Sharp Interface Limits of Phase Fields

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Contents

1 Review

1.1 Notation

Ω is an open subset of $\mathbb{R}^n$, usually open with Lipschitz boundary (a bounded Lipschitz set). $E \Subset \Omega$ denotes that $\overline{E} \subset \Omega$ and $\overline{E}$ is compact (even if Ω is not bounded).

1.2 Sobolev Spaces

Let u be an L^1_{loc}-function on Ω. Assume that there exists an L^1_{loc} function v such that the 'integration by parts' formula

$$\int_\Omega u\,\partial_i\phi\,\mathrm{d}x = -\int_\Omega v\phi \qquad \forall\ \phi \in C_c^\infty(\Omega)$$

holds. Then we say that v is the weak (or distributional) derivative of u in e_i-direction and write $v = \partial_i u$. Classical derivatives of smooth functions are weak derivatives and weak derivatives are unique (up to null sets, of course). The Sobolev space $W^{1,p}$ is the space of L^p-functions whose weak derivatives exist and also lie in L^p. It is a Banach space with the norm

$$||u||_{W^{1,p}} = \left(||u||_{L^p}^p + \sum_i ||\partial_i u||_{L^p}^p \right)^{\frac{1}{p}}$$

Recall the following result.

Theorem 1.1 (Rellich-Kondrakhov)**.** *Let Ω be a bounded Lipschitz set. For $p < n$, $W^{1,p}(\Omega)$ embeds continuously into $L^q(\Omega)$ with $q \leq p^* := \frac{np}{n-p}$ and the embedding is compact for $q < p^*$, i.e.*

$$||u||_{L^q(\Omega)} \leq C_{n,p,q,\Omega}\, ||u||_{W^{1,p}(\Omega)} \qquad \forall\ q \leq p^* \text{ and } u \in W^{1,p}(\Omega)$$

and for $q < p^*$ any sequence u_j which is bounded in $W^{1,p}$ has a subsequence which converges to some limit u in $L^q(\Omega)$ with respect to the L^q-norm. Note that $p^* > p$ for all n and that $p^* \to \infty$ as $p \nearrow n$.

For $p > n$, $W^{1,p}$ embeds continuously into the space of continuous functions, i.e. $u \in C^0(\overline{\Omega})$ and

$$||u||_{C^0(\overline{\Omega})} \leq C_{n,p,\Omega}\, ||u||_{W^{1,p}}$$

The embedding is also compact, i.e. if u_j is a sequence which is bounded in $W^{1,p}(\Omega)$, then there exists a continuous function u on $\overline{\Omega}$ such that $u_j \to u$ uniformly on $\overline{\Omega}$ (i.e. with respect to the C^0-norm).

For further information, see e.g. [Bre11, Dob10].

1.3 Functional Analysis and the Calculus of Variations

Let X be a Banach space and X^* its normed dual space (the vector space of all linear continuous maps from X to $\mathbb{R}$). Then we say that $x_j \rightharpoonup x$ (in words: x_n converges weakly to x) if $\phi(x_n) \to \phi(x)$ for all $\phi \in X^*$. Note that strong convergence (i.e. convergence with respect to the norm) implies weak convergence, but the converse is not true (except for finite dimensional and very pathological infinite dimensional spaces).

Remark 1.2. If $u_j, u \in L^p(\Omega)$, we have that

$$u_j \rightharpoonup u \qquad \Leftrightarrow \qquad \int_\Omega u_j \phi \to \int_\Omega u\phi \qquad \forall \phi \in C_c^\infty(\Omega).$$

Purely from the definition, there are more test functions since the dual space of $L^p(\Omega)$ is $L^{\frac{p}{p-1}}(\Omega)$ by the Riesz representation theorem. In particular, the function $\phi \equiv 1$ is an admissible test function if and only if Ω has finite measure.

If $u_j, u \in W^{1,p}(\Omega)$, we have that

$$u_j \rightharpoonup u \qquad \Leftrightarrow \qquad u_j \rightharpoonup u \text{ in } L^p(\Omega) \text{ and } \partial_i u_j \rightharpoonup \partial_i u \text{ in } L^p(\Omega)$$

for all $i = 1, \ldots, n$.

Theorem 1.3. *Let $1 < p < \infty$. Any sequence u_k which is uniformly bounded in $L^p(\Omega)$ or $W^{1,p}(\Omega)$ has a weakly convergent subsequence.*

This is a (weak) substitute of the Heine-Borel theorem in finite dimensional Banach spaces, which says that bounded closed sets are compact. This is *never* true in infinite-dimensional spaces, so we need to pass to weak topologies (more precisely, the weak* topology, but this makes no difference in L^p-spaces, at least for $p \in (1, \infty)$).

Definition 1.4. A functional $F : X \to \mathbb{R} \cup \{+\infty\}$ is called convex, if $F \not\equiv +\infty$ and

$$F(\lambda x + (1-\lambda)y) \leq \lambda\, F(x) + (1-\lambda)\, F(y)$$

for all $x, y \in X$ and $\lambda \in (0, 1)$.

Definition 1.5. A functional $F : X \to \mathbb{R} \cup \{+\infty\}$ is called weakly (sequentially) lower semi-continuous, if $F \not\equiv +\infty$ and

$$x_k \rightharpoonup x \quad \Rightarrow \quad F(x) \leq \liminf_{k\to\infty} F(x_k).$$

Theorem 1.6. *Convex functionals which are continuous with respect to the norm are weakly lower semi-continuous.*

Proof. A continuous convex function is the supremum of all continuous affine functions lying below it pointwise. In formulae, we have

$$\phi(x) = \sup\{Lx + \alpha \mid \text{where } L \in X^* \text{ and } \alpha \in \mathbb{R} \text{ such that } Ly + \alpha \leq \phi(y) \text{ for all } \ y \in \mathbb{R}\}.$$

(Proofs of this fact usually use the Hahn-Banach theorem. A graphical justification in one dimension is obvious.) Assume that $x_n \rightharpoonup x$, so

$$Lx_n + \alpha \to Lx + \alpha$$

for all $L \in X^*$ and $\alpha \in \mathbb{R}$ since elements of the dual space are continuous in the weak topology by the construction of the weak topology. Thus

$$\begin{aligned}
\phi(x) &= \sup_{(L,\alpha)\in\mathcal{M}} (Lx + \alpha) \\
&\leq L_0 x + \alpha_0 + \varepsilon \\
&= \lim_{n\to\infty} (L_0 x_n + \alpha_0) + \varepsilon \\
&\leq \liminf_{n\to\infty} \sup_{(L,\alpha)\in\mathcal{M}} (Lx_n + \alpha) + \varepsilon \\
&\leq \liminf_{n\to\infty} \phi(x_n) + \varepsilon
\end{aligned}$$

when we choose L_0, α_0 close enough to the supremum (measured in ε). Letting $\varepsilon \to 0$ proves the statement. The key point is that the parameter set

$$\mathcal{M} = \{L \in X^* \text{ and } \alpha \in \mathbb{R} \text{ such that } Ly + \alpha \leq \phi(y) \text{ for all } \ y \in \mathbb{R}\}$$

does not depend on the points x, x_n! $\square$

Example 1.7. The functional $F : L^p(\Omega) \to \mathbb{R}$ given by $F(u) = \int_\Omega f(u)\,\mathrm{d}x$ is lower semi-continuous if and only if f is convex.

Proof. If f is convex, then F is convex. Since f is convex on $\mathbb{R}$, it is continuous, and thus F is continuous (assuming some growth condition on f).

If f is not convex, we can construct a counterexample. For simplicity, take $\Omega = [0,1]$ and assume that $W(s) = (s^2-1)^2$ which has two zeros at $+1$ and -1 and is strictly positive in between. Consider

$$u_j(x) = \begin{cases} 1 & x \in [0, 2^{-j}) \\ 0 & x \in [2^{-j}, 2 * 2^{-j}) \\ 1 & x \in [2 * 2^{-j}, 3 * 2^{-j}) \\ \vdots & \vdots \end{cases}$$

where $u_j \rightharpoonup u \equiv 0$ (highly oscillating sequences weakly approach their mean). We have $F(u_j) \equiv 0$, but $F(u) = W(0) > 0$, so F is not lower semi-continuous. In general, we use the same idea of using oscillations for convex combination. $\square$

A great source for everything related to standard functional analysis and weak topologies is [Bre11].

1.4 The Modica-Mortola Functional

We assume from now on that Ω is bounded and has a Lipschitz boundary (think of the unit ball, unit cube, the area bounded by a torus...). The Modica-Mortola functional is given by

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \int_\Omega \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \, \mathrm{d}x & u \in W^{1,2}(\Omega) \cap L^4(\Omega) \\ +\infty & u \in L^1(\Omega) \setminus \left[W^{1,2}(\Omega) \cap L^4(\Omega)\right] \end{cases}$$

for $W(u) = \frac{(u^2-1)^2}{4}$. The functional is sometimes also referred to as the Ginzburg-Landau functional or the Cahn-Hilliard energy. Its time-normalised L^2-gradient flow is the Allen-Cahn equation

$$\varepsilon \, u_t = \varepsilon \, \Delta u - \frac{1}{\varepsilon} W'(u)$$

and its H^{-1}-gradient flow is the Cahn-Hillard equation

$$\varepsilon \, u_t = -\Delta \left(\varepsilon \, \Delta u - \frac{1}{\varepsilon} W'(u) \right).$$

The latter one is integral preserving, so as $t \to \infty$, we expect to see minimisers of $\mathcal{F}_\varepsilon$ in the class of functions with fixed integral. Let us prove rigorously that those exist.

Theorem 1.8. *For all $\alpha \in [-1,1]$ there exists a minimiser of $\mathcal{F}_\varepsilon$ in the set*

$$K_\alpha = \left\{ u \in L^1(\Omega) \, : \, \frac{1}{|\Omega|} \int_\Omega u \, \mathrm{d}x = \alpha \right\}.$$

Proof. We prove the theorem with the *direct method of the calculus of variations.* Let $u_k \in L^1(\Omega)$ be a sequence such that

$$\lim_{k \to \infty} \mathcal{F}_\varepsilon(u_k) = \inf_{u \in K_\alpha} \mathcal{F}_\varepsilon(u) < \infty.$$

The bounded energies induce uniform bounds in $W^{1,2}$ and L^4, since

$$W(u) = \frac{u^4}{4} - \frac{u^2}{2} + \frac{1}{4}$$

controls the L^4 norm, thus also the L^2-norm (because Ω is bounded). Here we use that only the highest order term of the polynomial matters for large arguments. So there is a subsequence which converges weakly simultaneously in both spaces to the *same* limit u. We split

$$\mathcal{F}_\varepsilon(w) = \int_\Omega \frac{\varepsilon}{2}\,|\nabla w|^2 + \frac{w^4}{4\varepsilon}\,\mathrm{d}x + \int_\Omega \frac{-2w^2+1}{4\varepsilon}\,\mathrm{d}x.$$

The first term is weakly lower semi-continuous because it is convex, the second is even weakly continuous because of the *compact* embedding $W^{1,2} \to L^2$. In total, $\mathcal{F}_\varepsilon$ is weakly lower semi-continuous because both terms are individually, and thus

$$\mathcal{F}_\varepsilon(u) \le \liminf_{k\to\infty} \mathcal{F}_\varepsilon(u_k) = \inf_{u\in K_\alpha} \mathcal{F}_\varepsilon(u) < \infty,$$

so u is a minimiser since also

$$\int_\Omega u\,\mathrm{d}x = \alpha|\Omega|$$

due to the compact embedding into L^1. □

Remark 1.9. The average condition is needed to make the problem non-trivial since otherwise the minimiser is just given by $u \equiv 1$ or $u \equiv -1$. We could do the same without prescribing an average but adding a forcing term to the functional and considering

$$\widetilde{\mathcal{F}}_\varepsilon(u) = \mathcal{F}_\varepsilon(u) - \int_\Omega f\,u\,\mathrm{d}x$$

for some $f \in L^\infty(\Omega)$.

Minimisers u_ε of $\mathcal{F}_\varepsilon$ are uniformly bounded in L^4, so there exists a weak limit u. By construction,

$$\alpha\,|\Omega| = \int_\Omega u_\varepsilon = \int_\Omega u_\varepsilon \cdot 1 \to \int_\Omega u \cdot 1 = \int_\Omega u$$

by weak convergence (using $\phi(x) \equiv 1$ as a test function). Can we say anything more about u? For deep results relating to this question and the convergence, see for example [CC06, LM89]. In the following we will lay the framework which these deeper results use.

2 Gamma-Convergence

We are interested in the limit of $\mathcal{F}_\varepsilon$ as $\varepsilon \to 0$. Note that

$$\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u) = \infty$$

for all $u \in W^{1,2}(\Omega) \cap L^4(\Omega)$ which are not either $u \equiv -$ or $u \equiv 1$ since the second term blows up (if Ω is connected, componentwise constant else). Thus the pointwise limit does not tell us anything about $\mathcal{F}_\varepsilon$ and we need a different concept of limits. We will use a notion which is particularly adapted to energy minimisation.

Definition 2.1. Let (X, d) be a metric space, $\mathcal{F}_\varepsilon, \mathcal{F} : X \to \mathbb{R} \cup \{+\infty\}$ functions, $x \in X$. Then we say that $\mathcal{F}_\varepsilon$ Γ-converges to $\mathcal{F}$ at x and write

$$\left[\Gamma - \lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon\right](x) = \mathcal{F}(x)$$

if the following two conditions are met.

1. For any sequence $x_\varepsilon \xrightarrow{d} x$, we have $\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(x_\varepsilon) \geq \mathcal{F}(x)$. ($\liminf$-inequality)
2. There exists a sequence $x_\varepsilon \xrightarrow{d} x$ such that $\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(x_\varepsilon) \leq \mathcal{F}(x)$. ($\limsup$-inequality)

We say that $\mathcal{F}_\varepsilon$ Γ-converges to $\mathcal{F}$ if it Γ-converges at every point. The sequence in the $\limsup$-inequality is called a *recovery sequence.*

This is the right kind of convergence as explained in the following lemma. Note that the sum of two Γ-convergent sequences need not be Γ-convergent! The $\liminf$-inequality is of course preserved, but we might not be able to find a simultaneous recovery sequence for both converging functionals, so that the $\limsup$-inequality might be violated.

Lemma 2.2. *1. If $\Gamma - \lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon = \mathcal{F}$ and $x_\varepsilon \to x_0$ is a sequence such that*

$$\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(x_\varepsilon) = \lim_{\varepsilon\to 0} \inf_{y\in X} \mathcal{F}_\varepsilon(y),$$

then $\mathcal{F}(x_0) = \inf_{y\in X} \mathcal{F}(y)$. ("Minimisers converge to minimisers")

2. If $\Gamma - \lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon = \mathcal{F}$ and $\mathcal{G}_\varepsilon \to \mathcal{G}$ uniformly, $\mathcal{G}$ is continuous, then

$$\Gamma - \lim_{\varepsilon\to 0}(\mathcal{F}_\varepsilon + \mathcal{G}_\varepsilon) = \mathcal{F} + \mathcal{G}.$$

3. Let $\mathcal{F} = \Gamma - \lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon$. Then $\mathcal{F}$ is lower semi-continuous.

Proof. **Ad (1).** By definition, $\mathcal{F}(x_0) \leq \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(x_\varepsilon) = \liminf_{\varepsilon\to 0} \inf_{x\in X} \mathcal{F}_\varepsilon(x)$. On the other hand, take a point $\bar{x}$ such that $\mathcal{F}(\bar{x}) \leq \inf_{x\in X} \mathcal{F}(x) + \delta$. Then there exists a sequence $\bar{x}_\varepsilon \to x$ such that

$$\inf_{x\in X} \mathcal{F}(x) + \delta \geq \mathcal{F}(\bar{x}) \geq \limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\bar{x}_\varepsilon) \geq \liminf_{\varepsilon\to 0} \inf_{y\in X} \mathcal{F}_\varepsilon(y) \geq \mathcal{F}(x_0).$$

Taking $\delta \to 0$ shows that x_0 is a minimiser.

Ad (2). Let $x_\varepsilon \to x$, then

$$\begin{aligned}\liminf_{\varepsilon\to 0}(\mathcal{F}_\varepsilon + \mathcal{G}_\varepsilon)(x_\varepsilon) &\geq \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(x_\varepsilon) + \liminf_{\varepsilon\to 0} \mathcal{G}(x_\varepsilon) + \liminf_{\varepsilon\to 0}(\mathcal{G}_\varepsilon - \mathcal{G})(x_\varepsilon)\\ &\geq \mathcal{F}(x) + \mathcal{G}(x)\end{aligned}$$

using both continuity (actually, lower semi-continuity) and uniform convergence. Now let x_ε be a recovery sequence for $\mathcal{F}_\varepsilon$, then

$$\limsup_{\varepsilon\to 0}(\mathcal{F}_\varepsilon + \mathcal{G}_\varepsilon)(x_\varepsilon) = \limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(x_\varepsilon) + \limsup_{\varepsilon\to 0} \mathcal{G}(x_\varepsilon) + (\mathcal{G}_\varepsilon - \mathcal{G})(x_\varepsilon) \geq (\mathcal{F} + \mathcal{G})(x)$$

using both continuity (actually, upper semi-continuity) and uniform convergence.

Ad (3). Let $x_k \to x$. We want to show that

$$\mathcal{F}(x) \leq \liminf_{k\to\infty} \mathcal{F}(x_k).$$

So take $\varepsilon = \varepsilon_k$ so small and $\tilde{x}_k$ such that $d(x_k, \tilde{x}_k) < 1/k$ and

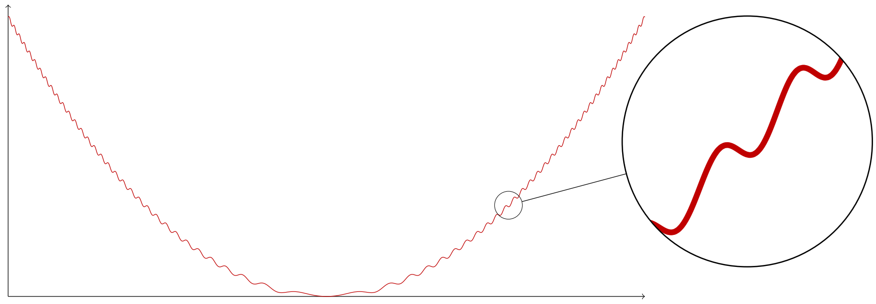
$$\mathcal{F}_{\varepsilon_k}(\tilde{x}_k) \leq \mathcal{F}(x_k) + 1/k.$$

Such ε_k and $\tilde{x}_k$ exist by the $\liminf$ inequality. Since $x_k \to x$ and $d(x_k, \tilde{x}_k) \to 0$, we have $\tilde{x}_k \to x$ and thus

$$\mathcal{F}(x) \leq \liminf_{k\to\infty} \mathcal{F}_{\varepsilon_k}(\tilde{x}_k) \leq \liminf_{k\to\infty} \mathcal{F}(x_k)$$

by construction. □

Remark 2.3. Note that local minimisers do not generally converge to local minimisers! The 'functionals' (i.e. functions) f_ε on the real line (with the usual metric) given by $f_\varepsilon(x) = x^2 + 2\varepsilon\sin(x^2/\varepsilon)$ have a large number of local minimisers on the whole real line, but their Γ-limit x^2 only has *one* minimiser at $x = 0$. Of course they converge under some *uniformity condition* on the domain where the functions are minimisers.



Remark 2.4. Fixing $\frac{1}{|\Omega|}\int_\Omega u\,\mathrm{d}x = \alpha$ is not a compact perturbation, but we adapt the space $X = K_\alpha$ and everything goes through. Adding a force term $\int_\Omega fu\,\mathrm{d}x$ is a compact perturbation.

Remark 2.5. We have shown that if the minimisers of Γ-convergent functionals converge to a point, then that point is a minimiser of the limit functional. For the existence of a convergent subsequence we need a mild 'equi-coercivity' condition. Assume that there exists a compact set $K \subset X$ such that

$$\inf_{y\in K} \mathcal{F}_\varepsilon(y) = \inf_{y\in X} \mathcal{F}_\varepsilon(y) \qquad \forall\, \varepsilon > 0,$$

then $\mathcal{F}$ has a minimiser since we can extract a convergent subsequence from an almost minimising sequence x_ε for $\mathcal{F}_\varepsilon$ that lies in the compact set K. If $\mathcal{F}_\varepsilon$ is lower semi-continuous for all ε, the functionals $\mathcal{F}_\varepsilon$ have minimisers in X which lie in K and the sequence of minimisers converges to a minimiser of $\mathcal{F}$.

Example 2.6.

1. Uniform Convergence of continuous functionals $\Rightarrow$ Γ-convergence.
2. Let us look at constant sequences where we just repeat one element of a function space all the time (which is not a constant function). Observe that $\chi_{\mathbb{Q}} \to 0$ in the sense of Γ-convergence,
3. $\chi_{\mathbb{R}\setminus\mathbb{Q}} \xrightarrow{\Gamma} 0$, but
4. $\Gamma - \lim_{\varepsilon\to 0}(\chi_{\mathbb{Q}} + \chi_{\mathbb{R}\setminus\mathbb{Q}}) \equiv 1$.
5. Γ-convergence can for example be used to deduce 2D elasticity from 3D elasticity since we don't need *pointwise* constructions.

Further properties and examples of Γ-convergence can be found for example in [Bra02], where also the example of Modica-Mortola in one dimension is treated in a similar way to our approach.

3 The Modica-Mortola Functional

3.1 The Modica-Mortola Functional in One Dimension

We first look at the convergence of the Modica-Mortola functional $\mathcal{F}_\varepsilon$ in one dimension. Let $I = [0,1]$ denote the unit interval. Note that

$$\begin{aligned}\int_x^y \frac{\varepsilon}{2}\,|u'|^2 + \frac{1}{\varepsilon}\,W(u)\,\mathrm{d}\xi &\geq \int_x^t \sqrt{2\,W(u)}\,u'\,\mathrm{d}\xi \\ &= \int_{u(x)}^{u(y)} \sqrt{2\,W(z)}\,\mathrm{d}z.\end{aligned}$$

Lemma 3.1 (equi-coercivity)**.** *Assume that* $\sup_{\varepsilon\to 0}\mathcal{E}_\varepsilon(u_\varepsilon) < \infty$*. Then there exists a piecewise constant function* $u : I \to \{-1,1\}$ *such that* $u_\varepsilon \to u$ *in* $L^1(I)$*.*

Here we take a piece-wise constant function to be a function that $u = \sum_{i=1}^N \alpha_i\,\chi_{I_i}$ where $\alpha_i \in \mathbb{R}$ and χ_{I_i} is the characteristic function of an interval $I_i = [a_i, b_i)$. So the function takes finitely many values and jumps a finite number of times. It is no restriction to assume that the intervals I_i are disjoint. We write $u \in PC(I)$ or $U \in PC(I, \{\alpha_1, \ldots, \alpha_n\})$ if we want to specify which values the function may take.

Proof of Lemma 3.1. Note that $\int_I (u_\varepsilon^2 - 1)^2\,\mathrm{d}x \leq \varepsilon\cdot\mathcal{F}_\varepsilon(u_\varepsilon) \to 0$, so $u_\varepsilon^2 \to 1$ in $L^2(I)$. In particular, we deduce that $|u_\varepsilon| \to 1$ pointwise almost everywhere (for a subsequence). Now partition I into N intervals $I_j = [\frac{j-1}{N}, \frac{j}{N}]$ of length $\frac{1}{N}$, $j = 1, \ldots, N$. Denote

$$\overline{u}_{j,\varepsilon} = \max_{x\in I_j} u_\varepsilon(x), \qquad \underline{u}_{j,\varepsilon} = \min_{x\in I_j} u_\varepsilon(x).$$

The maxima and minima exist because $W^{1,2}$-functions are continuous in one dimension (or using the density of smooth functions and choosing all u_ε to be smooth). We note that the set of indexes j where these differ significantly has uniformly bounded cardinality, namely set

$$J^\delta_{\varepsilon,N} = \big\{ j \in \{1, \ldots, N\} \;:\; \overline{u}_{j,\varepsilon} - \underline{u}_{j,\varepsilon} \geq \delta \big\}$$

and

$$\omega(\delta) = \min_{\xi\in\mathbb{R}}\,\min_{\eta\geq\xi+\delta} \int_\xi^\eta \sqrt{2\,W(z)}\,\mathrm{d}z.$$

Then by our initial observation

$$\#J^\delta_{\varepsilon,N} \leq \frac{\sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon)}{\omega(\delta)},$$

thus we only jump by an amount of at least δ only in a uniformly bounded number of subintervals, independently of N. We keep δ and N fixed and note that $J^\delta_{\varepsilon,N}$ is a sequence (in ε) with values in the (finite!) power set of $\{1, \ldots, N\}$. Therefore there exists a state in the power set that is hit infinitely often by $J^\delta_{\varepsilon,N}$. We pass to a subsequence in ε with this property so that $J^\delta_{\varepsilon,N} \equiv J^\delta_N$.

Let us fix $\delta = 1/2$ and write $J^\delta_N \equiv J_N$. We want to show that u_ε converges to some function $u \in PC(I)$ (the space of piecewise constant functions) in L^1 on

$$L_N := I \setminus \bigcup_{j\in J_N} I_j.$$

Fix η. For small enough ε we know that there exists $x \in I_j$ such that

$$|u_\varepsilon(x)| \geq 1 - \eta$$

due to the convergence pointwise almost everywhere. Since u_ε does not oscillate by more than $\delta = 1/2$ on the complement of the J_N-intervals and since u_ε is uniformly bounded (because $\omega(\delta) \to \infty$ as $\delta \to \infty$), we obtain convergence pointwise almost everywhere on L_N and L^1-convergence by the dominated convergence theorem. We can let $N \to \infty$ since $\# J_N^\delta$ is uniformly bounded and obtain a piecewise constant function. $\square$

We had argued that we get a weak L^4-limit of u_ε in any dimension. The strong L^1-convergence is a much stronger statement, and the precise characterisation of the limit is important in the computation of the Γ-limit.

Theorem 3.2 (Γ-limit)**.** *We have*

$$\Gamma(L^1) - \lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u) = \mathcal{F}(u) = c_0 \begin{cases} \#(\textit{jump set of } u) & \textit{if } u \in PC(I, \{-1,1\}) \\ +\infty & \textit{else} \end{cases}$$

where $c_0 = \int_{-1}^1 \sqrt{2\,W(z)}\,\mathrm{d}z$.

Proof. $\liminf$**-inequality.** The $\liminf$-inequality is only interesting if $\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon) < \infty$, so we see that $u_\varepsilon \to u$ for a piecewise constant function with values ± 1 and a finite number of jumps from the previous Lemma. Take a jump point x and choose an interval $(\underline{y}, \overline{y})$ such that u only jumps at x in $(\underline{y}, \overline{y})$.

Fix $\eta > 0$. For small enough ε, there exist $\underline{y} < \underline{y}_\varepsilon < \overline{y}_\varepsilon < \overline{y}$ such that

$$u_\varepsilon(\underline{y}_\varepsilon) < -1 + \eta, \qquad u_\varepsilon(\overline{y}_\varepsilon) > 1 - \eta,$$

so

$$\begin{aligned} \int_{\underline{y}}^{\overline{y}} \frac{\varepsilon}{2}\,|u'|^2 + \frac{1}{\varepsilon}\,W(u)\,\mathrm{d}\xi &\geq \int_{\underline{y}_\varepsilon}^{\overline{y}_\varepsilon} \sqrt{2\,W(u)}\,|u'|\,\mathrm{d}\xi \\ &= \int_{-1+\eta}^{1-\eta} \sqrt{2\,W(z)}\,\mathrm{d}z. \end{aligned}$$

We add up the different jump intervals, let $\eta \to 0$, and see that

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon) \geq \#(\text{jump set}) \cdot \int_{-1}^1 \sqrt{2\,W(z)}\,\mathrm{d}z.$$

$\limsup$**-inequality.** Let u be a piecewise constant function with values in $\{-1, 1\}$ (and finitely many jumps). For a recovery sequence, we just need a function u_ε such that

$$\frac{\varepsilon}{2}\,(u_\varepsilon')^2 + \frac{1}{\varepsilon}\,W(u_\varepsilon) \approx \sqrt{2\,W(u_\varepsilon)}\,|u_\varepsilon'|$$

with an error going to zero as $\varepsilon \to 0$. An explicit solution of this is $u_\varepsilon(x) = \tanh\left(\frac{x}{\sqrt{2}\,\varepsilon}\right)$. In general, this is constructed for $\varepsilon = 1$ and then appropriately rescaled as

$$u_\varepsilon(x) = u_1(x/\varepsilon).$$

Equality holds in Young's inequality if and only if both terms are equal, so we need to solve

$$(q')^2 = 2\,W(q) \qquad \Rightarrow \qquad q' = \sqrt{2\,W(q)}$$

if we focus on monotone solutions. We can solve this on some maximal interval forwards and backwards with $q(0) = 0$. If $q(x) \approx 1$, the equation is

$$\begin{aligned}(1-q)' &= -q' = -\sqrt{2\,W(q)}\\ &\approx -\sqrt{2\,W(1) + 2\,W'(1)\,(q-1) + W''(1)\,(q-1)^2} = -\sqrt{W''(1)}\,|q-1|,\end{aligned}$$

so q approaches the potential wells at ± 1 exponentially fast. This gives us a recovery sequence for one jump infinitely long jump since the transition happens only in infinite space. But since the function goes to ± 1 exponentially fast at $\pm\infty$ and the derivative does the same (by the ODE), we can (for small enough ε) piece several functions like this together around the different jumps. $\square$

Remark 3.3. Note that *uniform* convergence does not hold, even away from the jump points, but that staying away from ± 1 at any point costs a certain amount of energy. Thus the *δ-distant set*

$$A_\delta := \left\{x \in I \setminus (\text{jump set of } u) \;:\; \exists\, x_\varepsilon \to x, \limsup_{\varepsilon\to 0} |u_\varepsilon(x_\varepsilon) - u(x)| \geq \delta\right\}$$

is finite for all $\delta > 0$. We call this *morally uniform convergence.* Every *recovery sequence* on the other hand converges uniformly, since all energy is already in the jumps.

Big question: How do we generalise this to higher dimension?

3.2 Counting jumps - some heuristic motivation

Let $u \in PC(I, \{a_1, \ldots, a_N\})$ be a piecewise constant function. If we take a smooth approximation u_η of u which is monotone around jumps, we get the amount of almost jumps as

$$\text{total jump height} \approx \int_0^1 |u_\eta'|\,\mathrm{d}x = \sup\left\{\int_0^1 u_\eta'\,\phi\,\mathrm{d}x \;\middle|\; \phi \in C_c^\infty(I), \quad |\phi| \leq 1\right\}$$

where ϕ picks the sign ± 1 depending on whether we jump up or down. Since ϕ is smooth by definition and vanishes at ± 1, we get

$$\int_0^1 u_\eta'\,\phi\,\mathrm{d}x = -\int_0^1 u_\eta\,\phi'\,\mathrm{d}x$$

when we integrate by parts, so

$$\int_0^1 |u_\eta'|\,\mathrm{d}x = \sup\left\{\int_0^1 u_\eta\,\phi'\,\mathrm{d}x \;\middle|\; \phi \in C_c^\infty(I), \quad |\phi| \leq 1\right\}.$$

But here we can pass to the limit in η! So we now formally set

$$\int_0^1 |u'|\,\mathrm{d}x = \sup\left\{\int_0^1 u\,\phi'\,\mathrm{d}x \;\middle|\; \phi \in C_c^\infty(I), \quad |\phi| \leq 1\right\}.$$

also for piecewise constant functions. This generalises to higher dimension.

3.3 Functions of Bounded Variation

Definition 3.4. Let $u \in L^1(\Omega)$. We define the total variation of u in Ω by

$$TV(u,\Omega) := \sup\left\{ \int_\Omega u \operatorname{div}(\phi)\,\mathrm{d}x \,\middle|\, \phi \in C_c^\infty(\Omega,\mathbb{R}^n), \quad |\phi| \leq 1 \right\}.$$

Similarly as in one dimension, we can identify $TV(u,\Omega) = \int_\Omega |\nabla u|\,\mathrm{d}x$ by a partial integration, where this time ϕ should not just be ± 1, but also point in the same direction on S^{n-1} as the gradient of u in the smooth case.

Definition 3.5. The space of functions of bounded variation is defined as

$$BV(\Omega) = \left\{ u \in L^1(\Omega) \mid TV(u,\Omega) < \infty \right\}$$

and equipped with the norm

$$||u||_{BV(\Omega)} = ||u||_{L^1(\Omega)} + TV(u,\Omega).$$

Lemma 3.6. *[EG92, Section 5.2]*

1. *$BV(\Omega)$ is a Banach space.*
2. *If $u \in W^{1,1}(\Omega)$, then $TV(u,\Omega) = \int_\Omega |\nabla u|\,\mathrm{d}x$.*
3. *The smooth functions are weakly dense in $BV(\Omega)$: Let η_ε be a mollifier and $u_\varepsilon = u * \eta_\varepsilon \in L^1(\Omega')$ for $\Omega' \Subset \Omega$. Then*

$$u_\varepsilon \xrightarrow{L^1(\Omega')} u, \qquad \int_{\Omega'} |\nabla u_\varepsilon|\,\mathrm{d}x \leq TV(u,\Omega).$$

4. *$BV(\Omega)$ embeds continuously into $L^{\frac{n}{n-1}}(\Omega)$, i.e.*

$$||u||_{L^p} \leq C_{n,p,\Omega}\, ||u||_{BV} \qquad \forall\; u \in BV(\Omega), \quad p \leq \frac{n}{n-1}.$$

5. *$BV(\Omega)$ embeds compactly and with closed image into $L^p(\Omega)$ for $p < \frac{n}{n-1}$, i.e. if u_k is a bounded sequence in $BV(\Omega)$, then there exists a function $u \in BV(\Omega)$ with*

$$||u||_{BV} \leq \liminf_{k\to\infty} ||u_k||_{BV}$$

such that $u_k \to u$ with respect to the L^p-norm for all $p < \frac{n}{n-1}$.

Proof. **Ad (1).** Let u_k be a Cauchy-sequence in $BV(\Omega)$. Then u_k is a Cauchy sequence in $L^1(\Omega)$, so there exists $u \in L^1(\Omega)$ such that $u_k \to u$ in L^1. Furthermore

$$\int_\Omega u \operatorname{div}(\phi) = \lim_{k\to\infty} \int_\Omega u_k \operatorname{div}(\phi)\,\mathrm{d}x \leq \limsup_{k\to\infty} TV(u_k,\Omega),$$

for $\phi \in C_c^\infty(\Omega,\mathbb{R}^n)$ with $|\phi| \leq 1$, thus when we take the supremum over such ϕ (without effect on the right hand side), we find

$$TV(u,\Omega) \leq \liminf_{k\to\infty} TV(u_k,\Omega).$$

Furthermore

$$
\begin{aligned}
TV(u-u_k,\Omega) &= \sup\left\{\int_\Omega (u-u_k)\operatorname{div}(\phi)\,\mathrm{d}x \;\middle|\; \phi\right\} \\
&= \sup \lim_{m\to\infty}\left\{\int_\Omega (u_m-u_k)\operatorname{div}(\phi)\,\mathrm{d}x \;\middle|\; \phi\right\} \\
&\le \limsup_{m\to\infty} \sup\left\{\int_\Omega (u_m-u_k)\operatorname{div}(\phi)\,\mathrm{d}x \;\middle|\; \phi\right\} \\
&= \limsup_{m\to\infty} TV(u_m-u_k,\Omega) \\
&\to 0
\end{aligned}
$$

as $k\to 0$, so u_k converges to u in BV.

Ad (2). Direct calculation.

Ad(3). Direct calculation.

Ad (4). On $\Omega' \Subset \Omega$, we replace u by u_ε as in the previous point. For u_ε, the statement is known from the theory of Sobolev spaces. We can then let $\Omega' \nearrow \Omega$. (The last point is a bit more subtle – we actually have to work with *extension domains*, but its clear on convex domains where we can choose $\Omega' = \lambda\Omega$ for $\lambda < 1$ so that the embedding constants don't degenerate. C^2-domains are also ok, but very general domains may pose problems.)

Ad (5). Like point (4), this follows from the same property in $W^{1,1}$ by approximation with smooth functions. □

Example 3.7. 1. $u \in C_c^\infty(\Omega) \Rightarrow u \in BV(\Omega)$.

2. $u = \chi_E$ for $E \Subset \Omega$ and $\partial E \in C^1$. Then $TV(u,\Omega) = |\partial E|$ by the Gauss theorem and hence $u \in BV(\Omega)$.

Remark 3.8. BV is neither uniformly convex nor separable. This means that a Galerkin method cannot work in BV, since there is no good approximation by finite-dimensional spaces. $W^{1,1}$ is isometrically embedded in BV and agrees with the closure of $C^\infty \cap BV$ with respect to the BV-norm.

Definition 3.9. Let $E \subset \Omega$ be measurable. We say that E has finite perimeter in Ω if $\chi_E \in BV(\Omega)$ and write

$$
\operatorname{Per}(E,\Omega) = TV(\chi_E,\Omega).
$$

Corollary 3.10. *Let E_k be a sequence of open sets, $E_k \Subset \Omega$ with C^1-boundaries ∂E_k such that*

$$
\limsup_{k\to\infty} [|E_k| + |\partial E_k|] < \infty.
$$

Then there exists a set of finite perimeter E and a subsequence k_j such that $E = \lim_{j\to\infty} E_{k_j}$ in the sense that

$$
\lim_{j\to\infty} \chi_{E_{k_j}} = \chi_E \qquad \textit{in } L^1(\Omega) \qquad \textit{and} \quad \operatorname{Per}(E_{k_j}) \to \operatorname{Per}(E).
$$

The converse is also true.

Theorem 3.11. *Let E be a set of bounded perimeter. Then there exists a sequence of sets E_k such that $\partial E_k \in C^\infty$ and*

$$
E_k \xrightarrow{L^1} E, \qquad \operatorname{Per}(E_k) \to \operatorname{Per}(E).
$$

Idea of proof. Assume for simplicity that $E \Subset \Omega$. Consider the mollification $u_\varepsilon = \chi_E * \eta_\varepsilon$ for some small $\varepsilon > 0$. Then $u_\varepsilon \in C_c^\infty(\Omega)$ takes values in $[0,1]$. By Sard's theorem and the regular value theorem, we can take $\lambda_\varepsilon \in (0,1)$ such that $\partial\{u_\varepsilon > \lambda_\varepsilon\}$ is a C^∞-manifold – in fact, for fixed ε we can take almost every $\lambda \in (0,1)$. It is quite easy to show that

$$\chi_{E_{\lambda_\varepsilon}} \to \chi_E$$

in $L^1(\Omega)$ as $\varepsilon \to 0$, and we can extract a suitable subsequence. Furthermore, not all sets can have too large perimeter, because the total variation of u_ε is bounded by the total variation of χ_E and since the gradient of u_ε can be recovered from the perimeters of the sets $\{u_\varepsilon > \lambda\}$ (coarea formula). □

This property is sometimes denoted as *intermediate density* of smooth sets, since the convergence is weaker than norm convergence, but stronger than weak convergence.

Since BV-functions are only defined Lebesgue-almost everywhere, a set of finite perimeter is only defined in a weak sense, so one needs more analytic machinery to make sense of their boundary. In fact, a BV-function u has a weak gradient Du which is no longer a function, but a measure, and their (reduced) boundary $\partial_* E$ can be defined using the gradient measure $D\chi_E$. This boundary has many nice properties which resemble those of C^1-boundaries. In particular the perimeter of E is the $n-1$-dimensional Hausdorff measure of $\partial_* E$ – for this and more, see [EG92, Giu84].

3.4 The Modica-Mortola Functional in Higher Dimensions

Theorem 3.12.

$$\Gamma(L^1) - \lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u) = \begin{cases} c_0 TV(u, \Omega) & u \in BV(\Omega, \{-1, 1\}) \\ +\infty & \textit{else.} \end{cases}$$

where $c_0 = \frac{1}{2}\int_{-1}^{1} \sqrt{2\,W(z)}\,\mathrm{d}z$.

Our proof follows the original article [Mod87]; a different proof using a method called 'slicing' which reduces the n-dimensional case to the one-dimensional case, can be found in [Bra02]. The method is applicable in more situations, but also requires deeper knowledge of the properties of functions of finite perimeter and some additional measure theory, which is why we skip it here.

3.5 $\limsup$-construction

Assume that $E \Subset \Omega$ such that $\mathrm{Per}(E, \Omega) = \mathrm{Per}(E, \mathbb{R}^n) = \mathrm{Per}(E)$. We imagine the phase-field u_ε as an approximation of the characteristic function of a set E which makes a smoothed out transition on a length-scale ε at the boundary ∂E. Let us for the moment assume that $\partial E \in C^2$. Then we approximate the signed distance function

$$\mathrm{sdist}(x, \partial E) = \begin{cases} \mathrm{dist}(x, \partial E) & x \in E \\ -\mathrm{dist}(x, \partial E) & x \notin E \end{cases}$$

by a function r such that

1. there exists a neighbourhood $U_\delta = \{\mathrm{dist}(x, \partial E) < \delta\}$ of ∂E such that $r(x) = \mathrm{sdist}(x, \partial E)$ for all $x \in U$,

2. $r \geq \delta$ outside U_δ,

3. $r \in C^2(\mathbb{R}^n)$ and

4. $|\nabla r| \leq 1$.

Since $\partial E \in C^2$, sdist is C^2-smooth in a neighbourhood of ∂E and satisfies

$$\nabla \mathrm{sdist}(x) = \nu_{\partial E, \pi(x)}, \qquad \text{in particular } |\nabla \mathrm{sdist}| \equiv 1$$

on U_δ, where the closest point projection $\pi : U_\delta \to \partial E$ is C^2-smooth and uniquely defined (for small $\delta > 0$). We then set

$$u_\varepsilon(x) := q\left(\frac{r(x)}{\varepsilon}\right)$$

where $q : \mathbb{R} \to \mathbb{R}$ is the same function as in one dimension (the hyperbolic tangent). Now, using the co-area formula [EG92, Section 3.4], we see that

$$\begin{aligned}
\int_\Omega \frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\,W(u_\varepsilon)\,\mathrm{d}x - C\,\varepsilon^{-2}\,\exp(-\delta/\varepsilon) &\leq \int_{U_\delta} \frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\,W(u_\varepsilon)\,\mathrm{d}x \\
&= \int_{U_\delta} \frac{\varepsilon}{2}(q')^2\left(\frac{\mathrm{sdist}}{\varepsilon}\right)\left|\frac{\nabla \mathrm{sdist}}{\varepsilon}\right|^2 + \frac{1}{\varepsilon}\,W\left(q\left(\frac{\mathrm{sdist}}{\varepsilon}\right)\right)\,\mathrm{d}x \\
&= \int_{U_\delta} \frac{1}{2\varepsilon}(q')^2\left(\frac{\mathrm{sdist}}{\varepsilon}\right) + \frac{1}{\varepsilon}\,W\left(q\left(\frac{\mathrm{sdist}}{\varepsilon}\right)\right)\,\mathrm{d}x \\
&= \int_{U_\delta} \frac{1}{\varepsilon}\sqrt{2\,W\left(q\left(\frac{\mathrm{sdist}}{\varepsilon}\right)\right)}\,q'\left(\frac{\mathrm{sdist}}{\varepsilon}\right)\cdot 1\,\mathrm{d}x \\
&= \int_{U_\delta} \frac{1}{\varepsilon}\sqrt{2\,W\left(q\left(\frac{\mathrm{sdist}}{\varepsilon}\right)\right)}\,q'\left(\frac{\mathrm{sdist}}{\varepsilon}\right)\cdot |\nabla \mathrm{sdist}|\,\mathrm{d}x \\
&= \int_{-\delta}^{\delta}\left(\int_{\{\mathrm{sdist}=z\}} \frac{1}{\varepsilon}\sqrt{2\,W\left(q\left(\frac{\mathrm{sdist}}{\varepsilon}\right)\right)}\,q'\left(\frac{\mathrm{sdist}}{\varepsilon}\right)\,\mathrm{d}\mathcal{H}^{n-1}\right)\mathrm{d}z \\
&= \int_{-\delta}^{\delta} \frac{1}{\varepsilon}\sqrt{2\,W\left(\frac{z}{\varepsilon}\right)}q'\left(\frac{z}{\varepsilon}\right)\mathcal{H}^{n-1}\left(\{\mathrm{sdist} = z\}\right)\,\mathrm{d}z \\
&= \int_{-\delta/\varepsilon}^{\delta/\varepsilon} \sqrt{2\,W(q)}\,q'\,\left(\mathcal{H}^{n-1}(\partial E) + o(1)\right)\,\mathrm{d}z \\
&\to c_0\,H^{n-1}(\partial E) \\
&= c_0 \mathrm{Per}(E).
\end{aligned}$$

If E touches the boundary, we have the same construction on $\Omega' \Subset \Omega$, but as we let $\Omega' \nearrow \Omega$, we miss the part of the boundary of E which lies on $\partial\Omega$, so we get the perimeter relative to Ω. Finally, if E is not smooth, we can use the approximability of E by smooth sets E_k, approximate $-1+2\chi_{E_k}$ by suitable phase-fields $u_{\varepsilon,k}$ and extract a diagonal sequence $u_{\varepsilon_k,k}$ approaching $-1+2\chi_E$.

3.6 Compactness

Let us first prove the compactness result. Take any sequence $u_\varepsilon \in W^{1,2}(\Omega) \cap L^4(\Omega)$ such that

$$\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon) < \infty.$$

Then by Young's inequality with ε/completion of squares

$$\begin{aligned}\mathcal{F}_\varepsilon(u_\varepsilon) &= \int_\Omega \frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\,W(u_\varepsilon)\,\mathrm{d}x \\ &\geq \int_\Omega \sqrt{2\,W(u_\varepsilon)}\,|\nabla u_\varepsilon|\,\mathrm{d}x. \end{aligned} \tag{3.1}$$

Thus when we take G to be any primitive function of $\sqrt{2\,W}$, we see that the sequence $w_\varepsilon := G(u_\varepsilon)$ satisfies

$$\limsup_{\varepsilon\to 0} \int_\Omega |\nabla w_\varepsilon|\,\mathrm{d}x < \infty.$$

Furthermore $G(u_\varepsilon) \sim \frac{1}{6}|u_\varepsilon|^3$ clearly remains bounded in $L^1(\Omega)$, so overall in $BV(\Omega)$. Using the BV-compactness theorem and the compact embedding into $L^p(\Omega)$ for $1 \leq p < n/(n-1)$, we deduce that there exists $w \in BV(\Omega)$ such that (up to a subsequence) $w_\varepsilon \to w$ strongly in $L^p(\Omega)$ for all $p < n/(n-1)$ with

$$TV(w,\Omega) \leq \liminf_{\varepsilon\to 0} \int_\Omega |\nabla w_\varepsilon|\,\mathrm{d}x \leq \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon). \tag{3.2}$$

Since $G(u_\varepsilon) \to w$ in $L^1(\Omega)$, a subsequence converges pointwise almost everywhere. As G is strictly monotone increasing, we can take its inverse function and obtain that $u_\varepsilon \to G^{-1}(w)$ pointwise almost everywhere. Using $W(s) \geq s^4$ for all sufficiently large $|s|$, the bound on $\int_\Omega W(u_\varepsilon)\,\mathrm{d}x$ implies that u_ε is bounded in $L^4(\Omega)$. By a standard result on concentrations and weak compactness (see e.g. [Bre11, Exercise 4.16]) we have that (1) $u_\varepsilon \to G^{-1}(w)$ pointwise and (2) u_ε is bounded in $L^4(\Omega)$ together imply that $u_\varepsilon \to u = G^{-1}(w)$ strongly in $L^p(\Omega)$ for all $1 \leq p < 4$.

3.7 $\liminf$-inequality

If $u_\varepsilon \to u$ strongly in $L^1(\Omega)$ and $\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon) < \infty$, then $w = G(u)$ almost everywhere, since L^1-convergence implies convergence pointwise almost everywhere for a subsequence. Clearly u only takes the values ± 1 and w only the values $G(-1), G(+1)$, thus

$$u = -1 + 2\,\chi_E, \qquad w = G(-1) + (G(1) - G(-1))\cdot\chi_E = G(-1) + c_0\,\chi_E$$

one can easily relate their total variations by

$$c_0\,\mathrm{Per}(\{u=1\}) = \frac{c_0}{2}\,TV(u,\Omega) = TV(w,\Omega) \leq \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon)$$

due to (3.2), using that the total variation of a constant function vanishes and that the total variation is positively homogeneous of degree one. This concludes the proof of the $\liminf$-inequality.

4 Concluding Remarks

4.1 Convergence of the forced Allen-Cahn Equation

Consider the equation

$$\begin{cases} \varepsilon\,\partial_t u_\varepsilon &= \varepsilon\,\Delta u_\varepsilon - \frac{1}{\varepsilon}\,W'(u_\varepsilon) + g_\varepsilon \quad \text{in } \Omega_T = (0,T)\times\Omega \\ u_\varepsilon(0,\cdot) &= u_\varepsilon^0 \quad \text{in } \Omega \\ \nabla u_\varepsilon\cdot\nu_\Omega &= 0 \quad \text{on } (0,T)\times\partial\Omega. \end{cases}$$

The term g_ε is an additional forcing term, while $g_\varepsilon \equiv 0$ corresponds to the L^2-gradient flow of $\mathcal{F}_\varepsilon$. The ε in front of the time derivative comes in for time normalisation purposes. If we make the recovery sequence ansatz

$$u_\varepsilon(x) = q\left(\frac{\mathrm{sdist}(x, \partial E)}{\varepsilon}\right),$$

then (abbreviating $r = \mathrm{sdist}$) we get

$$\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\,W'(u_\varepsilon) = q'\left(\frac{r}{\varepsilon}\right)\,\Delta r + \frac{1}{\varepsilon}\,q''\left(\frac{r}{\varepsilon}\right)\,|\nabla r|^2 - \frac{1}{\varepsilon}\,W'\left(\frac{r}{\varepsilon}\right) = q'\left(\frac{r}{\varepsilon}\right)\,\Delta r = \sqrt{2W(u_\varepsilon)}\,\Delta r$$

since the other two terms cancel out along the optimal profile transition. The Laplacian of the signed distance function from a boundary is exactly the mean curvature evaluated at the closest point projection (see e.g. [GT83, Section 14.6]), so on the zero level set, the right hand side of the Allen-Cahn equation is exactly the mean curvature of the level set in this ansatz. We now need the time re-normalisation since the interface moves with speed $O(1)$ if the time derivative of order is $O(1/\varepsilon)$ because the transition is so steep.

As the Modica-Mortola functional approaches the perimeter, it makes sense to conjecture that the Allen-Cahn equation (somehow) approaches mean curvature flow, which is the L^2-gradient flow of the area functional (i.e. the perimeter on the space of boundaries), and that a perturbation might give rise to a forced mean curvature flow, i.e. a flow with normal velocity $H + g$ where g is a suitable limit of the g_ε. As long as everything is smooth, we can make the calculation above rigorous also in the parabolic case.

There are too many introductions to mean curvature flow to list them here, but see for example [Eck04] and the sources given in the introduction. For the interpretation of mean curvature as the L^2-gradient of the area functional (and thus of mean curvature flow as its L^2-gradient flow) see e.g. [Sim83].

There are several ways to make this rigorous. Since mean curvature flow develops singularities in finite time, we need weak concepts of solutions.

In [ESS92], the authors construct sub- and super-solutions to show convergence to level set mean curvature flow (see the excellent and easily readable article [ES91]) under the assumption that the zero-level set of the initial condition is non-fattening under level-set mean curvature flow. This approach uses the parabolicity of mean curvature flow and the maximum principle.

Ilmanen shows convergence to mean curvature flow [Ilm93] in the sense of Brakke [Bra78], where surfaces are interpreted as measures. Initial conditions have to be chosen in a suitable way and the proofs mimic those of the theory of mean curvature flow with slight additional complications when controlling the *discrepancy measures* which measure how far a phase-field is from making the physically sensible optimal profile transitions which we like to see.

An interesting approach which also applies to forced mean curvature flow is due to Mugnai and Röger [MR11, MR08]. The solution concept is again a (forced) version of Brakke flow. The methods have a Γ-convergence-y flavour, since the authors start by considering the *Allen-Cahn action functional*

$$S_\varepsilon(u) = \int_0^T \int_\Omega \left(\sqrt{\varepsilon}\partial_t u + \frac{1}{\sqrt{\varepsilon}}\left(-\varepsilon\Delta u + \frac{1}{\varepsilon}\,W'(u)\right)\right)^2 \,\mathrm{d}x.$$

Note that solutions to the Allen-Cahn equation have zero action functional energy, but that we can also consider the forced Allen-Cahn equation with g_ε such that

$$\sup_{\varepsilon>0} \int_0^T \int_\Omega \frac{1}{\varepsilon}\,g_\varepsilon^2 \,\mathrm{d}x\,\mathrm{d}t =: \Lambda < \infty.$$

We look at the weak mean curvature/Allen-Cahn right hand side $w_\varepsilon := -\varepsilon\,\Delta u_\varepsilon - \frac{1}{\varepsilon}\,W'(u_\varepsilon)$ and note that there is a (fairly technical) Γ-limit for the functional

$$\mathcal{W}_\varepsilon(u) = \frac{1}{c_0\varepsilon}\int_\Omega w_\varepsilon^2\,\mathrm{d}x,$$

at least in $n = 2, 3$ dimensions. Namely

$$\left[\Gamma(L^1) - \lim_{\varepsilon\to 0}\mathcal{W}_\varepsilon\right](-1 + 2\chi_E) = \int_{\partial E}|H|^2\,\mathrm{d}\mathcal{H}^2$$

if E is a set with C^2-boundary [RS06], i.e. we approach the integral of the mean curvature squared of the boundary, also known as Willmore's energy [KS12]. If the diffuse Willmore energy remains uniformly bounded along a sequence of phase-fields u_ε, then u_ε has to be relatively regular, and we see that the weak* limit μ of the Radon measures

$$\mu_\varepsilon = \frac{1}{c_0}\left(\frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\,W(u_\varepsilon)\right)\cdot\mathcal{L}^n$$

has to be an *integral varifold.* (A Radon measure is a 'nice' measure or equivalently, an object of the dual space of continuous functions with a similar duality as the one between L^1 and L^∞, see [EG92]. An integral varifold is a 'nice' Radon measure which resembles a surface of given dimension k, see the recent introductory article [Men17] and the sources cited there, in particular [Sim83].) In particular, μ is a measure such that

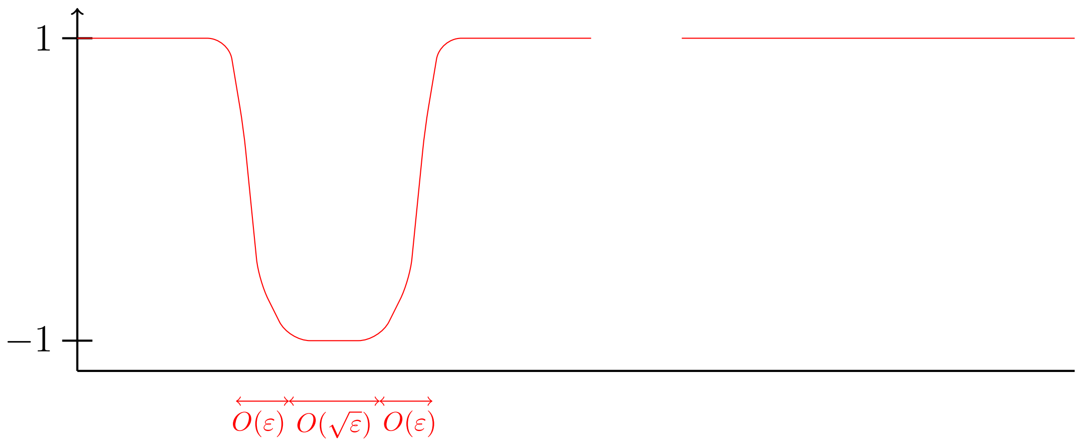
$$\lim_{\rho\searrow 0}\frac{\mu(B_\rho(x))}{\omega_{n-1}\,\rho^{n-1}} \in \mathbb{N}_0$$

for μ-almost all $x \in \mathrm{spt}(\mu)$. Here ω_n is the volume of the n-dimensional unit ball and

$$\mathrm{spt}(\mu) = \{x \in \mathbb{R}^n \mid \mu(B_r(x)) > 0 \quad \forall\, r > 0\}$$

is the *support* of the measure μ. In this sense, measure limits μ of phase-fields u_ε are integer-valued surfaces if Willmore's energy is bounded along the sequence, and for almost all times if the action functional is bounded. Thus the space where we get convergence to the forced Allen-Cahn equation is a space of (measure-theoretically generalised) surfaces with integer multiplicities.

Note that the measures μ carry more information about the behaviour of the phase-fields u_ε as $\varepsilon \to 0$ than the function limit u. While we only know that $u = \pm$ almost everywhere, the measures capture 'ghost interfaces' which we see on the phase field level, but cannot see in the function limit. In the picture, both functions have the same limit, but the one on the left has a ghost interface which disappears in the function limit, but not the measure limit!



4.2 Higher- and Lower-order Phase-Field Energies

A higher order energy. As stated above, the higher order phase-field energy

$$\mathcal{E}_\varepsilon(u) := \mathcal{F}_\varepsilon(u) + \mathcal{W}_\varepsilon(u)$$

converges to a sum of the perimeter functional and Willmore's energy, at least at smooth limit points and in dimensions $n \in \{2, 3\}$. The hardest part is to show that the limits of measures μ_ε is 'nice' (i.e. an integral varifold). As for the Modica-Mortola functional in *one* dimension, uniform bounds on $\mathcal{E}_\varepsilon$ imply morally uniform convergence [DW17].

A lower order energy. On a periodic interval, we can write

$$[u]^2_{H^1} = \int |u'|^2 \,\mathrm{d}x = \sum_{k\in\mathbb{Z}} |k|^2 \, |\hat{u}(k)|^2$$

where $\hat{u}(k)$ is the k-th Fourier coefficient of u. We can generalise this to

$$[u]^2_{H^s} = \sum_{k\in\mathbb{Z}} |k|^{2s} \, |\hat{u}(k)|^2$$

for $s \in (0, 1)$ and call this the H^s-semi norm. The generalised Modica-Mortola functionals

$$\mathcal{F}^s_\varepsilon(u) = c_{s,\varepsilon} [u]^2_{H^s} + \int_\Omega \frac{1}{\varepsilon} \, W(u) \,\mathrm{d}x$$

have different Γ-limits depending on s. For $s > 1/2$, we choose $c_\varepsilon \sim \varepsilon^{2s-1}$ and recover the perimeter. The same is true for $s = 1/2$ and $c_\varepsilon \sim \frac{1}{|\log \varepsilon|}$, while for $s < 1/2$, we can choose $c_\varepsilon \equiv 1$ and obtain a *non-local perimeter*. Also this generalises to many dimensions [SV12], but one has to use Sobolev spaces of fractional order [NPV12]. These can be defined either using Fourier-transforms as above, or as functions for which a suitable singular integral measuring the jumps in some sense remains finite.

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