Turnpike properties in the calculus of variations and optimal control

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May 2018

Let us consider the following problem of the calculus of variations:

$$\int_0^T f(v(t), v'(t))dt \to \min, \qquad (P_0)$$

$v : [0, T] \to R^n$ is an absolutely continuous function

$$\text{such that } v(0) = y,\ v(T) = z.$$

Here T is a positive number, y and z are elements of the n-dimensional Euclidean space R^n and an integrand $f : R^n \times R^n \to R^1$ is a continuous function.

We are interested in the structure of solutions of the problem (P_0) when y, z and T vary and T is sufficiently large.

Assume that the function f is strictly convex and differentiable and satisfies the following growth condition:

$$f(y, z)/(|y| + |z|) \to \infty \text{ as } |y| + |z| \to \infty.$$

Here we denote by $|\cdot|$ the Euclidean norm in R^n and by $< \cdot, \cdot >$ the scalar product in R^n.

In order to analyze the structure of minimizers of the problem (P_0) we consider the auxiliary minimization problem:

$$f(y, 0) \to \min, \ y \in R^n. \qquad (P_1)$$

It follows from the growth condition and the strict convexity of f that the problem (P_1) has a unique solution which will be denoted by $\bar{y}$. Clearly,

$$\partial f / \partial y(\bar{y}, 0) = 0.$$

Define an integrand $L : R^n \times R^n \to R^1$ by

$$L(y, z) = f(y, z) - f(\bar{y}, 0) - < \nabla f(\bar{y}, 0), (y, z) - (\bar{y}, 0) >$$

$$= f(y, z) - f(\bar{y}, 0) - < (\partial f / \partial z)(\bar{y}, 0), z > .$$

Clearly L is also differentiable and strictly convex and satisfies the same growth condition as f:

$$L(y, z)/(|y| + |z|) \to \infty \text{ as } |y| + |z| \to \infty.$$

Since f and L are strictly convex we obtain that

$$L(y,z) \geq 0 \text{ for all } (y,z) \in R^n \times R^n$$

and

$$L(y,z) = 0 \text{ if and only if } y = \bar{y},\ z = 0.$$

Consider the following auxiliary problem of the calculus of variations:

$$\int_0^T L(v(t), v'(t))dt \to \min, \qquad (P_2)$$

$v : [0,T] \to R^n$ is an absolutely continuous function

such that $v(0) = y,\ v(T) = z,$

where $T > 0$ and $y, z \in R^n$.

It is easy to see that for any absolutely continuous function $x : [0, T] \to R^n$ with $T > 0$,

$$\int_0^T L(x(t), x'(t))dt$$

$$= \int_0^T [f(x(t), x'(t))$$

$$-f(\bar{y}, 0) - < (\partial f/\partial z)(\bar{y}, 0), x'(t) >]dt$$

$$= \int_0^T f(x(t), x'(t))dt + Tf(\bar{y}, 0)$$

$$- < (\partial f/\partial z)(\bar{y}), x(T) - x(0) > .$$

These equations imply that the problems (P_0) and (P_2) are equivalent: a function $x : [0, T] \to R^n$ is a solution of the problem (P_0) if and only if it is a solution of the problem (P_2).

The integrand $L : R^n \times R^n \to R^1$ has the following property:

(C) If $\{(y_i, z_i)\}_{i=1}^{\infty} \subset R^n \times R^n$ satisfies

$$\lim_{i\to\infty} L(y_i, z_i) = 0,$$

then $\lim_{i\to\infty} y_i = \bar{y}$ and $\lim_{i\to\infty} z_i = 0$.

Indeed, assume that

$$\{(y_i, z_i)\}_{i=1}^{\infty} \subset R^n \times R^n \text{ and } \lim_{i\to\infty} L(y_i, z_i) = 0.$$

By the growth condition the sequence $\{(y_i, z_i)\}_{i=1}^{\infty}$ is bounded. Let (y, z) be a limit point of the sequence $\{(y_i, z_i)\}_{i=1}^{\infty}$. Then,

$$L(y, z) = \lim_{i\to\infty} L(y_i, z_i) = 0$$

$$\text{and } (y, z) = (\bar{y}, 0).$$

This implies that $(\bar{y}, 0) = \lim_{i\to\infty}(y_i, z_i)$.

Let $y, z \in R^n$, $T > 2$ and a function $\bar{x} : [0, T] \to R^n$ be an optimal solution of the problem (P_0). Then $\bar{x}$ is also an optimal solution of the problem (P_2). We will show that

$$\int_0^T L(\bar{x}(t), \bar{x}'(t))dt \leq 2c_0(|y|, |z|)$$

where $c_0(|y|, |z|)$ is a constant which depends only on $|y|$ and $|z|$.

Define a function $x : [0, T] \to R^n$ by

$$x(t) = y + t(\bar{y} - y),\ t \in [0, 1],$$

$$x(t) = \bar{y},\ t \in [1, T - 1],$$

$$x(t) = \bar{y} + (t - (T - 1))(z - \bar{y}),\ t \in [T - 1, T].$$

It follows from the definition of $\bar{x}$ and x that

$$\int_0^T L(\bar{x}(t), \bar{x}'(t))dt$$

$$\le \int_0^T L(x(t), x'(t))dt$$

$$= \int_0^1 L(x(t), \bar{y} - y)dt$$

$$+ \int_1^{T-1} L(\bar{y}, 0)dt$$

$$+ \int_{T-1}^T L(x(t), z - \bar{y})dt$$

$$= \int_0^1 L(x(t), \bar{y} - y)dt$$

$$+ \int_{T-1}^T L(x(t), z - \bar{y})dt.$$

It is not difficult to see that the integrals

$$\int_0^1 L(x(t), \bar{y} - y)dt \text{ and } \int_{T-1}^{T} L(x(t), z - \bar{y})dt$$

do not exceed a constant $c_0(|y|, |z|)$ which depends only on $|y|$, $|z|$.

Thus

$$\int_0^T L(\bar{x}(t), \bar{x}'(t))dt \leq 2c_0(|y|, |z|).$$

It is very important that in this inequality the constant $c_0(|y|, |z|)$ does not depend on T.

We denote by mes(E) the Lebesgue measure of a Lebesgue mesurable set $E \subset R^1$.

Now let ϵ be a positive number. By the property (C) there is $\delta > 0$ such that if $(y, z) \in R^n \times R^n$ and $L(y, z) \leq \delta$, then

$$|y - \bar{y}| + |z| \leq \epsilon.$$

Then by the choice of δ and the inequality

$$\int_0^T L(\bar{x}(t), \bar{x}'(t))dt \leq 2c_0(|y|, |z|),$$

$$\text{mes}\{t \in [0, T] :$$

$$|(\bar{x}(t), \bar{x}'(t)) - (\bar{y}, 0)| > \epsilon\}$$

$$\leq \text{mes}\{t \in [0, T] : L(\bar{x}(t), \bar{x}'(t)) > \delta\}$$

$$\leq \delta^{-1} \int_0^T L(\bar{x}(t), \bar{x}'(t))dt$$

$$\leq \delta^{-1} 2c_0(|y|, |z|)$$

and

$$\text{mes}\{t \in [0, T] :$$

$$|\bar{x}(t) - \bar{y}| > \epsilon\} \leq \delta^{-1} 2c_0(|y|, |z|).$$

Therefore the optimal solution $\bar{x}$ spends most of the time in an ϵ-neighbor- hood of the point $\bar{y}$. The Lebesgue measure of the set of all points t, for which $\bar{x}(t)$ does not belong to this ϵ-neighborhood, does not exceed the constant $2\delta^{-1}c_0(|y|,|z|)$ which depends only on $|y|,|z|$ and ϵ and does not depend on T. Following the tradition, the point $\bar{y}$ is called the turn-pike. Moreover we can show that the set

$$\{t \in [0,T] : |\bar{x}(t) - \bar{y}| > \epsilon\}$$

is contained in the union of two intervals $[0,\tau_1] \cup [T-\tau_2,T]$, where $0 < \tau_1, \tau_2 \le 2\delta^{-1}c_0(|y|,|z|)$.

Under the assumptions posed on f, the structure of optimal solutions of the problem (P_0) is rather simple and the turnpike $\bar{y}$ is calculated easily. On the other hand the proof is strongly based on the convexity of f and its time independence. The approach used in the proof cannot be employed to extend the turnpike result for essentially larger classes of variational problems. For such extensions we need other approaches and ideas. The question of what happens if the integrand f is nonconvex and nonautonomous seems very interesting. What kind of turnpike and what kind of convergence to the turnpike do we have for general nonconvex nonautonomous integrands? The following example helps to understand the problem.

Let

$$f(t,x,u) = (x - \cos(t))^2 + (u + \sin(t))^2,$$

$$(t,x,u) \in R^1 \times R^1 \times R^1$$

and consider the family of the variational problems

$$\int_{T_1}^{T_2} [(v(t) - \cos(t))^2$$

$$+(v'(t) + \sin(t))^2]dt \to \min, \qquad (P_3)$$

$v : [T_1, T_2] \to R^1$ is an absolutely continuous

function such that $v(T_1) = y,\ v(T_2) = z,$

where $y, z, T_1, T_2 \in R^1$ and $T_2 > T_1$. The integrand f depends on t, for each $t \in R^1$ the function $f(t, \cdot, \cdot) : R^2 \to R^1$ is convex, and for each $x, u \in R^1 \setminus \{0\}$ the functon $f(\cdot, x, u) : R^1 \to R^1$ is nonconvex. Thus the function $f : R^1 \times R^1 \times R^1 \to R^1$ is also nonconvex and depends on t.

Assume that $y, z, T_1, T_2 \in R^1$, $T_2 > T_1 + 2$ and $\widehat{v} : [T_1, T_2] \to R^1$ is an optimal solution of the problem (P_3). Note that the problem (P_3) has a solution since f is continuous and $f(t, x, \cdot) : R^1 \to R^1$ is convex and grows superlinearly at infinity for each $(t, x) \in [0, \infty) \times R^1$.

Define $v : [T_1, T_2] \to R^1$ by

$$v(t) = y + (\cos(T_1 + 1) - y)(t - T_1), \ t \in [T_1, T_1 + 1],$$

$$v(t) = \cos(t), \ t \in [T_1 + 1, T_2 - 1],$$

$$v(t) = \cos(T_2 - 1) + (t - T_2 + 1)(z - \cos(T_2)),$$

$$t \in [T_2 - 1, T_2].$$

It is easy to see that

$$\int_{T_1+1}^{T_2-1} f(t, v(t), v'(t))dt = 0$$

and

$$\int_{T_1}^{T_2} f(t, \widehat{v}(t), \widehat{v}'(t))dt$$

$$\leq \int_{T_1}^{T_2} f(t, v(t), v'(t))dt$$

$$= \int_{T_1}^{T_1+1} f(t, v(t), v'(t))dt +$$

$$\int_{T_2-1}^{T_2} f(t, v(t), v'(t))dt$$

$$\leq 2 \sup\{|f(t, x, u)| :$$

$$t, x, u \in R^1, \ |x|, |u| \leq |y| + |z| + 1\}.$$

Thus

$$\int_{T_1}^{T_2} f(t, \widehat{v}(t), \widehat{v}'(t))dt \leq c_1(|y|, |z|),$$

where

$$c_1(|y|,|z|) = 2\sup\{|f(t,x,u)| :$$

$$t,x,u \in R^1,\ |x|,|u| \le |y| + |z| + 1\}.$$

For any $\epsilon \in (0,1)$ we have

$$\text{mes}\{t \in [T_1, T_2] :$$

$$|\widehat{v}(t) - \cos(t)| > \epsilon\}$$

$$\le \epsilon^{-2} \int_{T_1}^{T_2} f(t, \widehat{v}(t), \widehat{v}'(t))dt$$

$$\le \epsilon^{-2} c_1(|y|,|z|).$$

Since the constant $c_1(|y|,|z|)$ does not depend on T_2 and T_1 we conclude that if $T_2 - T_1$ is sufficiently large, then the optimal solution $\widehat{v}(t)$ is equal to $\cos(t)$ up to ϵ for most $t \in [T_1, T_2]$.

Again, as in the case of convex time independent problems we can show that

$$\{t \in [T_1, T_2] : \ |x(t) - \cos(t)| > \epsilon\}$$

$$\subset [T_1, T_1 + \tau] \cup [T_2 - \tau, T_2]$$

where $\tau > 0$ is a constant which depends only on ϵ, $|y|$ and $|z|$.

This example shows that there exist nonconvex time dependent integrands which have the turnpike property with the same type of convergence as in the case of convex autonomous variational problems. The difference is that the turnpike is not a singleton but an absolutely continuous time dependent function defined on the infinite interval $[0, \infty)$. This leads us to the following definition of the turnpike property for general integrands.

Let us consider the following variational problem:

$$\int_{T_1}^{T_2} f(t, v(t), v'(t))dt \to \min, \qquad (P)$$

$v : [T_1, T_2] \to R^n$ is an absolutely continuous

function such that $v(T_1) = y$, $v(T_2) = z$.

Here $T_1 < T_2$ are real numbers, y and z are elements of the n-dimensional Euclidean space R^n and an integrand $f : [0, \infty) \times R^n \times R^n \to R^1$ is a continuous function.

We say that the integrand f has the *turnpike property* if there exists a locally absolutely continuous function $X_f : [0, \infty) \to R^n$ (called the "turnpike") which depends only on f and satisfies the following condition:

For each bounded set $K \subset R^n$ and each $\epsilon > 0$ there exists a constant $T(K, \epsilon) > 0$ such that for each $T_1 \geq 0$, each $T_2 \geq T_1 + 2T(K, \epsilon)$, each $y, z \in K$ and each optimal solution $v : [T_1, T_2] \to R^n$ of variational problem (P), the inequality $|v(t) - X_f(t)| \leq \epsilon$ holds for all $t \in [T_1 + T(K, \epsilon), T_2 - T(K, \epsilon)]$.

The turnpike property is very important for applications. Suppose that the integrand f has the turnpike property, K and ϵ are given, and we know a finite number of "approximate" solutions of the problem (P). Then we know the turnpike X_f, or at least its approximation, and the constant $T(K, \epsilon)$ which is an estimate for the time period required to reach the turnpike. This information can be useful if we need to find an "approximate" solution of the problem (P) with a new time interval $[T_1, T_2]$ and the new values $y, z \in K$ at the end points T_1 and T_2.

Namely instead of solving this new problem on the "large" interval $[T_1, T_2]$ we can find an "approximate" solution of problem (P) on the "small" interval $[T_1, T_1 + T(K, \epsilon)]$ with the values $y, X_f(T_1 + T(K, \epsilon))$ at the end points and an "approximate" solution of problem (P) on the "small" interval $[T_2 - T(K, \epsilon), T_2]$ with the values $X_f(T_2 - T(K, \epsilon)), z$ at the end points. Then the concatenation of the first solution, the function $X_f : [T_1 + T(K, \epsilon), T_2 - T(K, \epsilon)]$ and the second solution is an "approximate" solution of problem (P) on the interval $[T_1, T_2]$ with the values y, z at the end points.

The turnpike properties are discussed in A. Zaslavski, Turnpike Properties in the Calculus of Variations and Optimal Control, Springer, New York, 2006.

In Chapter 1 we introduce a space $\mathcal{M}$ of continuous integrands $f : [0, \infty) \times R^n \times R^n \to R^1$. This space is equipped with a natural complete metric. We show that for any initial condition $x_0 \in R^n$ there exists a locally absolutely continuous function $x : [0, \infty) \to R^n$ with $x(0) = x_0$ such that for each $T_1 \geq 0$ and $T_2 > T_1$ the function $x : [T_1, T_2] \to R^n$ is a solution of problem (P) with $y = x(T_1)$ and $z = x(T_1)$. We also establish that for every bounded set $E \subset R^n$ the $C([T_1, T_2])$ norms of approximate solutions $x : [T_1, T_2] \to R^n$ for the problem (P) with $y, z \in E$ are bounded by some constant which does not depend on T_1 and T_2.

In Chapter 2 we establish the turnpike property stated above for a generic integrand $f \in \mathcal{M}$. We establish the existence of a set $\mathcal{F} \subset \mathcal{M}$ which is a countable intersection of open everywhere dense sets in $\mathcal{M}$ such that for each $f \in \mathcal{F}$ the turnpike property holds. Moreover we show that the turnpike property holds for approximate solutions of variational problems with a generic integrand f and that the turnpike phenomenon is stable under small pertubations of a generic integrand f.

In Chapters 3-5 we study turnpike properties for autonomous problems (P) with integrands $f : R^n \times R^n \to R^1$ which do not depend on t. Since the turnpike theorems of Chapter 2 are of generic nature and the subset of $\mathcal{M}$ which consists of all time independent integrands are nowhere dense, the results of Chapter 2 can not be applied for this subset.

Moreover, we cannot expect to obtain the turnpike property stated above for the general autonomous case. Indeed, if an integrand f does not depend on t and has a turnpike, then this turnpike should also be time independent. It means that the turnpike is a stationary trajectory (a singleton). But it is not true when a time independent integrand f is not a convex function.

Consider the following example. Let

$$f(x_1, x_2, u_1, u_2)$$

$$= (x_1^2 + x_2^2 - 1)^2 + (u_1 + x_2)^2 + (u_2 - x_1)^2,$$

$$(x_1, x_2, u_1, u_2) \in R^2 \times R^2$$

and consider the family of the variational problems

$$\int_0^T f(v_1(t), v_2(t), v_1'(t), v_2'(t))dt \to \min, \quad (P_4)$$

$$(v_1, v_2) : [0, T] \to R^2$$

is an absolutely continuous function

such that $(v_1, v_2)(0) = y$, $(v_1, v_2)(T) = z$,

where $y = (y_1, y_2)$, $z = (z_1, z_2) \in R^2$ and $T > 0$.

The integrand f does not depend on t. Since f is continuous and for each $x = (x_1, x_2) \in R^2$ the function $f(x, \cdot) : R^2 \to R^1$ is convex and grows superlinearly at infinity, the problem (P_4) has a solution for each $T > 0$ and each $y, z \in R^2$. Clearly, if $T > 0$, $y = (\cos(0), \sin(0))$ and $z = (\cos(T), \sin(T))$, then the function

$$\widehat{x}_1(t) = \cos(t),\ \widehat{x}_2(t) = \sin(t),\ t \in [0, T]$$

is a solution of the problem (P_4). Thus, if the integrand f has a turnpike property, then the turnpike is not a singleton.

Let $T > 2$, $y, z \in R^2$ and let $\bar{v} = (\bar{v}_1, \bar{v}_2) : [0, T] \to R^2$ be a solution of the problem (P_4). Define a function $v = (v_1, v_2) : [0, T] \to R^n$ by

$$v(t) = y + t((\cos(1), \sin(1)) - y),\ t \in [0, 1],$$

$$v(t) = (\cos(t), \sin(t)),\ t \in [1, T - 1],$$

$$v(t) = (\cos(T - 1), \sin(T - 1))$$

$$+(t - T + 1)(z - (\cos(T - 1), \sin(T - 1)),$$

$$t \in [T - 1, T].$$

Then

$$\int_1^{T-1} f(v(t), v'(t))dt = 0$$

and

$$\int_0^T (\bar{v}_1(t)^2 + \bar{v}_2(t)^2 - 1)^2 dt$$

$$\leq \int_0^T f(\bar{v}(t), \bar{v}'(t))dt$$

$$\leq \int_0^T f(v(t), v'(t))dt$$

$$= \int_0^1 f(v(t), v'(t))dt$$

$$+ \int_{T-1}^T f(v(t), v'(t))dt$$

$$\leq \sup\{f(x_1, x_2, u_1, u_2) :$$

$$x_1, x_2, u_1, u_2 \in R^1$$

and $|x_i|, |u_i| \leq 2|y| + 2|z| + 2, \ i = 1, 2\}.$

Thus

$$\int_0^T (\bar{v}_1(t)^2 + \bar{v}_2(t)^2 - 1)^2 dt$$

$$\leq c_2(|y|, |z|)$$

with

$$c_2(|y|, |z|) = \sup\{f(x_1, x_2, u_1, u_2) :$$

$$x_1, x_2, u_1, u_2 \in R^1$$

$$\text{and } |x_i|, |u_i| \leq 2|y| + 2|z| + 2\}.$$

Here $c_2(|y|, |z|)$ depends only on $|y|, |z|$ and does not depend on T. For any $\epsilon \in (0, 1)$ we have

$$\text{mes}\{t \in [0, T] :$$

$$||(\bar{v}_1(t), \bar{v}_2(t))| - 1| > \epsilon\}$$

$$\leq \text{mes}\{t \in [0, T] :$$

$$|\bar{v}_1(t)^2 + \bar{v}_2(t)^2 - 1| > \epsilon^2\}$$

$$\leq \epsilon^{-4} \int_0^T (\bar{v}_1(t)^2 + \bar{v}_2^2 - 1)^2 dt$$

$$\leq \epsilon^{-4} c_2(|y|, |z|).$$

It means that for most $t \in [0, T]$, $\bar{v}(t)$ belongs to the ϵ-neighborhood of the set $\{x \in R^2 : |x| = 1\}$. Thus we can say that the integrand f has a weakened version of the turnpike property and the set $\{|x| = 1\}$ can be considered as the turnpike for f.

For a general autonomous nonconvex problem (P) we also have a version of the turnpike property in which a turnpike is a compact subset of R^n. This subset depends only on the integrand f.

Consider the following autonomous variational problem:

$$\int_0^T f(z(t), z'(t))dt \to \text{min},$$

$$z(0 = x,\ z(T) = y, \qquad (P_a)$$

$z : [0, T] \to R^n$ is an absolutely continuous function where $T > 0$, $x, y \in R^n$ and $f : R^{2n} \to R^1$ is an integrand.

We say that a time independent integrand $f = f(x, u) \in C(R^{2n})$ has the *turnpike property* if there exists a compact set $H(f) \subset R^n$ such that for each bounded set $K \subset R^n$ and each $\epsilon > 0$ there exist numbers $L_1 > L_2 > 0$ such that for each $T \geq 2L_1$, each $x, y \in K$ and an optimal solution $v : [0, T] \to R^n$ for the variational problem (P_a), the relation

$$\text{dist}(H(f), \{v(t) :$$

$$t \in [\tau, \tau + L_2]\}) \leq \epsilon$$

holds for each $\tau \in [L_1, T - L_1]$. (Here $\text{dist}(\cdot, \cdot)$ is the Hausdorff metric).

We also consider a weak version of this turnpike property for a time independent integrand $f(x,u)$. In this weak version, for an optimal solution of the problem (P_a) with $x, y \in R^n$ and large enough T, the relation

$$\text{dist}(H(f), \{v(t) :$$

$$t \in [\tau, \tau + L_2]\}) \leq \epsilon$$

with L_2, which depends on ϵ and $|x|, |y|$ and a compact set $H(f) \subset R^n$ depending only on the integrand f, holds for each $\tau \in [0,T] \setminus E$ where $E \subset [0,T]$ is a measurable subset such that the Lebesgue measure of E does not exceed a constant which depends on ϵ and on $|x|, |y|$.

These two turnpike properties for autonomous problems (P_a) are considered in Chapters 3-5.

In Chapter 3 we consider the space $\mathcal{A}$ of all time independent integrands $f \in \mathcal{M}$. We establish the existence of a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open everywhere dense sets in $\mathcal{A}$ such that for each $f \in \mathcal{F}$ the weakened version of the turnpike property holds.

The turnpike property for time independent integrands is established in Chapter 5 for a generic element of a subset $\mathcal{N}$ of the space $\mathcal{A}$. The space $\mathcal{N}$ is a subset of all integrands $f \in \mathcal{A}$ which satisfy some differentiability assumptions.

In the other chapters of the monograph we establish a number of turnpike results (generic and individual) for various classes of optimal control problems. We study optimal control of linear periodic systems with convex integrands (Chapter 6) and optimal solutions of linear systems with convex nonperiodic integrands (Chapter 7). In Chapter 8 we establish turnpike theorems for discrete-time control systems in Banach spaces and in complete metric spaces. Infinite-dimensional continuous-time optimal control problems in a Hilbert space are studied in Chapter 9. A turnpike theorem for a class of differential inclusions arising in economic dynamics is proved in Chapter 10 and structure of optimal trajectories of convex processes is studied in Chapter 11. In Chapter 12 we establish a turnpike property for a dynamic discrete-time zero-sum game.

The turnpike results mentioned above for variational problems were generalized in Structure of Approximate Solutions of Optimal Control Problems, Springer, 2013 by A. J. Zaslavski for optimal control problems.

Denote by $|\cdot|$ the Euclidean norm in the k-dimensional Euclidean space R^k. Let m, n be natural numbers.

We study a control system described by a differential equation

$$x'(t) = G(t, x(t), u(t)) \text{ a. e. } t \in \mathcal{I}, \qquad (1)$$

where $\mathcal{I}$ is either R^1 or $[T_1, \infty)$ or $[T_1, T_2]$ $(-\infty < T_1 < T_2 < \infty)$, and $x : \mathcal{I} \to R^n$ is an absolutely continuous (a. c.) function which satisfies

$$(t, x(t)) \in A \text{ for all } t \in \mathcal{I}, \qquad (2)$$

where A is a subset of R^{n+1}. The control function $u : \mathcal{I} \to R^m$ is Lebesgue measurable and satisfies the feedback control constraints

$$u(t) \in U(t, x(t)) \text{ a. e. } t \in \mathcal{I}, \qquad (3)$$

where $U : A \to 2^{R^m}$ is a point to set mapping with a graph

$$M = \{(t, x, u) : \ (t, x) \in A, \ u \in U(t, x)\}.$$

We suppose that M is a Borel measurable subset of R^{n+m+1} and that the function $G : M \to R^n$ is borelian.

For any $t \in R^1$ set

$$A(t) = \{x \in R^n : \ (t, x) \in A\}.$$

We assume that the set $A(t) \neq \emptyset$ for any $t \in R^1$.

The performance of the above control system is described by an integral functional

$$I^f(T_1, T_2, x, u) = \int_{T_1}^{T_2} f(t, x(t), u(t))dt,$$

where a borelian function $f : M \to R^1$ belongs to a complete metric space of functions $\mathfrak{M}$ defined below.

An a. c. function $x : \mathcal{I} \to R^n$, where $\mathcal{I}$ is either R^1 or $[T_1, \infty)$ or $[T_1, T_2]$ $(-\infty < T_1 < T_2 < \infty)$, will be called a *trajectory* if there exists a Lebesgue measurable function (referred to as a *control*) $u : \mathcal{I} \to R^m$ such that the pair (x, u) satisfies (1), (2), (3) and the function $t \to f(t, x(t), u(t))$ is locally Lebesgue integrable on $\mathcal{I}$.

For any $s \in R^1$ set $s_+ = \max\{s, 0\}$.

Let a_0 be a positive constant and let $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that

$$\psi(t) \to \infty \text{ as } t \to \infty.$$

Denote by $\mathfrak{M}$ the set of all borelian functions $f : M \to R^1$ which satisfy the following growth assumption:

(A)

$$f(t,x,u) \geq \max\{\psi(|x|),\ \psi(|u|),$$

$$\psi([|G(t,x,u)|-a_0|x|]_+)[|G(t,x,u)|-a_0|x|]_+\}-a_0$$

for each $(t,x,u) \in M$.

We equip the set $\mathfrak{M}$ with the uniformity which is determined by the following base:

$$E(N,\epsilon,\lambda) = \{(f,g) \in \mathfrak{M}\times\mathfrak{M} :\ |f(t,x,u)-g(t,x,u)|$$

$$\leq \epsilon \text{ for each } (t,x,u) \in M \text{ satisfying } |x|,|u| \leq N\}$$

$$\cap\{(f,g) \in \mathfrak{M}\times\mathfrak{M} :\ (|f(t,x,u)|+1)(|g(t,x,u)|+1)^{-1} \in$$

$$\text{for each } (t,x,u) \in M \text{ satisfying } |x| \leq N\},$$

where $N > 0$, $\epsilon > 0$ and $\lambda > 1$.

Clearly, the uniform space $\mathfrak{M}$ is Hausdorff and has a countable base. Therefore $\mathfrak{M}$ is metrizable. It is not difficult to show that the uniform space $\mathfrak{M}$ is complete.

We consider functionals of the form $I^f(T_1, T_2, x, u)$, where $f \in \mathfrak{M}$, $-\infty < T_1 < T_2 < \infty$ and $x : [T_1, T_2] \to R^n$, $u : [T_1, T_2] \to R^m$ is a trajectory-control pair.

For $f \in \mathfrak{M}$, a pair of numbers $T_1 \in R^1$, $T_2 > T_1$ and (T_1, y), $(T_2, z) \in A$ set

$$U^f(T_1, T_2, y, z) = \text{inf}\{I^f(T_1, T_2, x, u) :$$

$$x : [T_1, T_2] \to R^n, \; u : [T_1, T_2] \to R^m$$

is a trajectory-control pair satisfying

$$x(T_1) = y, \; x(T_2) = z\},$$

$$\sigma^f(T_1, T_2, y) = \text{inf}\{U^f(T_1, T_2, y, h) :$$

$$(T_2, h) \in A\}.$$

Here we assume that the infimum over empty set is ∞.

Denote by $\mathfrak{M}_{reg}$ the set of all functions $f \in \mathfrak{M}$ which satisfy the following assumption:

(B) there exist a trajectory-control pair

$$x_f : R^1 \to R^n,\ u_f : R^1 \to R^m$$

and a number $b_f > 0$ such that:

(i)

$$U^f(T_1, T_2, x_f(T_1), x_f(T_2)) = I^f(T_1, T_2, x_f, u_f)$$

for each $T_1 \in R^1$ and each $T_2 > T_1$;

(ii)

$\sup\{I^f(j, j+1, x_f, u_f) : j = 0, \pm 1, \pm 2, \dots\} < \infty$;

(iii) for each $S_1 > 0$ there exist $S_2 > 0$ and an integer $c > 0$ such that

$$I^f(T_1, T_2, x_f, u_f) \leq I^f(T_1, T_2, x, u) + S_2$$

for each $T_1 \in R^1$, each $T_2 \geq T_1 + c$ and each trajectory-control pair $x : [T_1, T_2] \to R^n$, $u : [T_1, T_2] \to R^m$ which satisfies $|x(T_1)|$, $|x(T_2)| \leq S_1$;

(iv) for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $(T, z) \in A$ which satisfies

$$|z - x_f(T)| \le \delta$$

there are

$$\tau_1 \in (T, T + b_f] \text{ and } \tau_2 \in [T - b_f, T),$$

and trajectory-control pairs

$$x_1 : [T, \tau_1] \to R^n,\ u_1 : [T, \tau_1] \to R^m,$$

$$x_2 : [\tau_2, T] \to R^n,\ u_2 : [\tau_2, T] \to R^m$$

which satisfy

$$x_1(T) = x_2(T) = z,$$

$$x_i(\tau_i) = x_f(\tau_i),\ i = 1, 2,$$

$$|x_1(t) - x_f(t)| \le \epsilon \text{ for all } t \in [T, \tau_1],$$

$$|x_2(t) - x_f(t)| \le \epsilon \text{ for all } t \in [\tau_2, T],$$

$$I^f(T, \tau_1, x_1, u_1) \le I^f(T, \tau_1, x_f, u_f) + \epsilon,$$

$$I^f(\tau_2, T, x_2, u_2) \le I^f(\tau_2, T, x_f, u_f) + \epsilon.$$

Note that assumption (B) means that the trajectory-control pair

$$x_f : R^1 \to R^n,\ u_f : R^1 \to R^m$$

is a solution of the corresponding infinite horizon optimal control problem associated with the integrand f and that certain controllability properties hold near this trajectory-control pair.

Theorem 1 *1. Let $f \in \mathfrak{M}_{reg}$ and $S_0 > 0$. Then there exists $S > 0$ such that for each pair of real numbers $T_1 < T_2$ and each trajectory-control pair*

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

which satisfies $|x(T_1)| \leq S_0$ the following inequality holds:

$$I^f(T_1, T_2, x_f, u_f) \leq I^f(T_1, T_2, x, u) + S.$$

2. Let $f \in \mathfrak{M}_{reg}$. Then for each $s \in R^1$ and each trajectory-control pair

$$x : [s, \infty) \to R^n,\ u : [s, \infty) \to R^m$$

one of the following relations holds:

(a)

$$I^f(s, t, x, u) - I^f(s, t, x_f, u_f) \to \infty \text{ as } t \to \infty;$$

(b)

$$sup\{|I^f(s,t,x_f,u_f) - I^f(s,t,x,u)| : \ t \in (s,\infty)\} < \infty.$$

Moreover, if the relation (b) holds, then

$$\sup\{|x(t)| : \ t \in [s,\infty)\} < \infty.$$

For each $f \in \mathfrak{M}_{reg}$ and each $r > 0$ we define a function $f_r \in \mathfrak{M}$ by

$$f_r(t,x,u) = f(t,x,u) + r \min\{|x - x_f(t)|,\ 1\}$$

$$\text{for all } (t,x,u) \in M.$$

It is easy to see that $f_r \in \mathfrak{M}_{reg}$ for each $f \in \mathfrak{M}_{reg}$ and each $r > 0$.

Let $\mathfrak{A}$ be a subset of $\mathfrak{M}_{reg}$ such that $f_r \in \mathfrak{A}$ for each $f \in \mathfrak{A}$ and each $r \in (0,1)$. Denote by $\bar{\mathfrak{A}}$ the closure of $\mathfrak{A}$ in the uniform space $\mathfrak{M}$ and consider the topological subspace $\bar{\mathfrak{A}} \subset \mathfrak{M}$ with the relative topology.

There exists a set $\mathcal{F} \subset \bar{\mathfrak{A}}$ which is a countable intersection of open everywhere dense sets in $\bar{\mathfrak{A}}$ and such the following theorems hold.

Theorem 2 *For each $f \in \mathcal{F}$ and each $S > 0$ there exist a neighborhood $\mathcal{U}$ of f in $\mathfrak{M}$ and positive numbers δ, Q such that the following assertions hold:*

$\inf\{U^g(T_1, T_2, y_1, y_2) : (T_i, y_i) \in A, \ i = 1, 2\} < \infty$ *for each $g \in \mathcal{U}$, each $T_1 \in R^1$ and each $T_2 > T_1$;*

for each $g \in \mathcal{U}$, each $T_1 \in R^1$, each $T_2 \geq T_1 + 1$ and each trajectory-control pair

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

which satisfies

$$I^g(T_1, T_2, x, u)$$

$$\leq \inf\{U^g(T_1, T_2, y_1, y_2) : \ (T_i, y_i) \in A, \ i = 1, 2\} + S$$

and

$$I^g(T_1, T_2, x, u) \leq U^g(T_1, T_2, x(T_1), x(T_2)) + \delta$$

the following inequality holds:

$$|x(t)| \leq Q \text{ for all } t \in [T_1, T_2].$$

Theorem 2 establishes uniform boundedness of approximate solutions of optimal control problems.

The next theorem is our first turnpike result.

Theorem 3 *Let $f \in \mathcal{F}$. Then there exists a bounded continuous function $X_f : R^1 \to R^n$ such that the following property holds.*

For each $S, \epsilon > 0$ there exist a neighborhood $\mathcal{U}$ of f in $\mathfrak{M}$ and real numbers $\Delta > 0$, $\delta \in (0, \epsilon)$ such that for each $g \in \mathcal{U}$, each $T_1 \in R^1$, each $T_2 \geq T_1 + 2\Delta$ and each trajectory-control pair

$x : [T_1, T_2] \to R^n$, $u : [T_1, T_2] \to R^m$ *satisfying*

$$I^g(T_1, T_2, x, u) \leq \inf\{U^g(T_1, T_2, y_1, y_2) :$$

$$(T_i, y_i) \in A,\ i = 1, 2\} + S,$$

$$I^g(T_1, T_2, x, u) \leq U^g(T_1, T_2, x(T_1), x(T_2)) + \delta$$

the following inequality holds:

$|x(t) - X_f(t)| \leq \epsilon$ *for all* $t \in [T_1 + \Delta, T_2 - \Delta]$.

Moreover, if $|x(T_1) - X_f(T_1)| \leq \delta$, *then*

$|x(t) - X_f(t)| \leq \epsilon$ *for all* $t \in [T_1, T_2 - \Delta]$

and if $|x(T_2) - X_f(T_2)| \leq \delta$*, then*

$$|x(t) - X_f(t)| \leq \epsilon \textit{ for all } t \in [T_1 + \Delta, T_2].$$

The next theorem is our second turnpike result.

Theorem 4 *Let* $f \in \mathcal{F}$*, let a bounded continuous function* $X_f : R^1 \to R^n$ *be as guaranteed by Theorem 3 and let* ϵ, M *be a pair of positive numbers. Then there exist a neighborhood* $\mathcal{U}$ *of* f *in* $\mathfrak{M}$*, real numbers* $l > 0$*,* $L > 0$ *and a natural number* p *such that for each* $g \in \mathcal{U}$*, each* $T_1 \in R^1$*, each* $T_2 \geq T_1 + L$ *and each trajectory-control pair* $x : [T_1, T_2] \to R^n$*,* $u : [T_1, T_2] \to R^m$ *which satisfies*

$$I^g(T_1, T_2, x, u) \leq \inf\{U^g(T_1, T_2, y_1, y_2) :$$

$$(T_i, y_i) \in A,\ i = 1, 2\} + M$$

there exist finite sequences

$$\{a_i\}_{i=1}^q,\ \{b_i\}_{i=1}^q \subset [T_1, T_2],$$

where $q \leq p$ *is a natural number, such that*

$$a_i \leq b_i \leq a_i + l \text{ for all integers } i = 1, \ldots, q$$

and

$$|x(t) - X_f(t)| \leq \epsilon \text{ for all } t \in [T_1, T_2] \setminus \cup_{i=1}^q [a_i, b_i].$$