Space-Time Discretization of Parabolic Time-Optimal Control Problems

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We consider the following time-optimal control problem subject to a linear parabolic equation,

$$\text{Minimize } j(T,q) := T + \frac{\alpha}{2}\int_0^T \|q(t)\|^2_{L^2(\Omega)}\mathrm{d}t,$$

$$\begin{aligned} T > 0,\ q \in Q_{ad}, & & \\ \partial_t u + Au = Bq, & \qquad \text{in } (0,T)\times\Omega, \\ u(0) = u_0, & \qquad \text{in } \Omega, \\ \|u(T) - u_d\|_{L^2(\Omega)} \le \delta, & \end{aligned} \tag{P}$$

with desired terminal state $u_d \in L^2(\Omega)$ and given $\delta > 0$. Moreover, Q_{ad} denotes the set of admissible controls and $\alpha > 0$ the regularization parameter.

Our aim is the numerical analysis for finite element discretizations of (P). It is worth mentioning that (P) is non-convex, possesses a non-linear dependence on T and is subject to state constraints. For these reasons the existence of Lagrange multipliers is non-trivial.

Thus, we first study the stability of the optimization problem with respect to perturbations of δ. Under reasonable assumptions on A, B and u_d we prove stability of (P). The stability condition allows to derive first order optimality conditions in qualified form.

Furthermore, we prove second order necessary as well as sufficient optimality conditions. The latter leads to a quadratic growth condition without two-norm discrepancy.

Employing the quadratic growth condition, we show *a priori* discretization error estimates for a discontinuous Galerkin discretization scheme in time and a continuous Galerkin scheme in space.

Moreover, we prove *a priori* discretization error estimates for the objective function values by employing similar arguments as for proving the stability condition. In particular, this technique does not rely on a second order sufficient optimality condition.