Fast reaction rates for a class of reaction diffusion systems

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1 Fast reaction rates and quasi-steady-state approximation

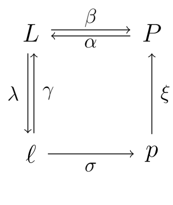
The fast reaction rates phenomena in a (chemical, physical, biological, etc) system, roughly speaking, means that one or many reactions in the system happen much faster comparing to other reactions. In this kind of phenomena, we can usually approximate the system by a new one with simpler structure (in a certain sense). For example, Bothe and Pierre in [2] proved if the following reaction chain

$$A + B \underset{k_2}{\overset{k_1}{\rightleftharpoons}} C \underset{k_4}{\overset{k_3}{\rightleftharpoons}} D + E$$

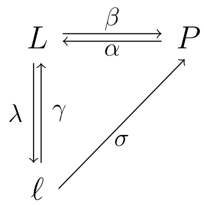
has $k_1 + k_4 \longrightarrow +\infty$ (intuitively, it means that C decays infinitely fast), then we can approximate the system (1) by the following, for some appropriate α and β,

$$A + B \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} D + E$$

Another example that in a preparing-work [4], we study for the a surface-volume reaction diffusion system which models the following reaction:



with the case $\xi \longrightarrow +\infty$ (intuitively the reaction $p \xrightarrow{\xi} P$ happens infinitely fast). Then, we obtain a new approximate system which can be simply described as



The approximated system is usually called *quasi-steady-state approximation* (or QSSA for short). The QSSA system is usually guessed by intuition. In some cases, however, fast reaction rates would lead to strange phenomenas. For example, considered in [3] a irreversible reaction

$$A + B \xrightarrow{k} C$$

with $k \longrightarrow +\infty$. In this case, the limiting system deduces to a nonlinear diffusion equation which models the spatial segregation, that is two substances lie in two separate sub-domains. Another example is a work of Bothe and Hilhorst [1], they studied the reversible chemical reaction as the reaction rate

$$\alpha A \underset{k}{\overset{k}{\rightleftarrows}} \beta B$$

Figure 1: A reversible chemical reaction

$k \to +\infty$. Then the system modeling the above reaction

$$\begin{cases} A_t - d_A \Delta A = -\alpha k (A^\alpha - B^\beta), \\ B_t - d_B \Delta B = \beta k (A^\alpha - B^\beta), \\ \partial_\nu A = \partial_\nu B = 0, \\ A(0,x) = A_0(x), B(0,x) = B_0(x) \end{cases} \tag{1}$$

approaches the nonlinear diffusion equation

$$\begin{cases} w_t - \Delta \phi(w) = 0, \\ \partial_\nu w = 0, \\ w(0,x) = w_0(x), \end{cases}$$

where $w_0 = \frac{u_0}{\alpha} + \frac{v_0}{\beta}$ and

$$\phi = \left(\frac{d_A}{\alpha}\mathrm{id} + \frac{d_B}{\beta}\eta\right) \circ \left(\frac{1}{\alpha}\mathrm{id} + \frac{1}{\beta}\eta\right)^{-1}$$

where $\eta(x) = x^{\beta/\alpha}$ for $x > 0$.

2 Goals

I propose two goals concerning the fast reaction rates phenomena.

2.1 Mixed surface-volume reaction diffusion systems

Consider now the reaction

$$\alpha U \underset{k}{\overset{k}{\rightleftarrows}} \beta V$$

but now U is a substance in the domain while V is a substance appearing only on the boundary. Taking into account the mass action kinetics, we derive

$$\begin{cases} U_t - d_U \Delta U = 0, & x \in \Omega, t > 0, \\ d_U \partial_\nu U = -\alpha k(U^\alpha - V^\beta), & x \in \partial\Omega, t > 0, \\ V_t - d_V \Delta_{\partial\Omega} V = \beta k(U^\alpha - V^\beta), & x \in \partial\Omega, t > 0, \\ U(0,x) = U_0(x), & x \in \Omega, \\ V(0,x) = V_0(x), & x \in \partial\Omega. \end{cases}$$

The question is: **What happens if we let $k \to +\infty$?** Because of the boundary reaction, the result would not be directly implied from the work of [1]. However, it is believed that we can use the ideas and arguments in that work to get the result.

2.2 Rate of convergence to the limiting system

This aim is more delicate. Since the proof of convergence to limiting system is usually based on a compactness method, that leads to a drawback of unknown rate of convergence. Thus, it is an interesting question to ask about the rate of convergence to limiting system for quasi-steady-state approximation.

References

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[3] D. Bothe, M. Pierre, *The instantaneous limit for reaction-diffusion systems with a fast irreversible reaction*, Dis. Cont. Dyn. Syst. – Series S, 5 (2012), 49–59.

[4] T.Q. Bao, K. Fellner, S. Rosenberger, *A reaction diffusion system modelling asymmetric stem cells division: existence and quasi-steady-state approximation*, work in preparation.