Tensor Analysis, Computation and Applications

by

Liqun Qi

Department of Applied Mathematics
The Hong Kong Polytechnic University

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1. Introduction

Tensors (hypermatrices) are extensions of matrices. The difference is that a matrix entry a_{ij} has two indices i and j, while a tensor entry $a_{i_1 \cdots i_m}$ has m indices $i_1, \cdots, i_m$. In the recent decade, major progresses have been made on the research of tensors. It is revealed that there are also profound theories on tensor analysis, just as matrix analysis. Today I will review progresses on six areas of Tensor Analysis, Computation and Applications. My review cannot be complete.

A book on spectral theory of tensors (hypermatrices) and special tensors (hypermatrices) is:

[A] L. Qi and Z. Luo, *Tensor Analysis: Spectral Properties and Special Tensors*, SIAM, Philadelphia, 2017.

1.1. A Book on Tensor (Hypermatrix) Analysis

1.2. Symmetric Tensors

Denote $[n] := \{1, \cdots, n\}$. An mth order n-dimensional real tensor (hypermatrix) $\mathcal{A} = (a_{i_1 \cdots i_m})$ is a multi-array of real numbers $a_{i_1 \cdots i_m}$, where $i_j \in [n]$ for $j \in [m]$. Denote the set of all mth order n-dimensional real tensors as $T_{m,n}$.

For an mth order n-dimensional tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$, if its entries $a_{i_1 \cdots i_m}$ are invariant under any permutation of its indices, then $\mathcal{A}$ is called a symmetric tensor. Denote the set of all mth order n-dimensional real symmetric tensors as $S_{m,n}$. Symmetric tensors arise from many applications.

Define the general Kronecker symbol as

$$\delta_{i_1 \cdots i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

Then $a_{i_1 \cdots i_m}$ is called a diagonal entry if $\delta_{i_1 \cdots i_m} = 1$, and an off-diagonal entry otherwise. The tensor with entries $\delta_{i_1 \cdots i_m}$ is called the **identity tensor** in $T_{m,n}$, denoted by $\mathcal{I}_{m,n}$, or simply $\mathcal{I}$ if its order and dimension are clear by the context.

For $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$, for $i \in [n]$, define

$$\Delta_i := \sum \{ |a_{ii_2 \cdots i_m}| : i_j \in [n] \text{ for } j = 2, \cdots, m, \delta_{ii_2 \cdots i_m} = 0 \}.$$

1.3. Examples of Symmetric Tensors

Example 1: Higher-order derivatives of sufficiently differentiable multi-variable functions. Let $f : \Re^n \to \Re$ have continuous mth order derivatives. Then its mth order derivatives $\nabla^{(m)} f(\mathbf{x})$ at any $\mathbf{x} \in \Re^n$ is an mth order n-dimensional real symmetric tensor.

Example 2: Coefficient tensors of multi-variate homogeneous polynomial forms. Let

$$f(\mathbf{x}) = \sum_{i_1,\cdots,i_m=1}^{n} a_{i_1\cdots i_m} x_{i_1} \cdots x_{i_m}$$

be a multi-variate homogeneous polynomial form. Then its coefficient tensor $\mathcal{A} = (a_{i_1\cdots i_m})$ is a symmetric tensor.

Example 3: Moment and cumulant tensors in signal processing. Let $\mathbf{x}$ be a random vector of dimension n, with components x_i. Then one defines its moment and cumulant tensors of order m as: $\mathcal{M}(\mathbf{x}) = (\mu_{i_i\cdots i_m})$ with $\mu_{i_1\cdots i_m} = \mathrm{E}\{x_{i_1} \cdots x_{i_m}\}$, and $\mathcal{C}(\mathbf{x}) = (c_{i_1\cdots i_m})$ with $c_{i_1\cdots i_m} = \mathrm{Cum}\{x_{i_1} \cdots x_{i_m}\}$. The cumulants of order 1 and 2 are better known under the names of statistical *mean* and *covariance*. Moment and cumulant tensors are symmetric tensors.

1.4. Examples 4: Higher Order Diffusion Tensors

Diffusion magnetic resonance imaging (D-MRI) has been developed in biomedical engineering for decades. It measures the apparent diffusivity of water molecules in human or animal tissues, such as brain and blood, to acquire biological and clinical information. In tissues, such as brain gray matter, where the measured apparent diffusivity is largely independent of the orientation of the tissue (i.e., isotropic), it is usually sufficient to characterize the diffusion characteristics with a single (scalar) apparent diffusion coefficient (ADC). However, in anisotropic media, such as skeletal and cardiac muscle and in white matter, where the measured diffusivity is known to depend upon the orientation of the tissue, no single ADC can characterize the orientation-dependent water mobility in these tissues. Because of this, a diffusion tensor model was proposed years ago to replace the diffusion scalar model. This resulted in **Diffusion Tensor Imaging** (DTI).

However, DTI is known to have a limited capability in resolving multiple fibre orientations within one voxel. This is mainly because the probability density function for random spin displacement is non-Gaussian in the confining environment of biological tissues and, thus, the modeling of self-diffusion by a second order tensor breaks down. Hence, researchers presented various **Higher Order Diffusion Tensor Imaging** models to overcome this problem. A higher order diffusion tensor is a higher order symmetric tensor.

1.5. Example 5: Adjacency and Laplacian Tensors

A uniform hypergraph is also called a k-graph. Let $G = (V, E)$ be a k-graph, where $V = \{1, 2, \cdots, n\}$ is the vertex set, $E = \{e_1, e_2, \cdots, e_m\}$ is the edge set, $e_p \subset V$ and $|e_p| = k$ for $p = 1, \cdots, m$, and $k \geq 2$. If $k = 2$, then G is an ordinary graph. We assume that $e_p \neq e_q$ if $p \neq q$.

The **adjacency tensor** $\mathcal{A} = \mathcal{A}(G)$ of G, is a kth order n-dimensional symmetric tensor, with $\mathcal{A} = (a_{i_1 i_2 \cdots i_k})$, where $a_{i_1 i_2 \cdots i_k} = \frac{1}{(k-1)!}$ if $(i_1, i_2, \cdots, i_k) \in E$, and 0 otherwise. Thus, $a_{i_1 i_2 \cdots i_k} = 0$ if two of its indices are the same.

For $i \in V$, its degree $d(i)$ is defined as $d(i) = |\{e_p : i \in e_p \in E\}|$. We assume that every vertex has at least one edge. Thus, $d(i) > 0$ for all i. The **degree tensor** $\mathcal{D} = \mathcal{D}(G)$ of G, is a kth order n-dimensional diagonal tensor, with its ith diagonal element as $d(i)$.

The **Laplacian tensor** $\mathcal{L}$ of G is defined by $\mathcal{D} - \mathcal{A}$. The **signless Laplacian tensor** $\mathcal{Q}$ of G is defined by $\mathcal{D} + \mathcal{A}$.

Adjacency tensors, degree tensors, Laplacian tensors and signless Laplacian tensors are real symmetric tensors.

1.6. Tensor Products

We use $\bullet$ to denote tensor inner product (full contraction). Let $\mathcal{A} = (a_{i_1\cdots i_m}), \mathcal{B} = (b_{i_1\cdots i_m}) \in T_{m,n}$. Then

$$\mathcal{A} \bullet \mathcal{B} = \sum_{i_1,\cdots,i_m=1}^{n} a_{i_1\cdots i_m} b_{i_1\cdots i_m}.$$

$\sqrt{\mathcal{A} \bullet \mathcal{A}}$ is called the Frobenius norm of $\mathcal{A}$, denoted as $\|\mathcal{A}\|_F$.

We use $\otimes$ to denote tensor outer product. Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$ and $\mathcal{B} = (b_{i_1\cdots i_p}) \in T_{p,n}$. Then $\mathcal{A} \otimes \mathcal{B} = \left(a_{i_1\cdots i_m} b_{i_{m+1}\cdots i_{m+p}}\right) \in T_{m+p,n}$.

Let $\mathbf{x} \in \Re^n$ and $\alpha \in \Re$. Denote $\alpha \mathbf{x}^m \equiv \alpha \mathbf{x}^{\otimes m} \equiv \alpha \underbrace{\mathbf{x} \otimes \cdots \otimes \mathbf{x}}_{m}$. Then $\alpha \mathbf{x}^m \in S_{m,n}$. We call it a *symmetric rank-one tensor*. In particular, we call $\mathbf{x}^m$ a *pure symmetric rank-one tensor.*

We will use small letters $x, y, a, b, \cdots$, for scalers, small bold letters $\mathbf{x}, \mathbf{y}, \cdots$, for vectors, capital letters $A, B, C, \cdots$, for matrices, calligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \cdots$, for tensors. We use $\mathbf{0}$ to denote the zero vector in $\Re^n$. Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$, we denote $|\mathcal{A}| = (|a_{i_1\cdots i_m}|)$.

1.7. Positive Definite and Semi-Definite Tensors

An n-dimensional homogeneous polynomial form of degree m, $f(\mathbf{x})$, where $\mathbf{x} \in \Re^n$, is equivalent to the tensor product of an n-dimensional tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ of order m, and the pure symmetric rank-one tensor x^m:

$$f(\mathbf{x}) \equiv \mathcal{A}\mathbf{x}^m \equiv \mathcal{A} \bullet \mathbf{x}^m := \sum_{i_1, \cdots, i_m = 1}^{n} a_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}.$$

The tensor $\mathcal{A}$ is called positive semi-definite (PSD) if $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \Re^n$; and positive definite (PD) if $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in \Re^n, \mathbf{x} \neq \mathbf{0}$. Clearly, when m is odd, there is no nontrivial positive semi-definite tensor.

When m is even, the positive (semi-)definiteness of such a tensor $\mathcal{A}$ or such a homogeneous polynomial form $f(\mathbf{x})$ plays an important role in polynomial theory, automatic control, stochastic process, magnetic resonance imaging, spectral hypergraph theory, etc.

2. Eigenvalues and Eigenvectors

An n-dimensional homogeneous polynomial form of degree m, $f(x)$, where $x \in \Re^n$, is equivalent to the tensor product of a **symmetric** n-dimensional tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ of order m, and the rank-one tensor x^m:

$$f(x) \equiv \mathcal{A}x^m := \sum_{i_1, \cdots, i_m = 1}^{n} a_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}.$$

The tensor $\mathcal{A}$ is called symmetric as its entries $a_{i_1 \cdots i_m}$ are invariant under any permutation of their indices. The tensor $\mathcal{A}$ is called positive definite (semidefinite) if $f(x) > 0$ ($f(x) \geq 0$) for all $x \in \Re^n$, $x \neq 0$. When m is even, the positive definiteness of such a homogeneous polynomial form $f(x)$ plays an important role in the stability study of nonlinear autonomous systems via Liapunov's direct method in **Automatic Control**. For $n \geq 3$ and $m \geq 4$, this issue is a hard problem in mathematics.

2.1. Eigenvalues of Tensors

[1]. L. Qi, "Eigenvalues of a real supersymmetric tensor", *Journal of Symbolic Computation* **40** (2005) 1302-1324,

defined eigenvalues and eigenvectors of a real symmetric tensor, and explored their practical applications in determining positive definiteness of an even degree multivariate form.
By the tensor product, $\mathcal{A}x^{m-1}$ for a vector $x \in \Re^n$ denotes a vector in $\Re^n$, whose ith component is

$$\left(\mathcal{A}x^{m-1}\right)_i \equiv \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}.$$

We call a number $\lambda \in C$ an **eigenvalue** of $\mathcal{A}$ if it and a nonzero vector $x \in C^n$ are solutions of the following homogeneous polynomial equation:

$$\left(\mathcal{A}x^{m-1}\right)_i = \lambda x_i^{m-1}, \ \ \forall\, i = 1, \cdots, n. \tag{1}$$

and call the solution x an **eigenvector** of $\mathcal{A}$ associated with the eigenvalue λ. We call an eigenvalue of $\mathcal{A}$ an **H-eigenvalue** of $\mathcal{A}$ if it has a real eigenvector x. An eigenvalue which is not an H-eigenvalue is called an **N-eigenvalue**. A real eigenvector associated with an H-eigenvalue is called an **H-eigenvector**.

The **resultant** of (1) is a one-dimensional polynomial of λ. We call it the **characteristic polynomial** of $\mathcal{A}$.

2.2. Theorem on Eigenvalues

Theorem 2.1 (**Qi 2005**)
We have the following conclusions on eigenvalues of an mth order n-dimensional symmetric tensor $\mathcal{A}$:
(a). A number $\lambda \in C$ is an eigenvalue of $\mathcal{A}$ if and only if it is a root of the characteristic polynomial ϕ.
(b). The number of eigenvalues of $\mathcal{A}$ is $d = n(m-1)^{n-1}$. Their product is equal to det$(\mathcal{A})$, the resultant of $\mathcal{A}x^{m-1} = 0$.
(c). The sum of all the eigenvalues of A is

$$(m-1)^{n-1}\mathrm{tr}(\mathcal{A}),$$

where tr$(\mathcal{A})$ denotes the sum of the diagonal elements of $\mathcal{A}$.
(d). If m is even, then $\mathcal{A}$ always has H-eigenvalues. $\mathcal{A}$ is positive definite (positive semidefinite) if and only if all of its H-eigenvalues are positive (non-negative).
(e). The eigenvalues of $\mathcal{A}$ lie in the following n disks:

$$|\lambda - a_{i,i,\cdots,i}| \leq \sum\{|a_{i,i_2,\cdots,i_m}| : i_2, \cdots, i_m = 1, \cdots, n, \{i_2, \cdots, i_m\} \neq \{i, \cdots, i\}\},$$

for $i = 1, \cdots, n$.

2.3. E-Eigenvalues

In the same paper, I defined another kind of eigenvalues for tensors. Their structure is different from the structure described by Theorem 2.1. Their characteristic polynomial has a lower degree.

Suppose that $\mathcal{A}$ is an mth order n-dimensional symmetric tensor. We say a complex number λ is an **E-eigenvalue** of $\mathcal{A}$ if there exists a complex vector x such that

$$\begin{cases} \mathcal{A}x^{m-1} = \lambda x, \\ x^T x = 1. \end{cases} \tag{2}$$

In this case, we say that x is an E-eigenvector of the tensor $\mathcal{A}$ associated with the E-eigenvalue λ. If an E-eigenvalue has a real E-eigenvector, then we call it a **Z-eigenvalue** and call the real E-eigenvector a **Z-eigenvector**.

2.4. The E-Characteristic Polynomial and Orthogonal Similarity

When m is even, the **resultant** of

$$\mathcal{A}x^{m-1} - \lambda(x^T x)^{\frac{m-2}{2}} x = 0$$

is a one dimensional polynomial of λ and is called the **E-characteristic polynomial** of $\mathcal{A}$. We say that $\mathcal{A}$ is regular if the following system has no nonzero complex solutions:

$$\begin{cases} \mathcal{A}x^{m-1} = 0, \\ x^T x = 0. \end{cases}$$

Let $P = (p_{ij})$ be an $n \times n$ real matrix. Define $\mathcal{B} = P^m \mathcal{A}$ as another mth order n-dimensional tensor with entries

$$b_{i_1,i_2,\cdots,i_m} = \sum_{j_1,j_2,\cdots,j_m=1}^{n} p_{i_1 j_1} p_{i_2 j_2} \cdots p_{i_m j_m} a_{j_1,j_2,\cdots,j_m}.$$

If P is an orthogonal matrix, then we say that $\mathcal{A}$ and $\mathcal{B}$ are **orthogonally similar**.

2.5. Theorem on E-eigenvalues

Theorem 2.2 (**Qi 2005**)
We have the following conclusions on E-eigenvalues of an mth order n-dimensional symmetric tensor $\mathcal{A}$:
(a). When $\mathcal{A}$ is regular, a complex number is an E-eigenvalue of $\mathcal{A}$ if and only if it is a root of its E-characteristic polynomial.
(b). Z-eigenvalues always exist. An even order symmetric tensor is positive definite if and only if all of its Z-eigenvalues are positive.
(c). If $\mathcal{A}$ and $\mathcal{B}$ are orthogonally similar, then they have the same E-eigenvalues and Z-eigenvalues.
(d). If λ is the Z-eigenvalue of $\mathcal{A}$ with the largest absolute value and x is a Z-eigenvector associated with it, then λx^m is the best rank-one approximation of $\mathcal{A}$, i.e.,

$$\|\mathcal{A}-\lambda x^m\|_F = \sqrt{\|\mathcal{A}\|_F^2 - \lambda^2} = \min\{\|\mathcal{A}-\alpha u^m\|_F : \alpha \in \Re, u \in \Re^n, \|u\|_2 = 1\},$$

where $\|\cdot\|_F$ is the Frobenius norm.

2.6. Invariants of Tensors

Tensors are practical physical quantities in relativity theory, fluid dynamics, solid mechanics and electromagnetism, etc. The concept of tensors was introduced by Gauss, Riemann and Christoffel, etc., in the 19th century in the study of differential geometry. In the very beginning of the 20th century, Ricci, Levi-Civita, etc., further developed tensor analysis as a mathematical discipline. But it was **Einstein** who applied tensor analysis in his study of general relativity in 1916. This made tensor analysis an important tool in theoretical physics, continuum mechanics and many other areas of science and engineering. The tensors in theoretical physics and continuum mechanics are physical quantities which are invariant under co-ordinate system changes. A scalar associated with a tensor is an **invariant** of that tensor, if it does not change under co-ordinate system changes. Theorem 2.2 (c) implies that E-eigenvalues and Z-eigenvalues are invariants of the tensor. Later research demonstrate that these eigenvalues, in particular Z-eigenvalues, have practical uses in physics and mechanics.

[2]. L. Qi, "Eigenvalues and invariants of tensors", *Journal of Mathematical Analysis and Applications* **325** (2007) 1363-1377,

also elaborated this. When m is odd, the E-characteristic polynomial was defined in [2] as the **resultant** of

$$\begin{cases} \mathcal{A}x^{m-1} - \lambda x_0^{m-2}x, \\ x^\top x - x_0^2. \end{cases}$$

2.7. The Best Rank-One Approximation

Theorem 2.2 (d) indicates that Z-eigenvalues play an important role in the best rank-one approximation. This also implies that Z-eigenvalues are significant in practice. The best rank-one approximation of higher order tensors has extensive engineering and statistical applications, such as **Statistical Data Analysis**. The following are some papers on this topic:

[3]. L. De Lathauwer, B. De Moor and J. Vandwalle, "On the best rank-1 and rank-$(R_1, R_2, \cdots, R_n)$ approximation of higher order tensors", *SIAM J. Matrix Anal. Appl*, **21** (2000) 1324-1342.

[4]. E. Kofidies and Ph.A. Regalia, "On the best rank-1 approximation of higher order supersymmetric tensors", *SIAM J. Matrix Anal. Appl*, **23** (2002) 863-884.

[5]. T. Zhang and G.H. Golub, "Rank-1 approximation to higher order tensors", *SIAM J. Matrix Anal. Appl*, **23** (2001) 534-550.

2.8. The Gelfand Formula

A survey papers on eigenvalues of nonnegative tensors is:

[6]. K.C. Chang, L. Qi and T. Zhang, “A survey on the spectral theory of nonnegative tensors”, *Numerical Linear Algebra with Applications* **20** (2013) 891-912. In that paper the description on the Gelfand formula is as follows: “A 2-order n-dimensional real tensor A is the $n \times n$ real matrix $A = (a_{ij})$. It can also be viewed as a linear endomorphism on $\Re^n$, hence the eigenvalue problem for A is a linear problem. In particular, the spectral radius $r(A)$ of A is defined to be

$$r(A) = \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } A\}.$$

According to Gelfand's formula, $r(A) = \lim_{k\to\infty} ||A^k||^{\frac{1}{k}}$, where $||\cdot||$ denotes the operator norm. Thus, $r(A)$ is an intrinsic property of A as it is entirely determined by A itself.”

Is Gelfand's formula still valid for a higher order tensor?

2.9. The Gelfand Formula

This question is answered recently in the following paper:

[7]. Y. Song and L. Qi, "Spectral properties of positively homogeneous operators induced by higher order tensors", *SIAM Journal on Matrix Analysis and Applications* **34** (2013) 1581-1595.

Abstract: The Fredholm alternative type results are proved for eigenvalues (E-eigenvalues, H-eigenvalues, Z-eigenvalues) of a higher order tensor $\mathcal{A}$. For the positively homogeneous operators $F_{\mathcal{A}}$ and $T_{\mathcal{A}}$ induced by a higher order tensor $\mathcal{A}$, we show some relationship between the Gelfand formula and the spectral radius, and present the upper bound of their spectral radii. Furthermore, for a nonnegative tensor $\mathcal{A}$, we obtain the practical relevance for the spectral radius of the operators $F_{\mathcal{A}}$ and $T_{\mathcal{A}}$ as well as the operator norms of $F_{\mathcal{A}}$ and $T_{\mathcal{A}}$.

2.10. Lek-Heng Lim's work

Independently, at Stanford University in 2005, Gene Golub's then Ph.D. student Lek-Heng Lim also defined eigenvalues for tensors in his paper:

[8]. L-H. Lim, "Singular values and eigenvalues of tensors: A variational approach", Proceedings of the 1st IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP), December 13-15, 2005, pp. 129-132.

Lim (2005) defined eigenvalues for general real tensors in the real field. The l^2 eigenvalues of tensors defined by Lim (2005) are Z-eigenvalues of Qi (2005), while the l^k eigenvalues of tensors defined by Lim (2005) are H-eigenvalues in Qi (2005) in the even order case. Notably, Lim (2005) proposed a multilinear generalization of the Perron-Frobenius theorem based upon the notion of l^k eigenvalues (H-eigenvalues) of tensors.

2.11. Algorithms for Calculating Eigenvalues

[9]. L. Qi, F. Wang and Y. Wang, "Z-eigenvalue methods for a global polynomial optimization problem", *Mathematical Programming* **118** (2009) 301-316.

[10]. T.G. Kolda and J.R. Mayo, "Shifted power method for computing tensor eigenpairs", *SIAM J. Matrix Analysis and Applications* **32** (2011) 1095-1124.

[11]. L. Han, "An unconstrained optimization approach for finding eigenvalues of even order real symmetric tensors", *Numerical Algebra, Control and Optimization* **3** (2013) 583-599.

[12]. S. Hu, Z. Huang and L. Qi, "Finding the Extreme Z-Eigenvalues of Tensors via a Sequential SDPs Method", *Numerical Linear Algebra with Applications* **20** (2013) 972-984.

[13]. T.G. Kolda and J.R. Mayo, "An Adaptive Shifted Power Method for Computing Generalized Tensor Eigenpairs", *SIAM J. Matrix Analysis and Applications* **35** (2014) 1563-1581.

[14]. C. Hao, C. Cui and Y. Dai, "A sequential subspace projection method for extreme Z-eigenvalues of supersymmetric tensors", *Numerical Linear Algebra with Applications* **22** (2015) 283-298.

[15]. C. Cui, Y. Dai and J. Nie, "All real eigenvalues of symmetric tensors", *SIAM J. Matrix Analysis and Applications* **35** (2014) 1582-1601.

2.12. Determinant

[16]. S. Hu, Z. Huang, C. Ling and L. Qi, "On determinants and eigenvalue theory of tensors", *Journal of Symbolic Computation* **50** (2013) 508-531.

We investigate properties of the determinants of tensors, and their applications in the eigenvalue theory of tensors. We show that the determinant inherits many properties of the determinant of a matrix. These properties include: solvability of polynomial systems, product formula for the determinant of a block tensor, product formula of the eigenvalues and Gershgorin's inequality. As a simple application, we show that if the leading coefficient tensor of a polynomial system is a triangular tensor with nonzero diagonal elements, then the system definitely has a solution in the complex space. We investigate the characteristic polynomial of a tensor through the determinant and the higher order traces. We show that the k-th order trace of a tensor is equal to the sum of the k-th powers of the eigenvalues of this tensor, and the coefficients of its characteristic polynomial are recursively generated by the higher order traces. Explicit formula for the second order trace of a tensor is given.
Another paper on determinants of tensors is:

[17]. J. Shao, H. Shan, L. Zhang, "On some properties of the determinants of tensors", *Linear Algebra and Its Applications* **439** (2013).

2.13. Eigenvalue Inclusion

We see one paper on this:

[18]. C. Li, Y. Li and X. Kung, "New eigenvalue inclusion sets for tensors", *Numerical Linear Algebra with Applications* **21** (2014) 39-50.

In this paper, the authors give a number of eigenvalue inclusion theorems, including a Taussky-type boundary result and a Brauer eigenvalue inclusion theorem for tensors, and their applications.
Further papers on eigenvalue inclusion.

[19]. C. Li, Z. Chen and Y. Li, "A New eigenvalue inclusion set for tensors and its applications", *Linear Algebra and Its Applications* **481** (2015) 36-53.

[20]. C. Bu, Y. Wei, L. Sun and J. Zhou, "Brualdi-type eigenvalue inclusion sets of tensors", *Linear Algebra and Its Applications* **480** (2015) 168-175.

[21]. G. Wang, G. Zhou and L. Caccett, "Sharp Brauer-type Eigenvalue Inclusion Theorems for Tensors", to appear in: *Pacific Journal of Optimization*.

2.14. Singular Values and Symmetric Embedding

We see the following two papers on this topic:

[22]. S. Ragnarsson and C.F. Van Loan, “Block tensors and symmetric em2eddings”, *Linear Algebra and Its Applications* **438** (2013) 853-874.

[23]. Z. Chen and L. Lu, “A tensor singular values and its symmetric embedding eigenvalues”, *Journal of Computational and Applied Mathematics* **250** (2013) 217-228.

2.15. Trace Formulas

Trace formulas were first introduced in

[24]. A. Morozov and S. Shakirov, "Analogue of the identity Log Det=Trace Log for resultants", *J. Geom. Phys.* **61** (2011) 708-726.

They were further discussed in the following three papers:

[25]. J. Cooper and A. Dutle, "Spectra of uniform hypergraphs", *Linear Alg. Appl.* **436** (2012) 3268-3292.

[17]. S. Hu, Z. Huang, C. Ling and L. Qi, "On Determinants and Eigenvalue Theory of Tensors", *Journal of Symbolic Computation* **50** (2013) 508-531.

[26]. J. Shao, L. Qi and S. Hu, "Some new trace formulas of tensors with applications in spectral hypergraph theory", *Linear and Multilinear Algebra* **63** (2015) 971-992.

In [26], we give some graph theoretical formulas for the trace $Tr_k(\mathcal{A})$ of a tensor $\mathcal{A}$ which do not involve the differential operators and auxiliary matrices, then apply them to spectral hypergraph theory.

2.16. Z-Spectral Radius

Let $\mathcal{A} \in S_{m,n}$. We may define its spectral radius as

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\},$$

its E-spectral radius as

$$\rho_E(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an E} - \text{eigenvalue of } \mathcal{A}\}.$$

Here, $|\lambda|$ means the modulus of λ as eigenvalues and E-eigenvalues of a symmetric tensor may not be real. In practice, it seems that the Z-spectral radius of $\mathcal{A}$:

$$\rho_Z(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an Z} - \text{eigenvalue of } \mathcal{A}\},$$

is more useful and meaningful. Here, $|\lambda|$ means the absolute value of λ as Z-eigenvalues are always real.

First application of Z-spectral radius is the best rank-one approximation to $\mathcal{A}$, as discussed earlier.

2.17. The Formula of Z-Spectral Radius

In the following paper,

[27]. L. Qi, "The Best Rank-One Approximation Ratio of a Tensor Space", *SIAM Journal on Matrix Analysis and Applications* **32** (2011) 430-442,

a formula for Z-spectral radius is given as follows:

$$\rho_Z(\mathcal{A}) = \max\{|\mathcal{A}\mathbf{x}^m| : \|\mathbf{x}\|_2 = 1, \mathbf{x} \in \Re^n\}.$$

Furthermore, it was shown there that $\rho_Z(\cdot)$ is a norm of $S_{m,n}$. The minimum ratio of $\min\{\frac{\rho_Z(\mathcal{A})}{\|\mathcal{A}\|_F} : \mathcal{A} \in S_{m,n}\}$ is positive and can be used to estimate the linear convergence rate of the successive best rank-one approximation method. It seems that Z-spectral radius is more essential. More research is needed on this topic.

2.18. Generalized Eigenvalues

Generalized Eigenvalues were introduced in the following paper:

[28]. K. C. Chang, K. Pearson and T. Zhang, "Perron Frobenius Theorem for nonnegative tensors", *Commu. Math. Sci.* **6** (2008) 507-520.

It has been further studied in the following papers:

[13]. T.G. Kolda and J.R. Mayo, "An Adaptive Shifted Power Method for Computing Generalized Tensor Eigenpairs", *SIAM J. Matrix Analysis and Applications* **35** (2014) 1563-1581.

[15]. C. Cui, Y. Dai and J. Nie, "All real eigenvalues of symmetric tensors", *SIAM J. Matrix Analysis and Applications* **35** (2014) 1582-1601.

[29]. W. Ding and Y. Wei, "Generalized tensor eigenvalue problems", *SIAM J. Matrix Analysis and Applications* **36** (2015) 1073-1099.

2.19. Perturbation bounds of Eigenvalues

Perturbation of eigenvalues was discussed in the following paper:

[30]. M. Che, L. Qi and Y. Wei, “Perturbation bounds of tensor eigenvalues and singular values problems”, *Linear Multilinear Algebra* **64** (2016) 622-652.

W. Li and M. Ng have several papers on perturbation bounds of eigenvalues of nonnegative tensors. This will be described in the next section.

3. Nonnegative Tensors

A tensor $\mathcal{A} \in T_{m,n}$ is called a nonnegative tensor if all of its entries are nonnegative. The eigenvalue problem of nonnegative tensors has applications in multilinear pagerank, spectral hypergraph theory and higher-order Markov chains, etc. The eigenvalue problem of general or symmetric tensors is in general NP-hard. On the other hand, recently, it was discovered that the largest eigenvalue problem of a nonnegative tensor has linearly convergent algorithms. The whole Perron-Frobenius theory of nonnegative matrices can be extended to nonnegative tensors, with more varieties: there are parallel theories based upon irreducible and weakly irreducible nonnegative tensors. A survey papers on eigenvalues of nonnegative tensors is:

[6]. K.C. Chang, L. Qi and T. Zhang, "A survey on the spectral theory of nonnegative tensors", *Numerical Linear Algebra with Applications* **20** (2013) 891-912.

Recently, study on eigenvalues of nonnegative tensors was further extended to essentially nonnegative tensors, M-tensors, copositive tensors, completely positive tensors.

3.1. The Perron-Frobenius Theorem

The following paper started the Perron-Frobenius Theory for irreducible nonnegative tensors:

[28]. K. C. Chang, K. Pearson and T. Zhang, "Perron Frobenius Theorem for nonnegative tensors", *Commu. Math. Sci.* **6** (2008) 507-520.

The following paper started the Perron-Frobenius Theory for weakly irreducible nonnegative tensors:

[31]. S. Friedland, S. Gaubert and L. Han, "Perron-Frobenius theorem for nonnegative multilinear forms and extensions", *Linear Algebra and Its Applications* **438** (2013) 738-749.

The other papers made contributions to the Perron-Frobenius Theory for nonnegative tensors include:

[32]. K.J. Pearson, "Essentially positive tensors", *International Journal of Algebra* **9** (2010) 421-427.

[33]. Y. Yang and Q. Yang, "Further results for Perron-Frobenius Theorem for nonnegative tensors", *SIAM Journal on Matrix Analysis and Applications* **31** (2010) 2517-2530.

[34]. K.C. Chang, K. Pearson and T. Zhang, "Primitivity, the convergence of the NQZ method, and the largest eigenvalue for nonnegative tensors", *SIAM Journal on Matrix Analysis and Applications* **32** (2011) 806-819.

[35]. Q. Yang and Y. Yang, "Further results for Perron-Frobenius Theorem for nonnegative tensors II", *SIAM Journal on Matrix Analysis and Applications* **32** (2011) 1236-1250.

3.2. The Perron-Frobenius Theorem

[36]. Y. Yang and Q. Yang, "Geometric simplicity of spectral radius of nonnegative irreducible tensors", *Frontiers of Mathematics in China* 8 (2013) 129-140.

[37]. S. Hu, Z. Huang and L. Qi, "Strictly nonnegative tensors and nonnegative tensor partition", *Science China Mathematics* **57** (2014) 181-195.

[38]. S. Hu and L. Qi, "A necessary and sufficient condition for existence of a positive Perron vector", *SIAM Journal on Matrix Analysis and Applications* **37** (2016) 1747-1770.

An eigenvalue of $\mathcal{A}$ is called real geometrically simple if it has only one independent real eigenvector.

In spectral hypergraph theory, it is discovered that adjacency tensors, Laplacian tensors, signless Laplacian tensors are all reducible, but they are weakly irreducible if the hypergraph is connected. Thus, in spectral hypergraph theory, weakly irreducible nonnegative tensor theory is more useful.

3.3. The Perron-Frobenius Theorem: The General Form

Theorem 3.1 (The Perron-Frobenius Theorem for Nonnegative Tensors) *If $\mathcal{A}$ is a nonnegative tensor of order m and dimension n, then $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with a nonnegative eigenvector $x \in \Re^n_+$. (Yang and Yang 2010)*

The spectral radius $\rho(\mathcal{A}) > 0$ if and only if $\mathcal{A}$ is nontrivially nonnegative. The spectral radius $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with a positive eigenvector $x \in \Re^n_{++}$ if and only if $\mathcal{A}$ is strongly nonnegative. If $\mathcal{A}$ is weakly irreducible, then $\mathcal{A}$ is strongly nonnegative. (Hu and Qi 2016)

Suppose that furthermore $\mathcal{A}$ is irreducible. If λ is an eigenvalue with a nonnegative eigenvector, then $\lambda = \rho(\mathcal{A})$. (Chang, Pearson and Zhang 2008)

In this case, if there are k distinct eigenvalues of modulus $\rho(\mathcal{A})$, then the eigenvalues are $\rho(\mathcal{A})e^{i2\pi j/k}$, where $j = 0, \cdots, k-1$. The number k is called the **cyclic index** *of $\mathcal{A}$. (Yang and Yang 2010)*

If moreover $\mathcal{A}$ is primitive, then its cyclic number is 1. (Chang, Pearson and Zhang 2011)

If $\mathcal{A}$ is essentially positive or even-order irreducible, then the unique positive eigenvalue is real geometrically simple. (Pearson 2010) (Yang and Yang 2011)

3.4. The Wielandt-Collatz characterization of $\rho(\mathcal{A})$

Chang, Pearson and Zhang (2008) extended the well-known Collatz minimax theorem for irreducible nonnegative matrices to irreducible nonnegative tensors. This theorem paves the way for constructing algorithms for calculating $\rho(\mathcal{A})$. This property may be also called the Wielandt-Collatz characterization of $\rho(\mathcal{A})$. Friedland, Gaubert and Han (2013) established this result for weakly irreducible nonnegative tensors.

Theorem 3.2 *Assume that $\mathcal{A}$ is a weakly irreducible nonnegative tensor of order m and dimension n. Then*

$$\mathrm{Min}_{x \in \Re^n_{++}} \mathrm{Max}_{x_i > 0} \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}} = \lambda_0 = \mathrm{Max}_{x \in \Re^n_{++}} \mathrm{Min}_{x_i > 0} \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}}, \qquad (3)$$

where $\lambda_0 = \rho(\mathcal{A})$ is the unique positive eigenvalue corresponding to the positive eigenvector.

3.5. Algorithms for Finding the Largest Eigenvalue of a Nonnegative Tensor

Based upon Theorem 3.2, Ng, Qi and Zhou proposed an algorithm for calculating $\rho(\mathcal{A})$ in:

[39]. M. Ng, L. Qi and G. Zhou, "Finding the largest eigenvalue of a nonnegative tensor", *SIAM Journal on Matrix Analysis and Applications* **31** (2009) 1090-1099.

Other papers on algorithms for finding the largest eigenvalue of a nonnegative tensor include:

[40]. Y. Liu, G. Zhou and N.F. Ibrahim, "An always convergent algorithm for the largest eigenvalue of an irreducible nonnegative tensor", *Journal of Computational and Applied Mathematics* **235** (2010) 286-292.

[41]. G. Zhou, L. Caccetta, K.L. Teo and S-Y. Wu, "Nonnegative polynomial optimization over unit spheres and convex programming relaxations", *SIAM Journal on Optimization* **22** (2012) 987-1008.

[42]. G. Zhou, L. Qi and S.Y. Wu, "Efficient algorithms for computing the largest eigenvalue of a nonnegative tensor", *Frontiers of Mathematics in China* 8 (2013) 155-168.

3.6. Convergence of The Ng-Qi-Zhou Algorithm

Chang, Pearson and Zhang (2011) proved the convergence of the Ng-Qi-Zhou algorithm for any starting point in $(\Re^n_+)_0$ if and only if the tensor is a primitive tensor.

Theorem 3.3 (Chang, Pearson and Zhang 2011) *The sequences generated by the Ng-Qi-Zhou algorithm converges to the unique positive eigenvalue $\rho(\mathcal{A})$ for any starting point $x^{(0)} \in (\Re^n_+)_0$ if and only if $\mathcal{A}$ is primitive.*

On the other hand, Friedland, Gaubert and Han (2013) proved the convergence of the Ng-Qi-Zhou algorithm for any starting point in $\Re^n_{++}$ if $\mathcal{A}$ is a weakly primitive tensor.

Theorem 3.4 (Friedland, Gaubert and Han 2013) *If $\mathcal{A}$ is a weakly primitive tensor, then the sequences generated by the Ng-Qi-Zhou algorithm converges to the unique positive eigenvalue $\rho(\mathcal{A})$ if $x^{(0)} \in \Re^n_{++}$.*

We see that these two theorems are consistent.

3.7. Linear Convergence of The Ng-Qi-Zhou Algorithm

Zhang and Qi established linear convergence of the Ng-Qi-Zhou algorithm for essentially positive tensors in the following paper:

[43]. L. Zhang and L. Qi, "Linear convergence of an algorithm for computing the largest eigenvalue of a nonnegative tensor", *Numerical Linear Algebra with Applications* **19** (2012) 830-841.

In the following paper, Zhang, Qi and Xu defined **weakly positive** tensors and established linear convergence of the Liu-Zhou-Ibrahim algorithm for weakly positive tensors.

[44]. L. Zhang, L. Qi and Y. Xu, "Linear convergence of the LZI algorithm for weakly positive tensors", *Journal of Computational Mathematics* **30** (2012) 24-33.

3.8. Symmetric Nonnegative Tensors

Let $\mathcal{A} \in S_{m,n}$ be a nonnegative tensor. In the following paper:

[45]. L. Qi, "Symmetric nonnegative tensors and copositive tensors", *Linear Algebra and Its Applications* **439** (2013) 228-238.

It was proved that

$$\rho(\mathcal{A}) = \max\{\mathcal{A}\mathbf{x}^m : \mathbf{x} \in \Re^n_+\}.$$

The following is another paper on symmetric nonnegative tensors:

[46]. G. Zhou, L. Qi and S.Y. Wu, "On the largest eigenvalue of a symmetric nonnegative tensor", *Numerical Linear Algebra with Applications* **20** (2013) 913-928.

3.9. Essentially Nonnegative Tensors

A tensor is called an essentially nonnegative tensor if its off-diagonal entries are nonnegative. The followings are two papers on essentially nonnegative tensors.

[47]. S. Hu, G. Li, L. Qi and Y. Song, "Finding the maximum eigenvalue of essentially nonnegative symmetric tensors via sum of squares programming", *Journal of Optimization Theory and Applications* **158** (2013) 717-738.

[48]. L. Zhang, L. Qi, Z. Luo and Y. Xu, "The dominant eigenvalue of an essentially nonnegative tensor", *Numerical Linear Algebra with Applications* **20** (2013) 929-941.

3.10. More papers on Computation of Eigenvalues of Nonnegative Tensors

The followings are some more papers on computation of eigenvalues of nonnegative tensors.

[49]. G. Zhou, L. Qi and S.Y. Wu, "Efficient algorithms for computing the largest eigenvalue of a nonnegative tensor", *Frontiers of Mathematics in China* 8 (2013) 155-168.

[50]. G. Zhou, L. Caccetta, K.L. Teo and S-Y. Wu, "Nonnegative polynomial optimization over unit spheres and convex programming relaxations", *SIAM Journal on Optimization* **22** (2012) 987-1008.

[51]. Z. Chen, L. Qi, Q. Yang and Y. Yang, "The solution methods for the largest eigenvalue (singular value) of nonnegative tensors and convergence analysis", *Linear Algebra and Its Applications* **439** (2013) 3713-3733.

[52]. Q. Ni and L. Qi, "A quadratically convergent algorithm for finding the largest eigenvalue of a nonnegative homogeneous polynomial map", *Journal of Global Optimization* **61** (2015) 627-641.

3.11. Perturbation bounds of Eigenvalues of Nonnegative Tensors

Li and Ng have worked on perturbation bounds of eigenvalues of nonnegative tensors.

[53]. W. Li, L. Cui and M. Ng, "The perturbation bound for the Perron vector of a transition probability tensor", *Numerical Linear Algebra with Applications* **20** (2013) 985-1000.

[54]. W. Li and M. Ng, "The perturbation bound for the spectral radius of a nonnegative tensor", *Advances of Numerical Analysis*, **2014**, Article ID 10952 (2014).

[55]. W. Li and M. Ng, "Some bounds for the spectral radius of nonnegative tensors", *Numerische Mathematik* **130** (2015) 315-335.

3.12. Higher-Order Markov Chains

Ng, Qi and Zhou (2009) discussed the application of the largest eigenvalue problem of a nonnegative tensor in higher-order Markov chains. They applied the method for finding the largest eigenvalue of a nonnegative tensor to compute the probability distribution of a higher-order Markov Chain. More papers of on this topic:

[56]. W. Li and M. Ng, "On the limiting probability distribution of a transition probability tensor", *Linear and Multilinear Algebra* **62** (2014) 362-385.

[57]. S. Hu and L. Qi, "Convergence of a second order Markov chain", *Applied Mathematics and Computation* **241** (2014) 183-192.

[58]. W. Li and M. Ng, "On the limiting probability distribution of a transition probability tensor", *Linear and Multilinear Algebra* **62** (2014) 362-385.

3.13. Z-Eigenvalues of Nonnegative Tensors

As we said, the higher-order Markov chain problem is related with Z-eigenvalues of nonnegative tensors. In the study of spectral hypergraphy theory, we also see some papers are linked with Z-eigenvalues of nonnegative tensors. Some papers reveal some similarities as well as differences between the Z-eigenvalues and H-eigenvalues of a nonnegative tensor:

[59]. K.C. Chang, K.J. Pearson and T. Zhang, "Some variational principles of the Z-eigenvalues for nonnegative tensors", *Linear Algebra and Applications* **438** (2013) 4166-4182.

[60]. K.C. Chang and T. Zhang, "On the uniqueness and non-uniqueness of the positive Z-eigenvector for transition probability tensors", *Journal of Mathematical Analysis and Applications* **408** (2013) 525-540.

3.14. Papers on Singular Values of Nonnegative Rectangular Tensors

[61]. K.C. Chang, L. Qi and G. Zhou, "Singular values of a real rectangular tensor", *Journal of Mathematical Analysis & Applications* **370** (2010) 284-294.

[62]. K. C. Chang and T. Zhang, "Multiplicity of singular values for tensors", *Commu. Math. Sci.* **7** (2009) 611-625.

[63]. Y. Yang and Q. Yang, "Singular values of nonnegative rectangular tensors", *Frontiers of Mathematics in China* **6** (2011) 363-378.

[64]. G. Zhou, L. Caccetta and L. Qi, "Convergence of an algorithm for the largest singular value of a nonnegative rectangular tensor", *Linear Algebra and Its Applications* **438** (2013) 959-968.

[65]. L. Zhang, "Linear convergence of an algorithm for the largest singular value of a real rectangular tensor", *Frontiers of Mathematics in China* **8** (2013) 141-153.

3.15. Nonlinear Positive Operators

In the following paper, a nonlinear version of Krein Rutman Theorem is established. A unified proof of the Krein Rutman Theorem for linear operators and for nonlinear operators, and of the Perron-Frobenius theorem for nonnegative matrices and for nonnegative tensors, is presented.

[66]. K.C. Chang, "A nonlinear Krein Rutman theorem", *Journal of Systems Science and Complexity* **22** (2009) 542-554.

Further papers on this topic:

[67]. Y. Song and L. Qi, "The existence and uniqueness of eigenvalue for monotone homogeneous mapping pairs", *Nonlinear Analysis* **75** (2012) 5283-5293.

[68]. Y. Song and L. Qi, "Positive eigenvalue-eigenvector of nonlinear positive mappings in a Banach space", *Frontiers of Mathematics in China* **9** (2014) 181-199.

4. Applications

The introduction of eigenvalues and E-eigenvalues of higher order tensors was motivated by the positive definiteness of multivariate homogeneous forms and the best rank-one approximation of higher order tensors. We now review the other applications and connections of eigenvalues of higher order tensors. The first notable application or connection is on magnetic resonance imaging.

4.1. Magnetic Resonance Imaging

The Nobel Prize in Physiology or Medicine 2003: Paul C Lauterbur and Peter Mansfield for their discoveries concerning "**Magnetic Resonance Imaging**".

Summary: Imaging of human internal organs with exact and non-invasive methods is very important for medical diagnosis, treatment and follow-up. This year's Nobel Laureates in Physiology or Medicine have made seminal discoveries concerning the use of magnetic resonance to visualize different structures. These discoveries have led to the development of modern magnetic resonance imaging, MRI, which represents a breakthrough in medical diagnostics and research.

Atomic nuclei in a strong magnetic field rotate with a frequency that is dependent on the strength of the magnetic field. Their energy can be increased if they absorb radio waves with the same frequency (resonance). When the atomic nuclei return to their previous energy level, radio waves are emitted. These discoveries were awarded the Nobel Prize in Physics in 1952. During the following decades, magnetic resonance was used mainly for studies of the chemical structure of substances. In the beginning of the 1970s, this year's Nobel Laureates made pioneering contributions, which later led to the applications of magnetic resonance in medical imaging.

4.2. Diffusion Tensor Imaging

Diffusion magnetic resonance imaging (D-MRI) has been developed in biomedical engineering for decades. It measures the apparent diffusivity of water molecules in human or animal tissues, such as brain and blood, to acquire biological and clinical information. In tissues, such as brain gray matter, where the measured apparent diffusivity is largely independent of the orientation of the tissue (i.e., isotropic), it is usually sufficient to characterize the diffusion characteristics with a single (scalar) apparent diffusion coefficient (ADC). However, in anisotropic media, such as skeletal and cardiac muscle and in white matter, where the measured diffusivity is known to depend upon the orientation of the tissue, no single ADC can characterize the orientation-dependent water mobility in these tissues. Because of this, a diffusion tensor model was proposed years ago to replace the diffusion scalar model. This resulted in **Diffusion Tensor Imaging** (DTI).

4.3. Diffusion Tensor

A diffusion tensor D is a second order three dimensional fully symmetric tensor. Under a Cartesian laboratory co-ordinate system, it is represented by a real three dimensional symmetric matrix, which has six independent elements $D = (d_{ij})$ with $d_{ij} = d_{ji}$ for $i, j = 1, 2, 3$. There is a relationship

$$\ln[S(b)] = \ln[S(0)] - \sum_{i,j=1}^{3} b d_{ij} x_i x_j. \tag{4}$$

Here $S(b)$ is the signal intensity at the echo time, $x = (x_1, x_2, x_3)$ is the unit direction vector, satisfying $\sum_{i=1}^{3} x_i^2 = 1$, the parameter b is given by

$$b = (\gamma\delta g)^2(\Delta - \frac{\delta}{3}),$$

γ is the proton gyromagnetic ratio, Δ is the separation time of the two diffusion gradients, δ is the duration of each gradient lobe. There are six unknown variables d_{ij} in the formula (10). By applying the magnetic gradients in six or more non-collinear, non-coplaner directions, one can solve (10) and get the six independent elements d_{ij}.

4.4. Eigenvalues of Diffusion Tensor

However, such elements d_{ij} cannot be directly used for biological or clinical analysis, as they vary under different laboratory coordinate systems. Thus, after obtaining the values of these six independent elements by MRI techniques, the biomedical engineering researchers will further calculate some characteristic quantities of this diffusion tensor D. These characteristic quantities are rotationally invariant, independent from the choice of the laboratory coordinate system. They include the three eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$ of D, the mean diffusivity (M_D), the fractional anisotropy (FA), etc. The largest eigenvalue λ_1 describes the diffusion coefficient in the direction parallel to the fibres in the human tissue. The other two eigenvalues describe the diffusion coefficient in the direction perpendicular to the fibres in the human tissue. The mean diffusivity is

$$M_D = \frac{\lambda_1 + \lambda_2 + \lambda_3}{3},$$

while the fractional anisotropy is

$$FA = \sqrt{\frac{3}{2}}\sqrt{\frac{(\lambda_1 - M_D)^2 + (\lambda_2 - M_D)^2 + (\lambda_3 - M_D)^2}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}},$$

where $0 \leq FA \leq 1$. If $FA = 0$, the diffusion is isotropic. If $FA = 1$, the diffusion is anisotropic.

4.5. Higher-Order Diffusion Tensor Imaging

However, DTI is known to have a limited capability in resolving multiple fibre orientations within one voxel. This is mainly because the probability density function for random spin displacement is non-Gaussian in the confining environment of biological tissues and, thus, the modeling of self-diffusion by a second order tensor breaks down. Hence, researchers presented various **Higher Order Diffusion Tensor Imaging** models to overcome this problem:

[H1]. J.H. Jensen, J.A. Helpern, A. Ramani, H. Lu and K. Kaczynski, "Diffusional kurtosis imaging: The quantification of non-Gaussian water diffusion by means of magnetic resonance imaging", *Magnetic Resonance in Medicine* **53** (2005) 1432-1440.

[H2]. H. Lu, J.H. Jensen, A. Ramani and J.A. Helpern, "Three-dimensional characterization of non-Gaussian water diffusion in humans using diffusion kurtosis imaging", *NMR in Biomedicine* **19** (2006) 236-247.

[H3]. E. Ozarslan and T.H. Mareci, "Generalized diffusion tensor imaging and analytical relationships between diffusion tensor imaging and high angular resolution diffusion imaging", *Magnetic Resonance in Medicine* **50** (2003) 955-965.

[H4]. D.S. Tuch, "Q-ball imaging", *Magnetic Resonance in Medicine* **52** (2004) 1358-1372,

[H5]. D.S. Tuch, T.G. Reese, M.R. Wiegell, N.G. Makris, J.W. Belliveau and V.J. Wedeen, "High angular resolution diffusion imaging reveals intravoxel white matter fiber heterogeneity", *Magnetic Resonance in Medicine* **48** (2002) 454-459.

4.6. Diffusion Kurtosis Tensor Imaging

The authors of [H1] and [H2] proposed to use a fourth order three dimensional fully symmetric tensor W, called the diffusion kurtosis (DK) tensor, to describe the non-Gaussian behavior. Under a Cartesian laboratory co-ordinate system, it is represented by a real fourth order three dimensional fully symmetric array, which has fifteen independent elements $W = (w_{ijkl})$ with w_{ijkl} being invariant for any permutation of its indices $i, j, k, l = 1, 2, 3$. The relationship (10) can be further expanded (adding a second Taylor expansion term on b) to:

$$\ln[S(b)] = \ln[S(0)] - \sum_{i,j=1}^{3} bd_{ij}x_i x_j + \frac{1}{6}b^2 M_D^2 \sum_{i,j,k,l=1}^{3} w_{ijkl}x_i x_j x_k x_l. \quad (5)$$

There are fifteen unknown variables w_{ijkl} in the formula (5). By applying the magnetic gradients in fifteen or more non-collinear, non-coplaner directions, one can solve (5) and get the fifteen independent elements w_{ijkl}.

Again, the fifteen elements w_{ijkl} vary when the laboratory co-ordinate system is rotated. What are the coordinate system independent characteristic quantities of the DK tensor W? Are there some type of eigenvalues of W, which can play a role here? Ed X. Wu and his group at Hong Kong University studied these questions. They searched by google possible papers on eigenvalues of higher order tensors. They found my paper [1]. This resulted in a surprising e-mail to me in February, 2007, and consequently, some collaborative studies on diffusion kurtosis tensor imaging.

4.7. D-Eigenvalues and Other Invariants

We proposed D-eigenvalues for the diffusion kurtosis tensor W and some other invariants to describe the DKI model.

[69]. L. Qi, Y. Wang and E.X. Wu, "D-eigenvalues of diffusion kurtosis tensors", *Journal of Computational and Applied Mathematics* **221** (2008) 150-157.

[70]. L. Qi, D. Han and E.X. Wu, "Principal invariants and inherent parameters of diffusion kurtosis tensors", *Journal of Mathematical Analysis and Applications* **349** (2009) 165-180.

[71]. D. Han, L. Qi and E.X. Wu, "Extreme diffusion values for non-Gaussian diffusions", *Optimization and Software* **23** (2008) 703-716.

[72]. E.S. Hui, M.M. Cheung, L. Qi and E.X. Wu, "Towards better MR characterization of neural tissues using directional diffusion kurtosis analysis", *Neuroimage* **42** (2008) 122-134.

[73]. E.S. Hui, M.M. Cheung, L. Qi and E.X. Wu, "Advanced MR diffusion characterization of neural tissue using directional diffusion kurtosis analysis", *Conf. Proc. IEEE Eng. Med. Biol. Soc. 2008* (2008) 3941-3944.

[74]. M.M. Cheung, E.S. Hui, K.C. Chan, J.A. Helpern, L. Qi, E.X. Wu, "Does diffusion kurtosis imaging lead to better neural tissue characterization? A rodent brain maturation study", *Neuroimage* **45** (2009) 386-392.

[75]. X. Zhang, C. Ling, L. Qi and E.X. Wu "The measure of diffusion skewness and kurtosis in magnetic resonance imaging", *Pacific Journal of Optimization* **6** (2010) 391-404.

4.8. Paper by Bloy and Verma

The authors of [H3], [H4] and [H5] suggested to use a single higher even order tensor to replace the second order diffusion tensor in (10). In particular, in [H4], Tuch proposed to reconstruct the diffusion orientation distribution function (ODF) of the underlying fiber population of a biological tissue. Experiments show that the QBI model may identify the underlying fiber directions well in the multiple fiber situations.

In 2008, Bloy and Verma in their paper cited my paper [1], and proposed to use Z-eigenvalues to identify principal directions of fibres in the Q-balling model.

[76]. L. Bloy and R. Verma, "On computing the underlying fiber directions from the diffusion orientation distribution function", in: *Medical Image Computing and Computer-Assisted Intervention – MICCAI 2008*, D. Metaxas, L. Axel, G. Fichtinger and G. Székeley, eds., (Springer-Verlag, Berlin, 2008) pp. 1-8.

4.9. Positive Semi-Definite Tensor Imaging

An intrinsic property of the diffusivity profile is positive semi-definite. Hence, the diffusion tensor, either second or higher order, must be positive semi-definite. For second order diffusion tensor, one may diagonalize the second order diffusion tensor and project it to the symmetric positive semi-definite cone by setting the negative eigenvalues to zero. Recently, some methods have been proposed to preserve positive semi-definiteness for a fourth order diffusion tensor. None of them is comprehensive to work for arbitrary high order diffusion tensors.

[77]. L. Qi, G. Yu and E.X. Wu, "Higher order positive semi-definite diffusion tensor imaging", *SIAM Journal on Imaging Sciences* **3** (2010) 416-433.

In the above paper, we propose a comprehensive model to approximate the ADC profile by a positive semi-definite diffusion tensor of either second or higher order. We call this model PSDT (positive semi-definite diffusion tensor).

4.10. The PSDT Model

We use $x = (x_1, x_2, x_3)^T$ to denote the magnetic field gradient direction. Assume that we use an mth order diffusion tensor. Then the diffusivity function can be expressed as

$$d(x) = \sum_{i=0}^{m} \sum_{j=0}^{m-i} d_{ij} x_1^i x_2^j x_3^{m-i-j}. \tag{6}$$

A diffusivity function d can be regarded as an mth order symmetric tensor. Clearly, there are

$$N = \sum_{i=1}^{m+1} i = \frac{1}{2}(m+1)(m+2)$$

terms in (13). Hence, each diffusivity function can also be regarded as a vector d in $\Re^N$, indexed by ij, where $j = 0, \cdots, m-i, i = 0, \cdots, m$.

With the discussion above, it is ready now to formulate the PSDT (positive semi-definite tensor) model. It is as follows:

$$P(d^*) = \min\{P(d) : \lambda_{\min}(d) \geq 0\}, \tag{7}$$

where P is a convex quadratic function of d, $\lambda_{\min}$ is the smallest Z-eigenvalue of d.

Theorem 4.1 $\lambda_{\min}(d)$ *is a continuous concave function. Hence, PSDT (7) is a convex optimization problem.*

4.11. Maps of Characteristic Quantities of PSDT

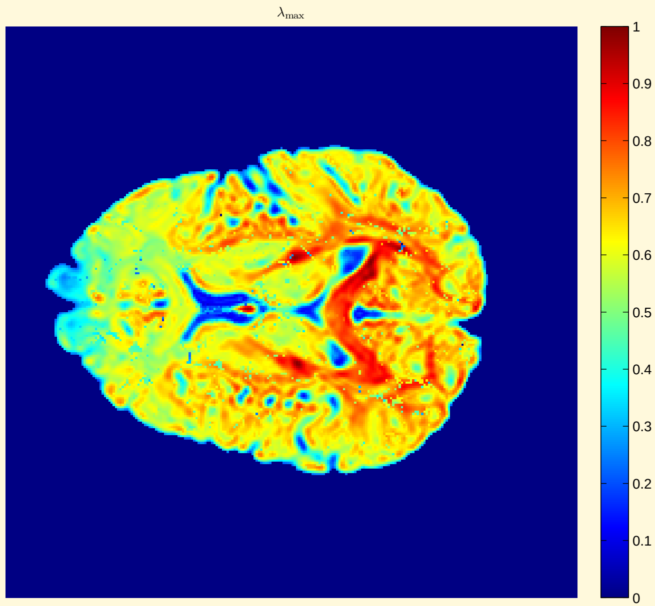

Figure 1 The map of $\lambda_{\max}$.

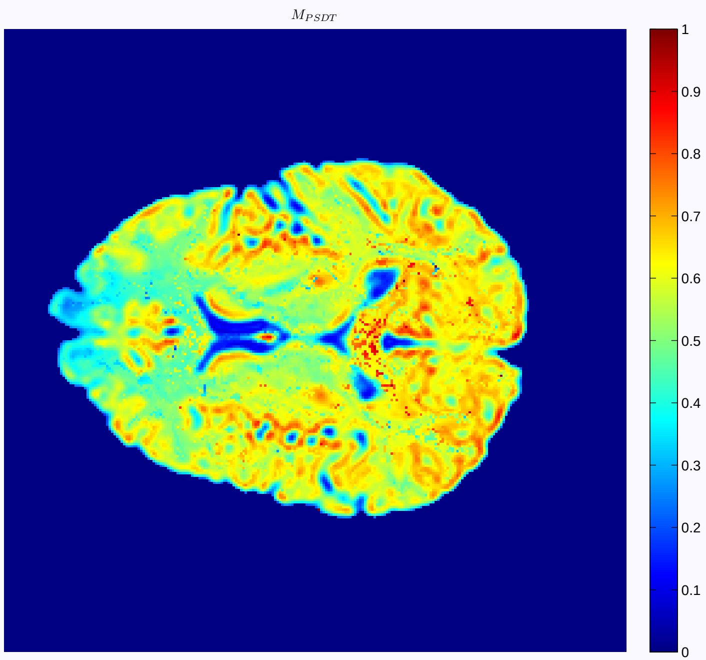

Figure 2 The map of M_{PSDT}.

4.12. Diffusion Orientation Distribution Function

In the introduction of [14], Bloy and Verma described themselves as within the QBI community. What is QBI? While it is highly regarded by some MRI researchers? Bloy and Verma apply Z-eigenvalues to ODF. What is ODF?

In the single fiber case, the DTI model works well. By diagonalization, the surface corresponding to the diffusion tensor is an ellipsoid with its long axis aligned with the fiber orientation. However, many voxels in diffusion MRI volumes potentially have multiple fibers with crossing, kissing or diverging configuration. Then ADC profile estimate from DTI fails to recover multiple fiber orientation.

In [H4], Tuch proposed to a HARDI technique that reconstruct the diffusion orientation distribution function (ODF) of the underlying fiber population of a biological tissue.

4.13. The Q-Ball Imaging

The ODF gives a good representation of underlying fiber distribution. Tuch [H4] showed that the ODF in a unit direction u, $\psi(u)$, could be estimated directly from the raw HARDI signal S by the Funk-Radon transformation:

$$\psi(u) = \int_{\Omega} \delta(u^\top x) S(x) dx, \tag{8}$$

where $\Omega = \{x \in \Re^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$, δ is the Dirac delta function.

As the QBI model may identify the underlying fiber directions well in the multiple fiber situations, it has been studied intensively. This forms the QBI community, mentioned in the introduction of [13], by Bloy and Verma.

4.14. Nonnegative Diffusion Orientation Distribution Function

The Funk-Radon transformation (8) involves the Dirac delta function, which is not a real valued function. The computation is not so easy. In 2007, Decoteaux et al proposed to use the Funk-Hecke theorem and the spherical harmonics to simplify the Funk-Radon transformation. In 2006, Descoteaux et all, showed that there is a close relation between spherical harmonics and higher order tensors. Thus, the QBI approach is connected to higher order tensors too. Bloy and Verma [13] thus applied Z-eigenvalues of higher order tensors, proposed in Qi [1], to this approach.
We propose a nonnegative diffusion orientation distribution function model in the following paper.

[78]. L. Qi, G. Yu and Y. Xu, "Nonnegative diffusion orientation distribution function", *Journal of Mathematical Imaging and Vision* **45** (2013) 103-113.

Two more paperson this topic:

[79]. S.L. Hu, Z.H. Huang, H.Y. Ni and L. Qi, "Positive definiteness of diffusion kurtosis imaging", *Inverse Problems and Imaging* **6** (2012) 57-75.

[80]. Chen, Y., Dai, Y., Han, D., Sun, W.: Positive semidefinite generalized diffusion tensor imaging via quadratic semidefinite programming. SIAM J. Imaging Sci. **6**, 1531-1552 (2013)

4.15. Space Tensor Conic Programming

Yinye Ye at Stanford University is a Visiting Chair Professor in our department. In April 2009, he visited our department. We discussed the PSDT model. He suggested to investigate this problem in a viewpoint of conic linear programming (CLP) problem. Actually, $S = \{d \in \Re^N : \lambda_{\min}(d) \geq 0\}$ is a convex cone in $\Re^N$. We worked together on the following paper on **Space Tensor Conic Programming**:

[81]. L. Qi and Y. Ye, "Space Tensor Conic Programming", *Computational Optimization and Applications* **59** (2014) 307-319.

During the last two decades, major developments in convex optimization was focusing on conic linear programming. Conic Linear programming (CLP) problems include linear programming (LP) problems, semi-definite programming (SDP) problems and second-order cone programming (SOCP).

Space tensors appear in physics and mechanics. They are real physical entities. Mathematically, they are tensors of three dimension. All the mth order symmetric space tensors, where m is a positive even integer, form an N-dimensional space, where $N = \frac{1}{2}(m+1)(m+2)$. Using the method given in [5], we identify in polynomial-time if an mth order symmetric space tensor is positive semi-definite or not.

4.16. More Papers on Space Tensor Programming

Three more papers on new algorithms for solving space tensor conic linear programming problems are as follows.

[82]. G. Li, L. Qi and G. Yu, "Semismoothness of the maximum eigenvalue function of a symmetric tensor and its application", *Linear Algebra and Its Applications* **438** (2013) 813-833.

[83]. L. Qi, Y. Xu, Y. Yuan and X. Zhang, "A cone constrained convex program: structure and algorithms", *Journal of Operations Research Society of China* **1** (2013) 37-53.

[84]. S. Hu, Z. Huang and L. Qi, "Finding the Extreme Z-Eigenvalues of Tensors via a Sequential SDPs Method", *Numerical Linear Algebra with Applications* **20** (2013) 972-984.

4.17. Image Authenticity Verification

Another notable application of eigenvalues of higher-order tensors is the image authenticity verification - to verify a photo or a picture is true or false. Clearly, this problem is also very significant in the real world. A well-known news story is that a hunter named Zhou recapatured a pasterboard tiger, and claimed to have discovered a rare species. This is called "Zhou tiger" on the internet. This attracted the attention of imaging scientists to invent adequate image authenticity verification methods to verify this. Two papers, Yu, Ng and Sun (2008) and Cao and Kot (2010), have addressed this problem.

A paper addressed this problem by further developing the concept of eigenvalue of higher-order tensors and applying it to this problem. Comparing to previous methods of Yu, Ng and Sun (2008) and Cao and Kot (2010), the new algorithm provides a more intuitive and reliable result. The paper is as follows.

[85]. F. Zhang, B. Zhou and L. Peng, "Gradient skewness tensors and local illumination detection for images", *Journal of Computational and Applied Mathematics volume* **237** (2013) 663-671.

4.18. The Abstract of The Zhang, Zhou and Peng Paper

In this paper, we propose the definition of D-eigenvalue for an arbitrary order tensor related with a second-order tensor D, and introduce the gradient skewness tensor which involves a three-order tensor derived from the skewness statistic of gradient images. As we happen to find out that the skewness value of oriented gradients of an image can measure the directional characteristic of illumination, the local illumination detection problem for an image can be abstracted as solving the largest D-eigenvalue of gradient skewness tensors. We study the properties of D-eigenvalue, and especially for gradient skewness tensors we provide the calculation methods of its D-eigenvalues and D-characteristic polynomial. Some numerical experiments show its application in illumination detection. Our method also presents excellent results in a class of image authenticity verification problem, which is to distinguish real and flat objects in a photograph.

4.19. Ellipticity in Solid Mechanics

In **Solid Mechanics**, the elasticity tensor $\mathcal{A} = (a_{ijkl})$ is partially symmetric in the sense that for any i, j, k, l, we have $a_{ijkl} = a_{kjil} = a_{ilkj}$. We say that they are strongly elliptic if and only if

$$f(x, y) \equiv \mathcal{A}xyxy \equiv \sum_{i,j,k,l=1}^{n} a_{ijkl}x_i y_j x_k y_l > 0,$$

for all unit vectors $x, y \in \Re^n, n = 2$ or 3. For an isotropic material, some inequalities have been established to judge the strong ellipticity. See

[S1]. J.K. Knowles and E. Sternberg, "On the ellipticity of the equations of non-linear elastostatics for a special material", *J. Elasticity*, 5 (1975) 341-361;

[S2]. J.K. Knowles and E. Sternberg, "On the failure of ellipticity of the equations for finite elastostatic plane strain", *Arch. Ration. Mech. Anal.* **63** (1977) 321-336;

[S3]. H.C. Simpson and S.J. Spector, "On copositive matrices and strong ellipticity for isotrropic elstic materials", *Arch. Rational Mech. Anal.* **84** (1983) 55-68;

[S4]. P. Rosakis, "Ellipticity and deformations with discontinuous deformation gradients in finite elastostatics", *Arch. Ration. Mech. Anal.* **109** (1990) 1-37:

[S5]. Y. Wang and M. Aron, "A reformulation of the strong ellipticity conditions for unconstrained hyperelastic media", *Journal of Elasticity* **44** (1996) 89-96.

4.20. M-Eigenvalues

In the following two papers, we studied conditions for strong ellipticity and introduced **M-eigenvalues** for the ellipticity tensor $\mathcal{A}$:

[86]. L. Qi, H.H. Dai and D. Han. "Conditions for strong ellipticity and M-eigenvalues", *Frontiers of Mathematics in China* **4** (2009) 349-364;

[87]. D. Han, H.H. Dai and L. Qi, "Conditions for strong ellipticity of anisotropic elastic materials", *Journal of Elasticity* **97** (2009) 1-13.

Denote $\mathcal{A}\cdot yxy$ as a vector whose ith component is $\sum_{j,k,l=1}^n a_{ijkl}y_jx_ky_l$, and $\mathcal{A}xyx\cdot$ as a vector whose lth component is $\sum_{i,j,k=1}^n a_{ijkl}x_iy_jx_k$. If $\lambda \in \Re$, $x \in \Re^n$ and $y \in \Re^n$ satisfy

$$\begin{cases} \mathcal{A}\cdot yxy = \lambda x, \ \mathcal{A}xyx\cdot = \lambda y, \\ \quad x^Tx = 1, \ y^Ty = 1, \end{cases} \tag{9}$$

we call λ an M-eigenvalue of $\mathcal{A}$, and call x and y left and right M-eigenvectors of $\mathcal{A}$, associated with the M-eigenvalue λ. Here, the letter "M" stands for mechanics.

Theorem 4.2 *M-eigenvalues always exist. The strong ellipticity condition holds if and only if the smallest M-eigenvalue of the elasticity tensor A is positive.*

5. PSD and SOS Tensors

We now have two checkable, conditionally sufficient and necessary conditions for positive definiteness and semi-definiteness of some classes of some even order symmetric tensors.

- An even order symmetric Z tensor is positive semi-definite (definite) if and only if it is a (strong) M tensor;
- An even order symmetric Cauchy tensor is positive semi-definite if and only if its generating vector is positive; it is positive definite if and only if furthermore the entries of its generating vector are mutually distinct.

5.1. M Tensors and Strong M Tensors

M Tensors are the tensor extensions of M matrices. It was introduced in the 2012 arXiv version of the following paper:

[88]. L. Zhang, L. Qi and G. Zhou, "M-tensors and some applications", *SIAM Journal on Matrix Analysis and Applications* **35** (2014) 437-452.

M tensors were further studied in

[89]. W. Ding, L. Qi and Y. Wei, "M-tensors and nonsingular M-tensors", *Linear Algebra and Its Applications* **439** (2013) 3264-3278.

[90]. J. He and T.Z. Huang, "Inequalities for M-tensors", *Journal of Inequality and applications* (2014) 2014:114.

A tensor in $T_{m,n}$ is called a **Z tensor** if all of its off-diagonal entries are non-positive. A Z tensor $\mathcal{A}$ is called a **M tensor** if it can be written as $\mathcal{A} = c\mathcal{I} - \mathcal{B}$, where $\mathcal{B}$ is a nonnegative tensor and $c \geq \rho(\mathcal{B})$. It is called a **strong M tensor** if furthermore we have $c > \rho(\mathcal{B})$.

5.2. Even Order Symmetric M Tensors

The following theorem was from [1].

Theorem 5.1 *An even order symmetric Z tensor is positive semi-definite if and only if it is an M tensor. An even order symmetric Z tensor is positive definite if and only if it is a strong M tensor.*

As pointed out in [1], one may modify an algorithm for finding the largest H-eigenvalue of a nonnegative tensor to find the smallest H-eigenvalue of a Z tensor. Thus, it is not difficult to identify an M tensor or a strong M tensor.

In the following paper, it is shown that the largest H-eigenvalue of a nonnegative tensor can be found by solving a semi-definite programming problem, which is in polynomial-time. Hence, the problem for identifying a Z-tensor is an M tensor or not can be solved in polynomial time.

[91]. S. Hu, G. Li, L. Qi and Y. Song, "Finding the maximum eigenvalue of essentially nonnegative symmetric tensors via sum of squares programming", *Journal of Optimization Theory and Applications* **158** (2013) 717-738.

Laplacian tensors of uniform hypergraphs are M tensors.

5.3. Hilbert Tensors

Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$. If for all $i_j \in [n]$ and $j \in [m]$, we have

$$a_{i_1 \cdots i_m} = \frac{1}{i_1 + \cdots + i_m - m + 1},$$

then $\mathcal{A}$ is called a Hilbert tensor. Hilbert tensors are extensions of Hilbert matrices. They are symmetric tensors. Hilbert matrices are positive definite. In the following paper, we showed that even order Hilbert tensors are positive definite.

[92]. Y. Song and L. Qi, "Infinite and finite dimensional Hilbert tensors", *Linear Algebra and Its Applications* **451** (2014) 1-14.

5.4. Cauchy Tensors

Hilbert tensors are Cauchy tensors.

In [94], we extended symmetric Cauchy matrices to symmetric Cauchy tensors, and gave sufficient and necessary conditions for positive semi-definiteness and positive definiteness of even order symmetric Cauchy tensors. This work extends Fielder's work [13] on symmetric Cauchy matrices in 2010 (note that M. Fielder was born in 1926!) Hilbert matrices are symmetric Cauchy tensors. In the following paper, for simplicity, we simply call symmetric Cauchy tensors as Cauchy tensors.

[93]. M. Fiedler, "Notes on Hilbert and Cauchy matrices", *Linear Algebra and Its Applications* **432** (2010) 351-356.

[94]. H. Chen and L. Qi, "Positive definiteness and semi-definiteness of even order symmetric Cauchy tensors", *Journal of Industrial Management and Optimization* **11** (2015) 1263-1274.

5.5. Cauchy Tensors

Let vector $\mathbf{c} = (c_1, c_2, \cdots, c_n) \in \Re^n$. Suppose that a real tensor $\mathcal{C} = (c_{i_1 i_2 \cdots i_m})$ is defined by

$$c_{i_1 i_2 \cdots i_m} = \frac{1}{c_{i_1} + c_{i_2} + \cdots + c_{i_m}}, \quad j \in [m],\ i_j \in [n].$$

Then, we say that $\mathcal{C}$ is an order m dimension n symmetric Cauchy tensor and the vector $c \in \mathbb{R}^n$ is called the generating vector of $\mathcal{C}$.

We should point out that, in this definition, for any m elements $c_{i_1}, c_{i_2}, \cdots, c_{i_m}$ in generating vector $\mathbf{c}$, it satisfies

$$c_{i_1} + c_{i_2} + \cdots + c_{i_m} \neq 0,$$

which implies that $c_i \neq 0$, $i \in [n]$.

5.6. Cauchy Tensors, Hilbert Tensors and Hankel Tensors

Suppose Cauchy tensor $\mathcal{C}$ and its generating vector $\mathbf{c}$ are defined as above. If

$$c_{i_1} + c_{i_2} + \cdots + c_{i_m} \equiv c_{j_1} + c_{j_2} + \cdots + c_{j_m}$$

whenever

$$i_1 + i_2 + \cdots + i_m = j_1 + j_2 + \cdots + j_m,$$

then Cauchy tensor $\mathcal{C}$ is a Hankel tensor. In general, a symmetric Cauchy tensor is not a Hankel tensor. We will discuss Hankel tensors more later.

If entries of $\mathbf{c}$ are defined such that

$$c_i = i - 1 + \frac{1}{m},\ i \in [n],$$

then Cauchy tensor $\mathcal{C}$ is a Hilbert tensor.

5.7. Positive Semi-Definiteness and Positive Definiteness of Cauchy Tensors

The following theorem was proved in [94].

Theorem 5.2 *Suppose that the order of a Cauchy tensor is even. Then it is positive semi-definite if and only if its generating vector* $\mathbf{c}$ *is positive, and it is positive definite if and only if its generating vector* $\mathbf{c}$ *is positive and the entries of* $\mathbf{c}$ *are mutually distinct.*

Clearly, these conditions are easily checkable. This result extends the result of Fielder [13]. From this result, we immediately conclude that an even order Hilbert tensor is positive definite.

5.8. Checkable Sufficient Conditions

We now have the following checkable sufficient conditions of positive definiteness and semi-definiteness for even order symmetric tensors.

- An even order (strictly) diagonally dominated symmetric tensor is positive semi-definite (definite);
- An even order (strictly) doubly diagonally dominated symmetric tensor is positive semi-definite (definite);
- An even order symmetric (strong) H^+ tensor is positive semi-definite (definite);
- An even order Hankel tensor is positive semi-definite if its associated Hankel matrix is positive semi-definite;
- An even order symmetric B_0 (B) tensor is positive semi-definite (definite);
- An even order symmetric double B_0 (B) tensor is positive semi-definite (definite);
- An even order symmetric quasi-double B_0 (B) tensor is positive semi-definite (definite).

5.9. Diagonally Dominated Tensors

A tensor $\mathcal{A} \in T_{m,n}$ is called **diagonally dominated** if for all $i \in [n]$,

$$a_{i\cdots i} \geq \Delta_i.$$

A tensor $\mathcal{A} \in T_{m,n}$ is called **strictly diagonally dominated** if for all $i \in [n]$,

$$a_{i\cdots i} > \Delta_i.$$

By Theorem 2.1, we may easily prove the following theorem.

Theorem 5.3 *An even order diagonally dominated symmetric tensor is positive semi-definite. An even order strictly diagonally dominated symmetric tensor is positive definite.*

Clearly, this is an easily checkable sufficient condition.

5.10. Doubly Diagonally Dominated Tensors

A tensor $\mathcal{A} \in T_{m,n}$ is called **doubly diagonally dominated** if for all $i \in [n]$,

$$a_{i\ldots i} \geq 0,$$

and for all $i \neq j \in [n]$,

$$a_{i\ldots i} a_{j\ldots j} \geq \Delta_i \Delta_j.$$

Furthermore, if for all $i \neq j \in [n]$, we have

$$a_{i\ldots i} a_{j\ldots j} > \Delta_i \Delta_j,$$

then $\mathcal{A}$ is called **strictly doubly diagonally dominated**. By Theorem **??**, we may easily prove the following theorem.

Theorem 5.4 *An even order doubly diagonally dominated symmetric tensor is positive semi-definite. An even order strictly doubly diagonally dominated symmetric tensor is positive definite.*

Clearly, this is also an easily checkable condition, and stronger than Theorem 2.2.

5.11. Symmetric H^+ Tensors and Strong H^+ Tensors

Discussing with Prof. Li Yaotang, we now further extend M tensors to H^+ tensors, which are extensions of H^+ matrices.

A tensor $\mathcal{A}$ in $T_{m,n}$ may always be denoted as $\mathcal{A} = \mathcal{D} - \mathcal{B}$, where $\mathcal{D}$ is a diagonal tensor consisting of the diagonal part of $\mathcal{A}$, while $\mathcal{B}$ consists of the off-diagonal part of $\mathcal{A}$. If $|\mathcal{D}| - |\mathcal{B}|$ is a (strong) M tensor, then $\mathcal{A}$ is called a (strong) H tensor. If furthermore $\mathcal{D} \geq \mathcal{O}$, then $\mathcal{A}$ is called a (strong) H^+ tensor. Then we have the following theorem.

Theorem 5.5 *An even order symmetric H^+ tensor is positive semi-definite. An even order symmetric strong H^+ tensor is positive definite.*

Similar results may be found in the following papers:

[95]. C. Li, F. Wang, J. Zhao, Y. Zhu and Y. Li, “Criterions for the positive definiteness of real supersymmetric tensors”, *Journal of Computational and Applied Mathematics* **255** (2014) 1-14.

[96]. M.R. Kannan, N. Shaked-Monderer, A. Berman, “Some properties of strong H-tensors and general H-tensors”, *Linear Algebra and its Applications* **476** (2015) 42-55.

5.12. Hankel Tensors

As a natural extension of Hankel matrices, Hankel tensors arise from applications such as signal processing.
Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$. If there is a vector $\mathbf{v} = (v_0, v_1, \cdots, v_{(n-1)m})^\top$ such that for $i_1, \cdots, i_m \in [n]$, we have

$$a_{i_1 \cdots i_m} \equiv v_{i_1+i_2+\cdots+i_m-m}, \tag{10}$$

then we say that $\mathcal{A}$ is an mth order **Hankel tensor**. Clearly, Hankel tensors are symmetric tensors. Denote the set of all real mth order n-dimensional Hankel tensors by $H_{m,n}$. Then $H_{m,n}$ is a linear subspace of $S_{m,n}$, with dimension $(n-1)m+1$.

Hankel tensors were introduced by Papy, De Lathauwer and Van Huffel in 2005 in the context of the harmonic retrieval problem, which is at the heart of many signal processing problems. In 2008, Badeau and Boyer proposed fast higher-order singular value decomposition (HOSVD) for third order Hankel tensors.

[97]. J.M. Papy, L. De Lauauwer and S. Van Huffel, "Exponential data fitting using multilinear algebra: The single-channel and multi-channel case", *Numerical Linear Algebra with Applications* **12** (2005) 809-826.

[98]. R. Badeau and R. Boyer, "Fast multilinear singular value decomposition for structured tensors", *SIAM J. Matrix Anal. Appl.* **30** (2008) 1008-1021.

5.13. Strong Hankel Tensors

In the following paper, I studied two subclasses of Hankel tensor: strong Hankel tensors and complete Hankel tensors. Even order strong Hankel tensors and complete Hankel tensors are positive semi-definite.

[99]. L. Qi, "Hankel tensors: Associated Hankel matrices and Vandermonde decomposition", *Communications in Mathematical Sciences* **13** (2015) 113-125.

Suppose that $\mathcal{A} \in H_{m,n}$ is defined by (13). Let $A = (a_{ij})$ be an $\lceil \frac{(n-1)m+2}{2} \rceil \times \lceil \frac{(n-1)m+2}{2} \rceil$ matrix with $a_{ij} \equiv v_{i+j-2}$, where $v_{2\lceil \frac{(n-1)m}{2} \rceil}$ is an additional number when $(n-1)m$ is odd. Then A is a Hankel matrix, associated with the Hankel tensor $\mathcal{A}$. Such an associated Hankel matrix is unique if $(n-1)m$ is even. If the Hankel matrix A is positive semi-definite, then we say that $\mathcal{A}$ is a **strong Hankel tensor**.

5.14. Generating Functions

Let $\mathcal{A}$ be a Hankel tensor defined by (13). Let $f(t)$ be an absolutely integrable real valued function on the real line $(-\infty, \infty)$ such that

$$v_k \equiv \int_{-\infty}^{\infty} t^k f(t) dt, \tag{11}$$

for $k = 0, \cdots, (n-1)m$. Then we say that f is a **generating function** of the Hankel tensor $\mathcal{A}$. We see that $f(t)$ is also the generating function of the associated Hankel matrix of $\mathcal{A}$. By the theory of Hankel matrices, $f(t)$ is well-defined.

Theorem 5.6 *A Hankel tensor $\mathcal{A}$ has a nonnegative generating function if and only if it is a strong Hankel tensor. An even order strong Hankel tensor is positive semi-definite.*

As we may check the associated Hankel matrix is positive semi-definite or not, it is checkable that a Hankel tensor is a strong Hankel tensor or not. This gives another checkable sufficient condition for positive semi-definite tensors when the order is even.

5.15. Vandermonde Decomposition and Complete Hankel Tensors

Suppose that $\mathbf{u} \in \Re^n$. If $\mathbf{u} = (1, u, u^2, \cdots, u^{n-1})^\top$, then $\mathbf{u}$ is called a **Vandermonde vector**. If

$$\mathcal{A} = \sum_{k=1}^{r} \alpha_k \left(\mathbf{u}_k\right)^m, \tag{12}$$

where $\alpha_k \in \Re$, $\alpha_k \neq 0$, $\mathbf{u}_k = (1, u_k, u_k^2, \cdots, u_k^{n-1})^\top \in \Re^n$ are Vandermonde vectors for $k = 1, \cdots, r$, and $u_i \neq u_j$ for $i \neq j$, then we say that tensor $\mathcal{A}$ has a **Vandermonde decomposition**.

Theorem 5.7 *Let $\mathcal{A} \in S_{m,n}$. Then $\mathcal{A}$ is a Hankel tensor if and only if it has a Vandermonde decomposition (12).*
Suppose that $\mathcal{A}$ has a Vandermonde decomposition (12). If m is even and $\alpha_k > 0$ for $i \in [r]$, then $\mathcal{A}$ is positive semi-definite.

In (12), if $\alpha_k > 0, k \in [r]$, then we say that $\mathcal{A}$ has a positive Vandermonde decomposition and call $\mathcal{A}$ a **complete Hankel Tensor**. Thus, Theorem 5.7 says that an even order complete Hankel tensor is positive semi-definite.
A further paper on Hankel tensor is as the following in.

[28]. W. Ding, L. Qi and Y. Wei, "Fast Hankel tensor-vector products and application to exponential data fitting", *Numerical Linear Algebra with Applications* **22** (2015) 814-832.

5.16. P and P_0 Tensors

P tensors and P_0 are extensions of P matrices and P_0 matrices respectively. They were introduced in the following paper.

[100]. Y. Song and L. Qi, "Properties of some classes of structured tensors", *Journal of Optimization: Theory and Applications* **165** (2015) 854-873.

Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$. We say that A is
(a) a **P_0 tensor** if for any nonzero vector $\mathbf{x}$ in $\Re^n$, there exists $i \in [n]$ such that $x_i \neq 0$ and

$$x_i \left(\mathcal{A}\mathbf{x}^{m-1}\right)_i \geq 0;$$

(b) a **P tensor** if for any nonzero vector $\mathbf{x}$ in $\Re^n$,

$$\max_{i\in[n]} x_i \left(\mathcal{A}\mathbf{x}^{m-1}\right)_i > 0.$$

5.17. P and P_0 Tensors

The following them was established in [29]. It extends the well-known result in matrix theory.

Theorem 5.8 *Let $\mathcal{A} \in T_{m,n}$ be a P (P_0) tensor. Then when m is even, all of its H-eigenvalues and Z-eigenvalues of $\mathcal{A}$ are positive (nonnegative). A symmetric tensor is a P (P_0) tensor if and only if it is positive (semi-)definite. There does not exist an odd order symmetric P tensor. If an odd order nonsymmetric P tensor exists, then it has no Z-eigenvalues. An odd order P_0 tensor has no nonzero Z-eigenvalues.*

5.18. B and B_0 Tensors

In the matrix literature, there is another easily checkable sufficient condition for positive definite matrices. It is easy to check a given matrix is a B matrix or not. In the following paper, it was proved that a B matrix is a P matrix. It is well-known that a symmetric matrix is a P matrix if and only if it is positive definite. Thus, a symmetric B matrix is positive definite.

[101]. J.M. Peña, "A class of P-matrices with applications to the localization of the eigenvalues of a real matrix", *SIAM Journal on Matrix Analysis and Applications* **22** (2001) 1027-1037.

P matrices and B matrices were extended to P tensors and B tensors in [100]. It is easy to check a given tensor is a B tensor or not, while it is not easy to check a given tensor is a P tensor or not. It was proved there that a symmetric tensor is a P tensor if and only it is positive definite. However, it was not proved in [100] if an even order B tensor is a P tensor or not, or if an even order symmetric B tensor is positive definite or not. As pointed out in [100], an odd order identity tensor is a B tensor, but not a P tensor. Thus we know that an odd order B tensor may not be a P tensor.

5.19. New Technique is Needed

The B tensor condition is not so strict compared with the strongly diagonal dominated tensor condition if the tensor is not sparse. A tensor in $T_{m,n}$ is strictly diagonally dominated tensor if every diagonal entry of that tensor is greater than the sum of the absolute values of all the off-diagonal entries in the same row. For each row, there are $n^{m-1} - 1$ such off-diagonal entries. Thus, this condition is quite strict when n and m are big and the tensor is not sparse. A tensor in $T_{m,n}$ is a B tensor if for every row of the tensor, the sum of all the entries in that row is positive, and each off-diagonal entry is less than the average value of the entries in the same row. An initial numerical experiment indicates that for $m = 4$ and $n = 2$, a symmetric B tensor is positive definite. Thus, it is possible that an even order symmetric B tensor is positive definite. If this is true, we will have an easily checkable, not very strict, sufficient condition for positive definite tensors.

However, the technique in [101] cannot be extended to the tensor case. It was proved in [101] that the determinant of every principal submatrix of a B matrix is positive. Thus, a B matrix is a P matrix. As we know, the determinant of every principal sub-tensor of a symmetric positive definite tensor is positive, but this is only a necessary, not a sufficient condition for symmetric positive definite tensors. Hence, the technique in [101] cannot be extended to the tensor case.

5.20. New Technique

In the following paper, we use a new technique to prove that an even order symmetric B tensor is positive definite. We show that a symmetric B tensor can always be decomposed to the sum of a strictly diagonally dominated symmetric M tensor and several positive multiples of partially all one tensors, and a symmetric B_0 tensor can always be decomposed to the sum of a diagonally dominated symmetric M tensor and several positive multiples of partially all one tensors. Even order partially all one tensors are positive semi-definite. As stated before, an even order diagonally dominated symmetric tensor is positive semi-definite, and an even order strictly diagonally dominated symmetric tensor is positive definite. Therefore, when the order is even, these imply that the corresponding symmetric B tensor is positive definite, and the corresponding symmetric B_0 tensor is positive semi-definite. Hence, this gives an easily checkable, not very strict, sufficient condition for positive definite and semi-definite tensors.

[102]. L. Qi and Y. Song, "An even order symmetric B tensor is positive definite", *Linear Algebra and Its Applications* **457** (2014) 303-312.

5.21. B and B$_0$ Tensors

Let $\mathcal{B} = (b_{i_1 \cdots i_m}) \in T_{m,n}$. We say that $\mathcal{B}$ is a **B tensor** if for all $i \in [n]$

$$\sum_{i_2,\cdots,i_m=1}^{n} b_{ii_2i_3\cdots i_m} > 0$$

and

$$\frac{1}{n^{m-1}} \left(\sum_{i_2,\cdots,i_m=1}^{n} b_{ii_2i_3\cdots i_m} \right) > b_{ij_2j_3\cdots j_m} \text{ for all } (j_2, j_3, \cdots, j_m) \neq (i, i, \cdots, i).$$

We say that $\mathcal{B}$ is a **B$_0$ tensor** if for all $i \in [n]$

$$\sum_{i_2,\cdots,i_m=1}^{n} b_{ii_2i_3\cdots i_m} \geq 0$$

and

$$\frac{1}{n^{m-1}} \left(\sum_{i_2,\cdots,i_m=1}^{n} b_{ii_2i_3\cdots i_m} \right) \geq b_{ij_2j_3\cdots j_m} \text{ for all } (j_2, j_3, \cdots, j_m) \neq (i, i, \cdots, i).$$

5.22. Partially All One Tensors

A tensor $\mathcal{C} \in T_{m,r}$ is called **a principal sub-tensor** of a tensor $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$ $(1 \leq r \leq n)$ if there is a set J that composed of r elements in $[n]$ such that

$$\mathcal{C} = (a_{i_1 \cdots i_m}), \text{ for all } i_1, i_2, \cdots, i_m \in J.$$

This concept was first introduced and used in [14] for symmetric tensor. We denote by $\mathcal{A}_r^J$ the principal sub-tensor of a tensor $\mathcal{A} \in T_{m,n}$ such that the entries of $\mathcal{A}_r^J$ are indexed by $J \subset [n]$ with $|J| = r$ $(1 \leq r \leq n)$.

Suppose that $\mathcal{A} \in S_{m,n}$ has a principal sub-tensor $\mathcal{A}_r^J$ with $J \subset [n]$ with $|J| = r$ $(1 \leq r \leq n)$ such that all the entries of $\mathcal{A}_r^J$ are one, and all the other entries of $\mathcal{A}$ are zero. Then $\mathcal{A}$ is called a **partially all one tensor**, and denoted by $\mathcal{E}^J$. If $J = [n]$, then we denote $\mathcal{E}^J$ simply by $\mathcal{E}$ and call it an **all one tensor**. An even order partially all one tensor is positive semi-definite. In fact, when m is even, if we denote by $\mathbf{x}_J$ the r-dimensional sub-vector of a vector $\mathbf{x} \in \Re^n$, with the components of $\mathbf{x}_J$ indexed by J, then for any $\mathbf{x} \in \Re^n$, we have

$$\mathcal{E}^J \mathbf{x}^m = \left(\sum \{x_j : j \in J\} \right)^m \geq 0.$$

5.23. An Even Order Symmetric B Tensor is Positive Definite

Theorem 5.9 *Suppose that $\mathcal{B} = (b_{i_1 \cdots i_m}) \in S_{m,n}$ is a symmetric B_0 tensor. Then either $\mathcal{B}$ is a diagonally dominated symmetric M tensor itself, or we have*

$$\mathcal{B} = \mathcal{M} + \sum_{k=1}^{s} h_k \mathcal{E}^{J_k}, \qquad (13)$$

where $\mathcal{M}$ is a diagonally dominated symmetric M tensor, s is a positive integer, $h_k > 0$ and $J_k \subset [n]$, for $k = 1, \cdots, s$, and $J_k \cap J_l = \emptyset$, for $k \neq l, k$ and $l = 1, \cdots, s$ when $s > 1$. If furthermore $\mathcal{B}$ is a B tensor, then either $\mathcal{B}$ is a strictly diagonally dominated symmetric M tensor itself, or we have (13) with $\mathcal{M}$ as a strictly diagonally dominated symmetric M tensor. An even order symmetric B_0 tensor is positive semi-definite. An even order symmetric B tensor is positive definite.

Three further papers on P and B tensors:

[103]. P. Yuan and L. You, "Some remarks on P, P_0, B and B_0 tensors", *Linear Algebra and Its Applications* **459** (2015) 511-521.

[104]. C. Li and Y. Li, "Double B-tensors and quasi-double B-tensors", *Linear Algebra and Its Applications* **466** (2015) 343-356.

[105]. C. Li, L. Qi and Y. Li, "MB-tensors and MB_0-tensors", *Linear Algebra and Its Applications* **484** (2015) 141-153.

5.24. SOS Tensors

SOS tensors were introduced in the following two papers.

[106]. S. Hu, G. Li and L. Qi, "A tensor analogy of Yuan's alternative theorem and polynomial optimization with sign structure", to appear in: *Journal of Optimization Theory and Applications*.

[107]. Z. Luo, L. Qi and Y. Ye, "Linear operators and positive semidefiniteness of symmetric tensor spaces", *Science China Mathematics* **58** (2015) 197-212.

A symmetric tensor $\mathcal{A} \in S_{m,n}$ uniquely define a homogeneous polynomial of n variable and degree m:

$$f(\mathbf{x}) = \mathcal{A}\mathbf{x}^m.$$

On the other hand, a homogeneous polynomial $f(\mathbf{x})$ of n variable and degree m, also uniquely determines a symmetric tensor $\mathcal{A}$ in $S_{m,n}$. Let $m = 2k$ be even. If $f(\mathbf{x})$ can be written as the sum of squares of homogeneous polynomials of degree k, then f is called an SOS (sum of squares) polynomial, and $\mathcal{A}$ is called an **SOS tensor**.

5.25. Properties of SOS tensors

Clearly, an SOS tensor is a PSD tensor, but not vice versa. The problem for determining a given general even order symmetric tensor $\mathcal{A}$ is PSD or not is NP-hard if the order is greater than 2. On the other hand, the problem for determining a given general even order symmetric tensor $\mathcal{A}$ is SOS or not is polynomial time solvable. It can be solved by solving a semi-definite linear programming problem. See the following two papers.

[108]. J.B. Lasserre, "Global optimization with polynomials and the problem of moments", *SIAM Journal on Optimization* **11** (2001) 796-817.

[109]. M. Laurent, "Sums of squares, moment matrices and optimization over polynomials", *Emerging Applications of Algebraic Geometry*, Vol. 149 of IMA Volumes in Mathematics and its Applications, M. Putinar and S. Sullivant eds., Springer, (2009) pp. 157-270.

Thus, SOS tensors are much easier to be identified compared with PSD tensors.

5.26. SOS Tensors and SOS Rank

In [110], we show that a complete Hankel tensor is a strong Hankel tensor, and an even order strong Hankel tensor is an SOS tensor. We also show that there are SOS Hankel tensors, which are not strong Hankel tensors. In [111], we show that an even order positive Cauchy tensor is an SOS tensor. In [112], we show that all the other even order symmetric PSD tensor classes, with easily checkable conditions, such as diagonally dominated tensors, B_0 tensors, doubly B_0 tensors, qausi-B_0 tensors, MB_0 tensors, H^+ tensors, are SOS tensors.

[110]. G. Li, L. Qi and Y. Xu, "SOS Hankel Tensors: Theory and Application", October 2014. arXiv:1410.6989.

[111]. H. Chen, G. Li and L. Qi, "Further results on Cauchy tensors and Hankel tensors", *Applied Mathematics and Computation* **275** (2016) 50-62.

[112]. H. Chen, G. Li and L. Qi, "SOS tensor decomposition: Theory and applications", *Communications in Mathematical Sciences* **14** (2016) 2073-2100.

5.27. A Question

An even order symmetric tensor is PSD if and only if its smallest H-(Z-)eigenvalue is nonnegative. For those easily checkable classes of PSD tensors, can we find their smallest H-(Z-)eigenvalues in polynomial time? For example, can we construct an efficient, polynomial-time algorithm to compute the smallest H-(Z-)eigenvalue of an even order symmetric B (B_0) tensor?

5.28. PNS Tensors and PNS-Free Zone

It was shown by young Hilbert in [113] that for homogeneous polynomial, only in the following three cases, a PSD polynomial definitely is an SOS polynomial: 1) $n = 2$; 2) $m = 2$; 3) $m = 4$ and $n = 3$. For tensors, the second case corresponds to the symmetric matrices, i.e., a PSD symmetric matrix is always an SOS matrix. Hilbert proved that in all the other possible combinations of $m = 2k$ and n, there are non-SOS PSD homogeneous polynomials [114].

[113]. D. Hilbert, "Über die Darstellung definiter Formen als Summe von Formenquadraten", *Mathematical Annals*, **32** (1888) 342-350.

[114]. B. Reznick, "Some concrete aspects of Hilbert's 17th problem", *Contemporary Mathematics* **253** (2000) 251-272.

5.29. PNS Tensors

However, Hilbert did not give an explicit example of non-SOS PSD (PNS) homogeneous polynomial. The first PNS homogeneous polynomial was given by Motzkin [115]:

$$f_M(\mathbf{x}) = x_3^6 + x_1^2x_2^4 + x_1^4x_2^2 - 3x_1^2x_2^2x_3^2.$$

By the Arithmetic-Geometric inequality, we see that it is a PSD polynomial. But it is not an SOS polynomial [114]. The other two PNS homogeneous polynomials with small m and n were given by Choi and Lam [116]

$$f_{CL1}(\mathbf{x}) = x_4^4 + x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 - 4x_1x_2x_3x_4$$

and

$$f_{CL2}(\mathbf{x}) = x_1^4x_2^2 + x_2^4x_3^2 + x_3^4x_1^2 - 3x_1^2x_2^2x_3^2.$$

Denote the set of all SOS tensors in $\mathrm{S}_{m,n}$ by $\mathrm{SOS}_{m,n}$. Then it is also a closed convex cone. Thus, $\mathrm{SOS}_{m,2} = \mathrm{PSD}_{m,2}$, $\mathrm{SOS}_{2,n} = \mathrm{PSD}_{2,n}$ and $\mathrm{SOS}_{4,3} = \mathrm{PSD}_{4,3}$. But for other $m = 2k \geq 4, n \geq 3$, we have $\mathrm{SOS}_{m,n} \subsetneq \mathrm{PSD}_{m,n}$.

[115]. T.S. Motzkin, "The arithmetic-geometric inequality", In: *Inequalities*, O. Shisha ed., Academic Press, New York, (1967) pp. 205-224.

[116]. M.D. Choi and T.Y. Lam, "Extremal positive semidefinite forms", *Mathematische Annalen* **231** (1977) 1-18.

5.30. The Hilbert-Hankel Problem

In [110], we raised a question, is a PSD Hankel tensor always an SOS tensor? If the answer to this question is "yes", then the problem for determining a given even order Hankel tensor is PSD or not is polynomial time solvable. Hence, this problem has important practical significance. On the other hand, theoretically, this is a Hilbert problem under the Hankel constraint. We discussed with the experts of the Hilbert problem: Bruce Reznick and Man-Duen Choi. The Hilbert-Hankel problem is a new open problem.

In the following papers, we consider two cases with the lowest dimension that there may be PNS tensors, such as the Motzkin tensor and the Choi-Lam tensor, in the general case (without the Hankel constraint), i.e., $m = 6$ and $n = 3$; $m = n = 4$.

[117]. G. Li, L. Qi and Q. Wang, "Are there sixth order three dimensional Hankel tensors?", November 2014. arXiv:1411.2368.

[118]. Y. Chen, L. Qi and Q. Wang, "Positive semi-definiteness and sum-of-squares property of fourth order four dimensional Hankel tensors", *Journal of Computational and Applied Mathematics* **302** (2016) 356-368.

We partially proved that in some subcases, there are no PNS tensors.

5.31. Generalized Anti-Circulant Tensors

In the following paper, we considered generalized anti-circulant tensors, which are special Hankel tensors.

[119]. G. Li, L. Qi and Q. Wang, "Positive semi-definiteness of generalized anti-circular tensors", *Journal of Computational and Applied Mathematics* **302** (2016) 356-368.

A Hankel tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ is generated by a generating vector $\mathbf{v} = (v_0, \cdots, v_{(n-1)m})^\top$, with

$$a_{i_1 \cdots i_m} = v_{i_1 + \cdots + i_m - m}.$$

If

$$v_i = v_{i+r}$$

for $i = 0, \cdots, (n-1)m - r$, with $r = n$, then $\mathcal{A}$ is called an anti-circulant tensor, which is a generalization of an anti-circulant matrix in the matrix theory. We further extend this concept to generalized anti-circulant tensors, by allowing $r \leq n$. We show that for the cases that $GCD(m, r) = 1$, $GCD(m, r) = 2$ and some other cases, including the matrix case that $m = 2$, when r is odd, $\mathcal{A}$ is PSD if and only if $v_0 = \cdots = v_r \geq 0$; and when r is even, $\mathcal{A}$ is PSD if and only if $v_0 = v_2 = \cdots = v_{r-2}$, $v_1 = v_3 = \cdots = v_{r-1}$ and $v_0 \geq |v_1|$; and in all these cases, $\mathcal{A}$ is PSD if and only if it is a strong Hankel tensor, thus an SOS Hankel tensor.

5.32. Strongly Symmetric Circulant Tensors

In the following paper, we considered three dimensional strongly symmetric circulant tensors. In some cases, we show that it is PNS-free. In the other cases, numerical tests detect that it is PNS free.

[120]. L. Qi, Y. Chen and Q. Wang, “Three dimensional strongly symmetric circulant tensors”, *Linear Algebra and Its Applications* **482** (2015) 207-220.

6. Spectral Graph Theory

Spectral graph theory is a well-studied and highly applicable subject. It studies the connection between properties of a graph, and the eigenvalues of a matrix associated with that graph. Comparing with the research of spectral graph theory, the research on spectral hypergraph theory is still on its beginning stage. Recently, due to the development of spectral theory of tensors, spectral hypergraph theory has also made its first stage progress. Several papers appeared on eigenvalues of the adjacency tensor and the Laplacian tensor of a uniform hypergraph. Three international workshops on Spectral Hypergraph Theory have been held at Fuzhou, Xining and Harbin in 2013, 2015 and 2016, respectively. A number of papers on spectral hypergraph theory appeared.

6.1. Spectral Graph Theory

On the website of SGT - Spectral Graph Theory, one may find 2154 researchers on spectral graph theory. This shows the popularity of this subject.

Here are some books on spectral graph theory:

[1]. D.M. Cvetković, M. Doob, I. Gutman and A. Torgašev, *Recent Results in the Theory of Graph Spectra*, North Holland, Amsterdam, 1988.

[2]. F.R.K. Chung, *Spectral Graph Theory*, Am. Math. Soc., Providence, RI, 1997.

[3]. D.M. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs, Theory and Application*, Academic Press, 1980.376 (2004) 173-186.

[4]. A.E. Brouwer and W.H. Haemers, *Spectra of Grapha*, Springer, 2011.

[5]. X.L. Li, Y.T. Shi and I. Gutman, *Graph Energy*, Springer, 2012.

The research on spectral graph theory in China is quite active. There are a good number of active Chinese researchers in spectral graph theory.

6.2. Hyperhraph

We may denote a hypergraph by $G = (V, E)$, where $V = \{1, 2, \cdots, n\}$ is the vertex set, $E = \{e_1, e_2, \cdots, e_m\}$ is the edge set, $e_p \subset V$ for $p = 1, \cdots, m$. If $|e_p| = k$ for $p = 1, \cdots, m$, and $k \geq 2$, then G is called a uniform hypergraph, or a k-graph. If $k = 2$, then G is an ordinary graph.

A book on hypergraph is:

Two books on hypergraph are:

[6]. C. Berge, *Hypergraphs, Combinatorics of Finite Sets*, 3rd edn., North-Holland, 1989.

[7]. A. Bretto, *Hypergraph Theory: An Introduction*, Springer, 2013.

6.3. Spectral Hypergraph Theory via Laplacian Matrices

The early works on spectral hypergraph theory are via Laplacian matrices. Here are some papers we found on this approach in the literature.

[8]. M. Bolla, "Spectra, Euclidean representations and clusterings of hypergraphs", *Discrete Mathematics*, 117 (1993) 19-39.

[9]. F.R.K. Chung, "The Laplacian of a Hypergraph", in *Expanding Graphs*, DIMACS Ser. Disc. Math. Theoret. Comput. Sci. 10, Am. Math. Soc., Providence, RI, 1993, pp. 21-36.

[10]. K. Feng and W. Li, "Spectra of hypergraphs and applications", *J. Number Theory*, 60 (1996) 1-22.

[11]. J.A. Rodríguez, "On the Laplacian eigenvalues and metric parameters of hypergraphs", *Linear Multilinear Algebra*, 50 (2002), 1-14.

[12]. J.A. Rodríguez, "On the Laplacian spectrum and walk-regular hypergraphs", *Linear Multilinear Algebra*, 51 (2003), 285-297.

[13]. J.A. Rodríguez, "Laplacian eigenvalues and partition problems in hypergraphs", *Appl. Math. Lett.* 22 (2009) 916-921.

[14]. L. Lu and X. Peng, "High-ordered random walks and generalized Laplacians on hypergraphs", in: Proceedings of Algorithms and Models for the Web-Graph: Eighth International Workshop, WAW2011, Atlanta, GA, USA, May 27-29, 2011.

[15]. L. Lu and X. Peng, "Loose Laplacian spectra of random hypergraphs", *Random Structures and Algorithms*, 41 (2012), 521-545.

6.4. Spectral Hypergraph Theory via Tensors I

In fact, the following early work of spectral hypergraph theory used the adjacency tensor and their Z-eigenvalues implicitly:

[16]. J. Friedman and A. Wigderson, "On the second eigenvalue of hypergraphs", *Combinatorica* 15 (1995) 43-965.

Then, in the early stage of spectral theory of tensors, Lim advocated to study spectral hypergraph theory via tensors:

[17]. L.H. Lim, "Eigenvalues of tensors and some very basic spectral hypergraph theory", Matrix Computations and Scientific Computing Seminar, April 16, 2008,
http://www.stat.uchicago.edu/ lekheng/work/mcsc2.pdf

In 2009, the following paper by Italian researchers attracted people to study spectral hypergraph theory via tensors:

[18]. S.R. Bulò and M. Pelillo, "New bounds on the clique number of graphs based on spectral hypergraph theory", in: T. Stützle ed., *Learning and Intelligent Optimization*, Springer Verlag, Berlin, (2009) pp. 45-48.

6.5. Spectral Hypergraph Theory via Tensors II

In 2012, the following two papers were published:

[19]. J. Cooper and A. Dutle, "Spectra of uniform hypergraphs", *Linear Algebra and Its Applications*, 436 (2012) 3268-3292.

[20]. S. Hu and L. Qi, "Algebraic connectivity of an even uniform hypergraph", *Journal of Combinatorial Optimization*, 24 (2012) 564-579.

6.6. Spectral Hypergraph Theory via Tensors III

In 2013, the following six papers were published:

[21]. G. Li, L. Qi and G. Yu, "The Z-eigenvalues of a symmetric tensor and its application to spectral hypergraph theory", *Numerical Linear Algebra with Applications*, 20 (2013) 1001-1029.

[22]. J. Xie and A. Chang, "On the H-eigenvalues of the signless Laplacian tensor for an even uniform hypergraph", *Frontiers of Mathematics in China*, 8 (2013) 107-128.

[23]. J. Xie and A. Chang, "On the Z-eigenvalues of the adjacency tensors for uniform hypergraphs", *Linear Algebra and Its Applications*, 430 (2013) 2195-2204.

[24]. J. Xie and A. Chang, "On the Z-eigenvalues of the signless Laplacian tensor for an even uniform hypergraph", *Numerical Linear Algebra with Applications*, 20 (2013) 1030-1045.

[25]. S. Hu, L. Qi and J. Shao, "Cored hypergraphs, power hypergraphs and their Laplacian eigenvalues", *Linear Algebra and Its Applications*, 439 (2013) 2980-2998.

[26]. K. Pearson and T. Zhang, "Eigenvalues on the adjacency tensor of products of hypergraphs", *International Journal on Contemporary Mathematical Sciences* **8** (2013) 151-158.

6.7. Spectral Hypergraph Theory via Tensors IV

In 2014, the following eight papers were published:

[27]. L. Qi, J. Shao and Q. Wang, "Regular uniform hypergraphs, s-cycles, s-paths and their largest Laplacian H-eigenvalues", *Linear Algebra and Its Applications*, 443 (2014) 215-227.

[28]. S. Hu and L. Qi, "The eigenvectors associated with the zero eigenvalues of the Laplacian and signless Laplacian tensors of a uniform hypergraph", *Discrete Applied Mathematics*, 169 (2014) 140-151.

[29]. L. Qi, "H^+-eigenvalues of Laplacian and signless Laplacian tensors", *Communications in Mathematical Sciences*, 12 (2014) 1045-1064.

[30]. C. Bu, J. Zhou and Y. Wei, "E-cospectral hypergraphs and some hypergraphs determined by their spectra", *Linear Algebra and Its Applications* **459** (2014) 397-403.

[31]. V. Nikiforov, "Analytic methods for uniform hypergraphs", *Linear Algebra and Its Applications* **457** (2014) 455-535.

[32]. V. Nikiforov, "Some extremal problems for hereditary properties of graphs", *The Electronic Journal of Combinatorics* **21** (2014) P1.17.

[33]. K. Pearson and T. Zhang, "On spectral hypergraph theory of the adjacency tensor", *Graphs and Combinatorics* **30** (2014) 1233-1248.

[34]. J. Zhou, L. Sun, W.Wang and C. Bu, "Some spectral properties of uniform hypergraphs", *The Electronic Journal of Combinatorics* **21** (2014) P4.24.

6.8. Spectral Hypergraph Theory via Tensors V

[35]. D. Ghoshdastidar, A. Dukkipati, “Consistency of spectral partitioning of uniform hypergraphs under planted partition model”, *Advances in Neural Information Processing Systems* **27** (NIPS 2014).

6.9. Spectral Hypergraph Theory via Tensors VI

In 2015, the following twelve papers were published:

[36]. J. Cooper and A. Dutle, "Computing hypermatrix spectra with the Poisson product formula", *Linear and Multilinear Algebra* **63** (2015) 956-970.

[37]. R. Cui, W. Li and M. Ng, "Primitive tensors and directed hypergraphs", *Linear Algebra and Its Applications* **471** (2015) 96-108.

[38]. S. Hu and L. Qi, "The Laplacian of a uniform hypergraph", *Journal of Combinatorial Optimization* **29** (2015) 331-366.

[39]. S. Hu, L. Qi and J. Xie, "The largest Laplacian and signless Laplacian H-eigenvalues of a uniform hypergraph", *Linear Algebra and Its Applications* **469** (2015) 1-27.

[40]. L. Kang, V. Nikiforov and X. Yuan, "The p-spectral radius of k-partite and k-chromatic uniform hypergraphs", *Linear Algebra and Its Applications* **478** (2015) 81-107.

[41]. M. Khan and Y. Fan, "On the spectral radius of a class of non-odd-bipartite even uniform hypergraphs", *Linear Algebra and Its Applications* **480** (2015) 93-106.

[42]. K. Pearson, "Spectral hypergraph theory of the adjacency hypermatrix and matroids", *Linear Algebra and Its Applications* **465** (2015) 176-187.

[43]. K. Pearson and T. Zhang, "The Laplacian tensor of a multi-hypergraph", *Discrete Mathematics* **338** (2015) 972-982.

6.10. Spectral Hypergraph Theory via Tensors VII

[44]. J. Shao, L. Qi and S. Hu, "Some new trace formulas of tensors with applications in spectral hypergraph theory", *Linear and Multilinear Algebra* **63** (2015) 971-992.

[45]. J. Xie and L. Qi, "The clique and coclique numbers' bounds based on the H-eigenvalues of uniform hypergraphs", *International Journal of Numerical Analysis & Modeling* **12** (2015) 318-327.

[46]. X. Yuan, M. Zhang and M. Lu, "Some bounds on the eigenvalues of uniform hypergraphs", *Linear Algebra and Its Applications* **484** (2015) 540-549.

[47]. D. Ghoshdastidar and A. Dukkipati, "A Provable Generalized Tensor Spectral Method for Uniform Hypergraph Partitioning", *Proceedings of the 32nd International Conference on Machine Learning*, Lille, France, 2015. JMLR: W&CP vol. 37.

6.11. Spectral Hypergraph Theory via Tensors VIII

In 2016, the following eighteen papers were published:

[48]. Z. Chen and L. Qi, "Circulant tensors with applications to spectral hypergraph theory and stochastic process", *Journal of Industrial and Management Optimization* **12** (2016) 1227-1247.

[49]. M. Khan, Y. Fan and Y. Tan, "The H-spectra of a class of generalized power hypergraphs", *Discrete Mathematics* **339** (2016) 1682-1689.

[50]. J. Xie and L. Qi, "Spectral directed hypergraph theoy via tensors", *Linear and Multilinear Algebra* **64** (2016) 780-794.

[51]. X. Yuan, L. Qi and J. Shao, "The proof of a conjecture on largest Laplacian and signless Laplacian H-eigenvalues of uniform hypergraphs", *Linear Algebra and Its Applications* **490** (2016) 18-30.

[52]. X. Yuan, J. Shao and H. Shan, "Ordering of some uniform supertrees with larger spectral radii", *Linear Algebra and Its Applications* **495** (2016) 206-222.

[53]. C. Bu, Y. Fan and J. Zhou, "Laplacian and signless Laplacian Z-eigenvalues of uniform hypergraphs", *Frontiers of Mathematics in China* **11** (2016) 511-520.

[54]. H. Li, J. Shao and L. Qi, "The extremal spectral radii of k-uniform supertrees", *Journal of Combinatorial Optimization* **32** (2016) 741-764.

6.12. Spectral Hypergraph Theory via Tensors IX

[55]. J. Shao, H. Shan and B. Wu, "Some spectral properties and characterizations of connected odd-bipartite uniform hypergraphs", *Linear and Multilinear Algebra* **63** (2016) 2359-2372.

[56]. J. Yue, L. Zhang and M. Lu, "The largest adjacency, signless Laplacian, and Laplacian H-eigenvalues of loose paths", *Frontiers of Mathematics in China* **11** (2016) 1-13.

[57]. Y. Fan, Y. Tan, X. Peng and A. Liu, "Maximizing spectral radii of uniform hypergraphs with few edges", *Discussiones Mathematicae Graph Theory* **36** (2016) 845-856.

[58]. M. Khan, Y. Fan and Y. Tan, "The H-spectra of a class of generalized power hypergraphs", *Discrete Mathematics* **339** (2016) 1682-1689.

[59]. L. Lu and S. Man, "Connected hypergraphs with small spectral radius", *Linear Algebra and Its Applications* **509** (2016) 206-227.

[60]. X. Yuan, L. Qi and J. Shao, "The proof of a conjecture on largest Laplacian and signless Laplacian H-eigenvalues of uniform hypergraphs", *Linear Algebra and Its Applications* **490** (2016) 18-30.

[61]. W. Li, A Chang and J Cooper, "Analytic connectivity of k-uniform hypergraphs", *Linear and Multilinear Algebra*.

6.13. Spectral Hypergraph Theory via Tensors X

[62]. A Banerjee, A Char and B Mondal, "Spectra of general hypergraphs", *Linear Algebra and Its Applications* **518** (2017) 14-30.

[63]. J. Chang, Y. Chen and L. Qi, "Computing eigenvalues of large scale sparse tensors arising from a hypergraph", *SIAM Journal on Scientific Computing* **36** (2016) A3618-A3643.

[64]. H. Lin, B. Mo, B. Zhou and W. Weng, "Sharp bounds for ordinary and signless Laplacian spectral radii of uniform hypergraphs", *Applied Mathematical Computation* **285** (2016) 217-227.

[65]. H. Lin, B. Zhou and B. Mo, "Upper bounds for h- and z-spectra radii of uniform hypergraphs", *Linear Algebra and Its Applications* **510** (2016) 205-221.

6.14. Laplacian Tensors and Signless Laplacian Tensors

A uniform hypergraph is also called a k-graph. Let $G = (V, E)$ be a k-graph, where $V = \{1, 2, \cdots, n\}$ is the vertex set, $E = \{e_1, e_2, \cdots, e_m\}$ is the edge set, $e_p \subset V$ and $|e_p| = k$ for $p = 1, \cdots, m$, and $k \geq 2$. If $k = 2$, then G is an ordinary graph. We assume that $e_p \neq e_q$ if $p \neq q$.

The **adjacency tensor** $\mathcal{A} = \mathcal{A}(G)$ of G, is a kth order n-dimensional symmetric tensor, with $\mathcal{A} = (a_{i_1 i_2 \cdots i_k})$, where $a_{i_1 i_2 \cdots i_k} = \frac{1}{(k-1)!}$ if $(i_1, i_2, \cdots, i_k) \in E$, and 0 otherwise. Thus, $a_{i_1 i_2 \cdots i_k} = 0$ if two of its indices are the same.

For $i \in V$, its degree $d(i)$ is defined as $d(i) = |\{e_p : i \in e_p \in E\}|$. We assume that every vertex has at least one edge. Thus, $d(i) > 0$ for all i. The **degree tensor** $\mathcal{D} = \mathcal{D}(G)$ of G, is a kth order n-dimensional diagonal tensor, with its ith diagonal element as $d(i)$.

The **Laplacian tensor** $\mathcal{L}$ of G is defined by $\mathcal{D} - \mathcal{A}$. The **signless Laplacian tensor** $\mathcal{Q}$ of G is defined by $\mathcal{D} + \mathcal{A}$.

Adjacency tensors, degree tensors, Laplacian tensors and signless Laplacian tensors are real symmetric tensors. Laplacian tensors are symmetric M tensors. Laplacian tensors and signless Laplacian tensors have H-eigenvalues. Even order Laplacian tensors and signless Laplacian tensors are positive semi-definite.

7. Application in Physics

Recently, European physicists have applied tensor eigenvalues, positive semi-definite tensors and copositive tensors to the study of quantum spin states, quantum field theory, liquid crystal and super-gravitation:

[H]. F. Bohnet-Waldraff, D. Braun and O. Giraud, "Tensor eigenvalues and entanglement of symmetric states", *Physical Review A* **94** (2016) 042324.

[I]. K. Kannike, "Vacuum stability of a general scalar potential of a few fields", *The European Physical Journal C* **76** (2016) 324.

[J]. E. G. Virga, "Octupolar order in two dimensions", *The European Physical Journal E* **38** (2015) 63.

[K]. G. Gaeta and E.G. Virga, "Octupolar order in three dimensions", *The European Physical Journal E* **39** (2016) 113.

[L]. D. Rathlev, "De-Sitter-Vakua in Supergravitationsmodellen", Master Thesis, Faculty of Physics, University of Göttingen, Germany, 2012

7.1. Collaboration

After seeing these, we explored these research topics, and conducted some joint works with them:

[M]. L. Qi, G. Zhang, D. Braun, F. Bohnet-Waldraff and O. Giraud, "Regularly decomposable tensors and classical spin states", to appear in: *Communications in Mathematical Sciences.*

[N]. Y. Chen, L. Qi and E.G. Virga, "Octupolar tensors for liquid crystals", April 2017. arXiv:1701.06761.

[O]. Y. Chen, A. Jákli and L. Qi, "Spectral analysis of piezoelectric tensors", March 2017, arXiv:1703.07937.

7.2. Notation in Quantum Mechanics

Quantum mechanics is a mathematical framework for the development of theories describing physical systems (in molecular or atomic scale). The modern form of quantum mechanics was formalized by Paul Dirac and John von Neumann in 1930s. Quantum mechanics is, as yet, the most accurate and complete description of the world known.

7.3. States

Associated to any isolated physical system there is a Hilbert space. The normalized column vectors in the Hilbert space are called **states** of the physical system. The physical system at time instant t is completely described by its states at time instant t. In quantum mechanics, a state is conventionally denoted by **Dirac ket**, like $|\phi\rangle$; its dual (complex conjugate transpose) state is denoted $\langle\psi|$. For example, given

$$|\phi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix},$$

we have

$$\langle\psi| = \frac{1}{\sqrt{2}} (1, -i) .$$

We use $\langle\phi|\psi\rangle$ to denote inner product. Thus, we always have

$$\| |\phi\rangle \|^2 \equiv \langle\phi|\phi\rangle = 1.$$

7.4. The Quantum Entanglement Problem

A general n-partite state (**composite state**) $|\Psi\rangle$ of a composite quantum system can be regarded as an element in a Hilbert tensor product space $\mathcal{H} = \otimes_{k=1}^{n}\mathcal{H}_k$, where the dimension of $\mathcal{H}_k$ is d_k for $k = 1, \cdots, n$.

If $d = d_1 = \ldots = d_n$, then the state $|\Psi\rangle$ is called a symmetric (multipartite) state.

A **separable (Hartree)** n-partite **state** $|\phi\rangle$ can be described by $|\phi\rangle = \otimes_{k=1}^{n}|\phi^{(k)}\rangle$ with $|\phi^{(k)}\rangle \in \mathcal{H}_k$.

A composite state which is not separable, is called **entangled**. An entangled state $|\Psi\rangle$ can be expressed as a weighted sum of some separable states, satisfying $\langle\Psi|\Psi\rangle = 1$.

The quantum entanglement problem is now regarded as a central problem in quantum information theory.

7.5. Our Paper

In March 2012, our group explored the relation between quantum information theory and the spectral theory of tensors. We found a close link between the quantum entanglement problem and Z-eigenvalues of nonnegative tensors. We wrote the following paper [1]. After almost four years, this paper was eventually published in a physics journal and got attention of physicists.

[1]. S. Hu, L. Qi and G. Zhang, "Computing the geometric measure of entanglement of multipartite pure states by means of non-negative tensors", *Physical Review A* **93** (2016) 012304.

7.6. The Paper of Bohnet-Waldraff, Braun and Giraud I

We now see the following paper:

[2]. F. Bohnet-Waldraff, D. Braun and O. Giraud, "Tensor eigenvalues and entanglement of symmetric states", *Physical Review A* **94** (2016) 042324.

Bohnet-Waldraff, Braun and Giraud wrote

"But the relevance of the spectral theory of tensors for the separability (or classicality) problem has just recently attracted some attention in the quantum information community. For example, in [25] (our paper [1]) it was shown that for pure states the largest tensor eigenvalue is equal to the geometric measure of entanglement i.e., the maximal overlap of the state with a pure separable state. This entanglement measure is in fact essentially equivalent to finding the best rank-one approximation of the tensor. Therefore, largest tensor eigenvalue is directly related to the entanglement of a state. In this paper, we will explore a new connection, which relates the smallest tensor eigenvalue to the entanglement of a pure or mixed state. This originates in the fact that the entanglement of a state is related to the positive-definiteness of a tensor, which in turn is linked to the sign of its smallest tensor eigenvalue."

Prof. Daniel Braun is a Professor at Institute of Physics, University of Tübingen, Germany. Fabian Bohnet-Waldraff is his Ph.D. student. Dr. Olivier Giraud, a researcher at Laboratory of Theoretical Physics and Statistical Models, University of Paris-Sud, France.

Section III of [2] is "Tensor Eigenvalues".

7.7. Collaboration

I communicated with Daniel, and had intensive discussion on this topic. This resulted our collaboration.

[3]. L, Qi, G. Zhang, D. Braun, F. Bohnet-Waldraff and and O. Giraud, "Regularly decomposable tensors and classical spin states", to appear in: *Communications in Mathematical Sciences.*

Here I summarize the content of our collaboration.

7.8. Bosons and Fermions

In quantum mechanics, as classified by Paul Dirac, there are two classes of particles, bosons and fermions. Bosons, including elementary particles such as photons, gluons, W and Z bosons, Higgs bosons, as well as composite particles such as mesons and stable nuclei of even mass number such as deuterium, follow Bose-Einstein statistics. Fermions, including electrons, quarks and leptons, as well as any composite particle made of an odd number of these, such as protons, baryons and many atoms and nuclei, follow Fermi-Dirac statistics.

According to the spin-statistics theorem of Wolfgang Pauli, particles with integer spin are bosons, while particles with half-integer spin are fermions. Thus, a spin-j state corresponds a boson if j is a positive integer, and corresponds a fermion if j is a positive half-integer.

For arbitrary pure spin states, a geometrical representation was developed by Ettore Majorana: a spin-j state is visualized as $N = 2j$ points on a unit sphere, called the Bloch sphere.

7.9. Tensor Representation of Spin States

In 2015, Giraud, Braun, Baguette, Bastin and Martin [38] proposed a tensor representation for spin states. The tensor representation of a spin-j state is a symmetric tensor of order $N = 2j$ and dimension 4. Thus, a boson corresponds an even order four dimensional tensor, while a fermion corresponds an odd order four dimensional tensor.

[4]. O. Giraud, D. Braun, D. Baguette, T. Bastin and J. Martin, "Tensor representation of spin states", *Physical Review Letters* **114** (2015) 080401.

Baguette, Bastin and Martin are Belgian physicists.

7.10. Regularly Decomposable Tensors

In 2016, Bohnet-Waldraff, Braun and Giraud [2] further studied the tensor representation of spin states. They showed that when j is an integer, i.e., N is an even number, if a spin-j state is classical, then its representative tensor is positive semi-definite (PSD) in the sense of Qi [1].

In [3], Braun, Qi, Zhang, Bohnet-Waldraff and Giraud introduced regularly decomposable tensors, and showed that a spin state is classical if and only if it is a regularly decomposable tensor. In the even order case, a regularly decomposable tensor a special PSD tensor. This result is also for the odd order case.

7.11. Copositive Tensors and Quantum Field Theory

The following paper is published last year:

[5]. K. Kannike, “Vacuum stability of a general scalar potential of a few fields”, *European Physical Journal C* **76** 324 (2016).

Kristjan Kannike, a researcher at National Institute of Chemical Physics and Biophysics, Estonia. In this paper, he wrote that

“A scalar potential has to be bounded from below to make physical sense. In the Standard Model (SM), it simply means that the self-coupling of the Higgs boson has to be positive. In an extended model with more scalar fields, the potential has to be bounded from below - the vacuum has to be stable - in the limit of large field values in all possible directions of the field space. In this limit, any terms with dimensionful couplings - mass or cubic terms - can be neglected in comparison with the quartic part of the scalar potential.”

Section 7 of [5] are on tensor eigenvalues and copositive tensors.

7.12. Prof. Epifanio Virga

In 2015, I noticed a paper:

[1] E.G. Virga "Octupolar order in two dimensions", *Eur. Phys. J. E* **38** (2015) 63.

Prof. Epifanio Virga is a mathematical physics professor at Italy, whose research is on liquid crystal. In [1], he applied tensor eigenvalues introduced by me to this research. I invited Prof. Virga to give a talk at the tensor conference at Nankai, May 2016. He has not been able to come. But later he sent me a new paper.

[2] G. Gaeta and E.G. Virga, "Octupolar order in three dimensions", *Eur. Phys. J. E* **39** (2016) 113.

7.13. Liquid Crystal

Liquid crystals are well-known for visualization applications in flat panel electronic displays. But beyond that, various optical and electronic devices, such as laser printers, light-emitting diodes, field-effect transistors, and holographic data storage, were invented with the development of bent-core (banana-shaped) liquid crystals. A **third-order** three dimensional symmetric traceless tensor was introduced in to characterize condensed phases exhibited by bent-core molecules. Based on such a tensor, the orientationally ordered octupolar (tetrahedratic) phase has been both predicted theoretically and confirmed experimentally. After that, the octupolar order parameters of liquid crystals have been widely studied. Generalized liquid crystal phases were also considered, which feature octupolar order tensors among so many others.

Virga [1] and Gaeta and Virga [2], in their studies of third-order octupolar tensors in two and three space dimensions, respectively, also introduced the *octupolar potential*, a scalar-valued function on the unit sphere obtained from the octupolar tensor. In particular, Gaeta and Virga [2] showed that the irreducible admissible region for the octupolar potential is bounded by a surface in the three-dimensional parameter space which has the form of a *dome* and, more importantly, that there are indeed *two* generic octupolar states, divided by a *separatrix* surface in paramter space. Physically, the latter surface was interpreted as representing a possible intra-octupolar transition.

7.14. A Joint Work

However, Gaeta and Virga [2] have not been able to give explicit algebraic expressions of the dome and the separatrix surface. This problem attracted the attention of Dr. Yannan Chen and me. Can we give explicit expressions of the dome and the separatrix surface by our theory of tensor eigenvalues? If we can do so, especially if we can give the explicit expression of the separatrix surface, then such a possible intra-octupolar transition state may be detected experimentally too. We worked with Epifanio Virga, and succeded. We wrote a joint paper:

[3] Y. Chen, L. Qi, and E.G. Virga, "Octupolar tensors for liquid crystals", arXiv:1701.06761, 2017.

In this paper, by using the resultant theory of algebraic geometry and the E-characteristic polynomial of the spectral theory of tensors, we give a closed-form, algebraic expression for both the dome and the separatrix. This turns the intra-octupolar transition envisioned in [2] into a quantitative, possibly observable prediction.