Error estimates for optimal control problems

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1 What could we do?

There are still a couple of open questions regarding error estimates for optimal control problems. Let us for instance consider an optimal control problem of the form

$$j(u) := \frac{1}{2}\|Su - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_X \to \min! \tag{1}$$

subject to

$$u \in U_{ad} := \{u \in X\colon u_a \le u \le u_b \text{ a. e. in } \Omega\},$$

where $S\colon X \to H^1(\Omega)$ is the solution operator of a boundary value problem. Here, X is some Hilbert space. It is easily possible to derive an error estimate for the finite element approximation in the norm of X and usually, it is also easy to obtain an estimate for the state $y = Su$ in some Banach space Y by introducing intermediate functions and applying stability properties of S_h from X to Y, i. e.

$$\|y - y_h\|_Y \le \|(S - S_h)u\|_Y + \|S_h(u - u_h)\|_Y \tag{2}$$

$$\le c\,(\|(S - S_h)u\|_Y + \|u - u_h\|_X) \tag{3}$$

For some pairs X and Y this technique leads to a sharp error estimates, but in some cases other techniques have to be applied. An example is considered in the next section.

2 Example: an optimal control problem in $H^1_0(\Omega)$

At the last summer workshop we already discussed problem (1) with the choice $X = H^1_0(\Omega)$. The *control–to–state* mapping S was the solution operator of the Poisson equation

$$-\Delta y = u \text{ in } \Omega, \qquad y = 0 \text{ on } \Gamma.$$

We derived the optimality system

$$(\nabla y, \nabla v) - (u, v) = 0 \qquad \forall v \in H^1_0(\Omega), \tag{4}$$

$$(\nabla p, \nabla v) - (y, v) = (-y_d, v) \qquad \forall v \in H^1_0(\Omega), \tag{5}$$

$$\alpha(\nabla u, \nabla(v - u)) + (p, v - u) \ge 0 \qquad \forall v \in U_{ad}, \tag{6}$$

and discretized this system with piecewise linear finite elements on a conforming triangulation $\mathcal{T}_h$ of Ω, i. e. we searched

$$\begin{aligned} y_h, p_h &\in V_h := \{v_h \in C_0(\overline{\Omega})\colon v_h \text{ is affine linear on all } T \in \mathcal{T}_h\}, \\ u_h &\in V_h \cap U_{ad}. \end{aligned}$$

If Ω is a convex domain with polygonal or polyhedral boundary we were able to prove the *a priori* estimate

$$|u - u_h|_{H^1(\Omega)} \le ch$$

using standard techniques, and from (2) we immediately get an estimate for the state

$$|y - y_h|_{H^1(\Omega)} \le ch.$$

If we are interested in an estimate for the state in $L^2(\Omega)$ the technique from (2) is not applicable since we would get

$$\|y - y_h\|_{L^2(\Omega)} \le c \left(\underbrace{\|(S - S_h)u\|_{L^2(\Omega)}}_{\le ch^2} + \underbrace{|u - u_h|_{H^1(\Omega)}}_{\le ch} \right) \le ch,$$

which is not optimal. There are two possibilities to improve this estimate:

1. We could exploit that S_h is even stable from $L^2(\Omega)$ to $L^2(\Omega)$ and (2) would then lead to
$$\|y - y_h\|_{L^2(\Omega)} \le c \left(\|(S - S_h)u\|_{L^2(\Omega)} + \|u - u_h\|_{L^2(\Omega)} \right).$$
It remains to prove an error estimate for the control in $L^2(\Omega)$. However, this means that we have to derive an estimate in $L^2(\Omega)$ for the obstacle problem (6) using some kind of Aubin-Nitsche method. But this is an open questions for about 40 years. We refer to Mosco [3] who proved the convergence rate h^2 for the one-dimensional obstacle problem, but he also notes that it is in general not possible to extend the result to higher dimensions. In a recent contribution of Steinbach [4] the Aubin-Nitsche method was applied to certain variational inequalities on the boundary, e. g. the contact problem, which might be useful to prove error estimates for certain boundary control problems.

2. Another approach has been presented by Meyer and Rösch [2], compare also [1] for Neumann boundary control problems. The idea is to exploit superconvergence properties of certain interpolation operators. In case of $X = L^2(\Omega)$ they solved (1) with piecewise constant controls and piecewise linear and continuous state and adjoint state. An error estimate for the state was obtained using the decomposition
$$\|y - y_h\|_{L^2(\Omega)} \le \|(S - S_h)u\|_{L^2(\Omega)} + \|S_h(u - R_h u)\|_{L^2(\Omega)} + \|S_h(R_h u - u_h)\|_{L^2(\Omega)}, \quad (7)$$
where R_h denotes the midpoint interpolant onto the space of piecewise constant functions. The key step of the convergence proof is the observation, that $R_h u$ is closer to u_h than u itself. This property is called *supercloseness*. Probably it is possible to extend this idea to the optimal control problem with $H_0^1(\Omega)$-regularization, but now, the control is discretized with piecewise linear and continuous functions and the operator R_h should then also map onto this space. The question arises whether it is possible to find such an interpolation operator which also possesses these superclosedness properties.

3 Summary

The following things should be investigated / have to be done:

Implementation: Determine the convergence rates we want to prove numerically.

Aubin-Nitsche for variational inequalities: Try to understand the key steps of the Aubin-Nitsche method presented in [4] and extend it to optimal control problems.

Error estimates for $H_0^1(\Omega)$-regularized problems: Prove an error estimate in $L^2(\Omega)$ for the state using (7).

We can of course also discuss about error estimates for different problems, under reduced regularity, taking the structure of the geometry into account or what ever.

References

[1] Mariano Mateos and Arnd Rösch. On saturation effects in the Neumann boundary control of elliptic optimal control problems. *Computational Optimization and Applications*, 49(2):359, 2011.

[2] Christian Meyer and Arnd Rösch. Superconvergence Properties of Optimal Control Problems. *SIAM J. Control and Optimization*, 43(3):970–985, 2004.

[3] U. Mosco. *Error estimates for some variational inequalities.* Springer, 1977.

[4] O. Steinbach. Boundary element methods for variational inequalities. *Numerische Mathematik*, 126(1):173, 2014.