Analysis and optimal control of Korteweg de Vries equation

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1 Some words about Korteweg de Vries equation

The nonlinear Korteweg de Vries (KdV for short) equation (with force)

$$\begin{cases} u_t + u_{xxx} + \frac{1}{2}uu_x + \lambda u + \mu u_x = f(t,x), x \in \mathbb{R}, t > 0, \\ u(0,x) = u_0(x), x \in \mathbb{R}, \end{cases} \tag{1}$$

is a mathematical model of waves on shallow water surfaces. It was introduced by D. J. Korteweg and G. de Vries in 1895 [1]. This equation also models the propagation of tsunamis [5] or ion-sound waves [4]. It contains a rich mathematical theory behind and is a topic of active mathematical research.

2 The first aspect: Longtime behaviour

Studying the asymptotic behaviour of dynamical systems arising from mechanics and physics is a capital issue. The long time behaviour of dynamical systems is usually captured by the concept so-called **attractors**. It was originated by Ladyzenskaya, J. Hale, J.P. LaSalle in 1970s. An attractor is simply understood as a compact set which attracts all orbits which started from a bounded set. The compactness of the attractor give us (much) more information about the dynamical systems. In many cases, the attractor is even finite dimensional (in fractal or Hausdorff sense), that means the dynamical system can be approximated by finite parameters.

Following this issue, we try to study the non-autonomous KdV equation (1). Let $u_0 \in H^2(\mathbb{R})$, we will try to

- prove the existence of a unique pullback attractor for the process generated by equation (1); [Since initial data has high regularity, the compact of the process is easier to treat]

- prove that the attractor is finite (in fractal or Hausdorff sense) dimensional; [Although the finite dimensional of non-autonomous attractor is never shown, the approach in [2] is believed to be applicable]

- prove the upper semi-continuity of the attractor with respect to the external for f;

- prove the smoothing property of the attractor. [That is solutions lie in the attractor are smoother than others]

3 The second aspect: Optimal control

There are two options up to now:

3.1 Inverse source problem

We consider the problem of reconstructing a spatially localized source $q = f$ from noisy measurements h_k at given points in space. The problem reads

$$\min J(q, u) = \sum_k \|u(x_k, \cdot) - h_k\|^2_{L^2_t} + \|q\|_{\mathcal{M}(L^2_t)} \quad \text{s.t. } (1). \tag{2}$$

Therefore we use the control space $\mathcal{M}(L^2_t)$ of L^2_t-valued measures which allows for Dirac functions in space. This is an improvement over the formulation considered in [6], where q is a linear combination of Dirac-measures at fixed positions.

During the workshop we need to address the following points:

- Existence and regularity of solutions for (1) with measure valued right-hand side. In particular, continuity of the solution in space is required in the formulation of (2).

- Existence of solutions of (2). Since (1) is nonlinear, the control problem is not convex, which might lead to difficulties.

- Formally deduce an optimality system. Due to the structure of the cost functional, the adjoint equation includes Dirac-measures, which can lead to low regularity of adjoint variable.
- Relaxation of the observation: Averaging on patches around the observation points
- Regularization of the control: Add a Tikhonov-term to the objective
- Solve the state equation (1) numerically.
- Solve optimal control problem numerically.

3.2 Control to attractor in finite time

We consider an "abstract" cost functional which penalizes the distance of the solution at time T to the attractor of the uncontrolled system. The optimal control problem looks like

$$\begin{aligned} &J(f_\mu, u) = \operatorname{dist}(u(T), \mathcal{A}) + \|q\|^2, \\ &\text{s.t. } \; u_t + u_{xxx} + \frac{1}{2} u u_x + \lambda u = f_0(x) + q(t,x) \end{aligned} \tag{3}$$

where $\mathcal{A}$ is the pullback attractor of

$$u_t + u_{xxx} + \frac{1}{2} u u_x + \lambda u = f_0(x).$$

The main difficulty is that $\mathcal{A}$ is not known in most cases, even its existence is often not clear. Therefore $\operatorname{dist}(u(T), \mathcal{A})$ needs to be approximated.

During the workshop we need to address the following points:

- Existence of solutions of (3) . Since $\operatorname{dist}(u(T), \mathcal{A})$ is not convex and (1) is nonlinear, the control problem is not convex, which might lead to difficulties.
- Formally deduce an optimality system.
- Find useful (heuristic) approximations of $\operatorname{dist}(u(T), \mathcal{A})$ and address the points above for this problem.
- Test approximation of the problem in ODE case with known attractors.
- Solve optimal control problem numerically.

References

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