An entropy-based method for the analysis of cross-diffusion PDEs

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Cross-diffusion PDEs...

Many multi-species systems in biology, chemistry, and physics can be described by reaction-diffusion systems:

$$\partial_t u - \operatorname{div}(A(u)\nabla u) = f(u) \quad \text{in } \Omega,\ t > 0, \qquad \text{(RD)}$$
$$(A(u)\nabla u)\cdot\nu = 0 \quad \text{on } \partial\Omega, \quad u(0) = u^0 \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain, and:

- $u = (u_1, \ldots, u_n)^\top$ is the vector of the densities or concentrations of the species;
- $A(u) = (A_{ij}(u)) \in \mathbb{R}^{n\times n}$ is the diffusion matrix;
- $f = (f_1, \ldots, f_n)$ is the vector of the reactions.

Cross-diffusion $\equiv$ the diffusion matrix $A(u)$ is not diagonal
$\Rightarrow$ some variables give contributions to the diffusion of other variables.

. . . and the trouble they make

Independent variables $u_1 \dots u_n$ represent densities, concentrations $\Rightarrow$ they should be nonnegative and bounded.

Cross-diffusion $\Rightarrow$ no maximum principle $\Rightarrow$ the proof of these properties is a challenging problem. Weak solutions may be unbounded.

Sometimes the diffusion matrix is neither symmetric nor positive definite $\Rightarrow$ even the local-in-time existence of solutions may be nontrivial.

The boundedness-by-entropy principle

Boundedness-by-entropy principle: a systematic method to prove the existence of global-in-time nonnegative bounded weak solutions to cross-diffusion systems possessing a formal **gradient-flow structure:**

$$\partial_t u - \operatorname{div}\left(B\nabla \frac{\delta \mathcal{H}}{\delta u}\right) = f(u).$$

- B is a positive semidefinite matrix;
- $\frac{\delta \mathcal{H}}{\delta u}$ is the variational derivative of the entropy $\mathcal{H}[u] = \int_\Omega h(u)dx$;
- $h : D \subset \mathbb{R}^n \to [0,\infty)$ is the entropy density.

Let us introduce the **entropy variable:** $w = Dh(u)$. The above system can be formulated as:

$$\partial_t u - \operatorname{div}(B(w)\nabla w) = f(u), \qquad B(w) = A(u)(D^2 h(u))^{-1}.$$

This formulation makes only sense if $Dh : D \to \mathbb{R}^n$ is invertible.

The boundedness-by-entropy principle

There are two consequences of this formulation.

1. If $f(u) \cdot w \leq 0$, the entropy $\mathcal{H}$ is a Lyapunov functional along solutions:

$$\frac{d\mathcal{H}}{dt} = \int_\Omega \partial_t u \cdot w dx \leq -\int_\Omega \nabla w : B(w) \nabla w dx$$
$$= -\int_\Omega \nabla u : (D^2 h) A(u) \nabla u dx \leq 0,$$

   taking into account that $B(w)$ (or equivalently $(D^2 h)A(u)$) is assumed to be positive semidefinite.

2. Because of the invertibility of Dh, the original variable satisfies $u(x,t) = (Dh)^{-1}(w(x,t)) \in D$, and if D is a bounded domain, we obtain automatically L^∞ bounds without the use of a maximum principle.

The boundedness-by-entropy principle: key result

Theorem (Boundedness-by-entropy principle)

Let $D \subset (0,1)^n$ be a bounded domain, $u^0 \in L^1(\Omega; D)$, and assume that:

- H1 *There exists a convex function $h \in C^2(D; [0,\infty))$ such that its derivative $Dh : D \to \mathbb{R}^n$ is invertible.*
- H2 *There exist $\alpha^* > 0$, $m_1, \ldots, m_n \in [0,1]$ such that:*

$$z^\top D^2 h(u) A(u) z \geq \alpha^* \sum_{i=1}^n u_i^{2(m_i-1)} z_i^2 \quad \forall z \in \mathbb{R}^n, \quad u \in D.$$

- H3 *It holds $A \in C^0(D; \mathbb{R}^{n\times n})$ and there exists $c_f > 0$ such that for all $u \in D$, $f(u) \cdot Dh(u) \leq c_f(1 + h(u))$.*

Then there exists a weak solution $u : \Omega \times (0,\infty) \to \overline{D}$ to (RD),

$$u \in L^2_{\mathrm{loc}}(0,\infty; H^1(\Omega; \mathbb{R}^2)), \quad \partial_t u \in L^2_{\mathrm{loc}}(0,\infty; H^1(\Omega; \mathbb{R}^2)').$$

For the proof see: A. Jüngel, *The boundedness-by-entropy principle for cross-diffusion systems* (2014).

Models with linear diffusivities

We show a class of cross-diffusion systems for $n = 2$ species, whose diffusivities depend linearly on the solution and for which global bounded weak solutions exist.

$$A_{ij}(u) = \alpha_{ij} + \beta_{ij} u_1 + \gamma_{ij} u_2 \quad i, j = 1, 2, \tag{LDIFF}$$

where α_{ij}, β_{ij}, γ_{ij} are real numbers.

Derivation: master equation for a random walk on a lattice in the diffusion limit. Transition rates depending linearly on the species' densities.

Models with linear diffusivities: examples

Examples of cross-diffusion systems with linear diffusivities:

- ion transport through narrow channels;
- population dynamics with complete segregation;
- **Most prominent example:** population systems of Shigesada-Kawasaki-Teramoto type (SKT) in several space dimensions:

$$A(u) = \begin{pmatrix} a_{10} + 2a_{11}u_1 + a_{12}u_2 & a_{12}u_1 \\ a_{21}u_2 & a_{20} + a_{21}u_1 + 2a_{22}u_2 \end{pmatrix},$$

$$f_i(u) = (b_{i0} - b_{i1}u_1 - b_{i2}u_2)u_i, \qquad i = 1, 2.$$

Models for two competitive species in heterogeneous evironments.
$a_{ij} \geq 0$, $i, j = 0, 1, 2$.
$a_{11}, a_{22} \equiv$ self-diffusion, $a_{12}, a_{21} \equiv$ cross-diffusion coefficients.
$b_{11}, b_{22} \equiv$ intra-specific, $b_{12}, b_{21} \equiv$ inter-specific competition constants.

Models with linear diffusivities: main result

Theorem (Case of linear diffusivities)

Let $D = \{(u_1, u_2) \in \mathbb{R}^2 : u_1 > 0,\ u_2 > 0,\ u_1 + u_2 < 1\}$,
$u^0 = (u_1^0, u_2^0) \in L^1(\Omega; \overline{D})$, *$A(u)$ given by (LDIFF) satisfying:*

$$\alpha_{12} = \alpha_{21} = \beta_{21} = \gamma_{12} = 0, \quad \beta_{22} = \beta_{11} - \gamma_{21}, \tag{C1}$$

$$\gamma_{11} = \gamma_{22} - \beta_{12}, \quad \gamma_{21} = \alpha_{22} - \alpha_{11} + \beta_{12}, \tag{C2}$$

$$\alpha_{11} > 0, \quad \alpha_{22} > 0, \quad \beta_{12} < \alpha_{11} + \min\{\beta_{11}, \gamma_{22}\}, \tag{C3}$$

$$\alpha_{11} + \beta_{11} \geq 0, \quad \alpha_{22} + \gamma_{22} \geq 0, \tag{C4}$$

and let $f_i(u) = u_i g_i(u)$, where $g_i(u)$ is continuous in $\overline{D}$ and nonpositive in $\{1 - \varepsilon < u_1 + u_2 < 1\}$ for some $\varepsilon > 0$ ($i = 1, 2$). Then there exists a weak solution $u = (u_1, u_2) : \Omega \times (0, \infty) \to \overline{D}$ to (RD) satisfying

$$u \in L^2_{\mathrm{loc}}(0, \infty; H^1(\Omega; \mathbb{R}^2)), \quad \partial_t u \in L^2_{\mathrm{loc}}(0, \infty; H^1(\Omega; \mathbb{R}^2)').$$

If the population approaches its total capacity $u_1 + u_2 = 1$, the reaction terms are nonpositive and lead to a decrease of the population.

Idea of the proof of Theorem 2: apply the general existence result stated in Theorem 1. $\Rightarrow$ Verify assumptions H1, H2, H3.

- Assumption H1 satisfied with entropy density:

  $$h(u) = u_1(\log u_1 - 1) + u_2(\log u_2 - 1) + (1 - u_1 - u_2)(\log(1 - u_1 - u_2) - 1),$$

  defined on $D = \{(u_1, u_2) \,:\, u_1 > 0,\ u_2 > 0,\ u_1 + u_2 < 1\}$.

- Assumption H3 satisfied: $f_i(u)(Dh)_i(u) \le c$ for $i = 1, 2$.

  $$f_i(u)(Dh)_i(u) = g_i(u)u_i \log u_i - u_i g_i(u) \log(1 - u_1 - u_2).$$

  $$g_i(u), u_i \log u_i \text{ bounded in } \overline{D} \ \Rightarrow\ g_i(u)u_i \log u_i \text{ bounded in } \overline{D}.$$

  $$-u_i g_i(u) \log(1 - u_1 - u_2) \le c_\varepsilon \text{ in } D^\varepsilon := \{0 < u_1 + u_2 \le 1 - \varepsilon\}.$$

  $$g_i(u) \le 0 \text{ in } D \backslash D^\varepsilon \ \Rightarrow\ -u_i g_i(u) \log(1 - u_1 - u_2) \le 0 \text{ in } D \backslash D^\varepsilon.$$

Proving that H2 holds

Assumption H2 requires the matrix $M(u) := (D^2h(u))A(u)$ to be positive definite for $u \in D$. We have to deal with twelve parameters α_{ij}, β_{ij}, and γ_{ij} $\Rightarrow$ the proof is not trivial .

In order to reduce the complexity of the problem, we assume that $M(u)$ is symmetric, motivated by the Onsager symmetry principle in non-equilibrium thermodynamics. This yields seven conditions, and we are left with five parameters.

Proving that H2 holds

By Sylvester's criterion, the positive semidefiniteness follows if the diagonal terms and the determinant of $M(u)$ are nonnegative.

$\Rightarrow$ Three multivariate quadratic polynomials in (u_1, u_2) have to be nonnegative.

$\Rightarrow$ Use the strong maximum principle applied to such polynomials to get conditions on the coefficients.

We stress the fact that **the maximum principle is <u>not</u> needed to prove the L^∞ bounds but to solve the algebraic problem.**

Verification of Hypothesis H2

We require that the matrix $M(u)$ is symmetric. This leads to the conditions:

$$\alpha_{12} = \alpha_{21} = \beta_{21} = \gamma_{12} = 0, \quad \beta_{22} = \beta_{11} - \gamma_{21},$$
$$\gamma_{11} = \gamma_{22} - \beta_{12}, \quad \gamma_{21} = \alpha_{22} - \alpha_{11} + \beta_{12},$$

and we are left with the five parameters α_{11}, α_{22}, β_{11}, β_{12}, and γ_{22}.

$$A(u) = \begin{pmatrix} \alpha_{11} & 0 \\ 0 & \alpha_{22} \end{pmatrix} + u_1 \begin{pmatrix} \beta_{11} & \beta_{12} \\ 0 & \beta_{11} - \gamma_{21} \end{pmatrix} + u_2 \begin{pmatrix} \gamma_{22} - \beta_{12} & 0 \\ \alpha_{22} - \alpha_{11} + \beta_{12} & \gamma_{22} \end{pmatrix}.$$

Verification of Hypothesis H2

Lemma (Positive semidefiniteness of M)

The matrix $M(u)$ is positive semidefinite for all $u \in D$ if and only if

$$\alpha_{11} \geq 0, \quad \alpha_{22} \geq 0, \quad \beta_{12} \leq \alpha_{11} + \min\{\beta_{11}, \gamma_{22}\}, \tag{A1}$$

$$\alpha_{11} + \beta_{11} \geq 0, \quad \alpha_{22} + \gamma_{22} \geq 0. \tag{A2}$$

Proof of lemma 3, step 1

Step 1: conditions (A1), (A2) are necessary. We first prove that the positive semidefiniteness of $M(u)$ implies (A1), (A2) by studying $M(u)$ close to the vertices of D. To this end, we define the matrix-valued functions, defined for $s \in (0, \frac{1}{2})$:

$$F_1(s) = sM(s,s), \quad F_2(s) = sM(1-2s,s), \quad F_3(s) = sM(s,1-2s).$$

We easily compute the limits:

$$F_i^0 \equiv \lim_{s \to 0^+} F_i(s) \qquad i = 1,2,3.$$

Since $M(u)$ is assumed to be positive semidefinite on D, also F_i^0 must be positive semidefinite for $i = 1,2,3$.
Sylvester's criterion applied to F_1^0, F_2^0, F_3^0 yields conditions (A1), (A2).

Proof of lemma 3, step 2

Step 2: Sign of the diagonal elements of M.

Claim. If (A1), (A2) hold then one of the two coefficients M_{11} or M_{22} is positive in D unless M is positive semidefinite in D.

$$
\begin{aligned}
f_1(u_2, u_3) &\equiv (1-u_2-u_3)u_3 M_{11}(1-u_2-u_3, u_2), \quad (u_2, u_3) \in D, \\
f_2(u_1, u_3) &\equiv (1-u_1-u_3)u_3 M_{22}(u_1, 1-u_1-u_3), \quad (u_1, u_3) \in D.
\end{aligned}
$$

Notice that:

- f_1 and f_2 are nonnegative on ∂D (easy direct computations).
- $\Delta_{(u_2,u_3)} f_1 = -\Delta_{(u_1,u_3)} f_2 = \text{constant}$ in D.

  $\Rightarrow$ either $\Delta_{(u_2,u_3)} f_1 \le 0$ or $\Delta_{(u_1,u_3)} f_2 \le 0$ in D.

By the **strong maximum principle**, there exists $i \in \{1, 2\}$ such that $f_i > 0$ in D unless $f_i \equiv 0$ in D. This means that $M_{ii} > 0$ in D unless $M_{ii} \equiv 0$ in D.

Proof of lemma 3, degenerate case

What if one of the coefficients M_{11} or M_{22} is identically zero in D?

In such a case M is positive semidefinite in D. In fact, straightforward computations lead to:

$$M_{11} \equiv 0 \quad \text{in } D \qquad \Rightarrow \qquad M = \alpha_{22} \begin{pmatrix} 0 & 0 \\ 0 & 1/u_2 \end{pmatrix},$$

$$M_{22} \equiv 0 \quad \text{in } D \qquad \Rightarrow \qquad M = \alpha_{11} \begin{pmatrix} 1/u_1 & 0 \\ 0 & 0 \end{pmatrix},$$

and since $\alpha_{11} \geq 0$ and $\alpha_{22} \geq 0$, the above matrices are positive semidefinite.

Proof of lemma 3, step 3

Step 3: Sign of the determinant of M.

By Step 2, we can assume that one of the two coefficients M_{11} or M_{22} is positive in D.

$$\det M = \det((D^2 h)A) = \det(D^2 h) \det A.$$

Since $\det D^2 h > 0$, then $\det M \geq 0$ if and only if $\det A \geq 0$.

Proof of lemma 3, step 3

It is straightforward to see that $\det A \geq 0$ on ∂D (direct computations).

Next, we consider the Hessian $C = D^2 \det A(u)$ with respect to u. Since $\det A$ is a (multivariate) quadratic polynomial in u, C is a symmetric constant matrix satisfying

$$\det C = -\big(\beta_{11}\beta_{12} + \gamma_{22}(\alpha_{11} - \alpha_{22} - \beta_{12})\big)^2 \leq 0.$$

Thus, one of the two eigenvalues of C is nonpositive, say $\lambda \leq 0$. Let $v \in \mathbb{R}^2 \backslash \{0\}$ be a corresponding eigenvector, i.e. $Cv = \lambda v$.

Proof of lemma 3, step 3

Let $u \in D$ be arbitrary. It exists a unique closed segment $\sigma \subset \overline{D}$,

$$\sigma \parallel v, \quad u \in \sigma, \quad \partial\sigma \subset \partial D.$$

We can write $\sigma = u + v\mathcal{I}$, $\mathcal{I} \subset \mathbb{R}$ suitable interval.

Let $\phi(s) = \det A(u + sv)$ for $s \in \mathcal{I}$.

$\Rightarrow$ $\phi''(s) = v^\top Cv = \lambda|v|^2 \leq 0$ for $s \in \mathcal{I}$.

$\Rightarrow$ ϕ is concave and attains its minimum at the extreme points of $\mathcal{I}$.

$\Rightarrow$ $\min_\sigma \det A = \min_{\partial\sigma} \det A$. But $\partial\sigma \subset \partial D$ and $\det A \geq 0$ on ∂D.

$\Rightarrow$ $\det A \geq 0$ on σ. $u \in \sigma \Rightarrow \det A(u) \geq 0$. $\square$

Verification of Hypothesis H2

Lemma (Strict positive definiteness of M)

Let conditions (C1)–(C4) hold. Then there exists $\varepsilon > 0$ such that for all $z \in \mathbb{R}^2$ and all $u \in D$,

$$z^\top M(u)z \geq \varepsilon \left(\frac{z_1^2}{u_1} + \frac{z_2^2}{u_2} \right).$$

In particular, Hypothesis H2 is fulfilled with $m_1 = m_2 = \frac{1}{2}$:

$$z^\top M(u)z \geq \alpha^* \sum_{i=1}^{2} u_i^{2(m_i-1)} z_i^2 \quad \forall z \in \mathbb{R}^2, \quad u \in D.$$

Proof of Lemma 4

The claim is equivalent to the positive semidefiniteness of the matrix $M^\varepsilon := M - \varepsilon\Lambda$ for a suitable $\varepsilon > 0$, where

$$\Lambda = \begin{pmatrix} 1/u_1 & 0 \\ 0 & 1/u_2 \end{pmatrix} = (D^2 h)P, \qquad P = \begin{pmatrix} 1-u_1 & -u_1 \\ -u_2 & 1-u_2 \end{pmatrix}.$$

Thus we can write $M - \varepsilon\Lambda = (D^2 h)A^\varepsilon$ with $A^\varepsilon = A - \varepsilon P$.

Proof of Lemma 4

We observe that A^ε has the same structure as A with perturbed parameters:

$$A = \begin{pmatrix} \alpha_{11} & 0 \\ 0 & \alpha_{22} \end{pmatrix} + u_1 \begin{pmatrix} \beta_{11} & \beta_{12} \\ 0 & \beta_{11} - \gamma_{21} \end{pmatrix} + u_2 \begin{pmatrix} \gamma_{22} - \beta_{12} & 0 \\ \alpha_{22} - \alpha_{11} + \beta_{12} & \gamma_{22} \end{pmatrix},$$

$$A^\varepsilon = \begin{pmatrix} \alpha_{11}^\varepsilon & 0 \\ 0 & \alpha_{22}^\varepsilon \end{pmatrix} + u_1 \begin{pmatrix} \beta_{11}^\varepsilon & \beta_{12}^\varepsilon \\ 0 & \beta_{11}^\varepsilon - \gamma_{21}^\varepsilon \end{pmatrix} + u_2 \begin{pmatrix} \gamma_{22}^\varepsilon - \beta_{12}^\varepsilon & 0 \\ \alpha_{22}^\varepsilon - \alpha_{11}^\varepsilon + \beta_{12}^\varepsilon & \gamma_{22}^\varepsilon \end{pmatrix},$$

$$\alpha_{11}^\varepsilon = \alpha_{11} - \varepsilon, \quad \alpha_{22}^\varepsilon = \alpha_{22} - \varepsilon, \quad \beta_{11}^\varepsilon = \beta_{11} + \varepsilon,$$
$$\beta_{12}^\varepsilon = \beta_{12} + \varepsilon, \quad \gamma_{22}^\varepsilon = \gamma_{22} + \varepsilon.$$

From Lemma 4 we conclude that M^ε is positive semidefinite if and only if (A1), (A2) holds for the parameters $(\alpha_{11}^\varepsilon, \alpha_{22}^\varepsilon, \beta_{11}^\varepsilon, \beta_{12}^\varepsilon, \gamma_{22}^\varepsilon)$ instead of $(\alpha_{11}, \alpha_{22}, \beta_{11}, \beta_{12}, \gamma_{22})$.

Proof of Lemma 4

The following conditions must hold:

$$\alpha_{11} - \varepsilon \geq 0, \quad \alpha_{22} - \varepsilon \geq 0, \quad \beta_{12} + \varepsilon \leq \alpha_{11} + \min\{\beta_{11}, \gamma_{22}\},$$
$$\alpha_{11} + \beta_{11} \geq 0, \quad \alpha_{22} + \gamma_{22} \geq 0.$$

Since $\varepsilon > 0$ is arbitrary:

$$\alpha_{11} > 0, \quad \alpha_{22} > 0, \quad \beta_{12} < \alpha_{11} + \min\{\beta_{11}, \gamma_{22}\},$$
$$\alpha_{11} + \beta_{11} \geq 0, \quad \alpha_{22} + \gamma_{22} \geq 0.$$

This means that M^ε is positive semidefinite for a suitable $\varepsilon > 0$ if and only if (C1)–(C4) hold. □

A particular case: the SKT model

Remember the population systems of Shigesada-Kawasaki-Teramoto type (SKT):

$$A(u) = \begin{pmatrix} a_{10} + 2a_{11}u_1 + a_{12}u_2 & a_{12}u_1 \\ a_{21}u_2 & a_{20} + a_{21}u_1 + 2a_{22}u_2 \end{pmatrix}, \qquad \text{(A-SKT)}$$

$$f_i(u) = (b_{i0} - b_{i1}u_1 - b_{i2}u_2)u_i, \qquad i = 1, 2.$$

Models for two competitive species in heterogeneous evironments.

$a_{ij} \geq 0,\ i, j = 0, 1, 2.$

$a_{11}, a_{22} \equiv$ self-diffusion, $a_{12}, a_{21} \equiv$ cross-diffusion coefficients.

$b_{11}, b_{22} \equiv$ intra-specific, $b_{12}, b_{21} \equiv$ inter-specific competition constants.

About the SKT model

- The existence of global weak solutions without any restriction on the diffusivities (except positivity) is known in several space dimensions.[1]
- Upper bounds are known if one space dimension if $a_{10} = a_{20}$.[2]
- If cross-diffusion is weaker than self-diffusion (i.e. $a_{12} < a_{22}$, $a_{21} < a_{11}$), weak solutions are bounded and Hölder continuous.[3]
- In the triangular case $a_{21} = 0$ the existence of global bounded solutions has been proved.[4]

[1] L. Chen and A. Jüngel. Analysis of a multi-dimensional parabolic population model with strong cross-diffusion. *SIAM J. Math. Anal.* (2004).
L. Chen and A. Jüngel. Analysis of a parabolic cross-diffusion population model without self-diffusion. *J. Diff. Eqs.* (2006).

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[3] D. Le. Global existence for a class of strongly coupled parabolic systems. *Ann. Math.* (2006).

[4] Y. Choi, R. Liu, and Y. Yamada. Existence of global solutions for the Shigesada-Kawasaki-Teramoto model with weak cross-diffusion. *Discrete Cont. Dynam. Sys.* (2003).

Corollary (Bounded weak solutions to the (SKT) model)

Let the assumptions of Theorem 2 hold except that the coefficients of A, defined in (A-SKT)*, are nonnegative and satisfy $a_{10} > 0$, $a_{20} > 0$ and:*

$$a_{21} = a_{11}, \quad a_{22} = a_{12}, \quad a_{20} - a_{10} = a_{11} - a_{22} \geq 0. \tag{C-SKT}$$

Furthermore, let $f(u)$ be given by the Lotka-Volterra terms:

$$f_i(u) = (b_{i0} - b_{i1}u_1 - b_{i2}u_2)u_i, \quad i = 1, 2,$$
$$b_{10} \leq \min\{b_{11}, b_{12}\}, \quad b_{20} \leq \min\{b_{21}, b_{22}\}.$$

Then there exists a bounded weak solution $u = (u_1, u_2)$ to (RD) satisfying u_1, $u_2 \geq 0$, $u_1 + u_2 \leq 1$ in $\Omega \times (0, \infty)$, and

$$u \in L^2_{\mathrm{loc}}(0, \infty; H^1(\Omega; \mathbb{R}^2)), \quad \partial_t u \in L^2_{\mathrm{loc}}(0, \infty; H^1(\Omega; \mathbb{R}^2)').$$

The SKT model: bounded weak solutions

The novelty of this corollary is not the global existence result but the uniform boundedness of weak solutions.

With conditions (C-SKT), the diffusion matrix becomes

$$A(u) = \begin{pmatrix} a_{10} + 2a_{11}u_1 + a_{12}u_2 & a_{12}u_1 \\ a_{11}u_2 & a_{20} + a_{11}u_1 + 2a_{12}u_2 \end{pmatrix},$$
$$a_{12} = a_{11} + a_{10} - a_{20},$$

i.e., we are left with three parameters a_{10}, a_{20}, and a_{11}.

- The cross-diffusion coefficient of one species is the same as the self-diffusion of the other species.
- The self-diffusion of species 1 is larger than that for species 2 .
- In the reaction term, the growth rates b_{10}, b_{20} are not larger than the intra- and inter-specific competition rates.

Proof Corollary 5

The corollary follows from Theorem 2 by specifying the diffusivities according to (A-SKT).

The requirement of the symmetry of $H(u)A(u)$ leads to the conditions $a_{11} = a_{21}$, $a_{22} = a_{12}$, and $a_{20} - a_{10} = a_{11} - a_{22}$, whereas (C3), (C4) become $a_{10} > 0$, $a_{20} > 0$, and $-a_{12} < a_{10} + 2\min\{a_{20} - a_{10}, 0\}$.

Taking into account that $a_{10} \le a_{20}$, the last condition is equivalent to $-a_{12} < a_{10}$, and this inequality holds since a_{10} is positive.

Finally, Hypothesis H3 follows from the inequality $g_i(u) = b_{i0} - b_{i1}u_1 - b_{i2}u_2 \le b_{i0} - \min\{b_{i1}, b_{i2}\}(u_1 + u_2) \le 0$ for $1 - \varepsilon < u_1 + u_2 < 1$, where $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and $\varepsilon_i = 1 - b_{i0}/\min\{b_{i1}, b_{i2}\} \in (0, 1)$.

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