Inverse point source location for the Helmholtz equation

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Abstract

To solve the problem of locating the spatial origin of a localized, time-periodic acoustic source from a number of pointwise measurements of the acoustic pressure, a sparsity regularized least squares problem formulation for the Helmholtz equation is considered.

1 The time-periodic wave equation

The propagation of acoustic waves in a homogeneous medium in a domain Ω can be described by the wave equation for the acoustic pressure p. Specifically it reads

$$\begin{aligned} \partial_{tt}p(t) - c^2\Delta p(t) &= f(t) \quad \text{in } \Omega, \\ \partial_t p(t) + c\partial_\nu p(t) &= 0 \quad \text{on } \Gamma_{\text{art}}, \\ p(t) &= 0 \quad \text{on } \Gamma_{\text{D}}. \end{aligned} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^d$ denotes a bounded computational domain with $\partial\Omega = \Gamma_{\text{art}} \cup \Gamma_{\text{D}}$, where $\Gamma_{\text{D}} \subset \partial D$ for $D \subset \mathbb{R}^d$ describes the surface of a solid reflecting object, and Γ_{art} describes an artificial boundary resulting from a truncation of the exterior domain $\Omega_0 = \mathbb{R}^d \setminus D$. The boundary conditions on Γ_{art} are referred to as zeroth-order absorbing boundary conditions [5, 6]. We assume that the forcing term f is given by

$$f(t) = \hat{u}(t)\delta_{\hat{x}},$$

which corresponds to a localized sound source at the point $\hat{x} \in \Omega_c \subset \Omega$ with time-dependent signal $\hat{u}$. Furthermore, we assume that $\hat{u}$ can be represented in the time-harmonic form

$$\hat{u} = \sum_{n=1}^{N} a_n \sin(\omega_n t + \varphi_n)) = \sum_{n=1}^{N} \text{Re}(\hat{u}_n \exp(i\omega_n t)),$$

where ω_n are some basic frequencies and $\hat{u}_n = a_n \exp(i\varphi_n) \in \mathbb{C}$ encodes the corresponding amplitude $a_n > 0$ and phase shift $\varphi_n \in \mathbb{R}$. By an application of the Fourier transform in t, the solution of (1.1) can now be reduced to the solution of the following elliptic boundary value problems

$$\begin{aligned} -\omega_n^2 p_n - c^2\Delta p_n &= \hat{u}_n\delta_{\hat{x}} \quad \text{in } \Omega, \\ i\omega_n p_n + c\partial_\nu p_n &= 0 \quad \text{on } \Gamma_{\text{art}}, \\ p_n &= 0 \quad \text{on } \Gamma_{\text{D}}, \end{aligned} \tag{1.2}$$

where $p_n \in H_0^1(\Omega \cup \Gamma_D, \mathbb{C})$ for $n = 1, \ldots, N$. The time dependent solution of (1.1) can then be obtained as

$$p(t) = \sum_{n=1}^{N} \text{Re}(p_n \exp(i\omega_n t)).$$

For the sake of concise notation, we denote the vector of frequencies by $\vec{\omega} \in \mathbb{R}^N$, and the solution of (1.2) by $\vec{p} \in H_0^1(\Omega \cup \Gamma_D, \mathbb{C}^N)$.

2 Solution of the inverse acoustic problem

We suppose that for some points $\{ x_k \}_{k=1,\dots,K}$ the values of the solution $\vec{p}(x_k) = p_d^k \in \mathbb{C}^N$ of (1.2) are given. To find the location of the sound source (and, at the same time, the amplitude and phase shift) we can consider the following regularized "least-squares" optimization problem

$$\begin{aligned} &\min_{x \in \Omega_c, \hat{u} \in \mathbb{C}^N} \quad \frac{1}{2} \sum_{k=1,\dots,K} \|\vec{p}(x_k) - p_d^k\|^2_{\mathbb{C}^K} + \alpha \|u\|_{\mathbb{C}^N}, \\ &\text{subject to} \quad -c^2 \Delta \vec{p} - \vec{\omega} \vec{p} = \hat{u} \delta_x, \quad (+\text{BC}) \end{aligned} \tag{2.1}$$

for a small regularization parameter $\alpha > 0$. A similar problem has been proposed and analyzed in [1] in the context of active noise control. Since $x \in \Omega_c$ is an optimization variable, the previous problem is non-convex. We will consider the following convex relaxation (which will be made precise during the workshop):

$$\begin{aligned} &\min_{u \in \mathcal{M}(\Omega_c, \mathbb{C}^N)} \quad \frac{1}{2} \sum_{k=1,\dots,K} \|\vec{p}(x_k) - p_d^k\|^2_{\mathbb{C}^K} + \alpha \|u\|_{\mathcal{M}(\Omega_c, \mathbb{C}^N)}, \\ &\text{subject to} \quad -c^2 \Delta \vec{p} - \vec{\omega} \vec{p} = u. \quad (+\text{BC}) \end{aligned} \tag{2.2}$$

In this problem, the solution is searched in the space of Radon measures. For this approach to inverse point source location, we refer to [3]. For a similar problem formulation with the wave equation, we refer to [7]. To justify this approach, several questions are of interest. Certainly, we want that the problem is well-posed in function space. Furthermore, structural questions are important: Is the solution given by a number of point sources? How many? Finally, in the case of noise-free measurements, it is known that for $\alpha \to 0$ the solutions converge (in a certain sense) to the solutions of a certain minimum norm problem; see [3]. Is this solution equal to the original solution?

3 Schedule for the workshop

We definitely want to accomplish the following goals:

- Develop a function space setting for the problem (2.2) and show well-posedness. Therefore, we can expect to combine the theory from [1] and [3] in a (more or less) straightforward way.
- Implement a numerical solution strategy for a finite-element discretization of (2.2). Compare the regularized semismooth Newton method (see, e.g., [4]) and the method proposed in [3]. If applicable, use a priori information on the maximal number of point sources.
- Evaluate in numerical tests whether the problem formulation delivers good results in artificial scenarios.

Other topics which we want to tackle are:

- Problem (2.2) has only a finite dimensional observation space. Therefore, its optimal solution is related to the Lagrange multiplier of a certain semi-infinite optimization problem; cf., e.g., [2]. Therefore, can we prove that the solution of (2.2) can be take as the linear combination of finitely many Dirac delta functions?
- Study the minimum norm problem ($\alpha \to 0$ in (2.2)). Therefore, we want to investigate first the free space case of (1.2) ($\Omega = \mathbb{R}^N$ with appropriate radiation conditions at infinity), where the solution for a sum of point sources can be given analytically.

References

[1] A. Bermúdez, P. Gamallo, and R. Rodríguez, *Finite element methods in local active control of sound*, SIAM J. Control Optim., 43 (2004), pp. 437–465 (electronic).

[2] J. F. Bonnans and A. Shapiro, *Perturbation analysis of optimization problems*, Springer Series in Operations Research, Springer, New York, 2000.

[3] K. Bredies and H. K. Pikkarainen, *Inverse problems in spaces of measures*, ESAIM Control Optim. Calc. Var., 19 (2013), pp. 190–218.

[4] C. Clason and K. Kunisch, *A measure space approach to optimal source placement*, Comput. Optim. Appl., 53 (2011), pp. 155–171.

[5] B. Engquist and A. Majda, *Absorbing boundary conditions for the numerical simulation of waves*, Math. Comp., 31 (1977), pp. 629–651.

[6] F. Ihlenburg, *Finite element analysis of acoustic scattering*, vol. 132 of Applied Mathematical Sciences, Springer-Verlag, New York, 1998.

[7] K. Kunisch, P. Trautmann, and B. Vexler, *Optimal control of the undamped linear wave equation with measure valued controls.* submitted, 2014.