = \left\{ u = \begin{pmatrix} \sum_{j=1}^{\infty}\langle b, e_j\rangle e_j \\ \sum_{j=1}^{\infty}\langle f, e_j\rangle e_j \\ \sum_{j=1}^{\infty}\langle n, e_j\rangle e_j \end{pmatrix} : \begin{cases} \sum_{j=1}^{\infty} |D_b\lambda_j|^{2\alpha}|\langle b, e_j\rangle|^2 < \infty \\ \sum_{j=1}^{\infty} |D_f\lambda_j|^{2\alpha}|\langle f, e_j\rangle|^2 < \infty \\ \sum_{j=1}^{\infty} |D_n\lambda_j|^{2\alpha}|\langle n, e_j\rangle|^2 < \infty \end{cases} \right\},$$

Reminder

and

$$A^\alpha u = \left(\sum_{j=1}^{\infty} D_b^\alpha \lambda_j^\alpha \langle b, e_j \rangle e_j, \sum_{j=1}^{\infty} D_f^\alpha \lambda_j^\alpha \langle f, e_j \rangle e_j, \sum_{j=1}^{\infty} D_n^\alpha \lambda_j^\alpha \langle n, e_j \rangle e_j \right)^t .$$

In these notations,

$$X^1 = (\mathcal{D}(-\Delta_D))^3,\ X^{\frac{1}{2}} = \mathcal{V},\ X^0 = \mathcal{H}, X^{-\frac{1}{2}} = \mathcal{V}'.$$

The operator A in X^0 can be extended or restricted, respectively, to a positive sectorial operator in X^α with domain $X^{\alpha+1}$, $\alpha \in \mathbb{R}$, and the corresponding semigroups $e^{-At}, t \geq 0$, in X^α are obtained from each other by natural restrictions and extensions. Moreover, if $\beta \leq \alpha$ we have $e^{-At}(X^\beta) \subset X^\alpha$ and

$$||e^{-At}||_{\mathcal{L}(X^\beta;X^\alpha)} \leq \frac{c_{\alpha,\beta}}{t^{\alpha-\beta}},\ t > 0,$$

for some constant $c_{\alpha,\beta} \geq 0$, where $||.||_{\mathcal{L}(V;W)}$ denotes the norm of a linear operator between the normed spaces V and W (see, e.g., Section 2.1.1 in [Temam, R. 1997])

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Existence of a global attractor

Theorem 2

The semigroup $\{S(t)\}_{t\in\mathbb{R}_+}$ associated with *(4-6)* processes a global attractor $\mathcal{A}$ in $\mathcal{V}^+$ provided

$$\lambda_1 D_b + \mu + \gamma - 2k > 0$$

where $\lambda_1 > 0$ is the first eigenvalue of the operator $-\Delta_D$ on the domain Ω with the homogeneous Dirichlet boundary condition.

Existence of a global attractor

Sketch of proof

(a) Existence of an absorbing set in $\mathcal{V}^+$.

+) absorbing set in $\mathcal{H}^+$

-) $$|b(t)|_{L^2}^2 \leq |b(0)|_{L^2}^2 e^{-\lambda t}, \lambda = -2(\lambda_1 D_b + \mu + \gamma - 2k).$$

-) $$|f(t)|_{L^2}^2 \leq \left(|f(0)|_{L^2}^2 - c_1\right) e^{-\lambda_1 D_f t} + c_1,\ c_1 = \frac{4rK^2}{27\lambda_1 D_f}.$$

-) $$|n(t)|_{L^2}^2 \leq \left(|n(0)|_{L^2}^2 - \frac{c_2}{\lambda_1 D_n - \lambda}|b(0)|_{L^2}^2\right) e^{-\lambda_1 D_n t} + \frac{c_2}{\lambda_1 D_n - \lambda}|b(0)|_{L^2}^2 e^{-\lambda t}.$$

=> the ball $B_{\mathcal{H}^+}(0, R)$ is a bounded absorbing set in $\mathcal{H}^+$.

Existence of a global attractor

Sketch of proof

(a) Existence of an absorbing set in $\mathcal{V}^+$.

+) absorbing set in $\mathcal{V}^+$

-) $$||b(t)||^2_{H^1_0} \leq c_3, \; c_3 = \frac{2(2k-\mu-\gamma)}{2D_b}R^2 + \frac{R^2}{2D_b} + \frac{(2k+\mu+\gamma)^2}{4D_b}R^2, \; \forall t \geq T+1.$$

-) $$||f(t)||^2_{H^1_0} \leq c_6, \; c_6 = (c_4+c_5)e^{2r}, \; \forall t \geq T+2.$$

-) $$||n(t)||^2_{H^1_0} \leq c_9, \; c_9 = c_8 + c_7, \; \forall t \geq T+1.$$

We choose $R_{\max} = \max\{2c_3, 2c_6, 2c_9\}$, it implies that $B_{\mathcal{V}^+}(0, R_{\max})$ is a bounded absorbing set in $\mathcal{V}^+$ for the semigroup $S(t)$.

Existence of a global attractor

Sketch of proof

(a) Existence of an absorbing set in $\mathcal{V}^+$.

(b) Asymptotic compactness of the semigroup.

Lemma 1 *Let* $B \subset X^{\frac{1}{2}}$ *be a bounded set. Then, for every* $T^* > 0$ *there exists a constant* κ *such that*

$$||S(T^*)u - S(T^*)v||_{X^{\frac{1}{2}}} \leq \kappa ||u - v||_{X^0}, \ \forall u, v \in B.$$

Existence of a global attractor

Sketch of proof

Now, we assume that the set $B \subset X^{\frac{1}{2}}$ is bounded.

Let $T > 0$ and $x_n \in B, t_n \geq 0, n \in \mathbb{N}$, be sequences such that $t_n \longrightarrow \infty$.

=> $\{S(t_n - T)x_n : t_n \geq T, n \in \mathbb{N}\}$ is bounded in $X^{\frac{1}{2}}$

=> $v_k := S(t_{n_k} - T)x_{n_k}$ converging weakly to v in $X^{\frac{1}{2}}$ and strongly to v in X^0

Lemma 1 => $||S(T)v_k - S(T)v||_{X^{\frac{1}{2}}} \leq \kappa ||v_k - v||_{X^0}$,

=> asymptotic compactness of the semigroup.

Existence of a global attractor

Fractal dimension estimates of the global attractor

The global attractor $\mathcal{A}$ has finite fractal dimension

[Efendiev et al. 2005]

Lemma 2 *Let V and W be Banach spaces such that the embedding $V \hookrightarrow W$ is dense and compact, and let $S(t), t \geq 0$, be a semigroup in V. We assume $\mathcal{A} \subset V$ is a compact invariant set and the semigroup satisfies the smoothing property: There exist $T^* > 0$ and a constant $\kappa \geq 0$ such that*

$$||S(T^*)u - S(T^*)v||_V \leq \kappa ||u - v||_W, \ \forall u, v \in \mathcal{A}.$$

Then, the fractal dimension of $\mathcal{A}$ in V is finite.

We now can apply Lemma 2 to the semigroup $S(t), t \geq 0$, with $V = X^{\frac{1}{2}}$ and $W = X^0$ to get the finiteness of the fractal dimension of the global attractor. □

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Local ultimate boundedness of global solutions

Theorem 3

The semigroup $\{S(t)\}_{t\in\mathbb{R}_+}$ associated with *(4-6)* processes local absorbing sets in $\mathcal{V}^+$.

Idea of proof

+) $|b+f+n|_{L^1} \leq c(K, |b_0|_{L^1}, |f_0|_{L^1}, |n_0|_{L^1})$.

+) $|b|^2_{L^2} \leq \varepsilon|\nabla b|^2_{L^2} + c(\varepsilon)|b|^2_{L^1}, \ \forall b \in H^1_0(\Omega), \ \forall \epsilon > 0.$ see in [Robinson J.C. 2001]

+) $|b(t)|^2_{L^2} \leq |b_0|^2_{L^2} e^{-(\lambda_1 D_b+\mu+\gamma)t} + c, \ c = c(K, K_b, k, |b_0|_{L^1}, |f_0|_{L^1}, |n_0|_{L^1})$.

Local ultimate boundedness of global solutions

Idea of proof

+) $|b+f+n|_{L^1} \leq c(K, |b_0|_{L^1}, |f_0|_{L^1}, |n_0|_{L^1})$.

$$\frac{d}{dt}|b+f+n|_{L^1} - \int_\Omega (D_b\Delta b + D_f\Delta f + D_n\Delta n)dx = (\beta\mu - \mu - \gamma)|b|_{L^1} + r\int_\Omega f dx - \frac{r}{K}\int_\Omega f^2 dx$$
$$\leq r\int_\Omega f dx - \frac{r}{K}\int_\Omega f^2 dx.$$

In addition, using Stokes' formula, we have

$$\int_\Omega (D_b\Delta b + D_f\Delta f + D_n\Delta n)dx = \int_{\partial\Omega}\left(\frac{\partial b}{\partial \nu} + \frac{\partial f}{\partial \nu} + \frac{\partial n}{\partial \nu}\right) ds \leq 0,$$

Hence,

$$\frac{d}{dt}|b+f+n|_{L^1} \leq r\int_\Omega f dx - \frac{r}{K}\int_\Omega f^2 dx.$$

Local ultimate boundedness of global solutions

Idea of proof

Using Cauchy's inequality, we have

$$-f^2 \leq -Kf + \frac{K^2}{4}.$$

Hence,

$$\frac{d}{dt}|b+f+n|_{L^1} \leq \frac{K^2}{4},$$

which yields

$$|b+f+n|_{L^1} \leq c(K, |b_0|_{L^1}, |f_0|_{L^1}, |n_0|_{L^1}).$$

OUTLINE

- Introduction
- Existence and Uniqueness of global mild solutions
- Existence of a finite-dimensional global attractor
- Local ultimate boundedness of global solutions
- Conclusions and Perspectives

Conclusions and perspectives

- $\lambda_1 D_b + \mu + \gamma - 2k > 0$: existence of a finite-dimensional global attractor
- When the condition is invalid: local ultimate boundedness of global solutions
- Optimal control, other biological systems...

Existence of a global attractor

Sketch of proof

(a) Existence of an absorbing set in $\mathcal{V}^+$.

+) absorbing set in $\mathcal{H}^+$

-) $$|b(t)|^2_{L^2} \leq |b(0)|^2_{L^2} e^{-\lambda t}, \lambda = -2(\lambda_1 D_b + \mu + \gamma - 2k).$$

-) $$|f(t)|^2_{L^2} \leq \left(|f(0)|^2_{L^2} - c_1\right) e^{-\lambda_1 D_f t} + c_1, \; c_1 = \frac{4rK^2}{27\lambda_1 D_f}.$$

-) $$|n(t)|^2_{L^2} \leq \left(|n(0)|^2_{L^2} - \frac{c_2}{\lambda_1 D_n - \lambda}|b(0)|^2_{L^2}\right) e^{-\lambda_1 D_n t} + \frac{c_2}{\lambda_1 D_n - \lambda}|b(0)|^2_{L^2} e^{-\lambda t}.$$

=> the ball $B_{\mathcal{H}^+}(0, R)$ is a bounded absorbing set in $\mathcal{H}^+$.

Conclusions and perspectives

- $\lambda_1 D_b + \mu + \gamma - 2k > 0$: existence of a finite-dimensional global attractor
- When the condition is invalid: local ultimate boundedness of global solutions
- Optimal control, other biological systems...

Thanks

Thank you very much for your attention!