Existence and blow-up of solutions for semilinear filtration problems

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joint work with K. Fellner, G. Pisante and D. Tzanetis

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The semilinear Filtration Problem

$$\begin{cases} u_t = \Delta K(u) + \lambda f(u), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial K(u)}{\partial n} + \beta(x) K(u) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0 & x \in \Omega, \end{cases}$$

Hypotheses

$$K(s),\ K'(s),\ K''(s) > 0$$

$$f(s) > 0,\ f'(s) > 0,\ f''(s) > 0$$

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The Osgood's type condition

$$\int_{s_0}^{\infty} \frac{K'(s)}{f(s)} ds < \infty, \qquad \int_{s_0}^{\infty} \frac{ds}{f(s)} < \infty.$$

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Aim of this work

- When $\lambda > \lambda^*$ the solution to Semilinear Filtration Problem becomes infinite in finite time (blow-up) independently of $u_0(x) \geq 0$.
- Blow-up of solutions for sufficiently large initial data and $\lambda \in (0, \lambda^*)$.
- Single point blow-up.
- Grow-up of solutions for $\lambda = \lambda^*$.

Physical Motivation (Flow of gas)

Flow of gas in a porous medium (Leibenzon, 1930; Muskat 1933)

$$\rho_t + \operatorname{div}(\rho\nu) = 0,$$

$$\nu = -\frac{k}{\mu}\nabla p, \quad p = p(\rho).$$

Second line left is the Darcy law for flows in porous media (Darcy, 1856).
To the right, put $p = p_0\rho^\gamma$, with $\gamma = 1$(isothermal), $\gamma > 1$ (adiabatic flow).

$$\rho_t = \operatorname{div}(\frac{k}{\mu}\rho\nabla p) = \operatorname{div}(\frac{k}{\mu}\rho\nabla p_0\rho^\gamma) = c\Delta\rho^{\gamma+1}.$$

Physical Motivation (Flow of gas)

A quite different approach is assuming that the state law is not power-like, but has the form $p = p(\rho)$, as happens in general barotropic gases, and also that k and μ may depend on ρ.

Filtration equation

In that case we get a final equation for the density of the form

$$\rho_t = \delta\Phi(\rho) + f,$$

where Φ is a given monotone increasing function of $\rho, \rho \geq 0$. The second term on the right-hand side, $f = f(x,t)$ represents mass sources or sinks distributed in the medium.

Physical Motivation (Plasma radiation)

Plasma radiation $m \geq 4$ (Zeldovich-Raizer, 1950)

Experimental fact: diffusivity at high temperatures is not constant as in Fourier's law, due to radiation.

$$\frac{d}{dt}\int_\Omega c\rho T dx = \int_{\partial\Omega} k(T)\nabla T \cdot n dS,$$

Inserting $k(T) = k_0 T^n$ and applying Gauss law we get

$$c\rho\frac{\partial T}{\partial t} = \operatorname{div}(k(T)\nabla T) = c_1 \Delta T^{n+1}.$$

- When k is not a power we get $T_t = \Delta\Phi(T)$ with $\Phi'(T) = k(T)$.

Physical Motivation (Population dynamics)

Gurtin and MacCamy (1977)

derive the filtration equation as the equation governing the density of a biological population which is allowed to migrate.

- The nonlinearity $K(u)$ arises in their model due to a crowding effect, i.e., individuals tend to migrate away from regions of high density.
- The source term $f(u)$ represents the contribution to the population supply due to births and deaths.

Mathematical Motivation-History

- **A.A. Lacey**, *Mathematical analysis of thermal runaway for spartially inhomogeneous reactions,* SIAM J. Appl. Math., (**1983**). (λ^* lies in the spectrum of the stationary problem)

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- **H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa**, *Blow-up for $u_t = \Delta u + \lambda g(u)$ revisited,* Adv. Diff. Eq., (**1996**).(energy method)

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Local existence of Classical Solutions

- Comparison techniques
- Iteration schemes
- Very weak solutions + regularity

$$\Downarrow$$

Theorem:

The problem

$$\begin{cases} u_t = \Delta K(u) + \lambda f(u), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial K(u)}{\partial n} + \beta(x) K(u) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \end{cases}$$

has a unique classical solution u with $C^{2,1}(\Omega_T)$ for some $T > 0$.

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The Steady State and the critical value λ^*

The corresponding steady-state:

$$\begin{cases} \Delta(K(w)) + \lambda f(w) = 0, & x \in \Omega \\ \mathcal{B}(K(w)) = 0 & x \in \partial\Omega \end{cases}$$

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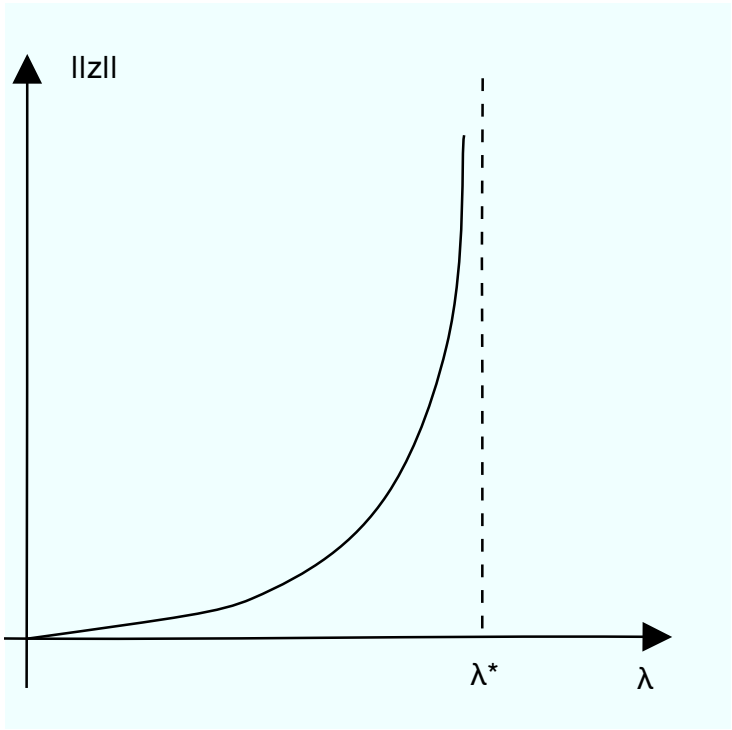

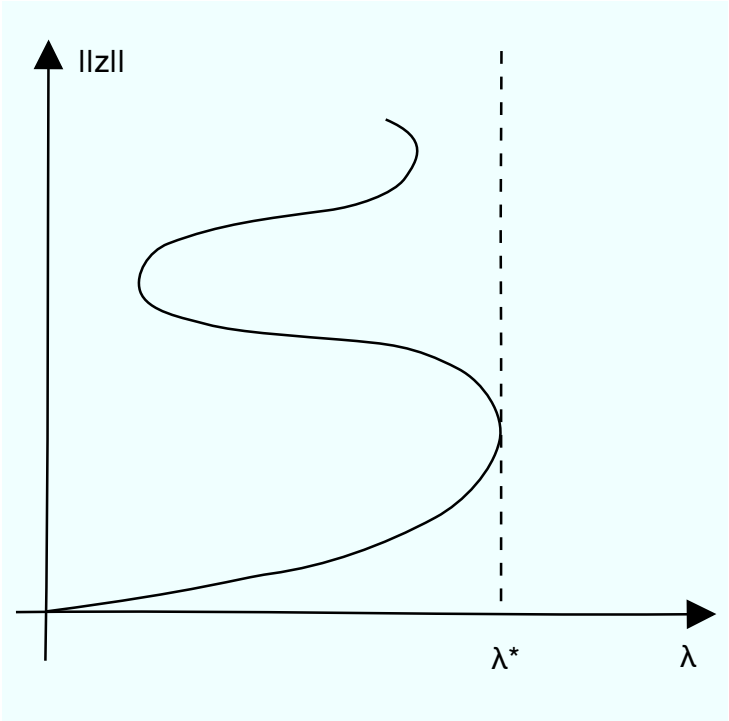

In what follows we consider the closed spectrum case; that is there exists a unique classical solution $z^* = K(w^*)$ at $\lambda = \lambda^*$.

Steady state

The corresponding steady-state problem is

$$\begin{cases} \Delta(K(w)) + \lambda f(w) = 0, & x \in \Omega \\ \mathcal{B}(K(w)) = 0 & x \in \partial\Omega \end{cases}$$

if $z(x) = K(w(x))$

$$\begin{cases} \Delta z + \lambda g(z) = 0, & x \in \Omega, \\ \mathcal{B}(z) = 0, & x \in \partial\Omega, \end{cases}$$

$g(z) = f(K^{-1}(z)) = f(w)$.

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$$g''(\sigma) = \frac{1}{K'(s)} \left(\frac{f'(s)}{K'(s)} \right)' > 0, \ s > 0.$$

$$g(\sigma) = f(s) > 0, \ \ g'(\sigma) = \frac{f'(s)}{K'(s)} > 0, \ \ g''(\sigma) > 0.$$

The linearized problem

$$\begin{cases} -\Delta[K'(w)\phi] = \lambda f'(w)\phi + \mu\phi & \text{in } \Omega \\ \mathcal{B}(K'(w)\phi) = 0 & \text{on } \partial\Omega. \end{cases}$$

Remark

We expand our equation according to $u(t,x) = w(x) + \Phi(t,x)$ and decompose $\Phi(t,x) = e^{-\mu t}\phi(x)$.

Remark

The linearized eigenvalue problem has a solution for each $\lambda \in (0, \lambda^*)$.

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First approach-Kaplan's method for $\boldsymbol{\lambda \gg 1}$

- A necessary condition for blow-up of solutions can be taken if we consider that $u(x,t)$ is uniform with respect to x, so $u(x,t) = v(t)$ and the spatial derivative zero.

$$\frac{dv}{dt} = \lambda f(v), \quad t > 0, \;\; v(0) = \sup_{\Omega} u_0(x),$$

then, $\lambda t < \int_{v(0)}^{v(t)} ds/f(s) \leq \int_{v(0)}^{\infty} ds/f(s) < \infty$.

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- A sufficient blow-up condition can be taken on using Kaplan's method. The difference with the next subsection is that here we have blow-up for λ large enough, namely for $\lambda > \mu > \lambda^*$.

Kaplan's method: A sufficient blow-up condition

Semilinear heat equation:

$$\begin{cases} u_t = \Delta u + f(u) & x \in Q \\ u(x,t) = 0 & x \in \partial\Omega \\ u(x,0) = u_0(x) \geq 0 & x \in \Omega \end{cases}, \qquad \begin{cases} \Delta\phi = -\lambda_1 \phi & x \in \Omega, \\ \phi(x) = 0 & x \in \partial\Omega \end{cases}$$

$$\Downarrow$$

$$\frac{d}{dt}\int_\Omega u\phi\,dx + \lambda_1 \int_\Omega u\phi\,dx = \int_\Omega f(u)\phi \geq f\left(\int_\Omega u\phi\,dx\right)$$

Define $A(t) := \int_\Omega u\phi\,dx$ to be the first Fourier coefficient of u.

$$A'(t) \geq f(A(t)) - \lambda_1 A(t)$$

Elementary blow-up criterion

Assume that f satisfies the Osgood's condition, Let t_0 be the largest zero of $f(s) - \lambda_1 s$. If $A(0) > t_0$ then u blows up in finite time.

Kaplan's method for Filtration problems

$$u_t = \Delta K(u) + \lambda f(u)$$

$$A(t) = \int_\Omega u(x,t)\varphi(x)dx\ , \quad A'(t) = \int_\Omega \varphi \Delta K(u)dx + \int_\Omega \varphi f(u)dx$$

$$\begin{cases} \Delta\phi = -\mu\phi & x \in \Omega, \\ \mathcal{B}(\phi) = 0 & x \in \partial\Omega, \end{cases} \qquad \int_\Omega \varphi dx = 1$$

$$\Downarrow$$

$$\boxed{A'(t) = -\mu \int_\Omega \varphi K(u)dx + \lambda \int_\Omega \varphi f(u)dx}$$

Kaplan's method for Filtration problems

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Kaplan's method for Filtration problems

$$\boxed{A'(t) = -\mu \int_\Omega \varphi K(u)dx + \lambda \int_\Omega \varphi f(u)dx}$$

$$+$$

$$\boxed{\int_\Omega [K(u(x,t)) - f(u(x,t))]\varphi(x)dx < 0}$$

$$+$$

$$\boxed{\lambda > \mu}$$

$$\Downarrow$$

$$\boxed{A'(t) \geq (\lambda - \mu) \int_\Omega f(u)\,\varphi dx \geq (\lambda - \mu) f(A)}$$

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The Supercritical case scenario : $\lambda > \lambda^*$

- The previous straight-forward proof is in general not sufficient to treat the full supercritical range $\lambda > \lambda^*$
- We have always $\mu \geq \lambda^*$

Supercritical Blow-Up

Assume additionaly that the function $g(\sigma) := f(K^{-1}(\sigma))$ is increasing and convex. Then, the solution to

$$\begin{cases} u_t = \Delta K(u) + \lambda f(u), & x \in \Omega, \quad t > 0, \\ \frac{\partial K(u)}{\partial n} + \beta(x) K(u) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0 & x \in \Omega, \end{cases}$$

blows up in finite time for any $\lambda > \lambda^*$ and any initial data $u_0 \geq 0$.

Proof of Blow-up for $\lambda > \lambda^*$ for any initial data

By various manipulations and for $\lambda > \lambda^*$ we derive:

$$A'(t) = \int_\Omega K'(w^*)\varphi^* v_t dx \geq \dots$$

$$\geq \lambda^* \int_\Omega [K'(w^*)(f(u) - f(w^*)) - f'(w^*)(K(u) - K(w^*))]\varphi^* dx,$$

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Idea: Construct

$h(s) : h(0) = 0,\ h(s) > 0$ for $s \in \mathbb{R}^*,\ h'' > 0$ and set $u = w^* + v$ such that,

$$A'(t) \geq \dots \geq \lambda^* h(A).$$

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$A'(t) \geq \lambda^* h(A)$, blow-up of $A(t)$ and since $A(t) \leq ||v(\cdot, t)||$, blow-up of v and hence of u ($u = w^* + v > v, w^*$ bounded).

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Blow-Up for large enough initial data

$$\int_\Omega (f^q(u(x,t)) - K(u(x,t))\,\phi(x)\,dx > 0, \qquad q > 1, \forall t > 0.$$

Remark

From the proof we get a first easy condition on u_0 which is sufficient for blow-up of the solutions $u(x,t)$ (under the previous assumptions). More precisely, by sufficiently large initial data we have blow-up if

$$A(0) = \int_\Omega u_0 \phi\,dx > f^{-1}(s_0)$$

where

$$s_0 = \max_{s \in \mathrm{R}} \{\lambda s \leq \mu s^q\}.$$

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Blow-Up above positive steady state

$$\begin{cases} u_t = \Delta K(u) + \lambda f(u), & x \in Q \\ \mathcal{B}(K(u)) = 0, & x \in \partial\Omega \\ u(x,0) = u_0(x) \geq 0 & x \in \Omega, \end{cases}, \quad \begin{cases} \Delta(K(w)) + \lambda f(w) = 0, & x \in \Omega \\ \mathcal{B}(K(w)) = 0 & x \in \partial\Omega \end{cases}$$

Theorem

$$\boxed{\begin{array}{c} g(\sigma) := f(K^{-1}(\sigma)) \ \ \textit{convex} \\ u_0 \geq w \ \textit{in} \ \Omega \\ u_0 \neq w \end{array}} \Rightarrow \boxed{u(x,t) \ \textit{blows-up in finite time}}$$

Thank you
for your attention!