International Research Training Group IGDK 1754

IGDK Students' Workshop

Characterization of I-sets and Besov spaces

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October 4.-9., 2015

For general existence theorems concerning systems of semilinear parabolic equations, subspaces of Sobolev spaces on a domain $\Omega \subset \mathbb{R}^d$, with $d > 0$, which represent the boundary conditions of the single equations, are very helpful. For one equation, the boundary $\partial\Omega$ consists of the Dirichlet part D and the rest of the boundary $\Gamma := \partial\Omega \backslash \mathrm{D}$. In this context, it turns out, that it makes sense to require that D is a $d-1$-set.

This term follows the subsequent definition of an I-set [1, cf. II.1.1/2] or [2, cf. Definition 4.1.].

Definition 1. Suppose that $0 < I \leq d$, $M \subset \mathbb{R}^d$ is closed, and ρ is the restriction of the I-dimensional Hausdorff measure $\mathcal{H}_I$ to M. In this case, M is an I-set, if there are constants $c_1, c_2 > 0$, such that

$$c_1 r^I \leq \rho\left(B(x,r) \cap M\right) \leq c_2 r^I, \ x \in M, \ r \in]0,1[, \tag{1}$$

where as usual $B(x,r)$ denotes the ball in $\mathbb{R}^d$, centered at x and with radius d.

It would be interesting to inspect conditions on the domain and on the boundary part Γ, which ensure that D is a $d-1$-set.

The space $\mathrm{W}^{1,p}_{\mathrm{D}}(\Omega)$ is defined as the completion in $\mathrm{W}^{1,p}(\Omega)$ of the subspace of $\mathrm{C}^\infty|_\Omega$-functions, which vanish in a neighborhood of D [2, cf. Definition 2.6].

Under the further assumption, that the part $\overline{\Gamma}$ of the boundary of Ω can be covered by relatively open sets, which fulfill a bi-Lipschitz condition [2, cf. Assumption 2.4.], the existence of an extension operator from the space $\mathrm{W}^{1,p}_{\mathrm{D}}(\Omega)$ to $\mathrm{W}^{1,p}_{\mathrm{D}}(\mathbb{R}^d)$ can be proven. This is needed to show resolvent estimates for elliptic differential operators, which in turn lead to the existence of solutions to many semilinear parabolic PDEs.

Another interesting question would thus be, in how far the mentioned class of domains and boundaries extends common regularity assumptions on the domain in PDE problems, and which domains are not contained in this class. This investigation can give an impression of how extensive the theory on PDEs with those new domains is, and might highlight new possibilities for semilinear parabolic or elliptic PDEs, as well as for optimal control problems including such as constraints.

One could also think about conditions on when unions or intersections of sets, which may or may not be $d-1$-sets themselves, result in a new set, which then is a $d-1$-set.

If D is a $d-1$-set, it can be shown, that there is a continuous restriction operator $\mathcal{R}_{\mathrm{D}}$ which maps every space $\mathrm{W}^{1,p}(\mathbb{R}^d)$ onto the Besov space $\mathrm{B}_{p,p}^{1-\frac{1}{p}}(\mathrm{D})$ for $p \in]1,\infty[$, and there also is an extension operator $\mathcal{E}_{\mathrm{D}}$, which maps $\mathrm{B}_{p,p}^{1-\frac{1}{p}}(\mathrm{D})$ continuously into $\mathrm{W}^{1,p}(\mathbb{R}^d)$, and which is right inverse to R_{D} [1, cf. Ch. VII].

The restriction of $f \in \mathrm{W}^{1,p}(\mathbb{R}^d)$ is established by the function

$$\lim_{r\to 0} \frac{1}{|B(y,r)|} \int\limits_{B(y,r)} f(x)dx \tag{2}$$

for $\mathcal{H}_{d-1}$-almost all $y \in \mathbb{R}^d$, and this function reproduces f in $\mathrm{W}^{1,p}(\mathbb{R}^d)$.

It is an interesting question, how this restriction and the representation work exactly, and where the importance of D to be a $d-1$-set comes into play. Also interesting in this context would be, if the property of D to be a $d-1$-set characterizes the spaces $\mathrm{B}_{p,p}^{1-\frac{1}{p}}(\mathrm{D})$ in a certain sense. This might lead to a characterization of the smoothness properties of $\mathrm{W}_{\mathrm{D}}^{1,p}(\mathbb{R}^d)$ or $\mathrm{W}_{\mathrm{D}}^{1,p}(\Omega)$, and give an idea for reasonable choices of objective functionals in optimal control problems.

References

[1] A.Jonsson, H.Wallin, Function spaces on subsets of $\mathbb{R}^n$, Math. Rep. 2, no. 1. (1984)

[2] A.F.M. ter Elst, J. Rehberg, Hölder estimates for second-order operators on domains with rough boundary, Adv. in Differential Equations 20, no. 3/4, 299–366, http://projecteuclid.org/euclid.ade/1423055203 (2015)