# Inverse Gravimetric problem: implementation of the Mumford-Shah regularization term with box constraint

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**Abstract**

The gravimetric inverse problem aims to determine Earth's density by measuring the gravity field on the surface or in boreholes. This problem is non-unique and numerically unstable. Indeed, a small perturbation of the datas can correspond in a sensible variation of the solution. Moreover, the dataset is in most of the cases incomplete and affected by noise, since they are obtained by an instrumental measure of the gravity field on discrete points. The stabilization of this inverse problem is usually achieved introducing a regularization term. In the literature, $H^1$ and $TV$ regularization are already appeared and successfully used for this kind of problem, but Mumford-Shah regularization term, which is more challenging from an implementation point of view, could perform better with respect to the previous methods due to its characteristics.

## 1 Formulation of the problem

Let $\Omega \subset \mathbb{R}^3$ be a subsurface domain with density $\rho(\mathbf{r})$. The gravitational potential $u(\mathbf{r})$ generated by the density function satisfies the Poisson's equation inside the domain $\Omega$:

$$\Delta u(\mathbf{r}) = -\mathbf{4}\pi\mathbf{G}\rho(\mathbf{r}), \qquad \mathbf{r} \in \mathbf{\Omega} \subset \mathbb{R}^{\mathbf{3}}, \tag{1}$$

where $G$ is the universal gravity constant, that will be omitted from now on.
The solution of (1) can be explicitly written using the fundamental solution $\mathcal{K}(\hat{\mathbf{r}}, \mathbf{r})$:

$$u(\hat{\mathbf{r}}) = \int_{\mathbf{\Omega}} \rho(\mathbf{r})\mathcal{K}(\hat{\mathbf{r}}, \mathbf{r})\, \mathbf{dr}, \tag{2}$$

where

$$\mathcal{K}(\hat{\mathbf{r}}, \mathbf{r}) = \frac{\partial}{\partial \mathbf{x_3}} \frac{\mathbf{1}}{|\mathbf{r} - \hat{\mathbf{r}}|} = \frac{\mathbf{x_3} - \hat{\mathbf{x}}_\mathbf{3}}{|\mathbf{r} - \hat{\mathbf{r}}|^\mathbf{3}}.$$

We need to solve the problem (2) with the known function $u$ on the surface. Unfortunately, this problem is in general non-unique. Unicity can be achieved restricting the set of function which can be considered as density functions or the geometry of the domain $\Omega$. Hence, restricting the domain $\Omega$ to a simply connected polygon, we obtain the uniqueness of the solution $\rho$ (see [4]).
Equation (2) can be rewritten in a operator form

$$K\rho = u$$

where the operator $K$ is compact. It can be shown that the injectivity of $K$ is not helping us in finding a solution, since the operator is not continuously invertible.
A standard technique to address the inverse problem (2) is minimizing the objective functional that combines the least-square data misfit with a regularization-stabilization term that penalize fast oscillating solution, i.e.

$$\min_{\rho} J(\rho, u) = \min_{\rho} \Big( \|K\rho - u\|^2_{L^2(\Omega)} + \alpha \mathcal{S}(\rho) \Big), \tag{3}$$

where $\mathcal{S}$ is a penalization functional.

## 2 The project

Historically, the solution of (3) has been successfully implemented for penalization functional used in the image segmentation such as $H^1$-norm and $TV$, but they somehow lack in physical reliability since the first tends to give blurred results and the second does not allow for big jumps in the functional.
**The main idea of the project is to use as regularization term the Mumford-Shah functional**

$$\mathcal{MS}(\rho,\Gamma) = \int_{\Omega\setminus\Gamma} |\nabla\rho|^2 \, d\mathbf{r} + \mathscr{H}^{\mathbf{2}}(\mathbf{\Gamma}), \tag{4}$$

where $\mathscr{H}^2$ is the 2-dimensional Hausdorff measure, and $\Gamma$ is the set where the density $\rho$ can jump.
Unfortunately, in [2], it is shown that the problem (3) in case $\mathcal{S}$ is the $\mathcal{MS}$ functional is not well-posed and without the introduction of additional constraints on the solution $\rho$. In [2], the authors prove that if a geometrical constraint is added on $\Gamma$, the problem is well-posed, while in [3] the well-posedness is obtained assuming the solution in $L^\infty(\Omega)$. Thus, in both the paper there is a poor (or absent) *numerical experiment* Session, where this constraint are implemented.
A natural and physical way to force the condition introduced in [3] can be the introduction of a box constraint for the density function. Indeed, it is realistic to suppose that we cannot have negative densities neither that a material on the earth or on any physical (not extreme like black holes) body in the universe has an infinite density. Then, **the problem we will face is to find**

$$\min_{\Gamma,\rho^{min}\leq\rho\leq\rho^{max}} \|K\rho - u\|^2_{L^2\Omega} + \mathcal{MS}(\rho,\Gamma).$$

## 3 The target of the workshop

- Understand the inverse gravimetric problem and replicate the experiments obtained for the Total Variation in [1];
- Replicate the same experiments using the MS-functional without implementing the box constraint;
  - $1^{st}$**strategy:** direct minimization of the MS-functional;
  - $2^{nd}$**strategy:** approximation via Ambrosio-Tortorelli functional;
- Introduce the box constraint and efficient numerical strategies to implement it;
- ...

## References


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[4] V.N. Strakhov, M.A. Brodsky, *On the uniqueness of the solution of the inverse logarithmic potential problem.* SIAM J. Appl. Math., 46, 324–344